Defining Logarithm as an Integral

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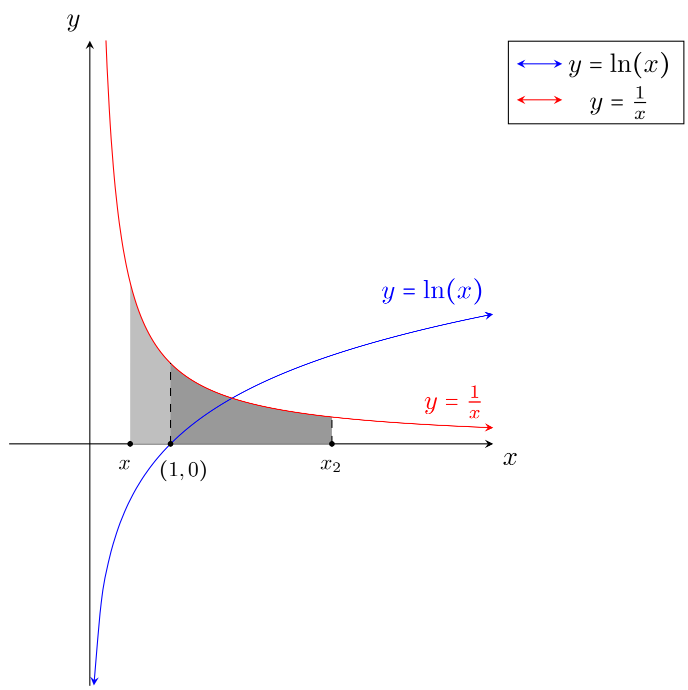
Definition $\ln(x)$ is a function given by:

$$\ln(x) = \int_1^x \frac{1}{t}\,\mathrm{d}t$$

This is true by the **Fundamental Theorem of Calculus**:

$$\mathrm{F}(x) = \int_a^{\mathrm{g}(x)} \mathrm{F}(t)\,\mathrm{d}t \Rightarrow \mathrm{F}'(x) = \mathrm{g}'(x) \cdot \mathrm{F}(\mathrm{g}(x))$$

1 Discussion



$$\begin{aligned}
&\text{Case } 1: && 0 < x < 1; && \ln(x) = -\int_x^1 \frac{1}{t}\,\mathrm{d}t \\
&\text{Case } 2: && x > 1; && \ln(x) = \int_1^x \frac{1}{t}\,\mathrm{d}t \\
&\text{Case } 2: && x = 1; && \ln(x) = \int_1^1 \frac{1}{t}\,\mathrm{d}t = 0
\end{aligned}$$

1.1 Definition:

$$\ln(e) = \int_1^e \frac{1}{t}\mathrm{d}t = 1$$

1.2 Properties of ln:

If $x > 0$, $y > 0$, $zy < 0$, then:

1. $\ln(xy) = \ln(x) + \ln(y)$
2. $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$
3. $\ln\left(x^k\right) = k\ln(x)$
4. $\ln(e^x) = x$

1.3 Derivatives of $y = \ln(x)$ or $y = \ln(\mathrm{u}(x))$:

$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln(\mathrm{u}(x))] = \frac{\mathrm{u}'(x)}{\mathrm{u}(x)} \;\Big|\; \frac{\mathrm{d}}{\mathrm{d}x}[\ln(x)] = \frac{1}{x}$$

1.4 Integrals:

$$\int \frac{\mathrm{u}'(x)}{\mathrm{u}(x)}\mathrm{d}x = \ln|\mathrm{u}(x)| + C$$

2 Examples: