Übungsblatt 12

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1 Aufgabe 2

1.1 a)

$f : \mathbb{R}^3 \to \mathbb{R}^3, (x, y, z) \to (x - y, y - z, z - x)$
Bild bestimmen:

$$\begin{pmatrix}1\\0\\1\end{pmatrix} \to \begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$\begin{pmatrix}0\\1\\0\end{pmatrix} \to \begin{pmatrix}-1\\1\\0\end{pmatrix}$$

$$\begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}0\\-1\\1\end{pmatrix}$$

$$\to \text{Matrix: } M = \begin{pmatrix}1 & -1 & 0\\0 & 1 & -1\\-1 & 0 & 1\end{pmatrix}$$

$\to$ Rang:

$$\begin{pmatrix}1 & -1 & 0\\0 & 1 & -1\\-1 & 0 & 1\end{pmatrix}_{|+\mathrm{I}} \to \begin{pmatrix}1 & -1 & 0\\0 & 1 & -1\\0 & -1 & 1\end{pmatrix}_{|+\mathrm{II}} \to \begin{pmatrix}1 & -1 & 0\\0 & 1 & -1\\0 & 0 & 0\end{pmatrix} \implies \text{Rang}(M) = 2$$

$$Im(f) = \left\langle \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\\0\end{pmatrix} \right\rangle, \text{ da } \begin{pmatrix}0\\-1\\1\end{pmatrix} \text{ linear abhänging}$$

Orthonormale Basis von $Im(f)$

$$\vec{w_1} = \begin{pmatrix}1\\0\\-1\end{pmatrix} \vec{w_2} = \begin{pmatrix}-1\\1\\0\end{pmatrix}$$

$$\vec{v_1} = \frac{\vec{w_1}}{||w_1||} = \frac{\begin{pmatrix}1\\0\\-1\end{pmatrix}}{\sqrt{1^2 + (-1)^2}} = \frac{1}{\sqrt{2}\begin{pmatrix}1\\0\\-1\end{pmatrix}}$$

$$\begin{aligned}
\vec{v_2'} &= \vec{w_2} = \langle w_2, \vec{v_1} \rangle = v_1 \\
&= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle * \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{2}} \left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle * \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle * \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2} \left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle * \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}(-1) * \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{2}(-1) * \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0,5 \\ 0 \\ -0,5 \end{pmatrix} \\
&= \begin{pmatrix} -0,5 \\ 1 \\ -0,5 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\vec{v_2} &= \frac{v_2'}{||v_2||} \\
&= \frac{1}{(-0,5)^2 + 1^2 + (-0,5)^2} * \begin{pmatrix} -0,5, 1, -0,5 \end{pmatrix} \\
&= \frac{1}{\sqrt{1,5}} * \begin{pmatrix} -0,5, 1, -0,5 \end{pmatrix}
\end{aligned}$$

1.1.1 Orthonormalbasis von $Im(f)^+$

$V = U \oplus U^+$ nach Proposition 9.10

$$\to \mathbb{R}^3 = Im(f) \oplus Im(f)^+$$

$Im(f)$ hat 2 Vektoren $\to$ 3. Vektor gesucht der linear unabhänging ist:

$$\vec{v_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \vec{v_3} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

zu zeigen: v_3 linear unabhänging:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right)^{|+II} \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)^{|*(-1)}$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)_{|-I} \begin{array}{l} \to a = 0 \\ \to a = 0 \\ \to a = 0 \end{array}$$

$\{v_1, v_2\} \oplus \{v_3\} \implies \{v_3\}$ ist Basis von $Im(f)^\perp$

Orthonormalisierung:
$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ist bereits orthonormal $\implies \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$ ist orthonormale Basis von $Im(f)^\perp$

1.2 b)

$$B = \begin{pmatrix} -2 & 4 & 3 \\ 0 & 0 & 0 \\ -1 & 5 & 2 \end{pmatrix}$$

zu zeigen:

$$B^5 93 - 2B^1 5 = B$$

1.2.1 Diagoanalisierung:

charakteristisches Polynom:

$$det \begin{pmatrix} -2-\lambda & 4 & 3 \\ 0 & -\lambda & 0 \\ -1 & 5 & 2-\lambda \end{pmatrix} = (-2-\lambda) det \begin{pmatrix} -\lambda & 0 \\ 5 & 2-\lambda \end{pmatrix} - 0 * det \begin{pmatrix} 4 & 3 \\ 5 & 2-\lambda \end{pmatrix} - 1 * det \begin{pmatrix} 4 & 3 \\ -\lambda & 0 \end{pmatrix}$$

$$\begin{aligned}
&= (-2-\lambda)((2-\lambda)(-\lambda) - 0 * 5) - 1 * ((4 * 0) - 3(-\lambda)) \\
&= (-2-\lambda)(-2\lambda + \lambda^2) - 1 * (3\lambda) \\
&= (-2-\lambda)(-2\lambda + \lambda^2) - (3\lambda) \\
&= 4\lambda - 2\lambda^2 + 2\lambda^2 - \lambda^3 - 3\lambda \\
&= -\lambda^3 + \lambda \\
&= \lambda(\lambda^2 + 1)
\end{aligned}$$

Eigenwerte:

$$\lambda(\lambda^2 + 1) = 0$$

$$\begin{aligned}
\lambda_1 &= 0 \\
-\lambda^2 + 1 &= 0 && | + \lambda^2 \\
\lambda^2 &= 1 && |\surd \\
\lambda_2 &= 1 \\
\lambda_3 &= -1
\end{aligned}$$

1.2.2 Eigenvektoren:

$$\implies \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = -1$$
$$\lambda_1 = 0$$

$$\left(\begin{array}{ccc|c} -2 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 5 & 2 & 0 \end{array}\right)^{|*(-0,5)} \to \left(\begin{array}{ccc|c} 1 & -2 & -1,5 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 5 & 2 & 0 \end{array}\right)_{|+\mathrm{I}}$$

$$\to \left(\begin{array}{ccc|c} 1 & -2 & -1,5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0,5 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & -1,5 & 0 \\ 0 & 3 & 0,5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad |\text{II und III vertauschen und II} : 3$$

$$\to \left(\begin{array}{ccc|c} 1 & -2 & -1,5 & 0 \\ 0 & 1 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)^{|+2\mathrm{II}} \to \left(\begin{array}{ccc|c} 1 & 0 & -\frac{7}{6} & 0 \\ 0 & 1 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\mathrm{I} : x - \frac{7}{6}z = 0$$
$$\mathrm{II} : y - \frac{1}{6}z = 0$$
$$y = -\frac{1}{6}z$$
$$x = \frac{7}{6}z$$
$$\implies \vec{a} = \begin{pmatrix} 7 \\ -1 \\ 6 \end{pmatrix}$$

$$\lambda_2 = 1$$
$$\begin{pmatrix} -3 & 4 & 3 \\ 0 & -1 & 0 \\ -1 & 5 & 1 \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathrm{I} : -3x + 4y + 3z = 0$$
$$\mathrm{II} : -y = 0 \implies y = 0$$
$$\mathrm{III} : -x + 5y + z = 0$$
$$y = 0 \implies \mathrm{I} : -3x + 3z = 0 \text{ und}$$
$$\mathrm{III} : -x + z = 0$$
$$\implies x = z$$
$$\implies \vec{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_3 = -1$$
$$\begin{pmatrix} -1 & 4 & 3 \\ 0 & 1 & 0 \\ -1 & 5 & 3 \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \mathrm{I}: -x+4y+3z&=0\\ \mathrm{II}: y&=0\\ \mathrm{III}: -x+5y+3z&=0\\ y=0 \implies \mathrm{I}: -x+3z&=0 \text{ und}\\ \mathrm{III}: -x+3z&=0\\ \implies x&=3z\\ \implies \vec{c}&=\begin{pmatrix}3\\0\\1\end{pmatrix} \end{aligned}$$

1.2.3 Eigenräume:

$$E_B(0)=\left\{\lambda * \begin{pmatrix}7\\-1\\6\end{pmatrix}\middle|\lambda\in\mathbb{R}\right\}$$

$$E_B(1)=\left\{\lambda * \begin{pmatrix}1\\0\\1\end{pmatrix}\middle|\lambda\in\mathbb{R}\right\}$$

$$E_B(-1)=\left\{\lambda * \begin{pmatrix}3\\0\\1\end{pmatrix}\middle|\lambda\in\mathbb{R}\right\}$$

$$D=\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix}$$

$$T=\begin{pmatrix}7&1&3\\-1&0&0\\6&1&1\end{pmatrix}, T^{-1}=\begin{pmatrix}0&-1&0\\-0,5&5,5&1,5\\0,5&0,5&-0,5\end{pmatrix}$$

$$B=TDT^{-1}$$

$$\begin{pmatrix}-2&4&3\\0&0&0\\-1&5&2\end{pmatrix}=\begin{pmatrix}7&1&3\\-1&0&0\\6&1&1\end{pmatrix}*\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix}*\begin{pmatrix}0&-1&0\\-0,5&5,5&1,5\\0,5&0,5&-0,5\end{pmatrix}$$

$$B^n=TD^nT^{-1}$$

zu zeigen: $B^{593}-2B^{15}=-B$

$$\begin{aligned} B^n=TD^nT^{-1} &\implies B^{593}=TD^{593}T^{-1}\\ &\implies D^{593}=\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix}^{593}=\begin{pmatrix}0^{593}&0&0\\0&1^{593}&0\\0&0&-1^{593}\end{pmatrix}=D\\ &\implies B^{593}=TDT^{-1}\\ &\implies B^{593}=B \end{aligned}$$

$$
\begin{aligned}
B^n = TD^nT^{-1} &\implies 2B^{15} = 2TD^{15}T^{-1} \\
&\implies D^{15} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}^{15} = \begin{pmatrix} 0^{15} & 0 & 0 \\ 0 & 1^{15} & 0 \\ 0 & 0 & -1^{15} \end{pmatrix} = D \\
&\implies 2B^{15} = 2TD^{15}T^{-1} = 2TDT^{-1} = 2B
\end{aligned}
$$

$$\implies B^{593} - 2B^{15} = -B \iff B - 2B = -B$$

$\square$