Exponential Change and Separable Differential Equations

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1 Exponential Change

Let y be the size of a population at time t.

$$\mathrm{y}(t)$$

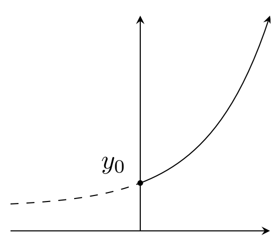
If rate of change of y is proportional to its size y; then

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky\,,\ k > 0 \text{ and the population is increasing.}$$

$$\Rightarrow\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t} = k\ \Rightarrow\ \frac{1}{y}\mathrm{d}y\ \Rightarrow\ \int \frac{1}{y}\mathrm{d}y = \int k\mathrm{d}t\ \Rightarrow\ \ln|y| = kt + C$$

$$\ln|y| = kt + C \to e^{\ln y} = e^{kt+C} = e^{c}\cdot e^{kt} = Ce^{kt}$$

$$|y| = Ce^{kt}\ \Rightarrow \boxed{y = \pm Ce^{kt}}$$



$$y(0) = y_0$$

2 Separable Differential Equations

Let $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x,y)$ where $y = \mathrm{y}(t)$.
$\mathrm{f}(x,y)$ is separable if $\mathrm{f}(x,y)$ can be expressed as $\mathrm{g}(x) \cdot \mathrm{h}(y)$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x,y) = \mathrm{g}(x) \cdot \mathrm{h}(y) \qquad \text{Multiply both sides by } \mathrm{d}x \tag{1}$$

$$\mathrm{d}y = \mathrm{f}(x,y)\mathrm{d}x = \mathrm{g}(x) \cdot \mathrm{h}(y) \cdot \mathrm{d}x \tag{2}$$

$$\mathrm{d}y = \mathrm{g}(x) \cdot \mathrm{h}(y)\mathrm{d}x \tag{3}$$

$$\frac{1}{\mathrm{h}(y)}\mathrm{d}y = \mathrm{g}(x)\mathrm{d}x \tag{4}$$

which gives us:

$$\int \frac{1}{\mathrm{h}(y)}\mathrm{d}y = \int \mathrm{g}(x)\mathrm{d}x$$