

Practice Problem Set 1

Laith

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1 Exponential Models

900 grams of a radioactive material is brought into a lab. The amount remaining after 4 days was $\frac{4}{9}$ of what remained after 2 days. Let $N(t)$ be the number of grams of radioactive material.

- (a) Find the number, in grams, of the remaining material t years after being brought into the lab.
- (b) How many grams remain after 2 days?
- (c) What is the half-life of the radioactive material?
- (d) How many days will it takes for only 1 gram to remain?

1.1 (a)

The basic exponential model looks like this:

$$f(t) = br^t$$

where b is the starting value, r is the rate of change, and t is time.

If $t_1 = 4$ days and $t_2 = 2$ days, in which the population at t_1 is $\frac{4}{9}$ of the population at t_2 , we can set up the following equation:

$$N(t_1) = \frac{4}{9}N(t_2) \tag{1.a1}$$

$$N(4) = \frac{4}{9}N(2) \tag{1.a2}$$

$$N(4) = \frac{4}{9}N(2) \tag{1.a3}$$

$$br^4 = \frac{4}{9}br^2 \tag{1.a4}$$

$$900r^4 = \frac{4}{9}900r^2 \tag{1.a5}$$

We know that $b = 900$, however b will cancel out.

$$\Rightarrow r^4 = \frac{4}{9}r^2 \tag{1.a6}$$

We can take the square root of both sides:

$$\sqrt{r^4} = \sqrt{\frac{4}{9}r^2} \Rightarrow r^2 = \frac{2}{3}r \tag{1.a7}$$

Divide by r .

$$\Rightarrow \boxed{r = \frac{2}{3}} \tag{1.a8}$$

Thus, our function $N(t)$ is:

$$N(t) = 900 \left(\frac{2}{3} \right)^t \tag{1.a9}$$

If we plug in $t = 2$:

$$N(2) = 900 \left(\frac{2}{3} \right)^2 = 900 \cdot \frac{4}{9} = 400$$

and if we plug in $t = 4$:

$$N(4) = 900 \left(\frac{2}{3} \right)^4 = 900 \cdot \frac{16}{81} = \frac{1600}{9}$$

Using a calculator, dividing $\frac{1600}{9}$ by 400 should gives us $\frac{4}{9}$:

$$\frac{\frac{1600}{9}}{400} = \frac{1600}{9 \cdot 400} = \frac{1600}{3600} = \frac{16}{36} = \frac{4}{9}$$

Now that we our function, and since t is in terms of days, we can multiply t by 365 (the average number of days per year) to get our answer for part (a):

$$N(t) = 900 \left(\frac{2}{3} \right)^{365t}$$

1.2 (b)

We can substitute 2 for t to get our answer:

$$N(2) = 900 \left(\frac{2}{3} \right)^2 = 900 \cdot \frac{4}{9} = \boxed{400 \text{ g}} \quad (1.b1)$$

Thus, **400 grams** of material remain after 2 days.

1.3 (c)

The half-life is basically how much time it takes for the population to reach half of its original size, thus we are just solving for time t when $N(t) = \frac{1}{2}b$:

$$N(t) = \frac{1}{2}900 = 450 \quad (1.c1)$$

$$900 \left(\frac{2}{3} \right)^t = 450 \quad (1.c2)$$

$$\Rightarrow \left(\frac{2}{3} \right)^t = \frac{45}{90} = \frac{1}{2} \quad (1.c3)$$

We can rewrite this using log of base $\frac{2}{3}$ since:

$$a^x = b \Rightarrow \log_a(b) = x$$

$$\log_{2/3}\left(\frac{1}{2}\right) = t \quad (1.c4)$$

$$\log_{2/3}(1) - \log_{2/3}(2) = t \quad (1.c5)$$

$$0 - \log_{2/3}(2) = t \quad (1.c6)$$

$$-\log_{2/3}(2) = t \quad (1.c7)$$

Using a calculator, we find:

$$\boxed{t \approx 1.71 \text{ days}} \quad (1.c8)$$

1.4 (d)

Now similar to part (c), we need to solve for time t , but this time:

$$N(t) = 1$$

so to solve for t :

$$900 \left(\frac{2}{3} \right)^t = 1 \tag{1.d1}$$

$$\Rightarrow \left(\frac{2}{3} \right)^t = \frac{1}{900} \tag{1.d2}$$

$$\log_{2/3}(1/900) = t \tag{1.d3}$$

$$\log_{2/3}(1) - \log_{2/3}(900) = t \tag{1.d4}$$

$$- \log_{2/3}(900) = t \tag{1.d5}$$

$$\boxed{t \approx 16.78 \text{ days}} \tag{1.d6}$$

2 Separable Differential Equations

Find the explicit solution to these differential equations that satisfy the given initial condition:

(a)

$$(1 + x^2)y' = \frac{1}{y^2}; \quad y(0) = 2$$

$$(1 + x^2)y' = \frac{1}{y^2} \implies (1 + x^2)\frac{dy}{dx} = \frac{1}{y^2} \quad (2.a1)$$

$$\implies (1 + x^2)dy = \frac{1}{y^2}dx$$

$$\implies y^2(1 + x^2)dy = dx$$

$$\implies y^2 dy = \frac{1}{1 + x^2} dx \quad (2.a2)$$

Now to integrate both sides:

$$\int y^2 dy = \int \frac{1}{1 + x^2} dx \implies \frac{y^3}{3} + C = \arctan(x) + C \quad (2.a3)$$

We will assume that there is only one C :

$$\implies \frac{y^3}{3} = \arctan(x) + C$$

$$\implies y^3 = 3(\arctan(x) + C)$$

$$\implies y = \sqrt[3]{3(\arctan(x) + C)} \quad (2.a4)$$

Now to solve for C such that $y(0) = 2$, in which $x = 0$:

$$2 = \sqrt[3]{3(\arctan(0) + C)} \implies 2^3 = 3(\arctan(0) + C) \quad (2.a5)$$

$$\implies 8 = 3(\arctan(0) + C)$$

$$\implies \frac{8}{3} = \arctan(0) + C$$

$$\implies \frac{8}{3} = C \quad (2.a6)$$

Thus, our equation is:

$$\boxed{y = \sqrt[3]{3\arctan(x) + \frac{8}{3}}}$$

(b)

$$x \, dy - (y + \sqrt{y}) \, dx = 0; \quad y(1) = 1$$

$$x \, dy - (y + \sqrt{y}) \, dx = 0 \tag{2.b1}$$

$$\Rightarrow x \, dy = (y + \sqrt{y}) \, dx \tag{2.b2}$$

$$\Rightarrow x \frac{dy}{dx} = y + \sqrt{y} \tag{2.b3}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y + \sqrt{y}}{x} \tag{2.b4}$$

$$\Rightarrow \frac{dy}{dx} \left(\frac{1}{y + \sqrt{y}} \right) = \frac{1}{x} \tag{2.b5}$$

$$\Rightarrow \frac{1}{y + \sqrt{y}} \, dy = \frac{1}{x} \, dx \tag{2.b6}$$

$$\Rightarrow \int \frac{1}{y + \sqrt{y}} \, dy = \int \frac{1}{x} \, dx \tag{2.b7}$$

$$\Rightarrow \int \frac{1}{\sqrt{y}(\sqrt{y} + 1)} \, dy = \int \frac{1}{x} \, dx \tag{2.b8}$$

Integrating the left side:

$$u = \sqrt{y} + 1 \quad du = \frac{1}{2\sqrt{y}} \, dx \rightarrow dx = 2\sqrt{y} \, du$$

$$\Rightarrow \int \frac{1}{\sqrt{y}(u)} 2\sqrt{y} \, du$$

$$\Rightarrow 2 \int \frac{1}{u} \, du$$

$$\Rightarrow 2 (\ln |u|) + C$$

$$\Rightarrow 2 (\ln |\sqrt{y} + 1|) + C$$

Integrating right side:

$$\int \frac{1}{x} = \ln |x| + C$$

Returning to equation 2.b8:

$$\Rightarrow 2 \ln |\sqrt{y} + 1| + C = \ln |x| + C \tag{2.b9}$$

We will get rid of the C on the left side since it is an arbitrary constant, thus we will assume there is only one constant:

$$\Rightarrow 2 \ln |\sqrt{y} + 1| = \ln |x| + C \tag{2.b10}$$

Raise e to the power of both sides in order to cancel out \ln :

$$\Rightarrow e^{2 \ln |\sqrt{y} + 1|} = e^{\ln |x| + C} \Rightarrow e^{2 \ln |\sqrt{y} + 1|} = e^{\ln |x|} \cdot e^C \tag{2.b11}$$

$$\Rightarrow (\sqrt{y} + 1)^2 = x \cdot e^C$$

$$\Rightarrow \sqrt{y} + 1 = \sqrt{x \cdot e^C}$$

$$\Rightarrow \sqrt{y} = \sqrt{x \cdot e^C} - 1$$

$$\Rightarrow y = x e^C - 2\sqrt{x e^C} + 1 \tag{2.b12}$$

Now to solve for C such that the condition $y(1) = 1$ is satisfied:

$$1 = (1) \cdot e^C - 2\sqrt{(1)e^C} + 1 \quad (2.b13)$$

$$\Rightarrow 1 = e^C - 2\sqrt{e^C} + 1$$

$$\Rightarrow 0 = e^C - 2\sqrt{e^C} \Rightarrow 2\sqrt{e^C} = e^C \quad (2.b14)$$

$$\Rightarrow 4e^C = (e^C)^2 \Rightarrow 4e^C = e^{2C} \quad (2.b15)$$

$$\Rightarrow 4 = e^C \Rightarrow \log_e(4) = C \quad (2.b16)$$

$$\Rightarrow \boxed{C = \ln(4)} \quad (2.b17)$$

Plugging this back into our original equation 2.b12:

$$y = xe^{\ln(4)} - 2\sqrt{xe^{\ln(4)}} + 1$$

$$e^{\ln(a)} = a$$

$$y = 4x - 2\sqrt{4x} + 1 = 4x - 4\sqrt{x} + 1$$

Thus, our answer is:

$$\boxed{y = 4x - 4\sqrt{x} + 1} \quad (2.b18)$$

(c)

$$\frac{dy}{dx} = \frac{y \ln(y)}{1 + x^2}; \quad y(0) = e^2$$

$$\frac{dy}{dx} = \frac{y \ln(y)}{1 + x^2} \Rightarrow \frac{1}{y \ln(y)} dy = \frac{1}{1 + x^2} dx \quad (2.c1)$$

Now we can integrate both sides:

$$\int \frac{1}{y \ln(y)} dy = \int \frac{1}{1 + x^2} dx \quad (2.c2)$$

Integrating left side:

$$\begin{aligned} u = \ln(y) \quad du = \frac{1}{y} dx \quad dx = y du \\ \int \frac{1}{y \ln(y)} dx &\Rightarrow \int \frac{1}{y u} y du \\ &\Rightarrow \int \frac{1}{u} du \\ &\Rightarrow \ln |u| \Rightarrow \ln |\ln(y)| \end{aligned} \quad (2.c3)$$

Integrating right side:

$$\int \frac{1}{1 + x^2} = \arctan(x) + C \quad (2.c4)$$

Returning to equation 2.c2:

$$\ln |\ln(y)| = \arctan(x) + C \quad (2.c5)$$

Raise e to the power of both sides to cancel out \ln :

$$e^{\ln |\ln(y)|} = e^{\arctan(x)+C} \implies \ln(y) = e^{\arctan(x)+C} \quad (2.c6)$$

Repeat:

$$e^{\ln(y)} = e^{(e^{\arctan(x)+C})} \implies y = e^{(e^{\arctan(x)+C})} \quad (2.c7)$$

Now to solve for C such that $y(0) = e^2$:

$$e^2 = e^{(e^{\arctan(0)+C})} \implies e^2 = e^{(e^C)} \quad (2.c8)$$

Since the bases are equal, we can assume that the exponents must also be equal, thus:

$$2 = e^C \implies \ln(2) = C \quad (2.c9)$$

Thus, going back to equation 2.c7, our equation is:

$$\begin{aligned} y &= e^{(e^{\arctan(x)+\ln(2)})} \implies y = e^{(e^{\arctan(x)} e^{\ln(2)})} \\ &\implies \boxed{y = e^{(2e^{\arctan(x)})}} \end{aligned} \quad (2.c10)$$

3 Integration by Substitution

(a)

$$\int \frac{[\log_3(x)]^2}{x} dx$$

$$\int \frac{[\log_3(x)]^2}{x} dx \Rightarrow \int \frac{1}{x} \left[\frac{\ln(x)}{\ln(3)} \right]^2 dx \quad (3.a1)$$

$$u = \ln(x) \quad du = \frac{1}{x} dx \quad dx = x du \quad (3.a2)$$

$$\Rightarrow \int \frac{1}{x} \left[\frac{u}{\ln(3)} \right]^2 x du$$

$$\Rightarrow \int \left[\frac{u}{\ln(3)} \right]^2 du$$

$$\Rightarrow \int \frac{u^2}{\ln^2(3)} du$$

$$\Rightarrow \frac{1}{\ln^2(3)} \int u^2 du$$

$$\Rightarrow \frac{1}{\ln^2(3)} \left[\frac{1}{3} u^3 \right] + C$$

$$\Rightarrow \frac{u^3}{3 \ln^2(3)} + C$$

$$\Rightarrow \boxed{\frac{\ln^3(x)}{3 \ln^2(3)} + C} \quad (3.a3)$$

(b)

$$\int \frac{[2 + \log_3(5x + 2)]^5}{15x + 6} dx$$

$$u = 5x + 2 \quad du = 5 dx \quad dx = \frac{1}{5} du$$

$$\Rightarrow \int \frac{[2 + \log_3(u)]^5}{3u} \cdot \frac{1}{5} du \quad (3.b1)$$

$$= \frac{1}{15} \int \frac{[2 + \log_3(u)]^5}{u} du \quad (3.b2)$$

$$v = 2 + \log_3(u) = 2 + \frac{\ln(u)}{\ln(3)} \quad dv = \frac{1}{\ln(3)} \cdot \frac{1}{u} du \quad du = \ln(3) u dv$$

$$\Rightarrow \frac{1}{15} \int \frac{v^5}{u} \ln(3) u \, dv \quad (3.b3)$$

$$= \frac{\ln(3)}{15} \int v^5 \, dv \quad (3.b4)$$

$$= \frac{\ln(3)}{15} \cdot \frac{v^6}{6} + C \quad (3.b5)$$

$$= \frac{\ln(3)v^6}{80} + C \quad (3.b6)$$

$$= \boxed{\frac{\ln(3) [2 + \log_3(5x + 2)]^6}{90} + C} \quad (3.b7)$$

(c)

$$\int 5x^{11} \sec(x^{12}) \tan(x^{12}) \, dx$$

$$u = x^{12} \quad du = 12x^{11} dx \quad dx = \frac{du}{12x^{11}} \quad (3.c1)$$

$$\Rightarrow \int 5x^{11} \sec(u) \tan(u) \frac{du}{12x^{11}} \quad (3.c2)$$

$$\Rightarrow \frac{5}{12} \int \sec(u) \tan(u) \, du \quad (3.c3)$$

$$\Rightarrow \frac{5}{12} [\sec(u)] + C \quad (3.c4)$$

$$\Rightarrow \boxed{\frac{5}{12} \sec(x^{12}) + C} \quad (3.c5)$$

(d)

$$\int \frac{1}{\sqrt{x} e^{-\sqrt{x}}} \sec^2(1 + \sqrt{x}) \, dx$$

$$u = \sqrt{x} \quad du = \frac{1}{2\sqrt{x}} dx \quad dx = 2\sqrt{x} du = 2u du$$

$$\Rightarrow \int \frac{1}{ue^{-u}} \sec^2(1 + u) 2u \, du \quad (3.d1)$$

$$\Rightarrow 2 \int \frac{1}{e^{-u}} \sec^2(1 + u) \, du \quad (3.d2)$$

$$\Rightarrow 2 \int e^u \sec^2(1 + u) \, du \quad (3.d3)$$

Now to do integration by parts:

$$v = e^u \quad dv = e^u \quad w = \int \sec^2(1 + u) = \tan(1 + u)$$

$$\Rightarrow e^u \tan(1+u) - \int e^u \sec^2(1+u) du \quad (3.d4)$$

Let:

$$I = \int e^u \sec^2(1+u)$$

$$I = e^u \tan(1+u) - I \quad (3.d5)$$

$$2I = e^u \tan(1+u) \quad (3.d6)$$

$$I = \frac{1}{2} e^u \tan(1+u) \quad (3.d7)$$

Returning back to our integral:

$$\int e^u \sec^2(1+u) du = \frac{1}{2} e^u \tan(1+u) \quad (3.d8)$$

Bring back our coefficient of 2:

$$2 \int e^u \sec^2(1+u) du = 2 \left(\frac{1}{2} e^u \tan(1+u) \right) \quad (3.d9)$$

$$2 \int e^u \sec^2(1+u) du = e^u \tan(1+u) \quad (3.d10)$$

Thus,

$$\int \frac{1}{\sqrt{x} e^{-\sqrt{x}}} \sec^2(1+\sqrt{x}) dx = e^u \tan(1+u) + C \quad (3.d11)$$

$$= e^{\sqrt{x}} \tan(1+\sqrt{x}) + C \quad (3.d12)$$

$$\boxed{e^{\sqrt{x}} \tan(1+\sqrt{x}) + C}$$