Integral Equations

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1 What is an Integral Equation?

When integrating by parts, there are some instances where the integral does not cancel out, or in other words, neither du nor dv approach 0. In order to solve such an integral, we need to set the integral equal to a variable. I will be using I to represent our integrals.

2 Examples

Let's evaluate some integrals:

2.1 Integral A

$$I = \int e^{2x} \cos(3x)\, dx \tag{A}$$

We will use the following substitutions:

$$\begin{aligned} u &= e^{2x} & du &= 2e^{2x} \\ v &= \int \cos(3x) = \frac{1}{3}\sin(3x) & dv &= \cos(3x) \end{aligned}$$

$$\int e^{2x} \cos(3x)\, dx = e^{2x}\frac{1}{3}\sin(3x) - \int 2e^{2x}\frac{1}{3}\sin(3x) \tag{1a}$$

$$= \frac{1}{3}e^{2x}\sin(3x) - \frac{2}{3}\int e^{2x}\sin(3x) \tag{1b}$$

Evaluating $\int e^{2x} \sin(3x)\, dx$:

$$\begin{aligned} u &= e^{2x} & du &= 2e^{2x} \\ v &= \int \sin(3x) = \frac{1}{3}\cos(3x) & dv &= \sin(3x) \end{aligned}$$

$$\begin{aligned} \int e^{2x} \sin(3x)\, dx &= e^{2x}\frac{1}{3}\cos(3x) - \int 2e^{2x}\frac{1}{3}\cos(3x)\, dx \\ &= \frac{1}{3}e^{2x}\cos(3x) - \frac{2}{3}\int e^{2x}\cos(3x)\, dx \end{aligned}$$

Notice that we have $\int e^{2x} \cos(3x)\, dx$, which is the integral we started with, I. Doing integration by parts again would only take us back to I, this we will substitue our integral in this expression with I.

$$I = \frac{1}{3}e^{2x}\sin(3x) - \frac{2}{3}\left[\frac{1}{3}e^{2x}\cos(3x) - \frac{2}{3}I\right]$$

$$= \frac{1}{3}e^{2x}\sin(3x) - \frac{2}{9}e^{2x}\cos(3x) + \frac{4}{9}I \tag{1c}$$

$$\Rightarrow -\frac{5}{9}I = \frac{1}{3}e^{2x}\sin(3x) - \frac{2}{9}e^{2x}\cos(3x) \tag{1d}$$

$$\Rightarrow I = -\frac{9}{5}\left[\frac{1}{3}e^{2x}\sin(3x) - \frac{2}{9}e^{2x}\cos(3x)\right] \tag{1e}$$

Thus:

$$I = \int e^{2x} \cos(3x)\, dx \tag{1f}$$

$$I = -\frac{9}{5}\left[\frac{1}{3}e^{2x}\sin(3x) - \frac{2}{9}e^{2x}\cos(3x)\right] \tag{1g}$$

$$\boxed{\int e^{2x}\cos(3x) = -\frac{9}{5}\left[\frac{1}{3}e^{2x}\sin(3x) - \frac{2}{9}e^{2x}\cos(3x)\right]} \tag{1h}$$