Trigonometric Integrals and Identities

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1 Useful Identities

Here are some useful identities:

$$(1) \quad \csc^2(x) - \cot^2(x) = 1$$

$$(2) \quad \sec^2(x) - \tan^2(x) = 1$$

$$(3) \quad \cos^2(x) + \sin^2(x) = 1$$

$$(4) \quad \sin(x \pm y) = \sin(x)\cos(y) \pm \sin(y)\cos(x)$$

$$(5) \quad \cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

$$(6) \quad \cos(2x) = \cos^2(x) - \sin^2(x)$$

$$(7) \quad \cos^2(x) = \frac{1 + \cos(2x)}{2}$$

$$(8) \quad \sin^2(x) = \frac{1 - \cos(2x)}{2}$$

1.1 Derivations

I will only be deriving equations 6, 7, and 8:

1.1.1 Equation 6

Equation 6 can be derived from equation 5:

$$\cos(x + x) = \cos(x)\cos(x) - \sin(x)\sin(x) \tag{1a}$$
$$\cos(2x) = \cos^2(x) - \sin^2(x) \tag{1b}$$

$$\boxed{\cos(2x) = \cos^2(x) - \sin^2(x)}$$

1.1.2 Equation 7

Equation 7 can be derived from equation 6:

$$\cos(2x) = \cos^2(x) - \sin^2(x) \tag{2a}$$

Using equation 3, we find that $\sin^2(x) = 1 - \cos^2(x)$:

$$\Rightarrow \cos(2x) = \cos^2(x) - (1 - \cos^2(x)) \tag{2b}$$

$$\Rightarrow \cos(2x) = \cos^2(x) - 1 + \cos^2(x) \tag{2c}$$

$$= 2\cos^2(x) - 1$$

$$\Rightarrow \cos(2x) + 1 = 2\cos^2(x) \tag{2d}$$

$$\Rightarrow \frac{\cos(2x) + 1}{2} = \cos^2(x) \tag{2e}$$

$$\boxed{\cos^2(x) = \frac{1 + \cos(2x)}{2}}$$

1.1.3 Equation 8

Equation 8 can be derived from equation 6:

$$\cos(2x) = \cos^2(x) - \sin^2(x) \tag{3a}$$

Using equation 3, we find that $\cos^2(x) = 1 - \sin^2(x)$:

$$\Rightarrow \cos(2x) = (1 - \sin^2(x)) - \sin^2(x) \tag{3b}$$

$$\Rightarrow \cos(2x) = 1 - 2\sin^2(x) \tag{3c}$$

$$\Rightarrow \cos(2x) - 1 = -2\sin^2(x) \tag{3d}$$

$$\Rightarrow \frac{\cos(2x) - 1}{-2} = \sin^2(x) \Rightarrow \frac{-\cos(2x) + 1}{2} = \sin^2(x) \tag{3e}$$

$$\boxed{\sin^2(x) = \frac{1 - \cos(2x)}{2}}$$

2 Trig Integrals

Using the identites, we can solve quite a few integrals:

2.1 Example 1

$$\int \cos^2(x)\sin^3(x)\,dx$$

There are multiple ways we can rewrite this expression using identites, but the goal is to rewrite it in a way that offers a good substituion u:

$$\implies \int \cos^2(x)\sin^2(x)\sin(x)\,dx = \int \cos^2(x)\left(1-\cos^2(x)\right)\sin(x)\,dx \tag{4a}$$

Seeing that we have $\sin(x)$ as the only other function, that leaves us with the substituion $u = \cos(x)$ since $du = -\sin(x)\,dx$:

$$\begin{aligned}
\implies \int u^2\left(1-u^2\right)\sin(x)\,\frac{du}{-\sin(x)} &= \int u^2(1-u^2)(-1)\,du && \text{(4b)}\\
&= -\int u^2(1-u^2)\,du && \text{(4c)}\\
&= -\int (u^2-u^4)\,du && \text{(4d)}\\
&= -\left[\int u^2\,du - \int u^4\,du\right] && \text{(4e)}\\
&= -\left[\frac{u^3}{3} - \frac{u^5}{5}\right] + C && \text{(4f)}\\
&= \frac{u^5}{3} - \frac{u^3}{5} + C && \text{(4g)}\\
&= \frac{\cos^5(x)}{3} - \frac{\cos^3(x)}{5} + C && \text{(4h)}
\end{aligned}$$

$$\boxed{\int \cos^2(x)\sin^3(x)\,dx = \frac{\cos^5(x)}{5} - \frac{\cos^3(x)}{3} + C}$$

2.2 Example 2

$$\int \sin^2(6x)\,dx$$

We could use the identity:

$$\sin^2(a) = \frac{1-\cos(2a)}{2}$$

where $a = 6x$.

$$\begin{aligned}
\implies \int \frac{1-\cos(2(6x))}{2}\,dx &= \int \frac{1-\cos(12x)}{2}\,dx && \text{(5a)}\\
&= \frac{1}{2}\int 1-\cos(12x)\,dx && \text{(5b)}\\
&= \frac{1}{2}\left[\int 1\,dx - \int \cos(12x)\,dx\right] && \text{(5c)}
\end{aligned}$$

Integrating $\cos(12x)$:

$$u = 12x \qquad du = 12\,dx \qquad dx = \frac{1}{12}du$$

$$\begin{aligned}\implies \int \cos(u)\frac{1}{12}\,du &= \frac{1}{12}\int \cos(u)\,du \\ &= \frac{1}{12}\sin(u) + C \\ &= \frac{1}{12}\sin(12x) + C\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\left[\int 1\,dx - \int \cos(12x)\,dx\right] &= \frac{1}{2}\left[x - \frac{1}{12}\sin(12x)\right] + C \\ &= \frac{x}{2} - \frac{\sin(12x)}{24} + C \qquad (5d)\end{aligned}$$

$$\boxed{\int \sin^2(6x)\,dx = \frac{x}{2} - \frac{\sin(12x)}{24} + C}$$

2.2.1 Proof

We can prove that:

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{\sin(2ax)}{4a} + C$$

$$\begin{aligned}\int \sin^2(ax)\,dx &= \int \frac{1-\cos(2ax)}{2} \qquad (6a) \\ &= \frac{1}{2}\int 1 - \cos(2ax) \qquad (6b) \\ &= \frac{1}{2}\left[\int 1\,dx - \int \cos(2ax)\,dx\right] \qquad (6c)\end{aligned}$$

Integrating $\cos(2ax)$:

$$u = 2ax \qquad du = 2a\,dx \qquad dx = \frac{1}{2a}\,du$$

$2a$ is treated is a single constant.

$$\begin{aligned}\implies \int \cos(2ax)\,dx &= \int \cos(u)\frac{1}{2a}\,du \\ &= \frac{1}{2a}\int \cos(u)\,du \\ &= \frac{1}{2a}\sin(u) + C \\ &= \frac{1}{2a}\sin(2ax) + C\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\left[\int 1\,dx - \int \cos(2ax)\,dx\right] &= \frac{1}{2}\left[x - \frac{1}{2a}\sin(2ax)\right] + C \\ &= \frac{x}{2} - \frac{\sin(2ax)}{4a} + C \qquad (6d) \\ & \qquad (6e)\end{aligned}$$

Thus,

$$\boxed{\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{\sin(2ax)}{4a} + C}$$