# Homework #3

assigned : 2 / 12 / 2020

due date : 2 / 19 / 2020
at 10:30 am

## Exercise 1

Let $m(t) = \cos(2\pi f_1 t) + 4\cos(6\pi f_1 t)$ and

$c(t) = 10\cos(2\pi f_0 t)$ with $f_0 >> f_1$ be the modulating signal

and the passband signal of a DSB-SC communication system,

respectively.

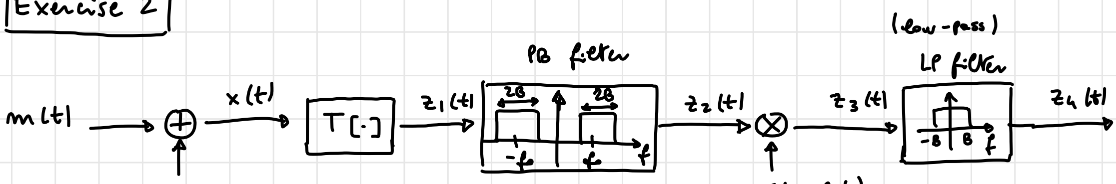
1) Find $\tilde{x}(t)$ and $x(t)$

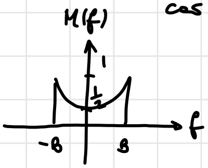
2) Find and plot $X(f)$

3) Compute the power $P_x$ of the passband signal and

compare it with the power of the complex envelope $P_{\tilde{x}}$

## Exercise 2




$y = T[x] = x^2$ : square-law device

1) Divide the above communication system into a transmitter and receiver part.

- What is the modulation implemented in the above scheme?
- What is the channel response $h(t)$?

2) Plot the spectrums : $X(f)$, $Z_1(f)$, $Z_2(f)$, $Z_3(f)$, and $Z_4(f)$

## Exercise 3

Let $m(t) = \frac{1}{2}\sin(2\pi f_1 t) - \frac{1}{2}\cos(4\pi f_1 t)$ with power $P_m$ be a modulating signal of an AM communication system with $m_A = 0.8$ $(0 \le m_A \le 1)$. Let also be $f_0$ the carrier frequency.

1) Find the passband signal $x(t)$ and complex envelope $\tilde{x}(t)$

2) Compute the modulation efficiency $\eta_{AM} = \dfrac{1}{1 + \frac{1}{m_A^2 P_m}}$

3) Compute the maximum amplitude deviation of modulating signal defined as

$$\Delta A_{AM} \triangleq \max_t A(t)$$

$$= \max_t m(t)$$

~~$A(t) = m(t)$ for AM~~

CORRECTION :

$A(t) = 1 + m_A\, m(t)$

thus $\Delta A_{AM} = 1 + m_A \max_t m(t)$

Hint: study $\frac{\partial}{\partial t} m(t) = 0$.

4) What are the stationary points $\{ t \in \mathbb{R} : \frac{\partial}{\partial t} m(t) = 0 \}$ ?

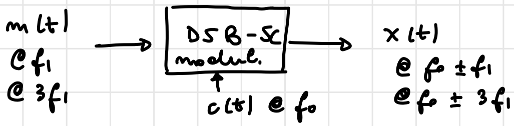
5) Find and plot $\tilde{X}(f)$ and $X(f)$

# Solutions

## Exercise 1

$$m(t) = \cos(2\pi f_1 t) + 4\cos(6\pi f_1 t)$$

$$x(t) = m(t)\cdot 10\cos(2\pi f_0 t) = m(t)\cdot c(t) \qquad \text{with } f_0 \gg f_1$$

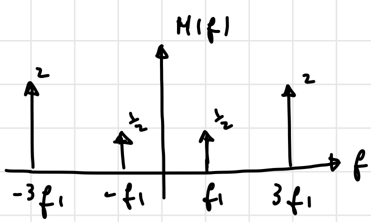



1) $$x(t) = \mathrm{Re}\left\{ \tilde{x}(t)\, e^{i2\pi f_0 t} \right\} \Longrightarrow \boxed{\tilde{x}(t) = 10\, m(t)}$$

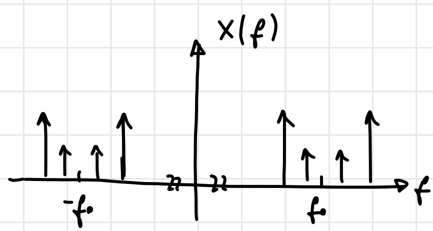
2) $$\underline{X(f)} = 10\, M(f) \otimes \frac{1}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right]$$

$$= \underline{5\left[M(f-f_0) + M(f+f_0)\right]}$$

with $$M(f) = \mathcal{F}\{m(t)\} = \frac{1}{2}\delta(f-f_1) + \frac{1}{2}\delta(f+f_1) + 2\,\delta(f-3f_1) + 2\,\delta(f+3f_1)$$




thus




see uploaded exercise in resource folder

3) $$P_{\tilde{x}} = \lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{T/2} |\tilde{x}(t)|^2\, dt = 100\, P_m = 100\left(\frac{1}{2} + \frac{1}{2}\cdot 16\right)$$

$$= 50 \cdot 17$$

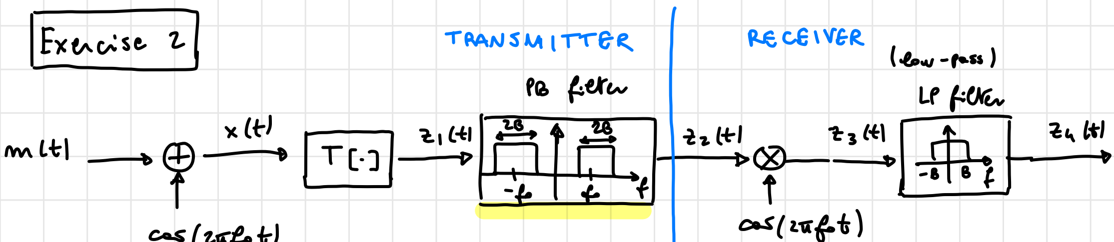
$$P_x = \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt = 25 \left[ \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \cdot 16 + \frac{1}{2} \cdot 16 \right] = 25 \cdot 17 = \frac{P_{\tilde{x}}}{2}$$

$$x(t) = m(t) \cdot c(t) = 10 \left[ \cos(2\pi f_1 t) + 4 \cos(6\pi f_1 t) \right] \cos(2\pi f_0 t)$$

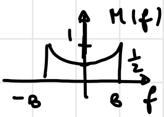
$$= 5 \Big[ \cos(2\pi (f_0 - f_1) t) + \cos(2\pi (f_0 + f_1) t) +$$

$$+ 4 \cos(2\pi (f_0 - 3 f_1) t) + 4 \cos(2\pi (f_0 + 3 f_1) t) \Big]$$

## Exercise 2



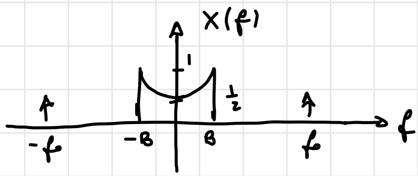

$y = T[x] = x^2$ : square-law device




1)-2) We've studied something similar in class...

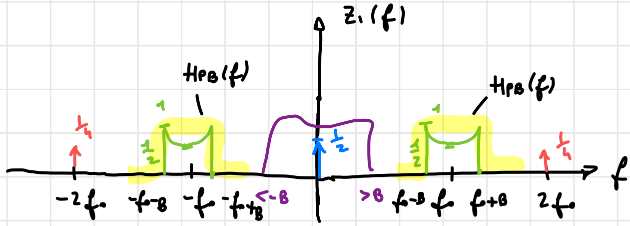
TRANSMITTER

$$x(t) = m(t) + \cos(2\pi f_0 t)$$



$$z_1(t) = T[x(t)] = x^2(t) = \left(m(t) + \cos(2\pi f_0 t)\right)^2 =$$

$$= m^2(t) + \frac{1}{2} + \frac{1}{2}\cos(4\pi f_0 t) + 2\, m(t)\cos(2\pi f_0 t)$$

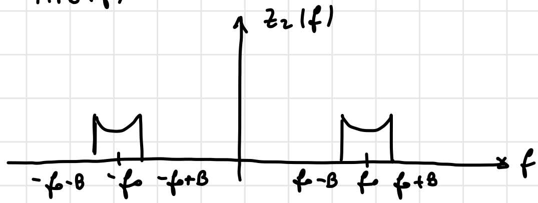



$$z_2(t) = z_1(t) \otimes h_{PB}(t) = 2\, m(t)\cos(2\pi f_0 t)$$

$\Updownarrow$

$$Z_2(f) = Z_1(f) \cdot H_{PB}(f)$$

it's a DSB-SC Passband signal!




## CHANNEL

transmitted PB signal

$$z_2(t) = s_T(t) \equiv r(t) \Leftarrow h(t) = \delta(t)$$ : ideal channel

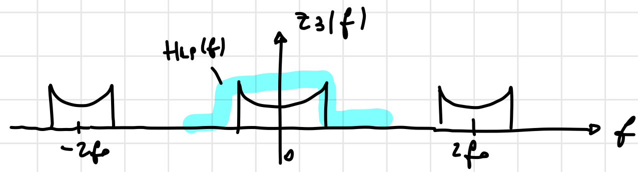
received PB signal

# RECEIVER

$$z_3(t) = z_2(t)\cos(2\pi f_0 t) = 2\,m(t)\cos^2(2\pi f_0 t)$$

$$= m(t) + m(t)\cos(4\pi f_0 t)$$

COHERENT
DEMODULATION
(perfect synch at RX)




$$z_4(t) = z_3(t) \otimes h_{LP}(t) = m(t) \qquad \left( z_4(t) \equiv \hat{m}(t) = m(t) \right)$$

$\Updownarrow$

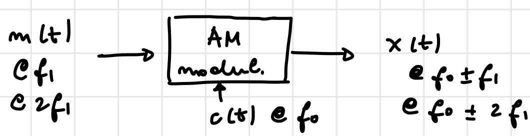
$$Z_4(f) = Z_3(f)\,H_{LP}(f)$$

- perfect synch at receiver
- ideal channel

OSS: This is the DSB-SC counterpart of the AM communication system we've seen in class.

## Exercise 3

$$m(t) = \frac{1}{2}\sin(2\pi f_1 t) - \frac{1}{2}\cos(4\pi f_1 t)$$




1) $\tilde{x}(t) = 1 + m_A\, m(t) = 1 + \frac{m_A}{2}\sin(2\pi f_1 t) - \frac{m_A}{2}\cos(4\pi f_1 t)$

$$x(t) = \mathrm{Re}\left\{ \tilde{x}(t)\, e^{i 2\pi f_0 t} \right\} = \tilde{x}(t) \cos(2\pi f_0 t) =$$

careful! It's not valid for all modulations (e.g. SSB has $\tilde{x}(t) \in \mathbb{C}$)

$$= \left(1 + \frac{m_A}{2} \sin(2\pi f_1 t) - \frac{m_A}{2} \cos(4\pi f_1 t)\right) \cos(2\pi f_0 t)$$

2) $$\eta_{AM} = \frac{P_{info}}{P_{TOT}} = \frac{1}{1 + \frac{1}{m_A^2 P_m}} =$$

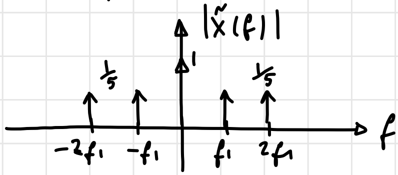
$$= \frac{1}{1 + \frac{1}{\left(\frac{8}{10}\right)^2 \cdot \frac{1}{4}}} = \frac{1}{1 + \frac{1}{\frac{16}{100}}} = \frac{16}{116} \approx 0.14 = 14\% \quad \left(\le 50\%_{AM}\right)$$

see uploaded exercise in resource folder

$$P_m = \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{T/2} m^2(t)\, dt = \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{4}$$

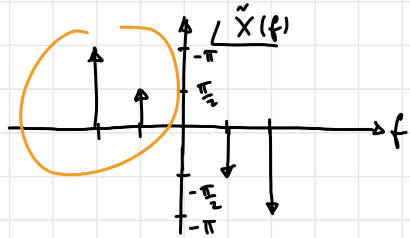
5) $$\tilde{X}(f) = \delta(f) + m_A M(f)$$

with $$M(f) = \frac{1}{4} e^{-i\frac{\pi}{2}} \left[\delta(f - f_1) - \delta(f + f_1)\right] - \frac{1}{4}\left[\delta(f - 2f_1) + \delta(f + 2f_1)\right]$$




$m_A = 0.8 = \frac{8}{10}$

due to Hermitian symmetry of $M(f)$



4) we look for $\max_t m(t)$

$$\frac{\partial}{\partial t} m(t) = \pi f_1 \cos(2\pi f_1 t) + 2\pi f_1 \sin(4\pi f_1 t) = 0$$

$\nearrow$ stationary points

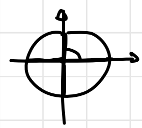
$f_1 > 0 \iff \cos(2\pi f_1 t) + 2 \sin(4\pi f_1 t) = 0$

$\sin 2\alpha = 2 \sin\alpha \cos\alpha$

$$\iff \cos(2\pi f_1 t)\big(1 + 4 \sin(2\pi f_1 t)\big) = 0$$

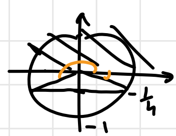
$$\iff \left[\begin{array}{ll} \cos(2\pi f_1 t) = 0 & (1) \\ \sin(2\pi f_1 t) = -\frac{1}{4} & (2) \end{array}\right.$$

(1) : $2\pi f_1 t = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$

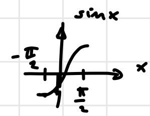


$$\iff \boxed{t = \frac{1}{4f_1} + \frac{k}{2f_1}, \quad k \in \mathbb{Z}}$$

(2) : $\sin(2\pi f_1 t) = -\frac{1}{4}$




$$\iff 2\pi f_1 t = \begin{cases} \pi - \sin^{-1}(-\frac{1}{4}) + 2k\pi \\ \sin^{-1}(-\frac{1}{4}) + 2k\pi \end{cases}, \quad k \in \mathbb{Z}$$




$$\iff \boxed{t = \begin{cases} \frac{1}{2f_1} + \frac{\sin^{-1}(\frac{1}{4})}{2\pi f_1} + \frac{k}{f_1} \\ \frac{-\sin^{-1}(\frac{1}{4})}{2\pi f_1} + \frac{k}{f_1} \end{cases}, \quad k \in \mathbb{Z}}$$

$$m(t) = \frac{1}{2}\sin(2\pi f_1 t) - \frac{1}{2}\cos(4\pi f_1 t)$$

① $$m(t) = \frac{1}{2}\sin(2\pi f_1 t) - \cos^2(2\pi f_1 t) + \frac{1}{2}$$

$\cos 2\alpha = 2\cos^2\alpha - 1$

when $\cos(2\pi f_1 t) = 0 \Rightarrow \sin(2\pi f_1 t) = \pm 1$

thus $$m(t) = \pm\frac{1}{2} + \frac{1}{2} = \begin{cases} 1 \\ 0 \end{cases}$$

② $$m(t) = \frac{1}{2}\sin(2\pi f_1 t) - \frac{1}{2} + \sin^2(2\pi f_1 t)$$

$\cos 2\alpha = 1 - 2\sin^2\alpha$

when $\sin(2\pi f_1 t) = -\frac{1}{4}$

$$m(t) = -\frac{1}{8} - \frac{1}{2} + \frac{1}{16} = \frac{-2-8+1}{16} = -\frac{9}{16}$$

therefore $$\boxed{\max_t m(t) = 1}$$