Practice Problem Set 1

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1 Exponential Models

900 grams of a radioactive material is brought into a lab. The amount remaining after 4 days was 4/9 of what remained after 2 days. Let $\mathrm{N}(t)$ be the number of grams of radioactive material.

(a) Find the number, in grams, of the remaining material t years after being brought into the lab.

(b) How many grams remain after 2 days?

(c) What is the half-life of the radioactive material?

(d) How many days will it takes for only 1 gram to remain?

1.1 (a)

The basic exponential model looks like this:

$$\mathrm{f}(t) = br^t$$

where b is the starting value, r is the rate of change, and t is time.

If $t_1 = 4$ days and $t_2 = 2$ days, in which the population at t_1 is $\frac{4}{9}$ of the population at t_2, we can set up the following equation:

$$\mathrm{N}(t_1) = \frac{4}{9}\mathrm{N}(t_2) \tag{1.1.1}$$

$$\mathrm{N}(4) = \frac{4}{9}\mathrm{N}(2) \tag{1.1.2}$$

$$\mathrm{N}(4) = \frac{4}{9}\mathrm{N}(2) \tag{1.1.3}$$

$$br^4 = \frac{4}{9}br^2 \tag{1.1.4}$$

$$900r^4 = \frac{4}{9}900r^2 \qquad \text{We know that } b = 900\text{, however } b \text{ will cancel out.} \tag{1.1.5}$$

$$\Rightarrow r^4 = \frac{4}{9}r^2 \qquad \text{We can take the square root of both sides:} \tag{1.1.6}$$

$$\sqrt{r^4} = \sqrt{\frac{4}{9}r^2} \Rightarrow r^2 = \frac{2}{3}r \qquad \text{Divide by } r. \tag{1.1.7}$$

$$\Rightarrow \boxed{r = \frac{2}{3}} \tag{1.1.8}$$

Thus, our function $\mathrm{N}(t)$ is:

$$\mathrm{N}(t) = 900\left(\frac{2}{3}\right)^t \tag{1.1.9}$$

If we plug in $t = 2$:

$$\mathrm{N}(2) = 900\left(\frac{2}{3}\right)^2 = 900 \cdot \frac{4}{9} = 400$$

and if we plug in $t = 4$:

$$\mathrm{N}(4) = 900\left(\frac{2}{3}\right)^4 = 900 \cdot \frac{16}{81} = \frac{1600}{9}$$

Using a calculator, dividing $\frac{1600}{9}$ by 400 should gives us $\frac{4}{9}$:

$$\frac{\frac{1600}{9}}{400} = \frac{1600}{9 \cdot 400} = \frac{1600}{3600} = \frac{16}{36} = \frac{4}{9}$$

Now that we our function, and since t is in terms of days, we can multiply t by 365 (the average number of days per year) to get our answer for part (a):

$$\boxed{\mathrm{N}(t) = 900\left(\frac{2}{3}\right)^{365t}}$$

1.2 (b)

We can substitue 2 for t to get our answer:

$$\mathrm{N}(2) = 900\left(\frac{2}{3}\right)^2 = 900 \cdot \frac{4}{9} = \boxed{400\,\mathrm{g}} \tag{1.2.1}$$

Thus, **400 grams** of material remain after 2 days.

1.3 (c)

The half-life is basically how much time it takes for the population to reach half of its original size, thus we are just solving for time t when $\mathrm{N}(t) = \frac{1}{2}b$:

$$\mathrm{N}(t) = \frac{1}{2}900 = 450 \tag{1.3.1}$$

$$900\left(\frac{2}{3}\right)^t = 450 \tag{1.3.2}$$

$$\Rightarrow \left(\frac{2}{3}\right)^t = \frac{45}{90} = \frac{1}{2} \tag{1.3.3}$$

We can rewrite this using log of base $\frac{2}{3}$ since:

$$a^x = b \Rightarrow \log_a(b) = x$$

$$\log_{2/3}(\frac{1}{2}) = t \tag{1.3.4}$$

$$\log_{2/3}(1) - \log_{2/3}(2) = t \tag{1.3.5}$$

$$0 - \log_{2/3}(2) = t \tag{1.3.6}$$

$$-\log_{2/3}(2) = t \tag{1.3.7}$$

Using a calculator, we find:

$$\boxed{t \approx 1.71\,\mathrm{days}} \tag{1.3.8}$$

1.4 (d)

Now similar to part (c), we need to solve for time t, but this time:

$$\mathrm{N}(t) = 1$$

so to solve for t:

$$900\left(\frac{2}{3}\right)^t = 1 \tag{1.4.1}$$

$$\Rightarrow \left(\frac{2}{3}\right)^t = \frac{1}{900} \tag{1.4.2}$$

$$\log_{2/3}(1/900) = t \tag{1.4.3}$$

$$\log_{2/3}(1) - log_{2/3}(900) = t \tag{1.4.4}$$

$$-\,log_{2/3}(900) = t \tag{1.4.5}$$

$$\boxed{t \approx 16.78\,\mathrm{days}} \tag{1.4.6}$$

2 Separable Differential Equations

Find the explicit solution to these differential equations that satisfy the given initial condition:

(a)

$$(1+x^2)y' = \frac{1}{y^2}; \quad y(0) = 2$$

$$\begin{aligned}
(1+x^2)y' = \frac{1}{y^2} &\implies (1+x^2)\frac{dy}{dx} = \frac{1}{y^2} && (2.1.1)\\
&\implies (1+x^2)dy = \frac{1}{y^2}dx\\
&\implies y^2(1+x^2)dy = dx\\
&\implies y^2\,dy = \frac{1}{1+x^2}\,dx && (2.1.2)
\end{aligned}$$

Now to integrate both sides:

$$\int y^2\,dy = \int \frac{1}{1+x^2}\,dx \implies \frac{y^3}{3} + C = \arctan(x) + C \tag{2.1.3}$$

We will assume that there is only one C:

$$\begin{aligned}
&\implies \frac{y^3}{3} = \arctan(x) + C\\
&\implies y^3 = 3(\arctan(x) + C)\\
&\implies y = \sqrt[3]{3(\arctan(x) + C)} && (2.1.4)
\end{aligned}$$

Now to solve for C such that $y(0) = 2$, in which $x = 0$:

$$\begin{aligned}
2 = \sqrt[3]{3(\arctan(0) + C)} &\implies 2^3 = 3(\arctan(2) + C) && (2.1.5)\\
&\implies 8 = 3(\arctan(0) + C)\\
&\implies \frac{8}{3} = \arctan(0) + C\\
&\implies \frac{8}{3} = C && (2.1.6)
\end{aligned}$$

Thus, our equation is:

$$\boxed{y = \sqrt[3]{3\arctan(x) + \frac{8}{3}}}$$

(b)

$$x\,dy - (y + \sqrt{y})dx = 0\,; \quad y(1) = 1$$

$$\begin{aligned} & x\,dy - (y + \sqrt{y})dx = 0 && (2.2.1)\\ \Rightarrow\ & x\,dy = (y + \sqrt{y})dx && (2.2.2)\\ \Rightarrow\ & x\frac{dy}{dx} = y + \sqrt{y} && (2.2.3)\\ \Rightarrow\ & \frac{dy}{dx} = \frac{y + \sqrt{y}}{x} && (2.2.4)\\ \Rightarrow\ & \frac{dy}{dx}\left(\frac{1}{y + \sqrt{y}}\right) = \frac{1}{x} && (2.2.5)\\ \Rightarrow\ & \frac{1}{y + \sqrt{y}}\,dy = \frac{1}{x}\,dx && (2.2.6)\\ \Rightarrow\ & \int \frac{1}{y + \sqrt{y}}\,dy = \int \frac{1}{x}\,dx && (2.2.7)\\ \Rightarrow\ & \int \frac{1}{\sqrt{y}(\sqrt{y} + 1)}\,dy = \int \frac{1}{x}\,dx && (2.2.8) \end{aligned}$$

Integrating the left side:

$$\begin{aligned} & u = \sqrt{y} + 1 \quad du = \frac{1}{2\sqrt{y}}dx \to dx = 2\sqrt{y}\,du\\ \Rightarrow\ & \int \frac{1}{\sqrt{y}(u)}\,2\sqrt{y}\,du\\ \Rightarrow\ & 2\int \frac{1}{u}\,du\\ \Rightarrow\ & 2\,(\ln|u|) + C\\ \Rightarrow\ & 2\,(\ln|\sqrt{y} + 1|) + C \end{aligned}$$

Integrating right side:

$$\int \frac{1}{x} = \ln|x| + C$$

Returning to equation 2.2.8:

$$\Rightarrow 2\ln|\sqrt{y} + 1| + C = \ln|x| + C \qquad (2.2.9)$$

We will get rid of the C on the left side since it is an arbitrary constant, thus we will assume there is only one constant:

$$\Rightarrow 2\ln|\sqrt{y} + 1| = \ln|x| + C \qquad (2.2.10)$$

Raise e to the power of both sides in order to cancel out ln:

$$\Rightarrow e^{2\ln\left|\sqrt{y}+1\right|} = e^{\ln|x|+C} \Rightarrow e^{2\ln\left|\sqrt{y}+1\right|} = e^{\ln|x|} \cdot e^{C} \tag{2.2.11}$$

$$\Rightarrow (\sqrt{y}+1)^2 = x \cdot e^C$$

$$\Rightarrow \sqrt{y}+1 = \sqrt{x \cdot e^C}$$

$$\Rightarrow \sqrt{y} = \sqrt{x \cdot e^C} - 1$$

$$\Rightarrow y = xe^C - 2\sqrt{xe^C} + 1 \tag{2.2.12}$$

Now to solve for C such that the condition $y(1) = 1$ is satisfied:

$$1 = (1) \cdot e^C - 2\sqrt{(1)e^C} + 1 \tag{2.2.13}$$

$$\Rightarrow 1 = e^C - 2\sqrt{e^C} + 1$$

$$\Rightarrow 0 = e^C - 2\sqrt{e^C} \;\Rightarrow\; 2\sqrt{e^C} = e^C \tag{2.2.14}$$

$$\Rightarrow 4e^C = \left(e^C\right)^2 \;\Rightarrow\; 4e^C = e^{2C} \tag{2.2.15}$$

$$\Rightarrow 4 = e^C \;\Rightarrow\; \log_e(4) = C \tag{2.2.16}$$

$$\Rightarrow \boxed{C = \ln(4)} \tag{2.2.17}$$

Plugging this back into our original equation 2.2.12:

$$y = xe^{\ln(4)} - 2\sqrt{xe^{\ln(4)}} + 1$$

$$e^{\ln(a)} = a$$

$$y = 4x - 2\sqrt{4x} + 1 = 4x - 4\sqrt{x} + 1$$

Thus, our answer is:

$$\boxed{y = 4x - 4\sqrt{x} + 1} \tag{2.2.18}$$