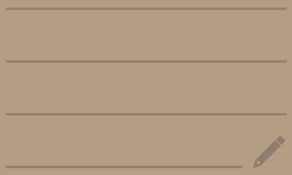
Introduction to Analysis and Optimization Techniques

Chapter 1: Statements, Sets, Counting

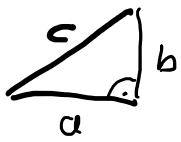


1.1 Variables and Statements

Mathematical statements are constructed using variables, sets, relations, and functions.

↖ placeholder for values, denoted by letters such as x, y, n, o, etc

$$a^2 + b^2 = c^2$$



There are two kinds: existential and universal

Existential variables can have one or more values but we don't know what they are.

Example (existential variable): "Is there a number with the following property: doubling it and adding 3 gives the same result as squaring it?"

We can introduce a variable x to replace the potentially ambiguous word "it"

"Is there a number x with the property that $2x + 3 = x^2$"

Universal variables are not restricted to have one particular value. What ever we say about it should be equally true for all values (in a given set).

Example (universal variable): "No matter what number might be chosen, if it is greater than 2, then its square is greater than 4."

"No matter what number x might be chosen: if $x > 2$, then $x^2 > 4$."

↑ temporary name enables us to keep generality of the statement

Universal statement establishes that a certain property is true for all elements in a set

Example: All positive numbers are greater than 0.

Conditional statement establishes that if one "thing" is true then some other thing has also to be true

Example: If 378 is divisible by 18, then 378 is divisible by 6.

Existential statement establishes that, given a property (that may or may not be true), there is at least one "thing" for which the property is true.

Example: There is a prime number that is even.

Combinations of these three statements are possible:

Example: "Every real number x has an additive inverse, or negative, $-x$" is a universal existential statement

Example: "For all animals x, if x is a dog, then x is a mammal" is a universal conditional statement

Many mathematical theorems are stated as universal conditional statements. by fixing a domain (what things do we talk about), a precondition for the theorem (what is the context), and a claimed property of the objects

Example (Lagrange's mean value theorem)

Let f be a differentiable function in an interval (a, b), which is continuous also in a and b. Then there exists at least one point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

A **mathematical theorem** is a statement that has been proved. There are some methods to attempt a proof.

Direct proof establishes the truth of a statement by logically combining axioms, definitions, and known theorems.

Example: "The sum of two even integers is always even."

Proof: Consider two even integers x and y.
Since they are even: $x = 2a$, $y = 2b$

$$x + y = 2a + 2b = 2(a+b) = 2c$$

has 2 as a factor

q.e.d.

Proof by contradiction establishes the truth of a statement by showing that the supposition the statement is false leads logically to a contradiction.

Example: "The equation $x^2 = 2$ has no solution in the set $\mathbb{Q}$ of rational numbers."

Proof: Suppose, by contradiction, that there are two positive integer numbers n and m such that $\left(\frac{m}{n}\right)^2 = 2$.

We can always assume m and n are prime each other. Since $m^2 = 2n^2$, it follows that m^2 is divisible by 2.

Then m is even, that is $m = 2k$ for some integer k.
We obtain $4k^2 = 2n^2$ and then also n^2 is divisible by 2. Therefore n is also even. But this is not possible since m and n are prime each other q.e.d.

Proof by mathematical induction is used to verify statements that describe procedures that repeat regularly according to defined patterns.

is used often to prove a proposition that depends on a natural number. Eg., the proposition "The sum of the first n odd numbers is n^2" (will see the proof later)

Intuition suggests that:

"If the proposition P is true for the number 0, and if moreover by assuming that P is true for a generic number $n > 0$ it follows that it is necessarily true also for the successor, that is $n+1$, then P will be true for all the natural numbers"

Remark: can be any number k

"If the proposition P is true for the number 0, and ...

E.g., "The sum of the interior angles of a polygon of n sides is $(n-2)\cdot 180°$" is meaningful for $n \geq 3$, so we can choose $k \geq 3$, e.g. $k = 3$

Formally (Principle of mathematical induction):

Let $P(n)$ be a property that is defined for integers n.
And suppose the following two statements are true:

1. $P(k)$ is true for a certain integer k (the induction base case)
2. every time P is true for $n > k$ then P is also true for $n+1$ (the induction hypothesis)

then the statement $P(k)$ is true for all integers $n \geq k$.

Example: "The sum of the first n odd numbers is n^2"

Proof (by induction): any odd number can be written in the form $(2k-1)$. We can enumerate all of the first n odd numbers by choosing $k = 1, 2, 3, \ldots, n$, whose sum can be written as

$$1 + 3 + 5 + \ldots + (2n-1)$$

For example, the sum of the first 6 odd numbers is: $1+3+5+7+9+11$, where it is $11 = 2\cdot 6 - 1$

The proof using the principle of induction is done in two steps:

1) the statement is true for $n = 1$ (trivial): $1^2 = 1$

2) we now show that the fact that 1) is true for some n, implies that it is true for its successor $n+1$.

The next odd number after $(2n-1)$ is $(2n-1+2) = (2n+1)$, hence the sum of first $n+1$ odds is

$$\underbrace{1 + 3 + 5 + \ldots + (2n-1)}_{\text{by assumption} = n^2} + (2n+1)$$

$$= n^2 + (2n+1) = (n+1)^2$$

i.e. the sum of first $n+1$ odd numbers is $(n+1)^2$

hence we have shown 2) q.e.d.

1.2 Sets

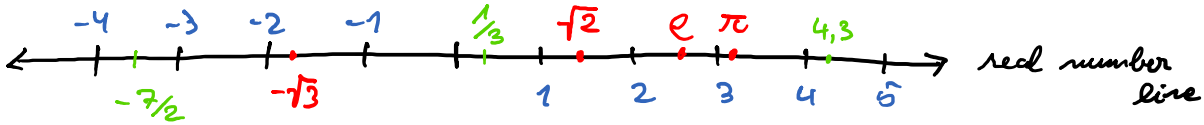
A **Set** is simply a **collection of elements**. The order of elements is irrelevant and duplicates are not counted.

Example: If C denotes the set of all countries that are currently in the United Nations, then Italy is an element of C, and we write $\text{Italy} \in C$

Example: If I denotes the set of all integers from 1 to 100, then the number 57 is an element of I: $57 \in I$ or $57 \in \{1, 2, 3, \dots, 100\}$ (set-roster notation)

Example: Let $A = \{1, 2, 3\}$, $B = \{3, 1, 2\}$, $C = \{1, 1, 2, 3, 3, 3\}$
A, B, C have exactly the **same three elements**: 1, 2, 3.
Therefore, they are simply different ways to represent the same set, and we can write $A = B = C$

Certain sets often referred to are given **special symbols**:
$\emptyset$ the empty set; $\mathbb{R}$ the set of all real numbers; $\mathbb{Z}$ the set of all integer numbers; $\mathbb{Q}$ the set of all rational numbers ($\subset$ = quotients of integers)



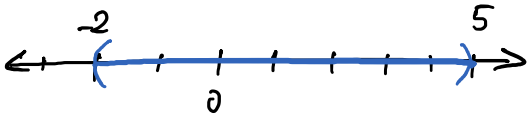
Rational numbers $\frac{n}{m}$ (with $n, m \in \mathbb{Z}$) can be written in decimal form, e.g. $\frac{3}{4} = 0.75$ or $\frac{1}{3} = 0.\bar{3}$, using:
- a finite number of digits after the comma, or
- an infinite number of digits which repeat periodically.

Real numbers can have any arrangement of digits in decimal form, including infinite and periodical
- e.g., $e \in \mathbb{R}$ but $e \notin \mathbb{Q}$; $\sqrt{2} \notin \mathbb{Q}$ (see example proof by contradiction)

Set-roster notation: $S = \{1, 2, 3\}$ (finite) or $\{1, 2, 3, 4, \ldots\}$ (infinite)

Set-builder notation: if $P(x)$ is a property that elements of the set S may or may not satisfy, then a set A may be defined by writing $A = \{x \in S \mid P(x)\}$ "The set of all x in S such that $P(x)$"

Example: $\{x \in \mathbb{R} \mid -2 < x < 5\}$ is the open interval of real numbers between -2 and 5



Example: $\{x \in \mathbb{Z} \mid -2 < x < 5\}$ is the set of all integers between -2 and 5. It is equal to $\{-1, 0, 1, 2, 3, 4\}$

Example: $\{x \in \mathbb{Z}^+ \mid -2 < x \leq 5\} = \{1, 2, 3, 4, 5\}$

↑ all positive elements of the set $\mathbb{Z}$ $\quad (\mathbb{Z}^+ = \{x \in \mathbb{Z} \mid x > 0\})$

If A and B are sets, then:

- A is called a **subset** of B, written $A \subseteq B$ if, and only if, every element of A is also a element of B.

$$A \subseteq B \iff \forall x, \text{ if } x \in A \text{ then } x \in B$$

- A is called a **proper subset** of B, written $A \subset B$ if, and only if, every element of A is in B but at least one is not

$$A \subset B \iff A \subseteq B \text{ and } \exists x \in B \text{ with } x \notin A$$

- A **equals** B, written $A = B$ if, and only if, every element of A is in B and every element of B is in A

$$A = B \iff A \subseteq B \text{ and } B \subseteq A$$

Important subsets of real numbers are intervals.
Given real numbers a and b with $a \leq b$

$$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$$
$$(a,b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$
$$[a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$
$$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

Symbols $+\infty$, $-\infty$ are used to indicate that intervals are unbounded either on the right or on the left

$$(a,+\infty) = \{x \in \mathbb{R} \mid x > a\}$$
$$(-\infty,b] = \{x \in \mathbb{R} \mid x \leq b\}$$

Operations on Sets. Let A, B be subsets of an universal set U:

- The union of A and B, denoted $A \cup B$, is the set of all elements that are in at least one of A and B

$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$$

- The intersection of A and B, denoted $A \cap B$, is the set of elements that are in both A and B

$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$$

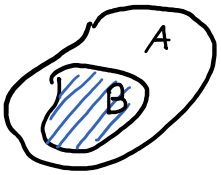
- The difference of B with A (or relative complement of A in B) denoted $B \setminus A$, is the set of all elements that are in B and not A

$$A \setminus B = \{x \in U \mid x \in A \text{ and } x \notin B\}$$

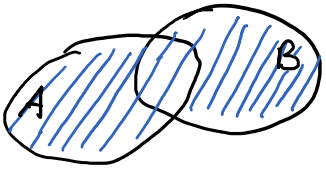
- The complement of A, denoted A^c, is the set of all elements in U that are not in A

$$A^c = \{x \in U \mid x \notin A\}$$

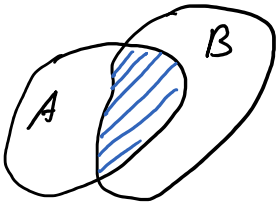
If A and B can be represented as regions in the plane, relationships can be represented by pictures, called **Venn diagrams**



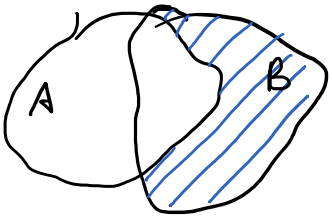
Inclusion $B \subset A$



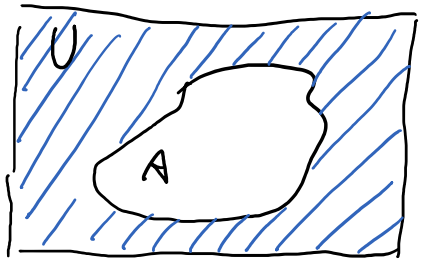
Union $A \cup B$



Intersection $A \cap B$



Difference $B \setminus A$



Complement A^c
$= U \setminus A$

Example: Let $U = \{a, b, c, d, e, f, g\}$ and $A = \{a, c, e, g\}$ and $B = \{d, e, f, g\}$

$A \cup B = \{a, c, d, e, f, g\}$; $A^c = \{b, d, f\}$

$A \cap B = \{e, g\}$; $B \setminus A = \{d, f\}$; $A \setminus B = \{a, c\}$

Example: Let $U = \mathbb{R}$, $A = \{x \in \mathbb{R} \mid -1 < x \leq 0\}$, $B = \{x \in \mathbb{R} \mid 0 \leq x < 1\}$

$A \cup B = \{x \in \mathbb{R} \mid x \in (-1, 0] \text{ or } x \in [0, 1)\} = (-1, 1)$

$A \cap B = \{x \in \mathbb{R} \mid x \in (-1, 0] \text{ and } x \in [0, 1)\} = \{0\}$

$B \setminus A = \{x \in \mathbb{R} \mid x \in [0, 1) \text{ and } x \notin (-1, 0]\} = (0, 1)$

$A^c = \{x \in \mathbb{R} \mid x \notin (-1, 0]\} = (-\infty, -1] \cup (0, +\infty)$

Properties of set intersection

- commutative : $A \cap B = B \cap A$
- associative : $(A \cap B) \cap C = A \cap (B \cap C)$
- idempotent : $A \cap A = A$

Properties of set union

- commutative $A \cup B = B \cup A$
- associative : $A \cup (B \cup C) = (A \cup B) \cup C$
- idempotent : $A \cup A = A$

Distributive properties

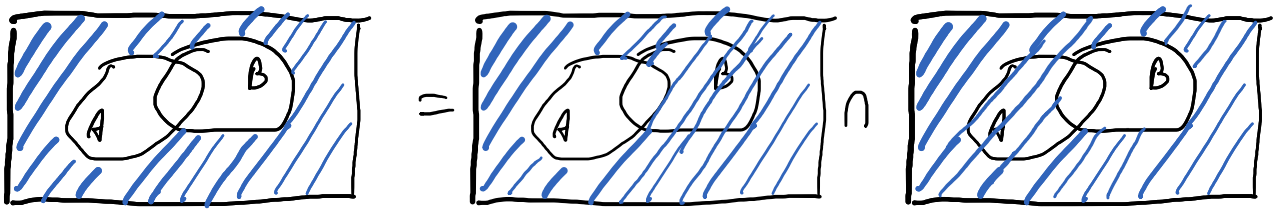
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

De Morgan's Laws

$$(A \cap B)^c = A^c \cup B^c$$

$$(A \cup B)^c = A^c \cap B^c$$

$$(A^c)^c = A$$



Set difference law:

$$A \setminus B = A \cap B^c$$

Unions and Intersections of an Indexed Collection of Sets

$$\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for at least one } i = 0, 1, 2, \ldots, n\}$$

$$\bigcap_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for all non-negative integers } i\}$$

Disjointness

Two sets are called disjoint if, and only if, they have no elements in common

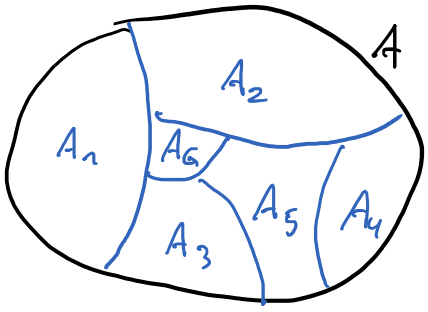
$$A \text{ and } B \text{ are disjoint} \iff A \cap B = \emptyset$$

Sets $A_1, A_2, A_3, \ldots$ are mutually disjoint if, and only if, no two sets A_i and A_j with distinct subscript have any element in common:

$$A_i \cap A_j = \emptyset \quad \text{whenever } i \neq j$$

A finite or infinite collection of non-empty sets $\{A_1, A_2, A_3, \ldots\}$ is a partition of a set A if, and only if

1. A is the union of all the A_i: $A = \bigcup_i A_i$
2. The sets $A_1, A_2, A_3 \ldots$ are mutually disjoint: $A_i \cap A_j = \emptyset$



A partition of a Set

Example:

$A = \{1, 2, 3, 4, 5, 6\}$

$A_1 = \{1, 2\}$, $A_2 = \{3, 6\}$,

$A_3 = \{4\}$, $A_4 = \{5\}$

Given a set A, the power set of A, denoted $\mathcal{P}(A)$, is the set of all subsets of A

Example. $\mathcal{P}(\{x,y,z\}) = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, \{x,y,z\}\}$

Given a set A, a k-combination of A is a subset of k distinct elements of A.

Example. $\{x,y\}, \{x,z\}, \{y,z\}$ are all the 2-combinations of the set $A = \{x,y,z\}$

The collection of all k-combinations of a set A with n elements, for $k = 0, 1, 2, \ldots n$, is a partition of $\mathcal{P}(A)$.

A permutation of a set is an arrangement of its elements into a sequence or linear order.

Example. There are 6 permutations of the set $\{1,2,3\}$, namely $(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)$.

Notation. $(1,2,3)$ denotes a 3-tuple, as opposed to $\{1,2,3\}$ which represents a 3-element set. A tuple is a finite ordered list, or sequence, of elements. An n-tuple is defined inductively using the construction of an ordered pair.

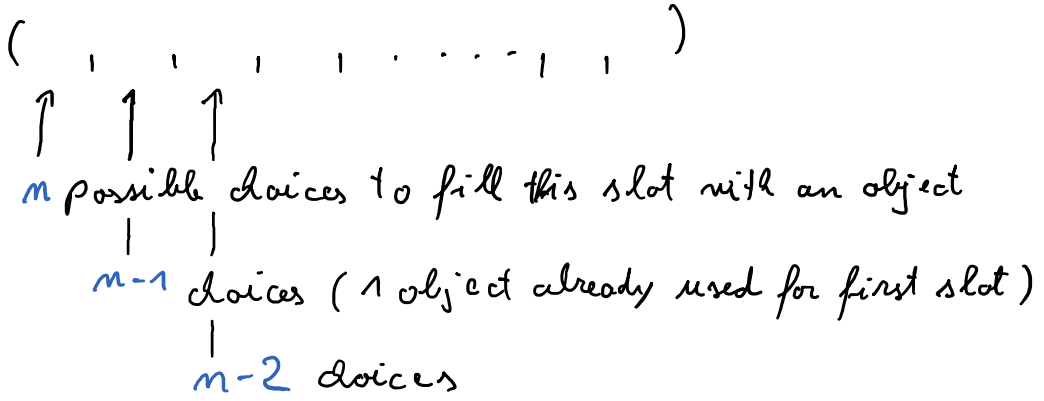
1.3 Counting

There are 6 permutations on any set of $n = 3$ objects. For a small enough n, we can find this number through enumeration (see example).

Example. We want to organize in a bookshelf 10 books. How many different arrangements are possible?

Enumerating them all becomes impossible (or at least tedious) as the number of objects grows.

Rather than enumerating, one can also count the number of permutations using a basic principle:



Number of permutations on an n-element set is therefore

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1 \overset{\text{def}}{=} n!$$

"n factorial"

Example. We want to organize in a bookshelf 10 books. How many different arrangements are possible?

We now understood that the answer is $n!$ with $n = 10$, that is

$$10! = 10 \cdot 9 \cdot 8 \cdot \ldots \cdot 2 \cdot 1 = 479.001.600$$

Example We want to organize in a bookshelf 10 books, which are divided into the following subjects: 4 are computer science, 3 are mathematics, 2 are statistics, and 1 is history. If we want all the books from the same subject to appear together, how many different arrangements are possible?

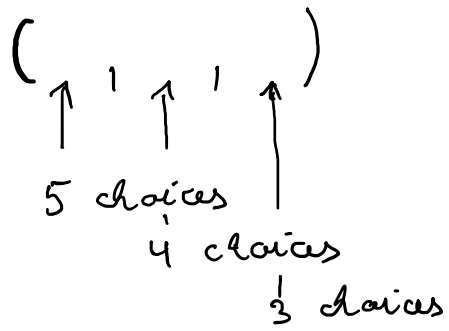
Solution:

- There are 4 subjects: we can choose any of the 4! orderings
- For each subject, we must choose the order of the books within the subject:
  - computer science: 4 books $\Rightarrow$ 4! orderings
  - mathematics: 3 books $\Rightarrow$ 3! orderings
  - statistics 2! orderings, history 1! orderings

$4!4!3!2!1! = 6.912$ possible ways

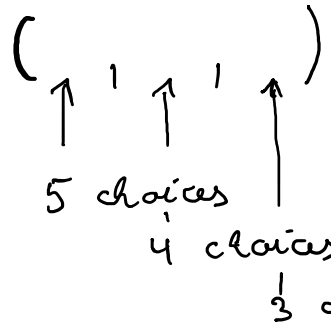
Rather than ordering the objects in a collection with n elements, we sometimes want to count the **number of groups with k objects** that can be formed. That is, we count the number of **k-selections from a n-element set.**

Example How many groups of 3 elements can be formed with the 5-element set $\{a, b, c, d, e\}$?



$\Rightarrow$ $5 \cdot 4 \cdot 3 = 60$ choices, including $(5,3,1)$, $(5,1,3)$, $(3,5,1)$, $(3,1,5)$ $(1,5,3)$, $(1,3,5)$ which are all **equivalent groups.**

To find the number of groups (= sets, not sequences), we need to count only 1 representative for each group, that is, we need to divide by the number of possible permutations to ignore their ordering.



$$\Rightarrow \quad \frac{5 \cdot 4 \cdot 3}{3!} = 10 \text{ groups}$$

Theorem 1.1

Let $k \leq n$. The combinations of k objects in an n-element set is

$$\binom{n}{k} \stackrel{\text{def}}{:=} \frac{n!}{(n-k)!\,k!}$$

$\binom{n}{k}$: binomial coefficient, "n choose k"

Note that $\frac{n!}{(n-k)!} = n \cdot (n-1) \cdot \ldots \cdot (n-k+1)$ which was $5 \cdot 4 \cdot 3 = 60$ in our example, providing the intuition for this formula.

A formal proof by induction can be done. It involves similar steps as in the proof of Theorem 1.3 in the appendix and is therefore left as an exercise.

Binomial coefficients are important in Probability Theory and Statistics. Think of the following to convince yourself: "If we randomly select k elements from a collection of n objects, what is the probability that a given object is among the k selected?" [solution is $\binom{n-1}{k-1}/\binom{n}{k} = \frac{k}{n}$]

They also appear in Analysis:

Theorem 1.2 (Newton's binomial theorem)

For each integer $n \geq 0$, with a and b real numbers,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Example $(a+b)^2 = \binom{2}{0} a^0 b^2 + \binom{2}{1} a^1 b^1 + \binom{2}{2} a^2 b^0 =$

$$= 1 \cdot 1 \cdot b^2 + 2 \cdot ab + 1 \cdot a^2 \cdot 1$$

$$= a^2 + 2ab + b^2$$

Example $(a-b)^3 = \binom{3}{0} a^0 (-b)^3 + \binom{3}{1} a^1 (-b)^2 + \binom{3}{2} a^2 (-b)^1 +$

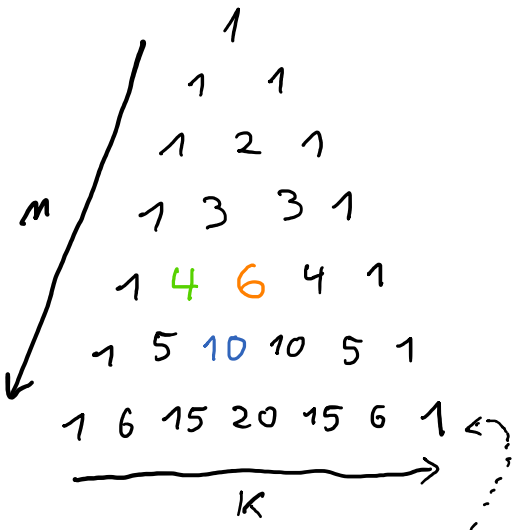
$$+ \binom{3}{3} a^3 (-b)^0 = \ldots = a^3 - 3a^2b + 3ab^2 - b^3$$

It may seem tedious to compute all the binomial coefficients when n gets larger. There is a handy method to compute them:

Pascal's triangle is built using the following recurrence formula:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Instead of quotient of factorials, it is computed recursively using summation.



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... row 6 gives all the $\binom{6}{k}$ coefficients:

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

The recursion at the basis of building Pascal's triangle can be proven using induction (left as an exercise).

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

We can use intuition as well to derive it: to choose k elements out of $n+1$, there are two possibilities:

- element $n+1$ is in the selection, or
- element $n+1$ not in the selection.

In the first case, we still have to choose $k-1$ from the remaining n elements, while in the second case we have to choose all k from the other n.

Appendix:

Proof of Newton's Binomial Theorem

Notice that Pascal's triangle is build row after row, that is, after having built row n we know how we build row $n+1$, the "successor row". This gives the intuition that Newton's binomial theorem can be proven by induction.

Proof of Theorem 1.2 (Newton's binomial theorem)

By induction on n.

Let $n=0$ our induction base case:

$$(a+b)^0 = \sum_{k=0}^{0} \binom{0}{k} a^k b^{(0-k)} = \binom{0}{0} a^0 b^0 = 1 \cdot 1 \cdot 1 = 1$$

which is an identity given that $(a+b)^0 = 1$ as well.

Lets now assume the statement is true for an arbitrary integer n. We need to prove that then it is true for $n+1$.

$$(a+b)^{n+1} = (a+b)(a+b)^n$$

$$= (a+b)\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} \qquad \text{(by the hypothesis)}$$

$$= \sum_{k=0}^{n}\binom{n}{k}a^{k+1}b^{n-k} + \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k+1}$$

index shifted by 1

$$= \sum_{k=1}^{n+1}\binom{n}{k-1}a^k b^{n-k+1} + \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k+1}$$

$$= a^{n+1} + \sum_{k=1}^{n}\binom{n}{k-1}a^k b^{n-k+1} + b^{n+1} + \sum_{k=1}^{n}\binom{n}{k}a^k b^{n-k+1}$$

in this last step we isolated the terms for $k=n+1$ and $k=0$

$$= a^{n+1} + b^{n+1} + \sum_{k=1}^{n}\left[\binom{n}{k-1} + \binom{n}{k}\right]a^k b^{n-k+1} =$$

recurrence formula (as with Pascal's triangle)

$$= a^{n+1} + b^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}a^k b^{n-k+1}$$

$$= \sum_{k=0}^{n+1}\binom{n+1}{k}a^k b^{n+1-k}$$

This last expression is in Newton's binomial form, and hence, the induction hypothesis is proven true.

q.e.d

Beyond the formal proof, it is also intuitive to use binomial coefficients for expansentuation of $(a+b)$:

$$\underbrace{(a+b)(a+b)(a+b)\cdots(a+b)}_{n \text{ times}} = (a+b)^n$$

To expand this, one has to take out one term from each factor.
For example, to obtain the term $a^k b^{n-k}$ in the expansion, we have to select a exactly k times, hence we "n choose k".

Appendix:

Cardinality of Powerset

Theorem 1.3

For all integers $n \geq 0$, if a set X has n elements, then its power set $\mathcal{P}(X)$ has 2^n elements.

Proof (by induction)

1. (base case) Let's consider the case $X = \emptyset$ that is, $n = 0$. $\mathcal{P}(\emptyset) = \{\emptyset\}$ which has $1 = 2^0$ elements.

2. (induction hypothesis) Let X have n elements and assume the above is true, hence $\mathcal{P}(X)$ has 2^n elements. If we add one new element, say y, to X we obtain the set $X \cup \{y\}$ which has $n+1$ elements. (the successor case)

Each of the 2^n subsets of X is also a subset of $X \cup \{y\}$, that is, $\mathcal{P}(X) \subset \mathcal{P}(X \cup \{y\})$.

Each subset containing y, that is, each element of $\mathcal{P}(X \cup \{y\}) \setminus \mathcal{P}(X)$, can be constructed by augmenting with y an element of $\mathcal{P}(X)$. And vice versa, each element of $\mathcal{P}(X)$ augmented with y ends up to be an element of $\mathcal{P}(X \cup \{y\}) \setminus \mathcal{P}(X)$. Hence, $\mathcal{P}(X \cup \{y\}) \setminus \mathcal{P}(X)$ and $\mathcal{P}(X)$ stay in 1-to-1 correspondence and therefore have the same number of elements, 2^n. Furthermore, $\{\mathcal{P}(X), \mathcal{P}(X \cup \{y\}) \setminus \mathcal{P}(X)\}$ is a partition of $\mathcal{P}(X \cup \{y\})$, and the number of elements is therefore $2^n + 2^n = 2 \cdot 2^n = 2^{n+1}$ q.e.d.