# Algorithmic Game Theory - HW1

Ido Kessler - 311398499, Jonathan Somer - 307923383

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# Part I
# Auctions

## Problem 1:

Denote $N$ to be the set of all bidder, such that $S \subseteq N$.

### Conditions:

- The bidder with the maximum private value is in $S$. We will denote her as $i_{max} = \underset{i \in S}{argmax}(v_i) = \underset{i \in N}{argmax}(v_i)$
- The second highest private value is also in $S$.
  $i_{second} = \underset{i \in S \setminus \{i_{max}\}}{argmax}(v_i) = \underset{i \in N \setminus \{i_{max}\}}{argmax}(v_i)$

### Sufficiency:

The item will always be allocated to $i_{max}$ , and she will pay the bid of the second highest bidder $i_{second}$. As $i_{second} \in S$, she can untruthfully lower her bid and improve the collective payoff. Bidder $i_{max}$ will pay less then he would have played if $i_{second}$ wouldn't have lied.

### Necessity:

**Assume by contradiction that $i_{max} \notin S$:** Then $i_{max} \in N \setminus S$ and as such no bidder in $S$ will get the item if bids are truthful. In this case $S$ collective payoff is 0. Any untruthful change on the bids in $S$ such that the maximum remains in $N \setminus S$ will not change the resulting payoff. Assume a bid exceeded $v_{i_{max}}$ $(b_{i_{max}})$, then the second is at least $v_{i_{max}}$, which is higher then any private value in $S$, resulting in a negative payoff. Thus if $i_{max} \notin S$ the collective payoff cannot be increased by untruthful bidding.
**We can now assume that $i_{max} \in S$; assume by contradiction that**

$i_{second} \notin S$**:** Any untruthful change in the bid of $i_{max}$ will not effect the resulting payoff as long as it's not less then $i_{second}$. Bidding less then $i_{second}$ will result in $i_{max}$ not getting the item and the payoff to be either 0 or negative (in case another bidder in $S$ bid a higher value then $i_{second}$). Any untruthful change in the bids of $S \backslash \{i_{max}\}$ will result in $i_{max}$ paying more, or a negative payoff. In this case the payoff cannot be increased. $\square$

# Problem 2:

## 1

Let there be some lopsided setting with an optimal allocation $T^*$ such that it holds that $\sum\limits_{i \in A} v_i(T_i^*) \geq \frac{1}{2} \sum_{i=1}^n v_i(T_i^*)$.

$$\frac{1}{2} \sum_{i=1}^n v_i(T_i^*) \leq \sum_{i \in A} v_i(T_i^*)$$
$$\frac{1}{2\sqrt{m}} \sum_{i=1}^n v_i(T_i^*) \leq \frac{1}{\sqrt{m}} \sum_{i \in A} v_i(T_i^*)$$
$$\frac{1}{2\sqrt{m}} \sum_{i=1}^n v_i(T_i^*) \leq \frac{1}{\sqrt{m}} \sum_{i \in A} v_i(T_i^*) \leq max_{i \in A} v_i(T_i^*)$$
$$\frac{1}{2\sqrt{m}} \sum_{i=1}^n v_i(T_i^*) \leq max_{i \in A} v_i(T_i^*) \leq max_i v_i(T_i^* \cup M \backslash T_i^*) = max_i v_i(M)$$

$\square$

## 2

Let there be some **none** lopsided setting with an optimal allocation $T^*$. Then it hold that $\sum\limits_{i \in A} v_i(T_i^*) < \frac{1}{2} \sum_{i=1}^n v_i(T_i^*)$.
Note that $\sum\limits_{i \in A} v_i(T_i^*) + \sum\limits_{i \notin A} v_i(T_i^*) = \sum_{i=1}^n v_i(T_i^*)$, so:

$$\sum_{i \in A} v_i(T_i^*) < \frac{1}{2} \sum_{i=1}^n v_i(T_i^*) \Rightarrow \sum_{i \notin A} v_i(T_i^*) > \frac{1}{2} \sum_{i=1}^n v_i(T_i^*)$$

Due to subadditivity:

$$\frac{1}{\sqrt{m}} \sum_{i \notin A} \sum_{t \in T_i^*} v_i(\{t\}) \geq \frac{1}{\sqrt{m}} \sum_{i \notin A} v_i(T_i^*) > \frac{1}{2\sqrt{m}} \sum_{i=1}^n v_i(T_i^*)$$

As $|T_i^*| < \sqrt{m}$:

$$\sum_{i\notin A} max_{t\in T_i^*}(v_i(\{t\})) \geq \frac{1}{\sqrt{m}}\sum_{i\notin A}\sum_{t\in T_i^*} v_i(\{t\}) > \frac{1}{2\sqrt{m}}\sum_{i=1}^{n} v_i(T_i^*)$$

Notice that the left size of the inequality is a description of a allocation of 1 item or less per bidder. □

## 3

We will use the results of the last two sections where we arrived at allocation $T$ that satisfied $SW_T = \Omega(\frac{OPT}{\sqrt{m}})$
We also know that $SW_T \leq OPT$, and as such $OPT \leq \sqrt{m}SW_T \leq \sqrt{m}OPT$. This means that $\sqrt{m}SW_T$ is $O(\sqrt{m})$-approximate by definition.

### Algorithm:

As we do not know if this is a lopsided setting or not. We will run two "sub-algorithms" that will produce a $O(\sqrt{m})$-approximate result for each setting. Due to the fact that the closer solution must be the larger we can simply return the maximum of both.
**For lopsided settings:**
Evaluate the $SW$ resulting from allocationg all goods to a single player for each of the players. Denote the resulting allocation as $T_{lop}$
**For none-lopsided settings:**
Due to the fact that in the non-lopsided setting there exists an $O(\sqrt{m})$-approximate allocation which allocates at most one item to each bidder we can use a "bipartite matching" solution. Create a bipartite graph with the $n$ bidders connected to the $m$ items with edges weighted $v_i(M_j)$ on the edge between the bidder $i$ and the item $j$. Using max maching algorithm we can compute the maximal match in $O(V^2E)$, which in our case is $O((m+n)^2 \cdot mn)$, which is poly-time. Denote the resulting allocation as $T_{non-lop}$
Running both algorithms is still poly-time, and choosing the maximal allocation between $T_{lop}$ and $T_{non-lop}$ guarantees a $O(\sqrt{m})$-approximate allocation.

## 4

We will use the result of section 3 in order to archive a $SW$ allocation in poly-time, and archive a truthful combinatorial auction using the VCG (Vickrey–Clarke–Groves) auction mechanism.

### Prices:

Denote $w^*$as the allocation that was chosen, $\Omega$ as the group of all possible allocation,$b$ as the vector of the bidder bids for all subgroups, and $b_i(w)$ as the bid that player $i$ was allocated with in the allocation $w$.

Using the VCG mechanism, we know that bidder $i$ will pay using the next expression:

$$p_i(b) = \underset{w \in \Omega}{max} \underset{j \neq i}{\Sigma} b_j(w) - \underset{j \neq i}{\Sigma} b_j(w^*)$$

**Caculating the values in polytime:**

Computing $\underset{j \neq i}{\Sigma} b_j(w^*)$ is poly-time, as it only sun $n$ values and as such calculating this is in poly-time - $O(n)$.
Computing $\underset{w \in \Omega}{max} \underset{j \neq i}{\Sigma} b_j(w)$ depend on the ability to find the maximum $w$ in $\Omega$. In our case, $\Omega$ is the available possible allocations, which contain two diffrent sets of allocation types: lopsided allocations and non-lopsided allocations.
**Lopsided alloctions:** Allocations containing only one bidder who recive all the items. There are only $n$ such allocations, and as such the maximum $w$ can be found in a $O(n^2)$ time using a naive approch - poly-time.
**Non-Lopsided Allocations:** Allocations that for each bidder allocate no more then 1 item. The number of such allocations is not in poly-space, and as such the "naive" computation of the expression is not in poly-time as well. To solve this we can use a "bipartite matching" solution to find the maximal match without the current bidder $i$. Create a bipartite graph with the $n - 1$ bidders (Without the current bidder $i$) connected to the $m$ items with edges weighted $v_k(M_j)$ on the edge between the bidder $k$ and the item $j$. Using max maching algorithm we can compute the maximal match in $O(V^2E)$, which in our case is $O((m+n)^2 \cdot mn)$, which is poly-time.
After computing the maximum value for both the lopsided case and the non-lopsided case, we can take the maximal value.

**Complexity:**

In order to compute the prices for all the bidders we need to run the algorithm $n$ times: $O(n * (n + n^2 + (m+n)^2 \cdot mn))$ which is in poly-time.

**Correctness:**

The VCG mechanism assure us that truthful strategy is a dominent strategy for this auction. While the bids for all the subset of the items that are bigget then 2 and are less then m, are not used in the allocation and prices, it still holds that the truthful strategy is dominant. This holds because while the bidder can lie in all the bids that are not used - it will hold no effect on the result, so the truthful strategy is one of many dominant strategies in this auction.

# Part II
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## Problem 3:

### Allocation rule:

Define $b'_i = v_i\beta_i = b_i\beta_i$. Our new allocation assigns the $i$th highest $\mathbf{b}'$ bidder to the $i$th best slot.

#### Claim: the new allocation rule is monotone

Recall the definition for a **monotone allocation rule:**

> An allocation rule $\mathbf{x}$ for a single-paramater environment is monotone if for every agent $i$ and bids $\mathbf{b_{-i}}$, the allocation $\mathbf{x_i}(z, \mathbf{b_{-i}})$ is non-decreasing in her bid $z$.

It is clear that $\mathbf{x_i}(z, \mathbf{b'_{-i}})$ is non-decreasing in $z$, because increasing $z$ will result in a higher place amongst the $b'$ bids (assuming $\forall i : \beta_i \geq 0$).

### DSIC Pricing:

Recall Myerson's payment formula:

$$p_i(\mathbf{b}) = \sum_{j=1}^{k} b_{j+1}(\alpha_j - \alpha_{j+1})$$

We define a new payment formula:

$$p_i(\mathbf{b}') = \sum_{j=1}^{k} b'_{j+1}(\alpha_j - \alpha_{j+1}) = \sum_{j=1}^{k} b_{j+1}\beta_{j+1}(\alpha_j - \alpha_{j+1})$$

#### Claim: the new payment formula is DSIC

Assume that in some other setting, the players' valuations correspond to $\mathbf{b}'$. That is, player $i'$ has a valuation and bid of $v'_i = b'_i = b_i\beta_i$. By Myerson's lemma our payment formula is DSIC, as the allocation rule is the same one used in the original proof and the payment is too.
Assume by contradiction that some player $i$ in the original setting can benefit from non-truthful bidding under the new payment scheme. Let $b_{i\text{-fake}} \neq b_i$ be the new and more benefitial bid for player $i$. Since $b'_i = b_i\beta_i \neq b_{i\text{-fake}}\beta_i$ it follows that player $i'$ could benefit by bidding $b_{i\text{-fake}}\beta_i$ instead of $b'_i$. In contradiction to Myerson's lemma.
□

# Part III
# VCG

## Problem 4:

### 1

We shall show a case where this is true.

**The initial state (truthful state):**

- Two items: $\{i_1, i_2\}$
- Two players: $\{1, 2\}$
- Valuations & bids:

$$b_1(\text{set}) = v_1(\text{set}) = \begin{cases} 2 & \text{set} = \{i_1, i_2\} \\ 0 & \text{else} \end{cases}$$

$$b_2(\text{set}) = v_2(\text{set}) = \begin{cases} 1 & \text{set} = \{i_1, i_2\} \\ 0 & \text{else} \end{cases}$$

In this case the $VCG$ allocates both items to player 1. And her payment is given by:

$$p_1 = \max_{\omega \in \Omega} \sum_{j \neq i} b_i(\omega) - (\sum_{j \neq i} b_i(\omega^*)) = 1 - 0 = 1$$

$$u_1 = v_1(\text{set}) - p_1 = 2 - 1 = 1$$

We now show how player 1 can bid non-truthfully and increase her utility:

**Player 1 bids untruthfully:**

- Two items: $\{i_1, i_2\}$
- Three players: $\{1_a, 1_b, 2\}$
- Bids:

$$b_{1_a}(\text{set}) = \begin{cases} 1 & \text{set} = \{i_1\} \\ 0 & \text{else} \end{cases}$$

$$b_{1_b}(\text{set}) = \begin{cases} 1 & \text{set} = \{i_2\} \\ 0 & \text{else} \end{cases}$$

$$b_2(\text{set}) = v_2(\text{set}) = \begin{cases} 1 & \text{set} = \{i_1, i_2\} \\ 0 & \text{else} \end{cases}$$

In this case $VCG$ allocates $i_1$ to $1_a$ and $i_2$ to $1_b$ acheiving a maximal welfare of 2. The payment of players $1_a, 1_b$ :

$$p_{1_a} = \max_{\omega \in \Omega} \sum_{j \neq i} b_i(\omega) - (\sum_{j \neq i} b_i(\omega^*)) = 1 - 1 = 0$$

$$p_{1_b} = \max_{\omega \in \Omega} \sum_{j \neq i} b_i(\omega) - (\sum_{j \neq i} b_i(\omega^*)) = 1 - 1 = 0$$

And in total:

$$u_1 = v_1(\{i_1, i_2\}) - (p_{1_a} + p_{1_b}) = 2 - 0 = 2$$

By bidding untruthfully we have improved the utility of player 1 by 1.

## 2

We shall disprove the claim. We assume the other bidders bid truthfully.

### CASE 1: con-bidder's value is less than the top bid made by another player

Utility will be 0 for con-bidder as the item is not allocated to him. Any non-truthful bid which does not change the top bid will have no effect on the resulting allocation. In addition any non-truthful which does change the top bid will result in con-bidder paying more than he values and this will result in a negative utility. Thus, in the case where the con-bidder's value is less than the top bid made by another player there is no way for her to improve her outcome via non-truthful bidding.

### CASE 2: con-bidder's value is more than the top bid made by another player:

In a second price auction, the two bids which affect the resulting allocation and pricing (the "result") are the first and second bids. We shall use this fact in order to define our cases.

#### case 2a: Highest bid is not made by the con-bidder

In this case the item will not be allocated to our player. By bidding truthfully our player would have acheived a positive utility, so this case does not improve the outcome for our player.

#### case 2b: Highest bid is made by the con-bidder

It does not matter what the exact value of the first bid is because its only effect is allocating the item to our player. Any non-truthful bid lower then the highest bid made by another player will not affect the pricing. Any non-truthful bid higher than the second highest bid will result in our player paying more but does not change the allocation and is a worse outcome overall.

□

# Part IV
# Online bipartite matching

## 1

### (i) $u_i^{\pi,\mathbf{p}} \geq 1-p$ for any $p_j$

If $p_j \geq p$ then item $j$ will not have an effect on the items available to player $i$ and player $i$ will have a utility of $1-p$. If $p_j \leq p$, item j' (the one player $i$ took under $\mathbf{p_{-j}}$) will still be available to player $i$ so $u_i^{\pi,\mathbf{p}} \geq 1-p$. (it is an easy induction to show that any player before $i$ might improve her utility but none will worsen it by taking $j'$)

### (ii)

Assume by contradiction that item $j$ is not matched. Then in player $i$'s turn $j$ was not matched. The set of of matched items at this point must be the same as under $\mathbf{p_{-j}}$ or else at some point some player made a sub-optimal decision. In player $i's$ turn she chooses item with $p > p_j$ in contradiction to the way a player chooses an item.

## 2

Calculating $\mathbb{E}_\omega[u_i]$, using (i):

$$\mathbb{E}_\omega[u_i] = [\text{fix some random order on } L \ \pi \text{ and some } \mathbf{p}_{-j}] = \mathbb{E}_\omega[u_i^{\pi,\mathbf{p}}] \geq 1-p$$

Where $p$ is the price of the item agent $i$ would take in $R \backslash \{j\}$ under $\mathbf{p}_{-j}$ (as in (i) we set $p=1$ if there is no such). The last inequality is simply due to $u_i^{\pi,\mathbf{p}} \geq 1-p$, so the expectation must be larger than $1-p$ too.
Calculating $\mathbb{E}_\omega[r_j]$ using (ii):

$$\mathbb{E}_\omega[r_j] = \mathbb{E}_\omega[\mathbf{1}^{\text{item j was taken by player i}} \cdot p_j] \underset{\text{from (ii)}}{=} \mathbb{E}_\omega[\mathbf{1}^{p_j<p} \cdot p_j] =$$

Notice that we can view this as the expectation of a continuous uniform random variable, between 0 to 1, and compute $\mathbb{E}_\omega[r_j]$ as the integral over all value from 0 to 1. Also notice that because we are working with a uniform distribution over $[0,1]$ the density is $\frac{1}{b-a}$ which in our case is 1.

We will denote $p = e^{\omega-1}$, $p_j = e^{\omega_j-1}$:

$$\int_0^1 1\cdot(\mathbf{1}^{\mathbf{p_j}<\mathbf{p}} \cdot e^{\omega_j-1})d\omega_j = \int_w^1 0\cdot e^{\omega_j-1}d\omega_j + \int_0^\omega 1\cdot e^{\omega_j-1}d\omega_j = e^{\omega-1} - e^{-1} = p - \frac{1}{e}$$

Finally, using the linearity of expecation:

$$\mathbb{E}_\omega[u_i + r_j] = \mathbb{E}_\omega[u_i] + \mathbb{E}_\omega[r_j] \geq (1-p) + (p - \frac{1}{e}) = 1 - \frac{1}{e}$$

□