Exponential Change and Separable Differential Equations

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1 Exponential Change

Let y be the size of a population at time t.

$$\mathrm{y}(t)$$

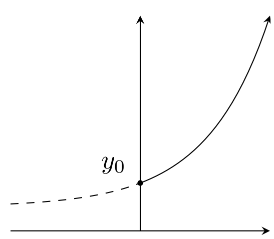
If rate of change of y is proportional to its size y; then

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky\,,\ k > 0 \text{ and the population is increasing.}$$

$$\Rightarrow\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t} = k\ \Rightarrow\ \frac{1}{y}\mathrm{d}y\ \Rightarrow\ \int \frac{1}{y}\mathrm{d}y = \int k\mathrm{d}t\ \Rightarrow\ \ln|y| = kt + C$$

$$\ln|y| = kt + C \to e^{\ln y} = e^{kt+C} = e^{c}\cdot e^{kt} = Ce^{kt}$$

$$|y| = Ce^{kt}\ \Rightarrow \boxed{y = \pm Ce^{kt}}$$



$$y(0) = y_0$$

2 Separable Differential Equations

Let $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x,y)$ where $y = \mathrm{y}(t)$.

f(x,y) is separable if f(x,y) can be expressed as $\mathrm{g}(x)\cdot\mathrm{h}(y)$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x,y) = \mathrm{g}(x)\cdot\mathrm{h}(y) \qquad \text{Multiply both sides by } \mathrm{d}x \tag{1}$$

$$\mathrm{d}y = \mathrm{f}(x,y)\mathrm{d}x = \mathrm{g}(x)\cdot\mathrm{h}(y)\cdot\mathrm{d}x \tag{2}$$

$$\mathrm{d}y = \mathrm{g}(x)\cdot\mathrm{h}(y)\mathrm{d}x \tag{3}$$

$$\frac{1}{\mathrm{h}(y)}\mathrm{d}y = \mathrm{g}(x)\mathrm{d}x \tag{4}$$

which gives us:

$$\int \frac{1}{\mathrm{h}(y)}\mathrm{d}y = \int \mathrm{g}(x)\mathrm{d}x$$

2.1 Examples

#9)

$$2\sqrt{xy}\frac{\mathrm{d}y}{\mathrm{d}x} = 1\,;\ x,\ y > 0$$

#20)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy + 3x - 2y - 6$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x(y+3) - 2(y+3) \tag{5.1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x-2)(y+3) \tag{5.2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}\cdot\mathrm{d}x = (x-2)(y+3)\cdot\mathrm{d}x \tag{5.3}$$

$$dy = (x-2)(y+3)\cdot\mathrm{d}x \tag{5.4}$$

$$\frac{1}{y+3}dy = (x-2)\cdot\mathrm{d}x \tag{5.5}$$

$$\int \frac{1}{y+3}\,dy = \int (x-2)\,\mathrm{d}x \tag{5.6}$$

$$\boxed{\ln|y+3| = \frac{1}{2}(x-2)^2 + C} \tag{5.7}$$

#16)

$$\sec x \frac{\mathrm{d}y}{\mathrm{d}x} = e^{y+\sin x}$$

$$\frac{1}{\cos x}\frac{\mathrm{d}y}{\mathrm{d}x} = e^{y+\sin x} \tag{5.8}$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \cos x \cdot e^{y+\sin x} \tag{5.9}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos x(e^{y})(e^{\sin x}) \tag{5.10}$$

$$\Rightarrow \frac{1}{e^{y}}\frac{\mathrm{d}y}{\mathrm{d}x} = \cos x(e^{\sin x}) \tag{5.11}$$

$$\frac{1}{e^{y}}\frac{\mathrm{d}y}{\mathrm{d}x} \cdot \mathrm{d}x = \cos x(e^{\sin x})\mathrm{d}x \tag{5.12}$$

$$\frac{1}{e^{y}}\, dy = \cos x(e^{\sin x})\mathrm{d}x \tag{5.13}$$

$$\int \frac{1}{e^{y}}\, dy = \int \cos x(e^{\sin x})\,\mathrm{d}x \tag{5.14}$$

$$\int e^{-y}\, dy = \int (e^{\sin x}) \cos x \,\mathrm{d}x \tag{5.15}$$

$$u = -y \;\; du = -dy \to -du = dy \;\Big|\; u = \sin x \;\; du = \cos x dx \tag{5.16}$$

$$\int e^{u}\, du = \int e^{u}\, du \tag{5.17}$$

$$-e^{u} = e^{u} + k \tag{5.18}$$

$$-e^{-y} = e^{\sin x} + k \tag{5.19}$$

$$\tag{5.20}$$

\# Show that each function $y = f(x)$ is a solution of the indicated DE.

$$2y' + 3y = e^{-x}$$

$$a) y = e^{-x}$$
$$b) y = e^{-x} + e^{\frac{-3}{2}x}$$

Example In a certain region, the population, $P(t)$, in thousand of people t year after census begin is approximated using an exponential growth model. The initial census showed $P_0 = 90$ and the population 2 years later was 120.

a) Find a formula for $P(t)$.

$$P(t) = P_0 e^{kt} = P_0 (e^k)^t \tag{6.1}$$
$$P(0) = 90,000 \tag{6.2}$$
$$\Rightarrow P(t) = 90,000 e^{kt} \tag{6.3}$$
$$\text{Two years later } t = 2 \; P(2) = 120,000 \tag{6.4}$$
$$120,000 = 90,000 e^{kt} \tag{6.5}$$
$$12 = 9 e^{k \cdot 2} \tag{6.6}$$
$$\frac{12}{9} = (e^k)^2 \tag{6.7}$$
$$\sqrt{\frac{12}{9}} = (e^k) \tag{6.8}$$
$$e^k = \frac{2}{\sqrt{3}} \tag{6.9}$$
$$P(t) = 90,000(\frac{2}{\sqrt{3}})^t \tag{6.10}$$

a) Find the population after 4 years $\Rightarrow P(4) = 90,000(\frac{2}{\sqrt{3}})^4$

$$P(4) = 90,000((\frac{2}{\sqrt{3}})^2)^2 \tag{6.11}$$
$$P(4) = 90,000(\frac{4}{3})^2 \tag{6.12}$$
$$P(4) = 90,000(\frac{16}{9}) \tag{6.13}$$
$$P(4) = 10,000(16) = 160,000 \tag{6.14}$$
$$\tag{6.15}$$

a)