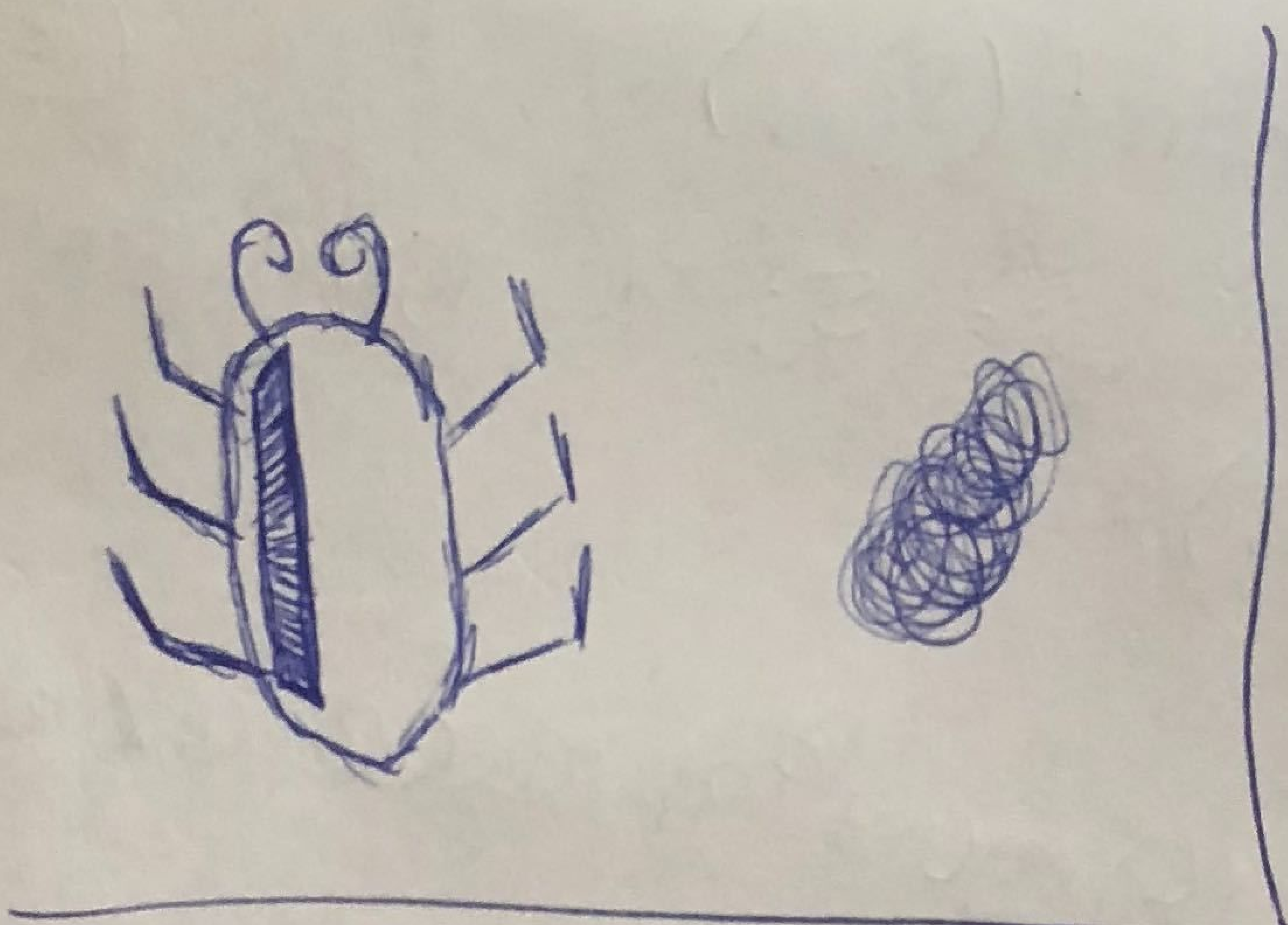


We can then simplify the summations on either side to

$$\sum_{i=0}^{m-1}\left(\sum_{c=0}^{s_i-1} f(s_{i,c})\right) < \sum_{i=0}^{m-1}\left(\sum_{c=0}^{s'_i-1} f(s'_{i,c})\right) \quad (16)$$

which is the def of I_f so $I_f(s) < I_f(s')$ $\square$

Note, this does not hold for "entire" positions.

It works for sequences RBRB etc from left to Right * R=0 B=1 as binary number

* bottom to top

| R | | = 0 $S=\{R\} \mapsto$ doesn't follow rules of game.

| B | H | = 1 $S=\{B\}$ Completed = (BRBR, BBRR, BRRB)

10 = | B | R | $S=\{B,R)$ completed = (BBRR or BRRB)

$$f(S_{j,s_j}) \leq \underbrace{\sum_{c=s_j}^{s_j-1} f(S_{j,c})}_{f(S_{j+1})} \Big\} \text{ less than } f(S'_{j+1})$$

③

Remember $S_j = S'_j$ b/c agree until j

so then $\forall c,\ S_{j,c} = S'_{j,c}$ (9)

$$\sum_{c=s_j}^{s'_j-1} f(S_{j,c}) = \sum_{c=s_j}^{s'_j-1} f(S'_{j,c}) = (10) \text{ just by (9)}$$

and then

$$\sum_{c=s_j}^{s'_j-1} f(S'_{j,c}) \leq \sum_{c=s_j}^{s'_j-1} + \overbrace{\sum_{i=j+1}^{m-1}\left(\sum_{c=0}^{s'_i-1} f(S'_{i,c})\right)}^{\text{zero or more}} = (11)$$

we use (7), (8), (10), (11) to put together (12):

$$\sum_{i=j+1}^{m-1}\left(\sum_{c=0}^{s_i-1}\right) < \sum_{c=s_j}^{s'_j-1} f(S'_{j,c}) + \sum_{i=j+1}^{m-1}\left(\sum_{c=0}^{s'_i-1} f(S'_{i,c})\right)$$

and then again from (9) $\sum_{c=0}^{s_j-1} f(S_{j,c}) = \sum_{c=0}^{s_j-1} f(S'_{j,c})$ (13)

so

$$\sum_{i=j}^{m-1}\left(\sum_{c=0}^{s_i-1} f(S_{i,c})\right) < \sum_{c=0}^{s_j-1} f(S'_{j,c}) + \sum_{c=s_j}^{s'_j-1} f(S'_{j,c}) + \sum_{i=j+1}^{m-1}\left(\sum_{c=0}^{s'_i+1} f(S'_{i,c})\right)$$

w/o sec term

so $< \sum_{i=j}^{m-1}\left(\sum_{c=0}^{s'_i-1} f(S'_{j,c})\right)$ (14) b/c remove < j^{th} term

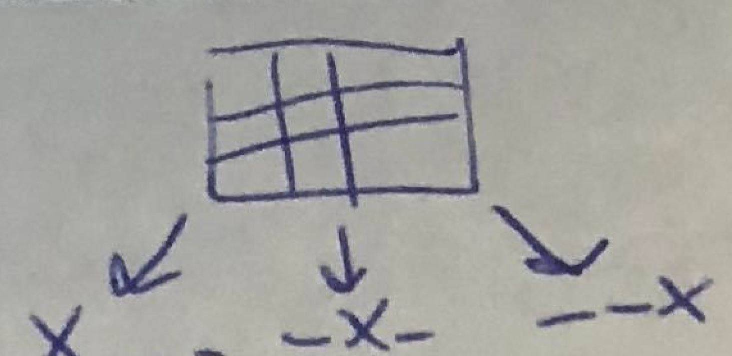
Because $i<j \Rightarrow S_i = S'_i$ and $i \leq j \Rightarrow S_i = S'_i \Rightarrow S_{i,c} = S'_{i,c}$

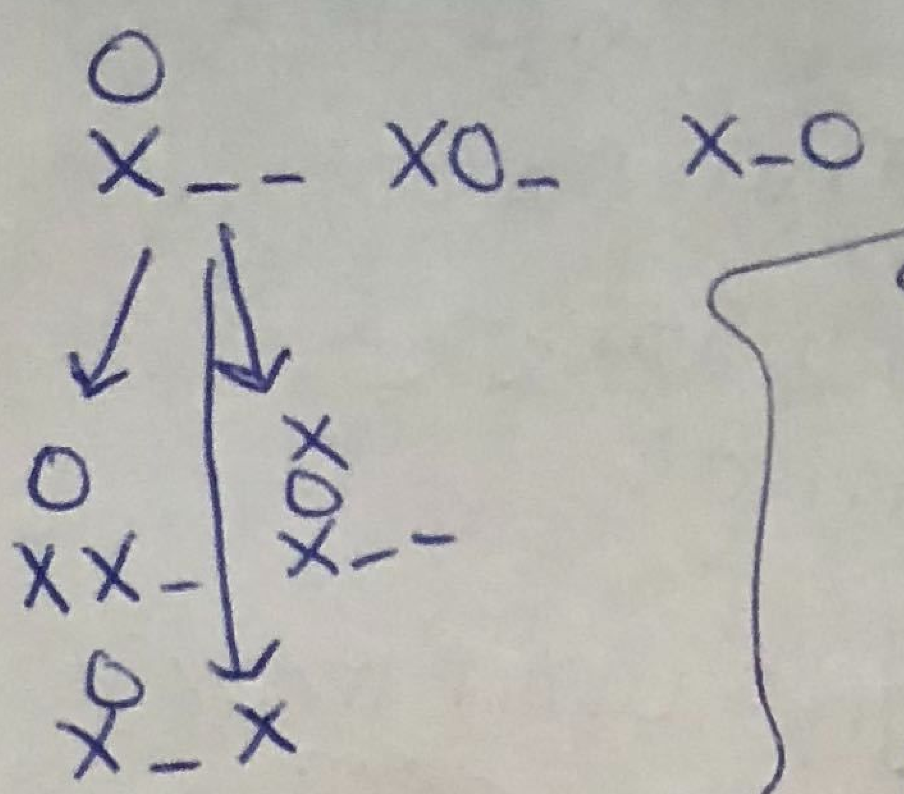
from (14) we can say

$$\sum_{i=0}^{j-1}\left(\sum_{c=0}^{s_i-1} f(S_{i,c})\right) + \sum_{i=j}^{m-1}\left(\sum_{c=0}^{s_i-1} f(S_{i,c})\right) < \sum_{i=0}^{j-1}\left(\sum_{c=0}^{s'_i-1} f(S'_{i,c})\right) + \sum_{i=j}^{m-1}\left(\sum_{c=0}^{s'_i-1} f(S'_{i,c})\right)$$

b/c sec term RHS > sec term LHS

→





$$\sum_{c=0}^{s_t-1} f(s_{t,c}) + \underbrace{f(s_{t+1})}_{\text{substitute}} \le \sum_{c=0}^{s_t+1} f(s_{t,c}) + \sum_{c=s_t}^{d} f(s_{t,c}) \quad (6)$$

$$(6) = \sum_{c=0}^{d} f(s_{t,c}) = f(s_t)$$

We substitute into the assumption from the step

$$\sum_{c=0}^{s_t-1} f(s_{t,c}) + \sum_{i=t+1}^{m-1}\left(\sum_{c=0}^{s_i-1} f(s_{i,c})\right) < f(s_t) \quad \text{by (2) and (6)}$$

so we condence LHS to $\sum_{i=t}^{m-1}\left(\sum_{c=0}^{s_i-1} f(s_{i,c})\right) < f(s_t)$ (1)

which was our first goal to prove □

We can now make the statement that

$$(7) = \sum_{i=j+1}^{m-1}\left(\sum_{c=0}^{s_i-1} f(s_{i,c})\right) < f(s_{j+1}) = f(s_{j,s_j}) \quad \text{by (1) and (3)}$$

because we know that $s_j < s'_j$

$$f(s_{j,s_j}) \le f(s_{j,s_j}) + \sum_{c=s_j+1}^{s'_j-1} f(s_{j,c}) = \sum_{c=s_j}^{s'_j-1} f(s_{j,c}) = (8)$$

hash(s): $\quad S = [s_0 \cdots s_{m-1}]$

$\forall$ length i (1)

$\forall$ num c less than or equal to s_i

total += ways to complete seq i ending in c

$I_f(s)$ is indexing function for some f } $\equiv$ hash f

$$I_f(s) = \sum_{i=0}^{m-1} \sum_{c=0}^{s_i - 1} f(S_{i,c}) \mapsto \text{seq } s \text{, terminating @ } S_{i-1} \text{ ending in } c$$

S and S' two seq which are valid and agree until index j S''s j^{th} item is bigger

$s_j \neq s'_j$. We further constrain $s_j < s'_j$

First, $0 \leq t \leq m$, show $\sum_{i=t}^{m-1} \left(\sum_{c=0}^{s_i - 1} f(S_{i,c}) < f(S_t) \right)$ for all such t

all the ways to finish sequences which are at least as long as t and end in a char $> S[t] < f(S_t)$

(1) let $t = m \mapsto 0 < 1$ left no terms & $f(S_m) = 1$ by def.

(2) inductively, assume $\sum_{i=t+1}^{m-1} \left(\sum_{c=0}^{s_i - 1} f(S_{i,c}) \right) < f(S_{t+1})$

note that $S_{t+1} = S_{t, s_t}$ } seq up to t followed by t^{th} item

so $(2) = f(S_{t, s_t}) \mapsto (4)$

then (5) is:

$$\sum_{c=s_t}^{d} f(S_{t,c}) = f(S_{t,s_t}) + \sum_{c=s_t+1}^{d} f(S_{t,c}) = f(S_{t+1}) + \sum_{c=s_t+1}^{d} f(S_{t,c})$$

$$f(S_{t+1}) \leq f(S_{t+1}) + \sum_{c=s_t+1}^{d} f(S_{t,c}) = \sum_{c=s_t}^{d} f(S_{t,c}) \text{ by (5) and (4)}$$