Homework Problem #48

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2/7/2023

Problem

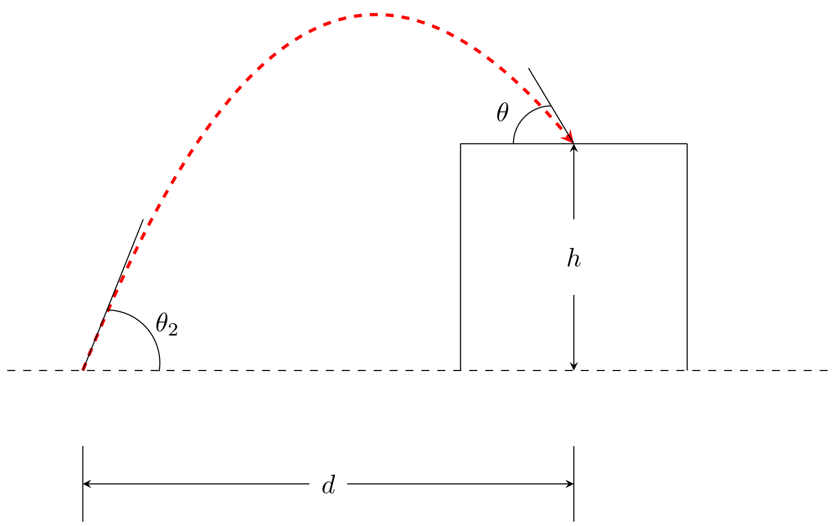
In Fig. 4-41, a ball is thrown up onto a roof, landing 4.00 s later at height $h = 20.0$ m above the release level. The ball's path just before landing is angled at $\theta = 60.0°$ with the roof.

(a) Find the horizontal distance d it travels. (See the hint to Problem 39.)

(b) What is the magnitude relative to the horizontal of the ball's initial velocity?

(c) What is the angle relative to the horizontal of the ball's initial velocity?

Figure 1: Figure 4-11

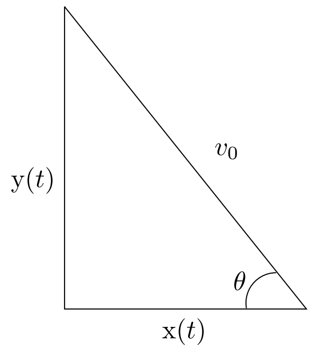


Solution

Part (a)

In order to find distance d, we can define a function as the horizontal position of the ball relative to where the ball landed. In other words, this function will tell us the position of the ball starting from the top of the building back to the ground.

In order to define this fucntion, let's draw a right triangle using θ:



We can use $\cos(\theta)$ to solve for $\mathrm{x}(t)$, in which we multiply by time t to cancel out initial velocity v_0:

$$\cos(\theta) = \frac{\mathrm{x}(t)}{v_0} \Rightarrow \boxed{\mathrm{x}(t) = v_0 \cos(\theta) t} \tag{1}$$

We can use $\sin(\theta)$ to solve for $\mathrm{y}(t)$, where we also multiply by t to cancel out velocity, but also subtract $\frac{1}{2}gt^2$ in order to account for gravity:

$$\sin(\theta) = \frac{\mathrm{y}(t)}{v_0} \Rightarrow \boxed{\mathrm{y}(t) = v_0 \cos(\theta) t - \frac{1}{2}gt^2} \tag{2}$$

The reason we also need a function for the vertical position $\mathrm{y}(t)$ is that we have an undefined variable v_0, which means we cannot yet solve for d.

Since we know that the ball lands at a height $h = 20$, we can say that our function $\mathrm{y}(t)$ at time $t = 4$ would equal -20:

$$\begin{aligned}
\mathrm{y}(4) &= -20 \\
\mathrm{y}(4) &= v_0 \sin(60)(4) - \frac{1}{2}(10)(4)^2 \\
-20 &= v_0 \sin(60)(4) - \frac{1}{2}(10)(4)^2 \\
-20 &= v_0 \cancelto{2\sqrt{3}}{\frac{\sqrt{3}}{2}(4)} - \cancelto{5}{\frac{1}{2}(10)}(16) \\
-20 &= v_0 2\sqrt{3} - 80
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow v_0 = \frac{\cancelto{30}{60}}{\cancel{2}\sqrt{3}} \\
&\Rightarrow v_0 = \frac{30 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} \\
&\Rightarrow v_0 = \frac{\cancelto{10}{30}\sqrt{3}}{\cancel{3}} = 10\sqrt{3}
\end{aligned}$$

Now that we have $v_0 = 10\sqrt{3}$, we can now solve for d:

$$\begin{aligned}
d &= \mathrm{x}(4) \\
\mathrm{x}(4) &= \cancelto{10\sqrt{3}}{v_0} \cos(60)(4) \\
\Rightarrow d &= (10\sqrt{3}) \cos(60)(4)
\end{aligned}$$

$$\begin{aligned}
d &= (10\sqrt{3}) \cancelto{2}{\frac{1}{2}4} \\
d &= (10\sqrt{3})2 \\
d &= 20\sqrt{3}
\end{aligned}$$

Thus, the horizontal distance d traveled by the ball is $20\sqrt{3}$ meters.

Part (b)

In order to find the magnitude of the initial velocity, we need to consider two things:

1. What is velocity in this scenario?
2. What functions can we use to model velocity?

Velocity will be a 2D vector (if you made it 3D, the z-component would just be 0 and have no effect on the magnitude):

$$\left|\vec{\mathbf{V}}\right| = (V_x,\, V_y)$$

Considering that, in terms of calculus, velocity is the derivative of position, we can derive the two functions $\mathrm{x}(t)$ and $\mathrm{y}(t)$ with respect to t to get the components of velocity respectively:

$$\begin{array}{l|l}
V_x(t) = \dfrac{\mathrm{d}}{\mathrm{d}t}[\mathrm{x}(t)] & V_y(t) = \dfrac{\mathrm{d}}{\mathrm{d}t}[\mathrm{y}(t)] \\
V_x(t) = \dfrac{\mathrm{d}}{\mathrm{d}t}[10\sqrt{3}\dfrac{1}{2}t] & V_y(t) = \dfrac{\mathrm{d}}{\mathrm{d}t}[10\sqrt{3}\dfrac{\sqrt{3}}{2}t - \dfrac{1}{2}gt^2] \\
V_x(t) = \dfrac{\mathrm{d}}{\mathrm{d}t}[5\sqrt{3}\,t] & V_y(t) = \dfrac{\mathrm{d}}{\mathrm{d}t}[5 \cdot 3t - \dfrac{1}{2}(10)t^2] \\
V_x(t) = 5\sqrt{3} & V_y(t) = \dfrac{\mathrm{d}}{\mathrm{d}t}[15t - 5t^2] \\
\rightarrow V_x = 5\sqrt{3} & V_y(t) = 15 - 10t
\end{array}$$

Now we can solve for V_y at time $t = 4$. Keep in mind that the functions we derived from start at where the ball landed, which occured at 4 seconds after the ball was launched. The value of these functions at $t = 4$ will still be the same as the functions starting from the ground at $t = 0$:

$$\begin{aligned}
V_y(4) &= 15 - 10(4) \\
V_y(4) &= 15 - 40 = -25 \\
V_y(4) &= -25
\end{aligned}$$

Now we can calculate the magnitude of the initial velocity, in which we know that the magnitude of a vector is the square root of the sum of the squared components:

$$\left|\vec{\mathbf{V}}\right| = \sqrt{{V_x}^2 + {V_y}^2}$$

So plugging in our values:

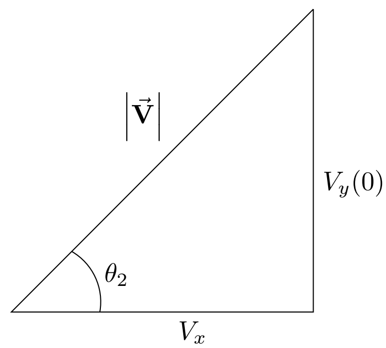
$$\begin{aligned}
\left|\vec{\mathbf{V}}\right| &= \sqrt{5\sqrt{3}^2 + -25^2} \\
&= \sqrt{25 \cdot 3 + 625} \\
&= \sqrt{75 + 625} = \sqrt{700} \\
&= \sqrt{100}\sqrt{7}
\end{aligned}$$

Thus the magnitude of our initial velocity is:

$$\boxed{\left|\vec{\mathbf{V}}\right| = 10\sqrt{7}\,\mathrm{m/s}}$$

Part (c)

We will designate θ_2 to be the angle of the initial velocity vector, and using that angle, we will draw a right triangle:



We can solve for θ_2 using any of the trig functions, but they key is to pass the value we get for the trig function into its respective inverse function:

$$\begin{aligned}
|\arctan(\tan(\theta_2))| &= \left|\arctan\left(\frac{V_y(0)}{V_x}\right)\right| \\
&= \left|\arctan\left(\frac{-25}{5\sqrt{3}}\right)\right| \\
&= \left|\arctan\left(\frac{-5}{\sqrt{3}}\right)\right| \\
&\approx 70.89^\circ \text{ *}{\scriptsize\text{Using a calculator.}}
\end{aligned}$$

$$\begin{aligned}
|\arcsin(\sin(\theta_2))| &= \left|\arcsin\left(\frac{V_y(0)}{\left|\vec{\mathbf{V}}\right|}\right)\right| \\
&= \left|\arcsin\left(\frac{-25}{10\sqrt{7}}\right)\right| \\
&= \left|\arcsin\left(\frac{-5}{2\sqrt{7}}\right)\right| \\
&\approx 70.89^\circ \text{ *}{\scriptsize\text{Using a calculator.}}
\end{aligned}$$

$$\begin{aligned}
|\arccos(\cos(\theta_2))| &= \left|\arccos\left(\frac{V_x}{v_0}\right)\right| \\
&= \left|\arccos\left(\frac{5\sqrt{3}}{10\sqrt{7}}\right)\right| \\
&= \left|\arccos\left(\frac{\sqrt{3}}{2\sqrt{7}}\right)\right| \\
&\approx 70.89^\circ \text{ *}{\scriptsize\text{Using a calculator.}}
\end{aligned}$$

Thus, the magnitude of the angle of our initial velocity is 70.89°.