1 Arbitrary Order, Arbitrary Dimensional Polynomial Solve With Smoothing

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Below I outline the mathematical foundation for higher order polynomial solving.

1.1 Generalised Indexing and Notation

The polynomial solve is a quite straightforward consequence of a simultaneous equation solve. The only really tricky thing is one of notation. For a polynomial in higher dimensional space, we have to be quite careful how things are indexed. For example, consider the generalised quadratic polynomial in two dimensions

$$y = a_{00} + a_{10}x_0 + a_{01}x_1 + a_{11}x_0x_1 + a_{20}x_0^2 + a_{02}x_1^2 \tag{1}$$

The polynomial coefficients a_{jk} have been indexed by the power on the corresponding products of x_i. Explicitly,

$$y = a_{00}x_0^0x_1^0 + a_{10}x_0^1x_1^0 + a_{01}x_0^0x_1^1 + a_{11}x_0^1x_1^1 + a_{20}x_0^2x_1^0 + a_{02}x_0^0x_1^2 \tag{2}$$

which is identical to equation 1. In the code, this is termed *index by power*. Occasionally we also use the concept of *index by vector*. In this indexing scheme, we index the i in the product of x_i, so for example the quadratic equation above becomes

$$y = b + b_0x_0 + b_1x_1 + b_{01}x_0x_1 + b_{00}x_0x_0 + b_{11}x_1x_1 \tag{3}$$

Conventionally, a quadratic equation in higher dimension is one in which the sum of the powers <= 2, i.e. the sum of *index by power* <= 2 and the length of *index by vector* <= 2.

For reasons discussed below, in this note a polynomial of order i will be one in which no term in *index by power* is more than i, or equivalently no integer in *index by vector* is repeated more than i times.

The coefficients $a_{00}, a_{01}, \ldots$ can be written in shorthand as $a_{\vec{p}}$ where $\vec{p}$ is a vector of integers as discussed above.

This notation can also be used to extend to derivatives. Consider the derivative

$$\frac{\partial^n y_i}{\partial x_0^{q_0}\partial x_1^{q_1}\ldots} \tag{4}$$

which can be written in shorthand as

$$y^{(\vec{q})} \tag{5}$$

1.2 Polynomial Solve with no Smoothing

Consider a polynomial in variables $\vec{x} = x_0, x_1, \ldots x_n$ such that

$$\vec{y}(x) = \sum_{\vec{p}} a_{\vec{p}} \prod_{j=1}^{j=n} x_j^{p_j} \tag{6}$$

If we have a set of known values $\vec{y}_{\vec{b}}(\vec{x}_{\vec{b}}$ at some known positions $\vec{x}_{\vec{b}}$ e.g. on a rectangular grid; then we can solve for $a_{\vec{p}}$ in the usual way as a linear system of simultaneous equations.

Then

$$\vec{y}_{\vec{b}}(\vec{x}_{\vec{b}}) = \sum_{\vec{p}} a_{\vec{p}} \prod_{j=1}^{j=n} x_{b_j}^{p_j} \tag{7}$$

where the measured values to be fitted, $\vec{y}_{\vec{b}}$ are known, the positions at which those values were measured, $\vec{x}$, are known but the polynomial coefficients $a_{\vec{p}}$ are not known.

This can be written as a system of linear equations like

$$\vec{G} = \mathbf{H}\vec{A} \tag{8}$$

where $\vec{G}$ is the vector of $\vec{y}_{\vec{b}}$, $\vec{a}$ is the vector of $a_{\vec{p}}$ and $\mathbf{H}$ is the matrix of $\prod x_{b_j}^{p_j}$.

In order for the polynomial fit to be successful, $\mathbf{H}$ must be invertible, so the points $\vec{x}_{\vec{b}}$ need to be chosen carefully. Otherwise no presumption has been made on the dimension or order of the fit.

1.3 Polynomial Solve with Smoothing

Derivatives of the polynomial may be known, in which case they can be used in addition to measured points.

Consider the measured derivatives $\vec{y}_{\vec{b}}^{(\vec{q})}(\vec{x}_{\vec{b}})$. Then

$$y_{\vec{b}}^{\vec{q}} = \sum_{\vec{p}} a_{\vec{p}} \prod^{j} \frac{p_j!}{(p_j - q_j)!} \vec{x}_{\vec{b}}^{(p_j - q_j)} \tag{9}$$

The simultaneous equation can then be reformulated as

$$\vec{G'} = \mathbf{H'}\vec{A} \tag{10}$$

where $\vec{G'}$ is the vector of $y^{\vec{p}}$ and $y^{(\vec{q})}$; $\mathbf{H'}$ is the matrix of $\prod x_{b_j}^{p_j}$ and $\prod p_j!/(p_j - q_j)!x_{b_j}^{p_j - q_j}$

In order for the polynomial fit to be successful, $\mathbf{H'}$ must be invertible.