Work and Spring Force

Laith Toom

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1 What is Work?

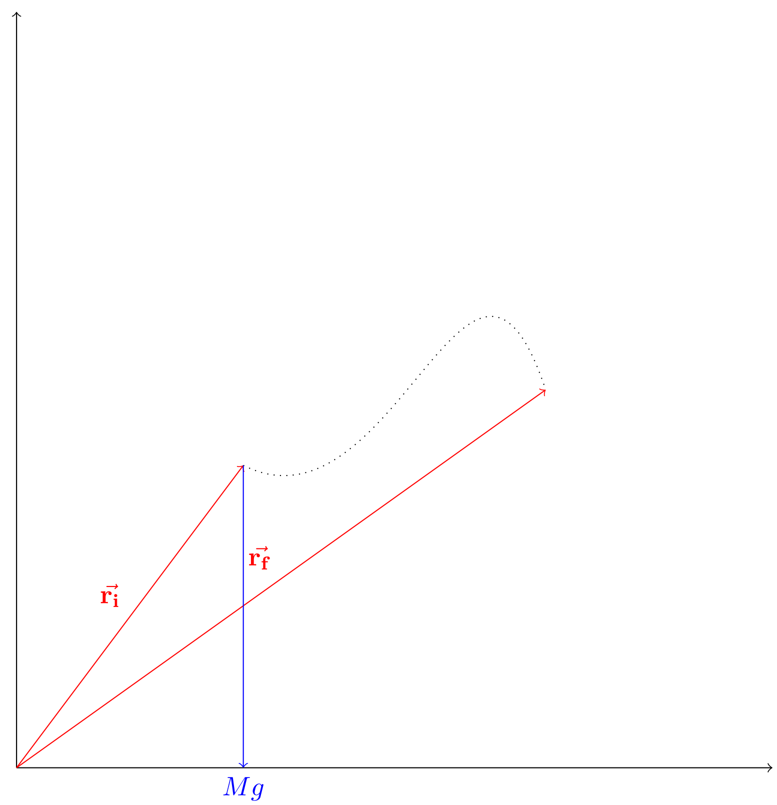
Work, W, is the change in kinetic energy:

$$W = \frac{1}{2}M|\vec{\mathbf{v_f}}|^2 - \frac{1}{2}M|\vec{\mathbf{v_i}}|^2 = \boxed{\frac{M}{2}\left(|\vec{\mathbf{v_f}}|^2 - |\vec{\mathbf{v_i}}|^2\right)}$$

Work is also equal to the dot product between a force vector and displacement vector:

$$W = \left|\vec{\mathbf{F}}\right| \cdot |\vec{\mathbf{r}}| \cdot \cos(\theta)$$

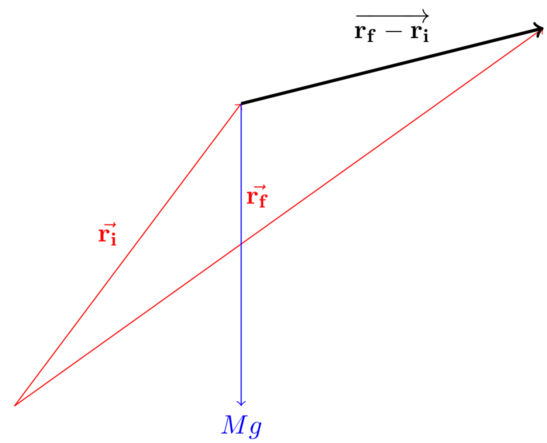
In the case of complex trajectory, such as:



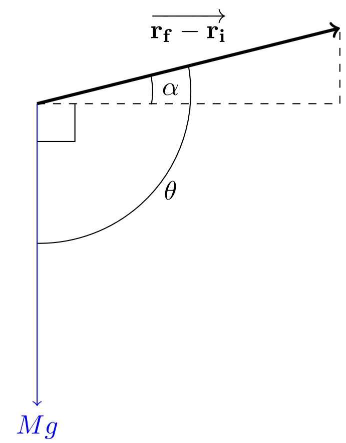
then work can be calculated by taking the integral of the trajectory, where the trajectory will be divided into very small vectors $\mathrm{d}\vec{\mathbf{r}}$:

$$\begin{align} W &= \int_{\vec{\mathbf{r_i}}}^{\vec{\mathbf{r_f}}} \left|\vec{\mathbf{F}}\right| \cdot \mathrm{d}\vec{\mathbf{r}} \tag{1} \\ &= \int_{\vec{\mathbf{r_i}}}^{\vec{\mathbf{r_f}}} Mg\,\hat{y} \cdot \mathrm{d}\vec{\mathbf{r}} \tag{2} \\ &= Mg\,\hat{y} \int_{\vec{\mathbf{r_i}}}^{\vec{\mathbf{r_f}}} \mathrm{d}\vec{\mathbf{r}} \tag{3} \\ &= Mg\,\hat{y}\,(\vec{\mathbf{r_f}} - \vec{\mathbf{r_i}}) \tag{4} \end{align}$$

Subtracting the two vectors $\vec{\mathbf{r_f}}$ and $\vec{\mathbf{r_i}}$ will result in a new vector, $\overrightarrow{\mathbf{r_f} - \mathbf{r_i}}$:



and if we look closer, we will find a right triangle:



We see that:

$$\alpha = \theta - 90 \iff \theta = \alpha + 90$$

Work is the dot product between the force vector and displacement vector. In our case, the force vector is Mg and the displacement vector is $\overrightarrow{\mathbf{r_f} - \mathbf{r_i}}$, and we see that θ is the angle between the two vectors, thus:

$$W_g = Mg \cdot \left|\overrightarrow{\mathbf{r_f} - \mathbf{r_i}}\right| \cdot \cos(\theta) \tag{5}$$

Since $\theta = \alpha + 90$, we can rewrite this as:

$$W_g = Mg \cdot \left|\overrightarrow{\mathbf{r_f} - \mathbf{r_i}}\right| \cdot \cos(\alpha + 90) \tag{6}$$

and since $\cos(A + 90) = -\sin(A)$, we can rewrite this again as:

$$W_g = Mg \cdot \left|\overrightarrow{\mathbf{r_f} - \mathbf{r_i}}\right| \cdot - \sin(\alpha) \tag{7}$$

Thus, we find the formula:

$$\boxed{W_g = -Mg\left|\overrightarrow{\mathbf{r_f} - \mathbf{r_i}}\right| \sin(\alpha)}$$