Practice Problem Set 1

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1 Exponential Models

900 grams of a radioactive material is brought into a lab. The amount remaining after 4 days was 4/9 of what remained after 2 days. Let N(t) be the number of grams of radioactive material.

(a) Find the number, in grams, of the remaining material t years after being brought into the lab.

(b) How many grams remain after 2 days?

(c) What is the half-life of the radioactive material?

(d) How many days will it takes for only 1 gram to remain?

Part (a)

1.0

The basic exponential model looks like this:

$$\mathrm{f}(t) = br^t$$

where b is the starting value, r is the rate of change, and t is time.

If $t_1 = 4$ days and $t_2 = 2$ days, in which the population at t_1 is $\frac{4}{9}$ of the population at t_2, we can set up the following equation:

$$\mathrm{N}(t_1) = \frac{4}{9}\mathrm{N}(t_2) \tag{1.1.1}$$

$$\mathrm{N}(4) = \frac{4}{9}\mathrm{N}(2) \tag{1.1.2}$$

$$\mathrm{N}(4) = \frac{4}{9}\mathrm{N}(2) \tag{1.1.3}$$

$$br^4 = \frac{4}{9}br^2 \tag{1.1.4}$$

$$900r^4 = \frac{4}{9}900r^2 \qquad \text{We know that } b = 900\text{, however } b \text{ will cancel out.} \tag{1.1.5}$$

$$\Rightarrow r^4 = \frac{4}{9}r^2 \qquad \text{We can take the square root of both sides:} \tag{1.1.6}$$

$$\sqrt{r^4} = \sqrt{\frac{4}{9}r^2} \Rightarrow r^2 = \frac{2}{3}r \qquad \text{Divide by } r. \tag{1.1.7}$$

$$\Rightarrow \boxed{r = \frac{2}{3}} \tag{1.1.8}$$

Thus, our function N(t) is:

$$\mathrm{N}(t) = 900\left(\frac{2}{3}\right)^t \tag{1.1.9}$$

If we plug in $t = 2$:

$$\mathrm{N}(2) = 900\left(\frac{2}{3}\right)^2 = 900 \cdot \frac{4}{9} = 400$$

and if we plug in $t = 4$:

$$\mathrm{N}(4) = 900\left(\frac{2}{3}\right)^4 = 900 \cdot \frac{16}{81} = \frac{1600}{9}$$

Using a calculator, dividing $\frac{1600}{9}$ by 400 should gives us $\frac{4}{9}$:

$$\frac{\frac{1600}{9}}{400} = \frac{1600}{9 \cdot 400} = \frac{1600}{3600} = \frac{16}{36} = \frac{4}{9}$$

Now that we our function, and since t is in terms of days, we can multiply t by 365 (the average number of days per year) to get our answer for part (a):

$$\boxed{\mathrm{N}(t) = 900\left(\frac{2}{3}\right)^{365t}}$$

Part (b)

We can substitue 2 for t to get our answer:

$$\mathrm{N}(2) = 900\left(\frac{2}{3}\right)^2 = 900 \cdot \frac{4}{9} = \boxed{400\,\mathrm{g}} \tag{1.2.1}$$

Thus, **400 grams** of material remain after 2 days.

Part (c)

The half-life is basically how much time it takes for the population to reach half of its original size, thus we are just solving for time t when $\mathrm{N}(t) = \frac{1}{2}b$:

$$\mathrm{N}(t) = \frac{1}{2}900 = 450 \tag{1.3.1}$$

$$900\left(\frac{2}{3}\right)^t = 450 \tag{1.3.2}$$

$$\Rightarrow \left(\frac{2}{3}\right)^t = \frac{45}{90} = \frac{1}{2} \tag{1.3.3}$$

We can rewrite this using log of base $\frac{2}{3}$ since:

$$a^x = b \Rightarrow \log_a(b) = x$$

$$\log_{2/3}(\frac{1}{2}) = t \tag{1.3.4}$$

$$\log_{2/3}(1) - \log_{2/3}(2) = t \tag{1.3.5}$$

$$0 - \log_{2/3}(2) = t \tag{1.3.6}$$

$$-\log_{2/3}(2) = t \tag{1.3.7}$$

Using a calculator, we find:

$$\boxed{t \approx 1.71\,\mathrm{days}} \tag{1.3.8}$$

Part (d)

Now similar to part (c), we need to solve for time t, but this time:

$$\mathrm{N}(t) = 1$$

so to solve for t:

$$900\left(\frac{2}{3}\right)^t = 1 \tag{1.4.1}$$

$$\Rightarrow \left(\frac{2}{3}\right)^t = \frac{1}{900} \tag{1.4.2}$$

$$\log_{2/3}(1/900) = t \tag{1.4.3}$$

$$\log_{2/3}(1) - log_{2/3}(900) = t \tag{1.4.4}$$

$$- log_{2/3}(900) = t \tag{1.4.5}$$

$$\boxed{t \approx 16.78\,\text{days}} \tag{1.4.6}$$

2 Separable Differential Equations

2.0Find the explicit solution to these differential equations that satisfy the given initial condition:

(b)

$$x\,dy - (y + \sqrt{y})dx = 0\,; \quad y(1) = 1$$

$$x\,dy - (y + \sqrt{y})dx = 0 \tag{2.1.1}$$

$$\Rightarrow x\,dy = (y + \sqrt{y})dx \tag{2.1.2}$$

$$\Rightarrow x\frac{dy}{dx} = y + \sqrt{y} \tag{2.1.3}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y + \sqrt{y}}{x} \tag{2.1.4}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1}{y + \sqrt{y}}\right) = \frac{1}{x} \tag{2.1.5}$$

$$\Rightarrow \frac{1}{y + \sqrt{y}}\,dy = \frac{1}{x}\,dx \tag{2.1.6}$$

$$\Rightarrow \int \frac{1}{y + \sqrt{y}}\,dy = \int \frac{1}{x}\,dx \tag{2.1.7}$$

$$\Rightarrow \int \frac{1}{\sqrt{y}(\sqrt{y} + 1)}\,dy = \int \frac{1}{x}\,dx \tag{2.1.8}$$

Integrating the left side:

$$u = \sqrt{y} + 1 \quad du = \frac{1}{2\sqrt{y}} dx \to dx = 2\sqrt{y}\, du$$

$$\Rightarrow \int \frac{1}{\sqrt{y}(u)} 2\sqrt{y}\, du$$

$$\Rightarrow 2 \int \frac{1}{u}\, du$$

$$\Rightarrow 2\left(\ln|u|\right) + C$$

$$\Rightarrow 2\left(\ln|\sqrt{y}+1|\right) + C$$

Integrating right side:

$$\int \frac{1}{x} = \ln|x| + C$$

Returning to equation 2.1.8:

$$\Rightarrow 2\ln|\sqrt{y}+1| + C = \ln|x| + C \tag{2.1.9}$$

We will get rid of the C on the left side since it is an arbitrary constant, thus we will assume there is only one constant:

$$\Rightarrow 2\ln|\sqrt{y}+1| = \ln|x| + C \tag{2.1.10}$$

Raise both sides to the power of e in order to cancel out ln:

$$\Rightarrow e^{2\ln\left|\sqrt{y}+1\right|} = e^{\ln|x|+C} \Rightarrow e^{2\ln\left|\sqrt{y}+1\right|} = e^{\ln|x|} \cdot e^{C} \tag{2.1.11}$$

$$\Rightarrow (\sqrt{y}+1)^2 = x \cdot e^C$$

$$\Rightarrow \sqrt{y} + 1 = \sqrt{x \cdot e^C}$$

$$\Rightarrow \sqrt{y} = \sqrt{x \cdot e^C} - 1$$

$$\Rightarrow y = xe^C - 2\sqrt{xe^C} + 1 \tag{2.1.12}$$

Now to solve for C such that the condition $y(1) = 1$ is satisfied:

$$1 = (1) \cdot e^C - 2\sqrt{(1)e^C} + 1 \tag{2.1.13}$$

$$\Rightarrow 1 = e^C - 2\sqrt{e^C} + 1$$

$$\Rightarrow 0 = e^C - 2\sqrt{e^C} \;\Rightarrow\; 2\sqrt{e^C} = e^C \tag{2.1.14}$$

$$\Rightarrow 4e^C = \left(e^C\right)^2 \;\Rightarrow\; 4e^C = e^{2C} \tag{2.1.15}$$

$$\Rightarrow 4 = e^C \;\Rightarrow\; \log_e(4) = C \tag{2.1.16}$$

$$\Rightarrow \boxed{C = \ln(4)} \tag{2.1.17}$$

Plugging this back into our original equation 2.1.12:

$$y = xe^{\ln(4)} - 2\sqrt{xe^{\ln(4)}} + 1$$
$$e^{\ln(a)} = a$$
$$y = 4x - 2\sqrt{4x} + 1 = 4x - 4\sqrt{x} + 1$$

Thus, our answer is:

$$\boxed{y = 4x - 4\sqrt{x} + 1} \tag{2.1.18}$$