Practice Problem Set 1

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1 Exponential Models

900 grams of a radioactive material is brought into a lab. The amount remaining after 4 days was 4/9 of what remained after 2 days. Let $\mathrm{N}(t)$ be the number of grams of radioactive material.

(a) Find the number, in grams, of the remaining material t years after being brought into the lab.

(b) How many grams remain after 2 days?

(c) What is the half-life of the radioactive material?

(d) How many days will it takes for only 1 gram to remain?

1.1 (a)

The basic exponential model looks like this:

$$\mathrm{f}(t) = br^t$$

where b is the starting value, r is the rate of change, and t is time.

If $t_1 = 4$ days and $t_2 = 2$ days, in which the population at t_1 is $\frac{4}{9}$ of the population at t_2, we can set up the following equation:

$$\mathrm{N}(t_1) = \frac{4}{9}\mathrm{N}(t_2) \tag{1.a1}$$

$$\mathrm{N}(4) = \frac{4}{9}\mathrm{N}(2) \tag{1.a2}$$

$$\mathrm{N}(4) = \frac{4}{9}\mathrm{N}(2) \tag{1.a3}$$

$$br^4 = \frac{4}{9}br^2 \tag{1.a4}$$

$$900r^4 = \frac{4}{9}900r^2 \qquad \text{We know that } b = 900\text{, however } b \text{ will cancel out.} \tag{1.a5}$$

$$\Rightarrow r^4 = \frac{4}{9}r^2 \qquad \text{We can take the square root of both sides:} \tag{1.a6}$$

$$\sqrt{r^4} = \sqrt{\frac{4}{9}r^2} \Rightarrow r^2 = \frac{2}{3}r \qquad \text{Divide by } r. \tag{1.a7}$$

$$\Rightarrow \boxed{r = \frac{2}{3}} \tag{1.a8}$$

Thus, our function $\mathrm{N}(t)$ is:

$$\mathrm{N}(t) = 900\left(\frac{2}{3}\right)^t \tag{1.a9}$$

If we plug in $t = 2$:

$$\mathrm{N}(2) = 900\left(\frac{2}{3}\right)^2 = 900 \cdot \frac{4}{9} = 400$$

and if we plug in $t = 4$:

$$\mathrm{N}(4) = 900\left(\frac{2}{3}\right)^4 = 900 \cdot \frac{16}{81} = \frac{1600}{9}$$

Using a calculator, dividing $\frac{1600}{9}$ by 400 should gives us $\frac{4}{9}$:

$$\frac{\frac{1600}{9}}{400} = \frac{1600}{9 \cdot 400} = \frac{1600}{3600} = \frac{16}{36} = \frac{4}{9}$$

Now that we our function, and since t is in terms of days, we can multiply t by 365 (the average number of days per year) to get our answer for part (a):

$$\boxed{\mathrm{N}(t) = 900\left(\frac{2}{3}\right)^{365t}}$$

1.2 (b)

We can substitue 2 for t to get our answer:

$$\mathrm{N}(2) = 900\left(\frac{2}{3}\right)^{2} = 900 \cdot \frac{4}{9} = \boxed{400\,\mathrm{g}} \tag{1.b1}$$

Thus, **400 grams** of material remain after 2 days.

1.3 (c)

The half-life is basically how much time it takes for the population to reach half of its original size, thus we are just solving for time t when $\mathrm{N}(t) = \frac{1}{2}b$:

$$\mathrm{N}(t) = \frac{1}{2}900 = 450 \tag{1.c1}$$

$$900\left(\frac{2}{3}\right)^{t} = 450 \tag{1.c2}$$

$$\Rightarrow \left(\frac{2}{3}\right)^{t} = \frac{45}{90} = \frac{1}{2} \tag{1.c3}$$

We can rewrite this using log of base $\frac{2}{3}$ since:

$$a^x = b \Rightarrow \log_a(b) = x$$

$$\log_{2/3}(\frac{1}{2}) = t \tag{1.c4}$$

$$\log_{2/3}(1) - \log_{2/3}(2) = t \tag{1.c5}$$

$$0 - \log_{2/3}(2) = t \tag{1.c6}$$

$$-\log_{2/3}(2) = t \tag{1.c7}$$

Using a calculator, we find:

$$\boxed{t \approx 1.71\,\mathrm{days}} \tag{1.c8}$$

1.4 (d)

Now similar to part (c), we need to solve for time t, but this time:

$$\mathrm{N}(t) = 1$$

so to solve for t:

$$900\left(\frac{2}{3}\right)^t = 1 \tag{1.d1}$$

$$\Rightarrow \left(\frac{2}{3}\right)^t = \frac{1}{900} \tag{1.d2}$$

$$\log_{2/3}(1/900) = t \tag{1.d3}$$

$$\log_{2/3}(1) - log_{2/3}(900) = t \tag{1.d4}$$

$$- log_{2/3}(900) = t \tag{1.d5}$$

$$\boxed{t \approx 16.78\,\text{days}} \tag{1.d6}$$

2 Separable Differential Equations

Find the explicit solution to these differential equations that satisfy the given initial condition:

(a)

$$(1+x^2)y' = \frac{1}{y^2}; \quad y(0) = 2$$

$$\begin{aligned}
(1+x^2)y' = \frac{1}{y^2} &\implies (1+x^2)\frac{dy}{dx} = \frac{1}{y^2} && \text{(2.a1)}\\
&\implies (1+x^2)dy = \frac{1}{y^2}dx\\
&\implies y^2(1+x^2)dy = dx\\
&\implies y^2\,dy = \frac{1}{1+x^2}\,dx && \text{(2.a2)}
\end{aligned}$$

Now to integrate both sides:

$$\int y^2\,dy = \int \frac{1}{1+x^2}\,dx \implies \frac{y^3}{3} + C = \arctan(x) + C \tag{2.a3}$$

We will assume that there is only one C:

$$\begin{aligned}
&\implies \frac{y^3}{3} = \arctan(x) + C\\
&\implies y^3 = 3(\arctan(x) + C)\\
&\implies y = \sqrt[3]{3(\arctan(x) + C)} && \text{(2.a4)}
\end{aligned}$$

Now to solve for C such that $y(0) = 2$, in which $x = 0$:

$$\begin{aligned}
2 = \sqrt[3]{3(\arctan(0) + C)} &\implies 2^3 = 3(\arctan(2) + C) && \text{(2.a5)}\\
&\implies 8 = 3(\arctan(0) + C)\\
&\implies \frac{8}{3} = \arctan(0) + C\\
&\implies \frac{8}{3} = C && \text{(2.a6)}
\end{aligned}$$

Thus, our equation is:

$$\boxed{y = \sqrt[3]{3\arctan(x) + \frac{8}{3}}}$$

(b)

$$x\,dy - (y+\sqrt{y})dx = 0\,; \quad y(1) = 1$$

$$x\,dy - (y+\sqrt{y})dx = 0 \tag{2.b1}$$
$$\Rightarrow x\,dy = (y+\sqrt{y})dx \tag{2.b2}$$
$$\Rightarrow x\frac{dy}{dx} = y+\sqrt{y} \tag{2.b3}$$
$$\Rightarrow \frac{dy}{dx} = \frac{y+\sqrt{y}}{x} \tag{2.b4}$$
$$\Rightarrow \frac{dy}{dx}\left(\frac{1}{y+\sqrt{y}}\right) = \frac{1}{x} \tag{2.b5}$$
$$\Rightarrow \frac{1}{y+\sqrt{y}}\,dy = \frac{1}{x}\,dx \tag{2.b6}$$
$$\Rightarrow \int \frac{1}{y+\sqrt{y}}\,dy = \int \frac{1}{x}\,dx \tag{2.b7}$$
$$\Rightarrow \int \frac{1}{\sqrt{y}(\sqrt{y}+1)}\,dy = \int \frac{1}{x}\,dx \tag{2.b8}$$

Integrating the left side:

$$u = \sqrt{y}+1 \quad du = \frac{1}{2\sqrt{y}}dx \to dx = 2\sqrt{y}\,du$$
$$\Rightarrow \int \frac{1}{\sqrt{y}(u)}\,2\sqrt{y}\,du$$
$$\Rightarrow 2\int \frac{1}{u}\,du$$
$$\Rightarrow 2\,(\ln|u|) + C$$
$$\Rightarrow 2\,(\ln|\sqrt{y}+1|) + C$$

Integrating right side:

$$\int \frac{1}{x} = \ln|x| + C$$

Returning to equation 2.b8:

$$\Rightarrow 2\ln|\sqrt{y}+1| + C = \ln|x| + C \tag{2.b9}$$

We will get rid of the C on the left side since it is an arbitrary constant, thus we will assume there is only one constant:

$$\Rightarrow 2\ln|\sqrt{y}+1| = \ln|x| + C \tag{2.b10}$$

Raise e to the power of both sides in order to cancel out ln:

$$\Rightarrow e^{2\ln\left|\sqrt{y}+1\right|} = e^{\ln|x|+C} \Rightarrow e^{2\ln\left|\sqrt{y}+1\right|} = e^{\ln|x|}\cdot e^{C} \tag{2.b11}$$
$$\Rightarrow (\sqrt{y}+1)^2 = x\cdot e^{C}$$
$$\Rightarrow \sqrt{y}+1 = \sqrt{x\cdot e^{C}}$$
$$\Rightarrow \sqrt{y} = \sqrt{x\cdot e^{C}} - 1$$
$$\Rightarrow y = xe^{C} - 2\sqrt{xe^{C}} + 1 \tag{2.b12}$$

Now to solve for C such that the condition $y(1) = 1$ is satisfied:

$$1 = (1) \cdot e^C - 2\sqrt{(1)e^C} + 1 \tag{2.b13}$$

$$\Rightarrow 1 = e^C - 2\sqrt{e^C} + 1$$

$$\Rightarrow 0 = e^C - 2\sqrt{e^C} \;\Rightarrow\; 2\sqrt{e^C} = e^C \tag{2.b14}$$

$$\Rightarrow 4e^C = \left(e^C\right)^2 \;\Rightarrow\; 4e^C = e^{2C} \tag{2.b15}$$

$$\Rightarrow 4 = e^C \;\Rightarrow\; \log_e(4) = C \tag{2.b16}$$

$$\Rightarrow \boxed{C = \ln(4)} \tag{2.b17}$$

Plugging this back into our original equation 2.b12:

$$y = xe^{\ln(4)} - 2\sqrt{xe^{\ln(4)}} + 1$$

$$e^{\ln(a)} = a$$

$$y = 4x - 2\sqrt{4x} + 1 = 4x - 4\sqrt{x} + 1$$

Thus, our answer is:

$$\boxed{y = 4x - 4\sqrt{x} + 1} \tag{2.b18}$$

(c)

$$\frac{dy}{dx} = \frac{y\ln(y)}{1+x^2}\,; \quad y(0) = e^2$$

$$\frac{dy}{dx} = \frac{y\ln(y)}{1+x^2} \Longrightarrow \frac{1}{y\ln(y)}dy = \frac{1}{1+x^2}dx \tag{2.c1}$$

Now we can integrate both sides:

$$\int \frac{1}{y\ln(y)}dy = \int \frac{1}{1+x^2}dx \tag{2.c2}$$

Integrating left side:

$$u = \ln(y) \quad du = \frac{1}{y}dx \quad dx = y\,du$$

$$\begin{aligned} \int \frac{1}{y\ln(y)}dx &\Longrightarrow \int \frac{1}{y\,u}y\,du \\ &\Longrightarrow \int \frac{1}{u}du \\ &\Longrightarrow \ln|u| \Longrightarrow \ln|\ln(y)| \end{aligned} \tag{2.c3}$$

Integrating right side:

$$\int \frac{1}{1+x^2} = \arctan(x) + C \tag{2.c4}$$

Returning to equation 2.c2:

$$\ln|\ln(y)| = \arctan(x) + C \tag{2.c5}$$

Raise e to the power of both sides to cancel out ln:

$$e^{\ln|\ln(y)|} = e^{\arctan(x)+C} \implies \ln(y) = e^{\arctan(x)+C} \tag{2.c6}$$

Repeat:

$$e^{\ln(y)} = e^{\left(e^{\arctan(x)+C}\right)} \implies y = e^{\left(e^{\arctan(x)+C}\right)} \tag{2.c7}$$

Now to solve for C such that $y(0) = e^2$:

$$e^2 = e^{\left(e^{\arctan(0)+C}\right)} \implies e^2 = e^{\left(e^C\right)} \tag{2.c8}$$

Since the bases are equal, we can assume that the exponents must also be equal, thus:

$$2 = e^C \implies \ln(2) = C \tag{2.c9}$$

Thus, going back to equation 2.c7, our equation is:

$$\begin{aligned} y = e^{\left(e^{\arctan(x)+\ln(2)}\right)} &\implies y = e^{\left(e^{\arctan(x)}e^{ln(2)}\right)} \\ &\implies \boxed{y = e^{\left(2e^{\arctan(x)}\right)}} \end{aligned} \tag{2.c10}$$

3 Integration by Substituion

(a)

$$\int \frac{[\log_3(x)]^2}{x}\,dx$$

$$\int \frac{[\log_3(x)]^2}{x}\,dx \Longrightarrow \int \frac{1}{x}\left[\frac{\ln(x)}{\ln(3)}\right]^2\,dx \tag{3.a1}$$

$$u = \ln(x) \quad du = \frac{1}{x}dx \quad dx = x\,du \tag{3.a2}$$

$$\begin{aligned}
&\Longrightarrow \int \frac{1}{x}\left[\frac{u}{\ln(3)}\right]^2 x\,du \\
&\Longrightarrow \int \left[\frac{u}{\ln(3)}\right]^2\,du \\
&\Longrightarrow \int \frac{u^2}{\ln^2(3)}\,du \\
&\Longrightarrow \frac{1}{\ln^2(3)}\int u^2\,du \\
&\Longrightarrow \frac{1}{\ln^2(3)}\left[\frac{1}{3}u^3\right] + C \\
&\Longrightarrow \frac{u^3}{3\ln^2(3)} + C
\end{aligned}$$

$$\Longrightarrow \boxed{\frac{\ln^3(x)}{3\ln^2(3)} + C} \tag{3.a3}$$

(b)

$$\int \frac{[2+\log_3(5x+2)]^5}{15x+6}\,dx$$

$$u = 5x+2 \qquad du = 5\,dx \qquad dx = \frac{1}{5}\,du$$

$$\Longrightarrow \int \frac{[2+\log_3(u)]^5}{3u}\cdot\frac{1}{5}\,du \tag{3.b1}$$

$$= \frac{1}{15}\int \frac{[2+\log_3(u)]^5}{u}\,du \tag{3.b2}$$

$$v = 2+\log_3(u) = 2+\frac{\ln(u)}{\ln(3)} \qquad dv = \frac{1}{\ln(3)}\cdot\frac{1}{u}\,du \qquad du = \ln(3)u\,dv$$

$$\implies \frac{1}{15}\int \frac{v^5}{u}\ln(3)u\,dv \tag{3.b3}$$

$$= \frac{\ln(3)}{15}\int v^5\,dv \tag{3.b4}$$

$$= \frac{\ln(3)}{15}\cdot\frac{v^6}{6} + C \tag{3.b5}$$

$$= \frac{\ln(3)v^6}{80} + C \tag{3.b6}$$

$$= \boxed{\frac{\ln(3)\left[2+\log_3(5x+2)\right]^6}{90} + C} \tag{3.b7}$$

(c)

$$\int 5x^{11}\sec\left(x^{12}\right)\tan\left(x^{12}\right)dx$$

$$u = x^{12} \quad du = 12x^{11}dx \quad dx = \frac{du}{12x^{11}} \tag{3.c1}$$

$$\implies \int 5x^{11}\sec(u)\tan(u)\,\frac{du}{12x^{11}} \tag{3.c2}$$

$$\implies \frac{5}{12}\int \sec(u)\tan(u)\,du \tag{3.c3}$$

$$\implies \frac{5}{12}\left[\sec(u)\right] + C \tag{3.c4}$$

$$\implies \boxed{\frac{5}{12}\sec\left(x^{12}\right) + C} \tag{3.c5}$$

(d)

$$\int \frac{1}{\sqrt{x}\,e^{-\sqrt{x}}}\sec^2(1+\sqrt{x})\,dx$$

$$u = \sqrt{x} \qquad du = \frac{1}{2\sqrt{x}}\,dx \qquad dx = 2\sqrt{x}\,du = 2u\,du$$

$$\implies \int \frac{1}{ue^{-u}}\sec^2(1+u)\,2u\,du \tag{3.d1}$$

$$\implies 2\int \frac{1}{e^{-u}}\sec^2(1+u)\,du \tag{3.d2}$$

$$\implies 2\int e^u\sec^2(1+u)\,du \tag{3.d3}$$

$$\tag{3.d4}$$

Now to do integration by parts:

$$v = e^u \qquad dv = e^u \qquad w = \int \sec^2(1+u) = \tan(1+u)$$

$$\implies e^u \tan(1+u) - \int e^u \sec^2(1+u)\,du \tag{3.d5}$$

Let:

$$I = \int e^u \sec^2(1+u)$$

$$I = e^u \tan(1+u) - I \tag{3.d6}$$

$$2I = e^u \tan(1+u) \tag{3.d7}$$

$$I = \frac{1}{2} e^u \tan(1+u) \tag{3.d8}$$

$$\tag{3.d9}$$

Returning back to our integral:

$$\int e^u \sec^2(1+u)\,du = \frac{1}{2} e^u \tan(1+u) \tag{3.d10}$$

Bring back our coefficient of 2:

$$2\int e^u \sec^2(1+u)\,du = 2\left(\frac{1}{2} e^u \tan(1+u)\right) \tag{3.d11}$$

$$2\int e^u \sec^2(1+u)\,du = e^u \tan(1+u) \tag{3.d12}$$

Thus,

$$\int \frac{1}{\sqrt{x}\, e^{-\sqrt{x}}} \sec^2(1+\sqrt{x})\,dx = e^u \tan(1+u) + C \tag{3.d13}$$

$$= e^{\sqrt{x}} \tan\big(1+\sqrt{x}\big) + C \tag{3.d14}$$

$$\boxed{e^{\sqrt{x}} \tan\big(1+\sqrt{x}\big) + C}$$