Friction and Circular Motion

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1 Friction

When a object is moving along a surface, there is a force acting against that movement, and that force is friction $\vec{\mathbf{f}}$.

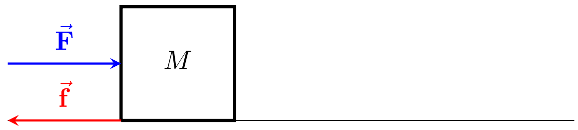
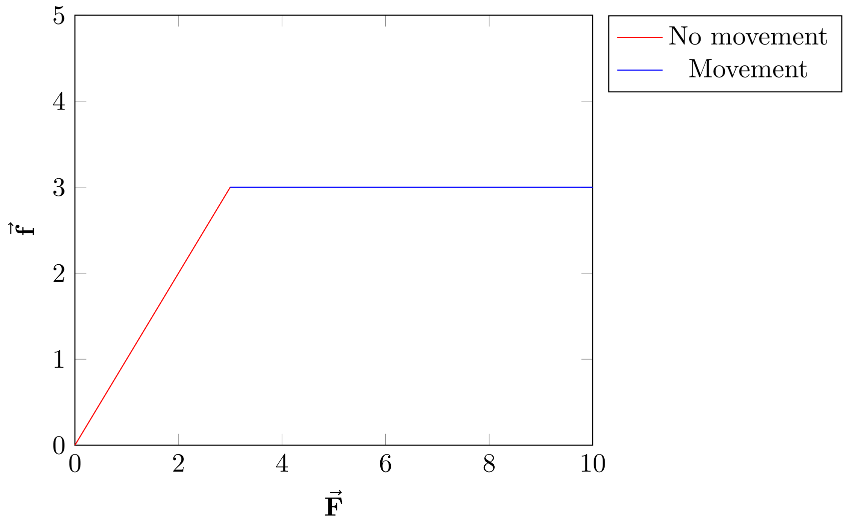


Figure 1: Box moving along ground with friction $\vec{\mathbf{f}}$

The relationship between $\vec{\mathbf{f}}$ and $\vec{\mathbf{F}}$ is linear with a slope of 1, meaning that there is a 1:1 ratio between friction and the force moving the box. As a result, the two forces cancel out and no movement occurs.

However, this only holds true until friction reaches its maximum force:

Figure 2: $\vec{\mathbf{f}}$ with respect to $\vec{\mathbf{F}}$



The maximum force of friction, $\vec{\mathbf{f}}_{\text{max}}$, is defined as the product of the **coefficient of friction**, μ, and the normal force, $\vec{\mathbf{N}}$, acting on the object:

$$\vec{\mathbf{f}}_{\text{max}} = \mu\vec{\mathbf{N}}$$

μ will be a given constant, and it depends on the type of surface where movement is occuring. Slippery surfaces such as ice will have a lower coefficient while rought surfaces such as asphalt or concrete will have a higher coefficient.

2 Cricular Motion

Does a object in circular motion with a constant speed have a force acting on it?

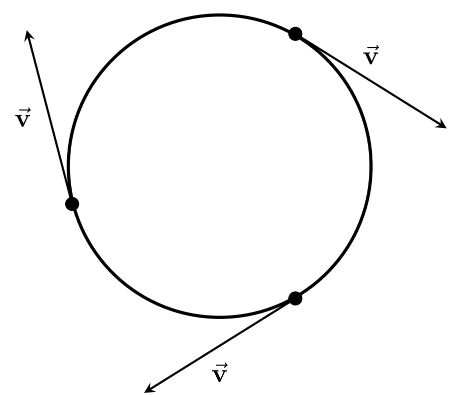


Figure 3: Velocity of object in circular motion

An object in circular motion does have a force acting on it because the direction of velocity changes, which means the componenets of velocity change, and for there to be change in velocity, there must be acceleration.

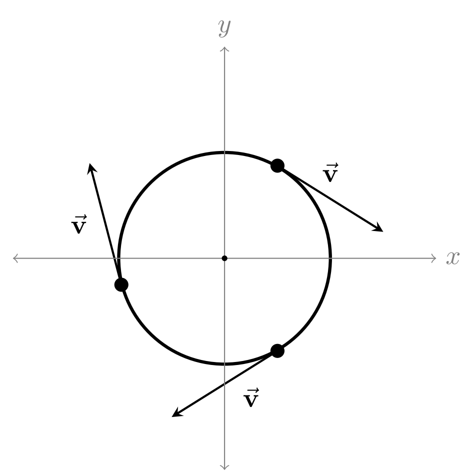
However, the magnitude of velocity, **speed**, in this scenario is costant, it is only the componenets that are changing:

$$|\vec{\mathbf{v}}| = \sqrt{{v_x}^2 + {v_y}^2}$$
$$\frac{\mathrm{d}|\vec{\mathbf{v}}|}{\mathrm{d}t} = 0$$
$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = a_x$$
$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = a_y$$

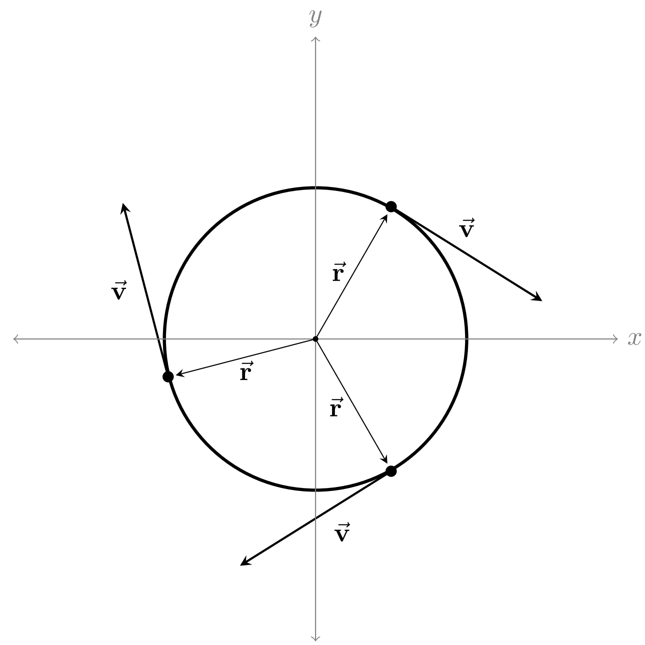
if you forgot, acceleration $\vec{\mathbf{a}}$ is also a vector, with x and y componenets acting on the respective x and y componenets of velocity $\vec{\mathbf{v}}$.

2.1 Determing the Componenets of Velocity

We can redraw Figure 3 with axes x and y:



We can also draw displacement vectors $\vec{\mathbf{r}}$ from the origin to the start of the velocity vectors:



Decomposing the displacement vectors into r_x and r_y will form triangles, which will allow us to solve for those componenets of displacement:

$$r_x = r\cos(\theta)$$
$$r_y = r\sin(\theta)$$

we can then differeniate to get the componenets of velocity:

$$v_x = \frac{\mathrm{d}r_x}{\mathrm{d}t} = -r\sin(\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t}$$
$$v_y = \frac{\mathrm{d}r_y}{\mathrm{d}t} = r\cos(\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

if you can visualize it, as the object moves around the circle, the displacement vector $\vec{\mathbf{r}}$ follows it, and so componenets of the displacement vector change. Thus, a new triangle is formed, which means a new value for θ, thus θ does have a rate of change, which we can call **angular velocity** and comes out during differentiation as seen above.

The rate of change of θ will be represented as omega ω:

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

we will assume that ω is a constant, meaning the rate of change of the angle does not change over time, thus we can rewrite the componenets as:

$$v_x = -r\,\omega\sin(\theta)$$
$$v_y = r\,\omega\cos(\theta)$$

2.2 Finding the Components of Acceleration

We can differeniate the components velocity to get acceleration:

$$a_x = \frac{\mathrm{d}v_x}{\mathrm{d}t} = -r\,\omega\cos(\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t}$$
$$a_y = \frac{\mathrm{d}v_x}{\mathrm{d}t} = -r\,\omega\sin(\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

we can simplify this to:

$$a_x = -r\,\omega^2\cos(\theta)$$
$$a_y = -r\,\omega^2\sin(\theta)$$

2.3 Relationship Between Displacement, Velocity, and Acceleration

Since:

$$r_x = r\cos(\theta) \qquad r_y = r\sin(\theta) \tag{1}$$

then,

$$a_x = -\omega^2 \cdot r\cos(\theta) \implies \boxed{a_x = -\omega^2 r_x}$$
$$a_y = -\omega^2 \cdot r\sin(\theta) \implies \boxed{a_y = -\omega^2 r_y}$$

We can also relate the vectors as a whole with these relations:

$$\begin{aligned}
\vec{\mathbf{a}} &= \sqrt{{a_x}^2 + {a_y}^2} \\
&= \sqrt{-\omega^2 {r_x}^2 + -\omega^2 {r_y}^2} \\
&= \sqrt{(-\omega^2)^2(r_x)^2 + (-\omega^2)^2(r_y)^2} \\
&= \sqrt{(-\omega^2)^2\left[(r_x)^2 + (r_y)^2\right]} \\
&= \sqrt{(-\omega^2)^2}\sqrt{\left[(r_x)^2 + (r_y)^2\right]} \\
&= (-\omega^2)\sqrt{(r_x)^2 + (r_y)^2}
\end{aligned}$$

We know that:

$$|\vec{\mathbf{r}}| = \sqrt{{r_x}^2 + {r_y}^2}$$

thus:

$$\vec{\mathbf{a}} = -\omega^2\vec{\mathbf{r}}$$

$$\boxed{\vec{\mathbf{a}} = -\omega^2\vec{\mathbf{r}}}$$

We can also use the same substitutions 1 with velocity:

$$v_x = -\omega \cdot r\sin(\theta) \implies v_x = -\omega r_y$$
$$v_y = \omega \cdot r\cos(\theta) \implies v_x = \omega r_x$$

Which also means:

$$
\begin{aligned}
\vec{\mathbf{v}} &= \sqrt{{v_x}^2 + {v_y}^2} \\
&= \sqrt{(-\omega r_y)^2 + (\omega r_x)^2} \\
&= \sqrt{(-\omega)^2 (r_y)^2 + (\omega)^2 (r_x)^2} \\
&= \sqrt{\omega (r_y)^2 + \omega (r_x)^2} \\
&= \sqrt{\omega^2 (r_y)^2 + \omega^2 (r_x)^2} \\
&= \sqrt{\omega^2} \sqrt{(r_y)^2 + (r_x)^2} \\
&= \omega \sqrt{(r_x)^2 + (r_y)^2} \\
&= \omega \vec{\mathbf{r}}
\end{aligned}
$$

$$
\boxed{\vec{\mathbf{v}} = \omega \vec{\mathbf{r}}}
$$