Determining the Number of Unique Positions in Topitop

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Video Walkthrough: Click here

1 Introduction

The goal of GamesCrafters is to strongly solve different games through the use of computational game theory. However, in order for solving a game to be practical, it must have a state space that is small enough to explore in a reasonable amount of time and to store once data about each game state is generated. It is the goal of the theory team to determine the size of the state space for different games so that the research and development group can prioritize which games to solve.

In this paper, we seek to count the number of unique positions in the board game Topitop. Topitop is a unique board game that is dart-board-like, impartial, and loopy. All of these qualities make Topitop a difficult game to fully solve, which is why this paper will not be going to such lengths. However, we do want to know if Topitop fits our criteria for being a game that practically could be fully solved by our research and development group one day. The rest of this paper explores the size of the state space of Topitop.

2 Rules of Topitop

The standard Topitop board is a 3x3 grid of squares, starting out with nothing on the board. Each player chooses a one color between red or blue and gets 2 buckets of that color. There are four different basic pieces in Topitop - red buckets, blue buckets, small sand piles, and large sand piles. The sand piles belong to any player. A *castle piece* can be made by placing a small sand pile on a large sand pile. A *bucket piece* can be made by placing a bucket on a small sand pile. Lastly, a red or blue *sand castle* can be made when a bucket is on top of a castle piece. Only buckets and pieces with a specific color bucket on it are in control of the respective player.

Each player takes turns making one valid move and cannot reverse the opponent's move that was just made. Legal moves fall under one of the three following variations:

1. Place one of your buckets or one of the small or large sand piles on any free space on the board. You cannot place a piece on top of another piece (from off the board to on top of a piece already on the board). Pieces must be moved on top of other pieces.
2. Move one of your buckets or any small or large sand pile already placed on the board from one square to another (one space at any time in any direction)
   (a) Any piece already on the board can be moved to an adjacent free space.
   (b) A bucket can go on top of a small sand pile.
   (c) A small sand pile can go on top of a large sand pile.
   (d) A small sand pile with a bucket on its top can go on top of a large sand pile.
   (e) A bucket can go on top of a sand castle.
3. Move any combination of sand piles with your buckets on top, or any sand castle, to any free space.

The blue player goes first. The players alternate making legal moves. Whenever a player cannot make a legal move, he or she passes the turn to the other player.

The first player to have two buckets on top of two sand castles wins.

3 Possible Variations

There are several possible variations to the rules of Topitop that can make for slightly different versions of the game. The simplest is a *misere* variation, where the first player to have to buckets on top of two sand castles loses instead of wins. There are also variations where one can change the board dimensions, number of buckets, number of small sand piles, number of large sand piles, and the number of sand castles with buckets on top needed to win. The code provided along with this paper has infrastructure in place that allows for these changes, but these variations will not be explored in this paper.

4 Mathematical Count

We will first attempt to come up with a closed form solution for the state space of Topitop. With any given board state, we can first consider the number of large sand piles on the board. There are 9 possible locations for a large sand pile, and there can be up to 4 large sand piles on the board. The large sand piles are all identical, so given that there are L large sand piles on the board, there are then $\binom{9}{L}$ possible rearrangements.

In counting the number of ways to rearrange the small sand piles, it is important to keep track of whether they are placed on top of large sand piles to form castle pieces, because buckets cannot be placed on large sand piles. As a result, it is then helpful to predetermine the number of bucket pieces prior to rearranging the locations of the small sand piles. There are can anywhere from 0 to L bucket pieces. The castle pieces are all identical, so given that there are C castle pieces on the board, there are then $\binom{L}{C}$ possible rearrangements. After that, there are then up to $4 - C$ remaining small sand piles that could be placed anywhere there isn't a castle piece or large sand pile, as the number of castle pieces has already been determined. The sand piles are also all identical, so given that there are S small sand piles on the board, there are then $\binom{9-L}{S}$ possible rearrangements.

For the rearrangements for the buckets, we have to consider the red and blue buckets separately. Without loss of generality, we can start with the red buckets. There are $L - C$ large sand piles in their own spaces, so there are $9 - (L - C)$ remaining spaces where buckets could be placed. There can be up to 2 red buckets on the board, and they are both identical, so given that there are R red buckets on the board, there are then $\binom{9-(L-C)}{R}$ possible rearrangements. For the blue buckets, they also cannot be placed in spaces where there are only large sand piles, and they also cannot be placed in spaces where red buckets have already been placed. There are then $9 - (L - C) - R$ remaining spaces where blue buckets could be placed. There can be up to 2 blue buckets on the board, and they are both identical, so given that there are B blue buckets on the board, there are then $\binom{9-(L-C)-R}{B}$ possible rearrangements.

It is important to note that the above counting scheme allows for the case where there are 4 sand castles on the board, 2 for each player. 4 sand castles is an impossible position, as the game would've ended when either player constructed 2 sand castles, and they cannot do so simultaneously. We then have to subtract these cases out of our total count for the scheme above. The number of rearrangements for 4 sand castles is given by the expression $\frac{9 \cdot 8 \cdot 7 \cdot 6}{2 \cdot 2} = 756$, as the 2 red sand castles are identical and the 2 blue sand castles are identical.

The total number of unique board states in Topitop is then given by the following expression:

$$\sum_{L=0}^{4} \binom{9}{L} \sum_{C=0}^{L} \binom{L}{C} \sum_{S=0}^{4-C} \binom{9-L}{S} \sum_{R=0}^{2} \binom{9-L+C}{R} \sum_{B=0}^{2} \binom{9-L+C-R}{B} - 756$$

Writing a program to evaluate this expression, the total number of unique board states in Topitop comes out to 36,250,090.

5 Final comments

It should be noted that while the above number and expression does reflect the amount of unique board states in Topitop, it does not necessarily reflect the amount of unique game states. Topitop is not an impartial game. Before all bucket pieces are placed on the board, the player whose turn it is can only place his/her own color buckets on the board, not the buckets of the opposing player. Additionally, when bucket pieces are placed on the board, the player can only move his/her own buckets, pieces with his/her own buckets on top of them, and sand piles with no buckets on top of them. As a result, it is important for the game state to keep track of which player's move is next. That information effectively doubles the mathematical count above.

In addition, because players cannot undo an opponents previous move, it is important for the game state to keep track of the previous move or the previous position, as that limits what moves are available to the player whose move it currently is. Each unique board state can have multiple previous states that lead to it, as Topitop is somewhat loopy. However, in a 3x3 board, there is only 40 possible combinations where a piece is moved from one square to an adjacent square, and in Topitop, most board states will not have that many possible moves that could have lead to it. That puts an absolute upper bound on the number of game states in Topitop at $80 \cdot 36250090 = 2{,}900{,}007{,}200$. This is a number that is well within our reach in terms of fully solving.