9 Distributions and Copulas for Integrated Risk Management

1 Chapter Overview

In Chapter 7 we considered multivariate risk models that rely on the normal distribution. In Chapter 6 we saw that the univariate normal distribution provides a poor description of asset return distributions—even for well-diversified indexes such as the S&P 500. The normal distribution is convenient but underestimates the probability of large negative returns. The multivariate normal distribution has similar problems. It underestimates the joint probability of simultaneous large negative returns across assets. This in turn means that risk management models built on the multivariate normal distribution are likely to exaggerate the benefits of portfolio diversification. This is clearly not a mistake we want to make as risk managers.

In Chapter 6 we built univariate standardized nonnormal distributions of the shocks

$$z_t \sim D(0, 1)$$

where $z_t = r_t/\sigma_t$ and where $D(*)$ is a standardized univariate distribution.

In this chapter we want to build multivariate distributions for our shocks

$$z_t \sim D(0, \Upsilon_t)$$

where z_t is now a vector of asset specific shocks, $z_{i,t} = r_{i,t}/\sigma_{i,t}$, and where Υ_t is the dynamic correlation matrix. We are assuming that the individual variances have already been modeled using the techniques in Chapters 4 and 5. We are also assuming that the correlation dynamics have been modeled using the DCC model in Chapter 7.

The material in this chapter is relatively complex for two reasons: First, we are departing from the convenient world of normality. Second, we are working with multivariate risk models. The chapter proceeds as follows:

- First, we define and plot threshold correlations, which will be our key graphical tool for detecting multivariate nonnormality.
- Second, we review the multivariate standard normal distribution, and introduce the multivariate standardized symmetric t distribution and the asymmetric extension.

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- Third, we define and develop the copula modeling idea.
- Fourth, we consider risk management and in particular, integrated risk management using the copula model.

2 Threshold Correlations

Just as we used QQ plots to visualize univariate nonnormality in Chapter 6 we need a graphical tool for visualizing nonnormality in the multivariate case. Bivariate threshold correlations are useful in this regard. Consider the daily returns on two assets, for example the S&P 500 and the 10-year bond return introduced in Chapter 7. Threshold correlations are conventional correlations but computed only on a selected subset of the data. Consider a probability p and define the corresponding empirical percentile for asset 1 to be $r_1(p)$ and similarly for asset 2, we have $r_2(p)$. These empirical percentiles, or thresholds, can be viewed as the unconditional *VaR* for each asset. The threshold correlation for probability level p is now defined by

$$\rho(r_{1,t}, r_{2,t}; p) = \begin{cases} Corr\left(r_{1,t}, r_{2,t} | r_{1,t} \leq r_1(p) \text{ and } r_{2,t} \leq r_2(p)\right) & \text{if } p \leq 0.5 \\ Corr\left(r_{1,t}, r_{2,t} | r_{1,t} > r_1(p) \text{ and } r_{2,t} > r_2(p)\right) & \text{if } p > 0.5 \end{cases}$$

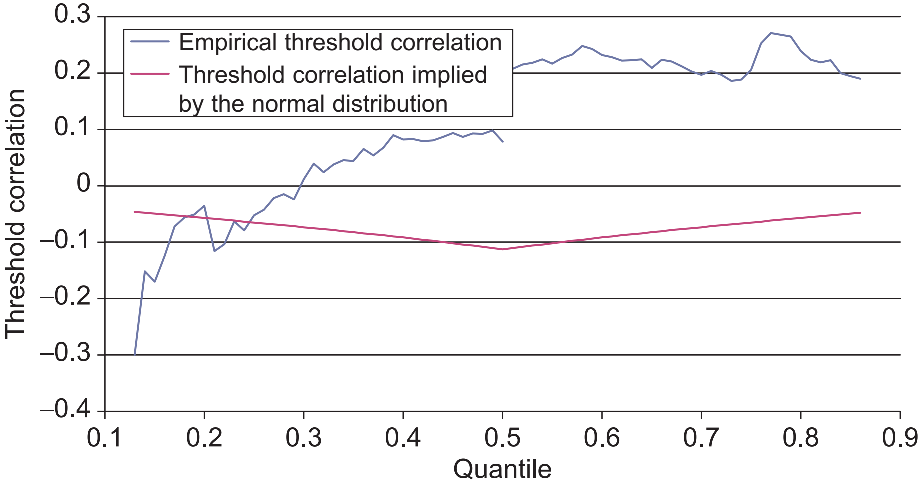
In words, we are computing the correlation between the two assets conditional on both of them being below their pth percentile if $p < 0.5$ and above their pth percentile if $p > 0.5$. In a scatterplot of the two assets we are including only the data in square subsets of the lower-left quadrant when $p < 0.5$ and we are including only the data in square subsets of the upper-right quadrant when $p > 0.5$. If we compute the threshold correlation for a grid of values for p and plot the correlations against p then we get the threshold correlation plot.

The threshold correlations are informative about the dependence across asset returns conditional on both returns being either large and negative or large and positive. They therefore tell us about the tail shape of the bivariate distribution.

The blue line in Figure 9.1 shows the threshold correlation for the S&P 500 return versus the 10-year treasury bond return. When p gets close to 0 or 1 we run out of observations and cannot compute the threshold correlations. We show only correlations where at least 20 observations were available. We use a grid of p values in increments of 0.01. Clearly the most extreme threshold correlations are quite variable and so should perhaps be ignored. Nevertheless, we see an interesting pattern: The threshold correlations get smaller when we observe large negative stock and bond returns simultaneously in the left side of the figure. We also see that large positive stock and bond returns seem to have much higher correlation than the large negative stock and bond returns. This suggests that the bivariate distribution between stock and bond returns is asymmetric.

The red line in Figure 9.1 shows the threshold correlations implied by the bivariate normal distribution when using the average linear correlation coefficient implied by the two return series. Clearly the normal distribution does not match the threshold correlations found in the data.

Figure 9.1 Threshold correlation for S&P 500 versus 10-year treasury bond returns.



Notes: We use daily returns on the S&P 500 index and the 10-year treasury bond index. The blue line shows the threshold correlations from the returns data and the red line shows the threshold correlations implied by the normal distribution with a correlation matching that of the returns data.

Given that we are interested in constructing distributions for the return shocks, rather than the returns themselves we next compute threshold correlations for the shocks as follows:

$$\rho(z_{1,t}, z_{2,t}; p) = \begin{cases} Corr\left(z_{1,t}, z_{2,t} | z_{1,t} \leq z_1(p) \text{ and } z_{2,t} \leq z_2(p)\right) & \text{if } p \leq 0.5 \\ Corr\left(z_{1,t}, z_{2,t} | z_{1,t} > z_1(p) \text{ and } z_{2,t} > z_2(p)\right) & \text{if } p > 0.5 \end{cases}$$

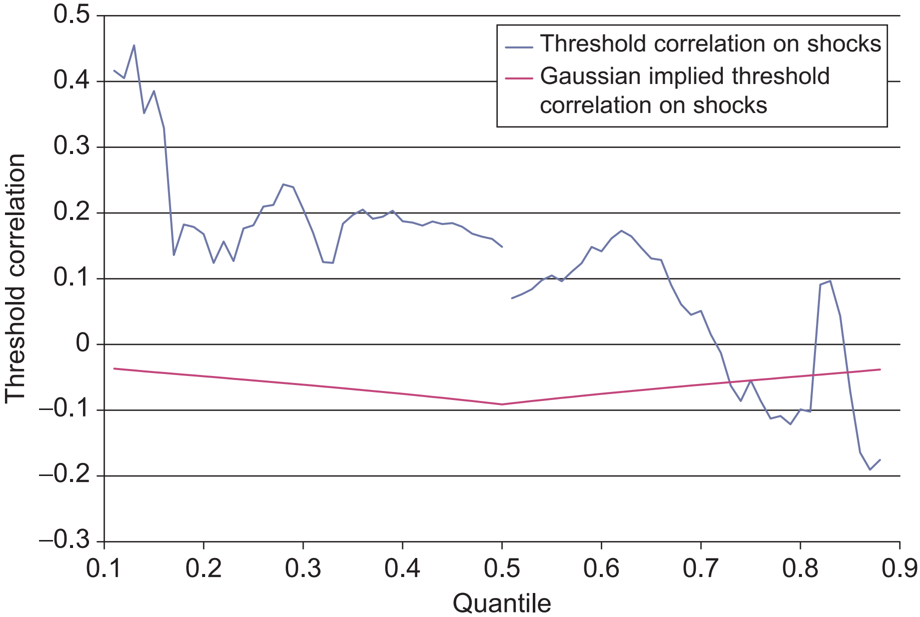
Figure 9.2 shows the threshold correlation plot using the GARCH shocks rather than the returns themselves.

Notice that the patterns are quite different in Figure 9.2 compared with Figure 9.1. Figure 9.2 suggests that the shocks have higher threshold correlations when both shocks are negative than when they are both positive. This indicates that stocks and bonds have important nonlinear left-tail dependencies that risk managers need to model. The threshold correlations implied by the bivariate normal distribution again provide a relatively poor match of the threshold correlations from the empirical shocks.

3 Multivariate Distributions

In this section we consider multivariate distributions that can be combined with GARCH (or RV) and DCC models to provide accurate risk models for large systems of assets. Because we have already modeled the covariance matrix, we need to develop standardized multivariate distributions. We will first review the multivariate standard normal distribution, then we will introduce the multivariate standardized symmetric

Figure 9.2 Threshold correlation for S&P 500 versus 10-year treasury bond GARCH shocks.



Notes: We use daily GARCH shocks on the S&P 500 index and the 10-year treasury bond index. The blue line shows the threshold correlations from the empirical shocks and the red line shows the threshold correlations implied by the normal distribution with a correlation matching that of the empirical shocks.

t distribution, and finally an asymmetric version of the multivariate standardized t distribution.

3.1 The Multivariate Standard Normal Distribution

In Chapter 8 we simulated returns from the normal distribution. In the bivariate case we have the standard normal density with correlation ρ defined by

$$f(z_{1,t}, z_{2,t}; \rho) = \Phi_\rho\left(z_{1,t}, z_{2,t}\right) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{z_{1,t}^2 + z_{2,t}^2 - 2\rho z_{1,t} z_{2,t}}{2(1-\rho^2)}\right)$$

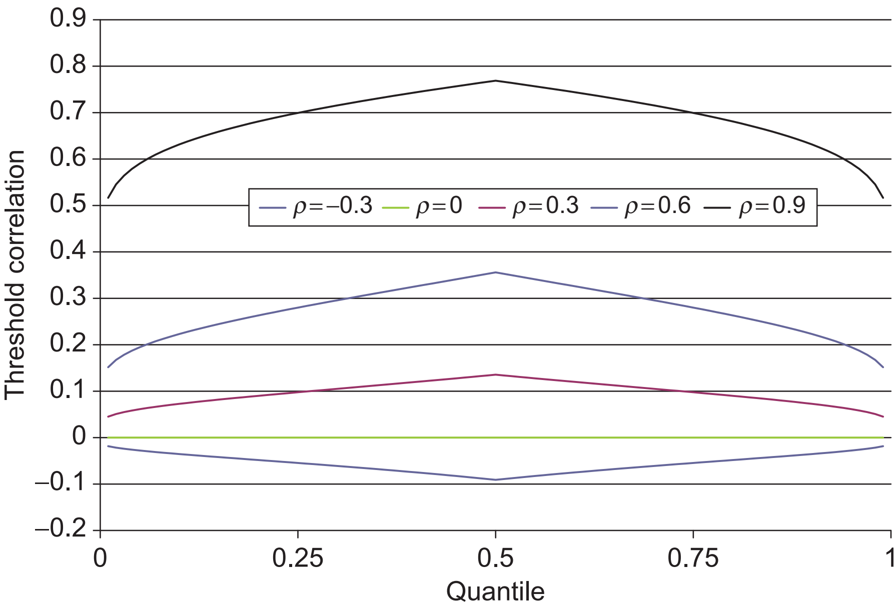
where $1-\rho^2$ is the determinant of the bivariate correlation matrix

$$|\Upsilon| = \begin{vmatrix} 1 & \rho \\ \rho & 1 \end{vmatrix} = 1 - \rho^2$$

We can of course allow for the correlation ρ to be time varying using the DCC models in Chapter 7.

Figure 9.3 shows the threshold correlation for a bivariate normal distribution for different values of ρ. The figure has been constructed using Monte Carlo random

Figure 9.3 Simulated threshold correlations from bivariate normal distributions with various linear correlations.



Notes: The threshold correlations from the bivariate normal distribution are plotted for various values of the linear correlation parameter.

numbers as in Chapter 8. Notice that regardless of ρ the threshold correlations go to zero as the threshold we consider becomes large (positive or negative). The bivariate normal distribution cannot accurately describe data that has large threshold correlations for extreme values of p.

In the multivariate case with n assets we have the density with correlation matrix Υ

$$f(z_t;\Upsilon) = \Phi_\Upsilon(z_t) = \frac{1}{(2\pi)^{n/2}|\Upsilon|^{1/2}}\exp\left(-\tfrac{1}{2}z_t'\Upsilon^{-1}z_t\right)$$

which also will have the unfortunate property that each pair of assets in the vector z_t will have threshold correlations that tend to zero for large thresholds. Again we could have a dynamic correlation matrix.

Because of the time-varying variances and correlations we had to use the simulation methods in Chapter 8 to construct multiday *VaR* and *ES*. But we saw in Chapter 7 that the 1-day *VaR* is easily computed via

$$VaR^p_{t+1} = -\sigma_{PF,t+1}\Phi_p^{-1}, \quad \text{where } \sigma_{PF,t+1} = \sqrt{w_t'D_{t+1}\Upsilon_{t+1}D_{t+1}w_t}$$

where we have portfolio weights w_t and the diagonal matrix of standard deviations D_{t+1}.

The 1-day ES is also easily computed using

$$ES_{t+1}^{p} = \sigma_{PF,t+1} \frac{\phi\left(\Phi_p^{-1}\right)}{p}$$

The multivariate normal distribution has the convenient property that a linear combination of multivariate normal variables is also normally distributed. Because a portfolio is nothing more than a linear combination of asset returns, the multivariate normal distribution is very tempting to use. However the fact that it does not adequately capture the (multivariate) risk of returns means that the convenience of the normal distribution comes at a too-high price for risk management purposes. We therefore now consider the multivariate t distribution.

3.2 The Multivariate Standardized t Distribution

In Chapter 6 we considered the univariate standardized t distribution that had the density

$$f_{\tilde{t}(d)}(z;d) = C(d)\left(1 + z^2/(d-2)\right)^{-(1+d)/2}, \quad \text{for } d > 2$$

where the normalizing constant is

$$C(d) = \frac{\Gamma((d+1)/2)}{\Gamma(d/2)\sqrt{(d-2)\pi}}$$

The bivariate standardized t distribution with correlation ρ takes the following form:

$$f_{\tilde{t}(d,\rho)}(z_1, z_2; d, \rho) = C(d,\rho)\left(1 + \frac{z_1^2 + z_2^2 - 2\rho z_1 z_2}{(d-2)\left(1-\rho^2\right)}\right)^{-(d+2)/2}, \quad \text{for } d > 2$$

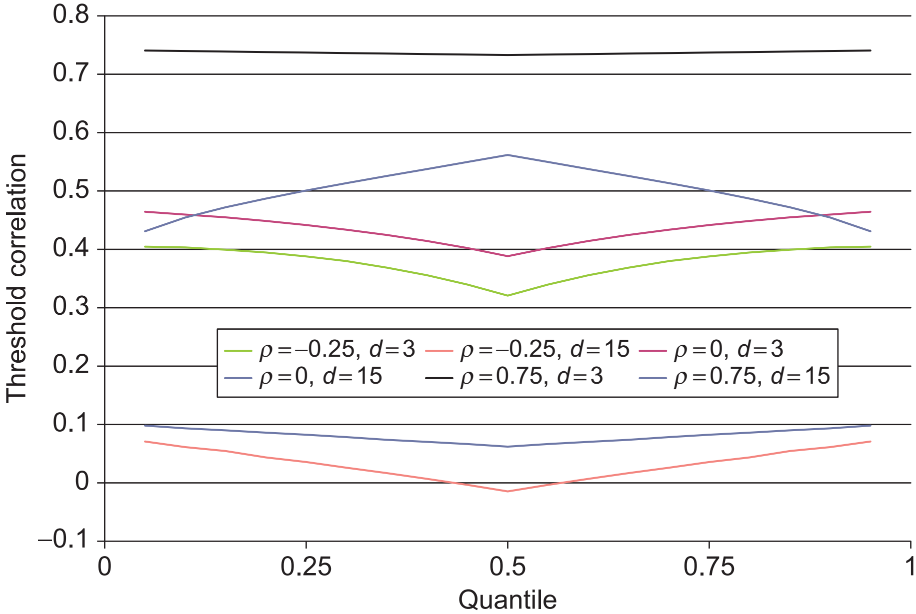
where

$$C(d,\rho) = \frac{\Gamma((d+2)/2)}{\Gamma(d/2)(d-2)\pi\left(1-\rho^2\right)^{1/2}}$$

Note that d is a scalar here and so the two variables have the same degree of tail fatness.

Figure 9.4 shows simulated threshold correlations of the bivariate standard t distribution for different values of d and ρ. Notice that we can generate quite flexible degrees of tail dependence between the two variables when using a multivariate t distribution. However, we are constrained in one important sense: Just as the univariate t distribution is symmetric so is the multivariate t distributions. The threshold correlations are therefore symmetric in the vertical axis.

Figure 9.4 Simulated threshold correlations from the symmetric t distribution with various parameters.



Notes: We simulate a large number of realizations from the bivariate symmetric t distribution. The figure shows the threshold correlations from the simulated data when using various values of the correlation and d parameters.

In the case of n assets we have the multivariate t distribution

$$f_{\tilde{t}(d,\Upsilon)}(z; d, \Upsilon) = C(d, \Upsilon)\left(1 + \frac{z'\Upsilon^{-1}z}{(d-2)}\right)^{-(d+n)/2}, \quad \text{for } d > 2$$

where

$$C(d, \Upsilon) = \frac{\Gamma((d+n)/2)}{\Gamma(d/2)\,((d-2)\pi)^{n/2}\,|\Upsilon|^{1/2}}$$

Using the density definition we can construct the likelihood function

$$\ln L = \sum_{t=1}^{T} \ln\left(f_{\tilde{t}(d,\Upsilon)}(z_t; d, \Upsilon)\right)$$

which can be maximized to estimate d. The correlation matrix can be preestimated using

$$\Upsilon = \frac{1}{T}\sum_{t=1}^{T} z_t z_t'$$

The correlation matrix Υ can also be made dynamic, which can be estimated in a previous step using the DCC approach in Chapter 7.

Following the logic in Chapter 6, an easier estimate of d can be obtained by computing the kurtosis, ζ_2, of each of the n variables. Recall that the relationship between excess kurtosis and d is

$$\zeta_2 = \frac{6}{d-4}$$

Using all the information in the n variables we can estimate d using

$$d = \frac{6}{\frac{1}{n}\sum_{i=1}^{n}\zeta_{2,i}} + 4$$

where $\zeta_{2,i}$ is the sample excess kurtosis of the ith variable.

A portfolio of multivariate t returns does not itself follow the t distribution unfortunately. We therefore need to rely on Monte Carlo simulation to compute portfolio *VaR* and *ES* even for the 1-day horizon.

The standardized symmetric n dimensional t variable can be simulated as follows:

$$z = \sqrt{\frac{d-2}{d}}\sqrt{W}U$$

where W is a univariate inverse gamma random variable, $W \sim IG\left(\frac{d}{2}, \frac{d}{2}\right)$, and U is a vector of multivariate standard normal variables, $U \sim N(0, \Upsilon)$, and where U and W are independent. This representation can be used to simulate standardized multivariate t variables. First, simulate a scalar random W, then simulate a vector random U (as in Chapter 8), and then construct z as just shown.

The simulated z will have a mean of zero, a standard deviation of one, and a correlation matrix Υ. Once we have simulated MC realizations of the vector z we can use the techniques in Chapter 8 to simulate MC realizations of the vector of asset returns (using GARCH for variances and DCC for correlations), and from this the portfolio *VaR* and *ES* can be computed by simulation as well.

3.3 The Multivariate Asymmetric *t* Distribution

Just as we developed a relatively complex asymmetric univariate t distribution in Chapter 6, we can also develop a relatively complex asymmetric multivariate t distribution.

Let λ be an $n \times 1$ vector of asymmetry parameters. The asymmetric t distribution is then defined by

$$f_{asy\tilde{t}}(z; d, \lambda, \Upsilon)$$
$$= \frac{C_{asy}\left(d, \dot{\Upsilon}\right) K_{\frac{d+n}{2}}\left(\sqrt{\left(d + (z-\dot{\mu})'\,\dot{\Upsilon}^{-1}\,(z-\dot{\mu})\right)\lambda'\dot{\Upsilon}^{-1}\lambda}\right)\left(1 + \frac{1}{d}\,(z-\dot{\mu})'\,\dot{\Upsilon}^{-1}\,(z-\dot{\mu})\right)^{-(d+n)/2}}{\exp\left(-\,(z-\dot{\mu})'\,\dot{\Upsilon}^{-1}\lambda\right)\left(\sqrt{\left(d + (z-\dot{\mu})'\,\dot{\Upsilon}^{-1}\,(z-\dot{\mu})\right)\lambda'\dot{\Upsilon}^{-1}\lambda}\right)^{-(d+n)/2}}$$

where

$$\dot{\mu} = -\frac{d}{d-2}\lambda,$$

$$\dot{\Upsilon} = \frac{d-2}{d}\left(\Upsilon - \frac{2d^2}{(d-2)^2(d-4)}\lambda\lambda'\right), \text{ and}$$

$$C_{asy}\left(d, \dot{\Upsilon}\right) = \frac{2^{(1-(d+n)/2)}}{\Gamma(d/2)(d\pi)^{n/2}|\dot{\Upsilon}|^{\frac{1}{2}}}$$

and where $K_{\frac{d+n}{2}}(x)$ is the so-called modified Bessel function of the third kind, which can be evaluated in Excel using the formula besselk(x, $(d+n)/2$).

Note that the vector $\dot{\mu}$ and matrix $\dot{\Upsilon}$ are constructed so that the vector of random shocks z will have a mean of zero, a standard deviation of one, and the correlation matrix Υ. Note also that if $\lambda = 0$ then $\dot{\mu} = 0$ and $\dot{\Upsilon} = \frac{d-2}{d}\Upsilon$.

Although it is not obvious from this definition of $f_{asy\tilde{t}}(z; d, \lambda)$, we can show that the asymmetric t distribution will converge to the symmetric t distribution as the asymmetry parameter vector λ goes to a vector of zeros.

Figure 9.5 shows simulated threshold correlations of the bivariate asymmetric t distribution when setting $\lambda = 0.2$ for both assets, $d = 10$, and when considering different values of ρ. Look closely at Figure 9.5. Note that the asymmetric t distribution is able to capture asymmetries in the threshold correlations and gaps in the threshold correlation around the median (the 0.5 quantile on the horizontal axis), which we saw in the stock and bond thresholds in Figures 9.1 and 9.2.

From the density $f_{asy\tilde{t}}(z; d, \lambda, \Upsilon)$ we can construct the likelihood function

$$\ln L = \sum_{t=1}^{T} \ln\left(f_{asy\tilde{t}}(z_t; d, \lambda, \Upsilon)\right)$$

which can be maximized to estimate the scalar d and and vector λ. As before, the correlation matrix can be preestimated using

$$\Upsilon = \frac{1}{T}\sum_{t=1}^{T} z_t z_t^{'}$$

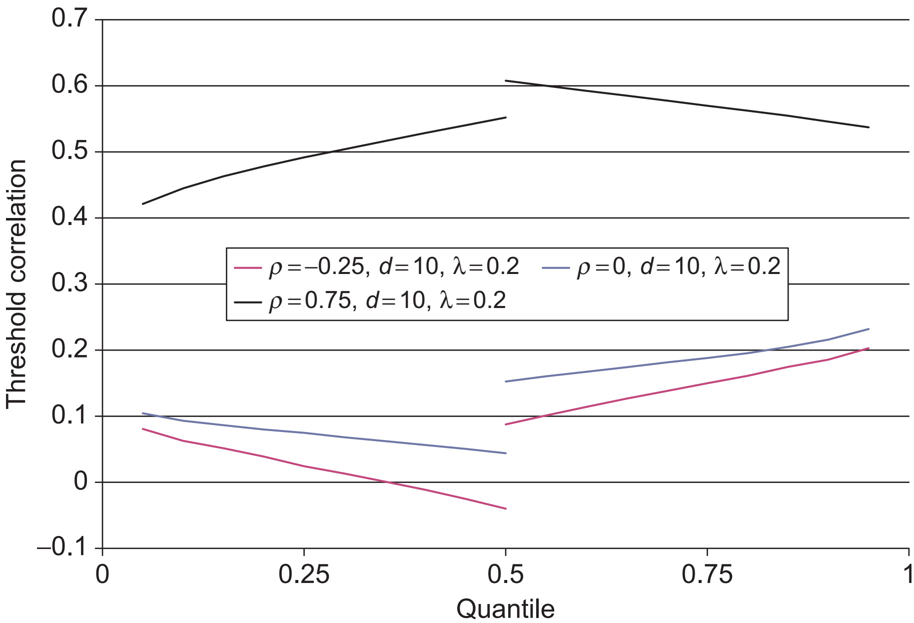
The correlation matrix Υ can also be made dynamic, Υ_t, which can be estimated in a previous step using the DCC approach in Chapter 7 as mentioned earlier.

Simulated values of the (nonstandardized) asymmetric t distribution can be constructed from inverse gamma and normal variables. We now have

$$z = \dot{\mu} + \sqrt{W}U + \lambda W$$

where W is again an inverse gamma variable $W \sim IG\left(\frac{d}{2}, \frac{d}{2}\right)$, U is a vector of normal variables, $U \sim N\left(0, \dot{\Upsilon}\right)$, and U and W are independent. Note that the asymmetric t distribution generalizes the symmetric t distribution by adding a term related to the same inverse gamma random variable W, which is now scaled by the asymmetry vector λ.

Figure 9.5 Simulated threshold correlations from the asymmetric t distribution with various linear correlations.



Notes: We simulate a large number of realizations from the bivariate asymmetric t distribution. The figure shows the threshold correlations from the simulated data when using various correlation values.

The simulated z vector will have the following mean:

$$E[z] = \dot{\mu} + \frac{d}{d-2}\lambda = 0$$

where we have used the definition of $\dot{\mu}$ from before. The variance-covariance matrix of the simulated shocks will be

$$Cov(z) = \frac{d}{d-2}\dot{\Upsilon} + \frac{2d^2}{(d-2)^2(d-4)}\lambda\lambda' = \Upsilon$$

where we have used the definition of $\dot{\Upsilon}$ from before.

The asymmetric t distribution allows for much more flexibility than the symmetric t distribution because of the vector of asymmetry parameters, λ. However in large dimensions (i.e., for a large number of assets, n) estimating the n different λs may be difficult.

Note that the scalar d and the vector λ have to describe the n univariate distributions as well as the joint density of the n assets. We may be able to generate even more flexibility by modeling the univariate distributions separately using for example the asymmetric t distribution in Chapter 6. In this case each asset i would have its own $d_{1,i}$ and its own $d_{2,i}$ (using Chapter 6 notation) capturing univariate skewness and kurtosis. But we then need a method for linking the n distributions together. Fortunately, this is exactly what copula models do.

4 The Copula Modeling Approach

The multivariate normal distribution underestimates the threshold correlations typically found in daily returns. The multivariate t distribution allows for larger threshold correlations but the condition that the d parameter is the same across all assets is restrictive. The asymmetric t distribution is more flexible but it requires estimating many parameters simultaneously.

Ideally we would like to have a modeling approach where the univariate models from Chapters 4 through 6 can be combined to form a proper multivariate distribution. Fortunately, the so-called copula functions have been developed in statistics to provide us exactly with the tool we need.

Consider n assets with potentially different univariate (also known as marginal) distributions, $f_i(z_i)$ and cumulative density functions (CDFs) $u_i = F_i(z_i)$ for $i = 1, 2, \ldots, n$. Note that u_i is simply the probability of observing a value below z_i for asset i. Our goal is to link the marginal distributions across the assets to generate a valid multivariate density.

4.1 Sklar's Theorem

Sklar's theorem provides us with the theoretical foundation we need. It states that for a very general class of multivariate cumulative density functions, defined as $F(z_1, \ldots, z_n)$, with marginal CDFs $F_1(z_1), \ldots, F_n(z_n)$, there exists a unique copula function, $G(\bullet)$ linking the marginals to form the joint distribution

$$\begin{aligned} F(z_1, \ldots, z_n) &= G(F_1(z_1), \ldots, F_n(z_n)) \\ &= G(u_1, \ldots, u_n) \end{aligned}$$

The $G(u_1, \ldots, u_n)$ function is sometimes known as the copula CDF.

Sklar's theorem then implies that the multivariate probability density function (PDF) is

$$\begin{aligned} f(z_1, \ldots, z_n) &= \frac{\partial^n G(F_1(z_1), \ldots, F_n(z_n))}{\partial z_1 \cdots \partial z_n} \\ &= \frac{\partial^n G(u_1, \ldots, u_n)}{\partial u_1 \cdots \partial u_n} \times \prod_{i=1}^{n} f_i(z_i) \\ &= g(u_1, \ldots, u_n) \times \prod_{i=1}^{n} f_i(z_i) \end{aligned}$$

where the copula PDF is defined in the last equation as

$$g\left(u_1, \ldots, u_n\right) \equiv \frac{\partial^n G(u_1, \ldots, u_n)}{\partial u_1 \cdots \partial u_n}$$

Consider now the logarithm of the PDF

$$\ln f(z_1, \ldots, z_n) = \ln g(u_1, \ldots, u_n) + \sum_{i=1}^{n} \ln f_i(z_i)$$

This decomposition shows that we can build the large and complex multivariate density in a number of much easier steps: First, we build and estimate n potentially different marginal distribution models $f_1(z_1), \ldots, f_n(z_n)$ using the methods in Chapters 4 through 6. Second, we decide on the copula PDF $g(u_1, \ldots, u_n)$ and estimate it using the probability outputs u_i from the marginals as the data.

Notice how Sklar's theorem offers a very powerful framework for risk model builders. Notice also the analogy with GARCH and DCC model building: The DCC correlation model allows us to use different GARCH models for each asset. Similarly copula models allow us to use a different univariate density model for each asset.

The log likelihood function corresponding to the entire copula distribution model is constructed by summing the log PDF over the T observations in our sample

$$\ln L = \sum_{t=1}^{T} \ln g(u_{1,t}, \ldots, u_{n,t}) + \sum_{t=1}^{T} \sum_{i=1}^{n} \ln f_i(z_{i,t})$$

But if we have estimated the n marginal distributions in a first step then the copula likelihood function is simply

$$\ln L_g = \sum_{t=1}^{T} \ln g(u_{1,t}, \ldots, u_{n,t})$$

The upshot of this is that we only have to estimate the parameters in the copula PDF function $g(u_{1,t}, \ldots, u_{n,t})$ in a single step. We can estimate all the parameters in the marginal PDFs beforehand. This makes high-dimensional modeling possible. We can for example allow for each asset to follow different univariate asymmetric t distributions (from Chapter 6) each estimated one at a time. Taking these asset-specific distributions as given we can then link them together by estimating the parameters in $g(u_{1,t}, \ldots, u_{n,t})$ in the second step.

Sklar's theorem is very general: It holds for a large class of multivariate distributions. However it is not very specific: It does not say anything about the functional form of $G(u_1, \ldots, u_n)$ and thus $g(u_{1,t}, \ldots, u_{n,t})$. In order to implement the copula modelling approach we need to make specific modeling choices for the copula CDF.

4.2 The Normal Copula

After Sklar's theorem was published in 1959 researchers began to search for potential specific forms for the copula function. Given that the copula CDF must take as inputs marginal CDFs and deliver as output a multivariate CDF one line of research simply took known multivariate distributions and reverse engineered them to take as input probabilities, u, instead of shocks, z.

The most convenient multivariate distribution is the standard normal, and from this we can build the normal copula function. In the bivariate case we have

$$\begin{aligned} G(u_1, u_2; \rho^*) &= \Phi_{\rho^*}(\Phi^{-1}(u_1), \Phi^{-1}(u_2)) \\ &= \Phi_{\rho^*}(\Phi^{-1}(F_1(z_1)), \Phi^{-1}(F_2(z_2))) \end{aligned}$$

where ρ^* is the correlation between $\Phi^{-1}(u_1)$ and $\Phi^{-1}(u_2)$ and we will refer to it as the copula correlation. As in previous chapters, $\Phi^{-1}(\bullet)$ denotes the univariate standard normal inverse CDF.

Note that if the two marginal densities, F_1 and F_2, are standard normal then we get

$$\begin{aligned} G(u_1, u_2; \rho^*) &= \Phi_{\rho^*}(\Phi^{-1}(\Phi(z_1)), \Phi^{-1}(\Phi(z_2))) \\ &= \Phi_{\rho^*}(z_1, z_2) \end{aligned}$$

which is simply the bivariate normal distribution. But note also that if the marginal distributions are NOT the normal then the normal copula does NOT imply the normal distribution. The normal copula is much more flexible than the normal distribution because the normal copula allows for the marginals to be nonnormal, which in turn can generate a multitude of nonnormal multivariate distributions.

In order to estimate the normal copula we need the normal copula PDF. It can be derived as

$$\begin{aligned} g(u_1, u_2; \rho^*) &= \frac{\phi_{\rho^*}(\Phi^{-1}(u_1), \Phi^{-1}(u_2))}{\phi\left(\Phi^{-1}(u_1)\right)\phi\left(\Phi^{-1}(u_2)\right)} \\ &= \frac{1}{\sqrt{1-\rho^{*2}}} \exp\left\{-\frac{\Phi^{-1}(u_1)^2 + \Phi^{-1}(u_2)^2 - 2\rho^*\Phi^{-1}(u_1)\Phi^{-1}(u_2)}{2(1-\rho^{*2})}\right. \\ &\qquad \left. + \frac{\Phi^{-1}(u_1)^2 + \Phi^{-1}(u_2)^2}{2}\right\} \end{aligned}$$

where $\phi_{\rho^*}(*)$ denotes the bivariate standard normal PDF and $\phi(*)$ denotes the univariate standard normal PDF. The copula correlation, ρ^*, can now be estimated by maximizing the likelihood

$$\ln L_g = \sum_{t=1}^{T} \ln g(u_{1,t}, u_{2,t}) = -\frac{T}{2} \ln\left(1 - \rho^{*2}\right)$$
$$- \sum_{t=1}^{T} \frac{\Phi^{-1}(u_{1,t})^2 + \Phi^{-1}(u_{2,t})^2 - 2\rho^* \Phi^{-1}(u_{1,t})\Phi^{-1}(u_{2,t})}{2(1-\rho^{*2})}$$
$$+ \frac{\Phi^{-1}(u_{1,t})^2 + \Phi^{-1}(u_{2,t})^2}{2}$$

where we have $u_{1,t} = F_1\left(z_{1,t}\right)$ and $u_{2,t} = F_2\left(z_{2,t}\right)$.

In the general case with n assets we have the multivariate normal copula CDF and copula PDF

$$G(u_1, \ldots, u_n; \Upsilon^*) = \Phi_{\Upsilon^*}(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n))$$
$$g(u_1, \ldots, u_2; \Upsilon^*) = \frac{\phi_{\Upsilon^*}(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n))}{\prod_{i=1}^{n} \phi\left(\Phi^{-1}(u_i)\right)}$$
$$= \left|\Upsilon^*\right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\Phi^{-1}(u)'(\Upsilon^{*-1} - I_n)\Phi^{-1}(u)\right\}$$

where u is the vector with elements $(u_1, \ldots, u_n)$, and where I_n is an n-dimensional identity matrix that has ones on the diagonal and zeros elsewhere. The correlation matrix, Υ^*, in the normal copula can be estimated by maximizing the likelihood

$$\ln L_g = \sum_{t=1}^{T} \ln g(u_{1,t}, \ldots, u_{n,t})$$
$$= -\frac{1}{2}\sum_{t=1}^{T} \ln\left|\Upsilon^*\right| - \frac{1}{2}\sum_{t=1}^{T} \Phi^{-1}(u_t)'(\Upsilon^{*-1} - I_n)\Phi^{-1}(u_t)$$

If the number of assets is large then Υ^* contains many elements to be estimated and numerical optimization will be difficult.

Let us define the copula shocks for asset i on day t as follows:

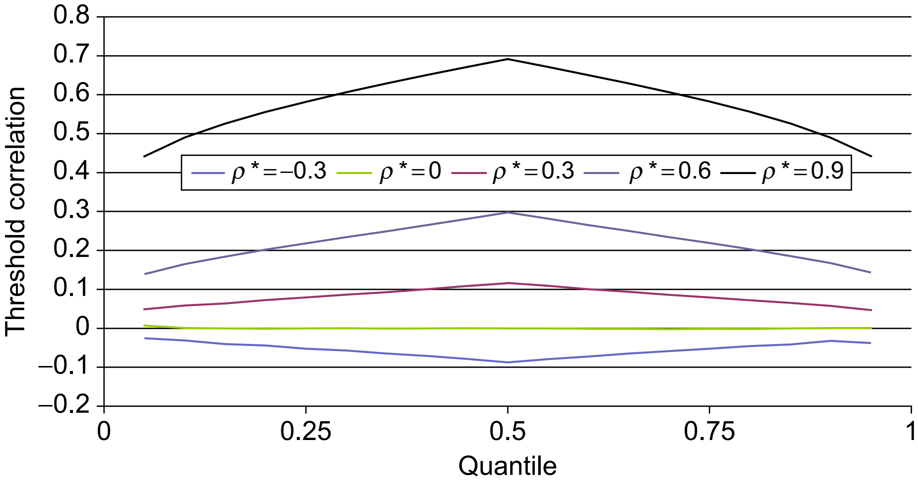
$$z_{i,t}^* = \Phi^{-1}(u_{i,t}) = \Phi^{-1}\left(F_i(z_{i,t})\right)$$

An estimate of the copula correlation matrix can be obtained via correlation targeting

$$\Upsilon^* = \frac{1}{T}\sum_{t=1}^{T} z_t^* z_t^{*\prime}$$

In small dimensions this can be used as starting values of the MLE optimization. In large dimensions it provides a feasible estimate where the MLE is infeasible.

Figure 9.6 Simulated threshold correlations from the bivariate normal copula with various copula correlations.



Notes: We simulate a large number of realizations from the bivariate normal copula. The figure shows the threshold correlations from the simulated data when using various values of the copula correlation parameter.

Consider again the previous bivariate normal copula. We have the bivariate distribution

$$\begin{aligned} F(z_1, z_2) &= G(u_1, u_2) \\ &= \Phi_{\rho^*}(\Phi^{-1}(u_1), \Phi^{-1}(u_2)) \end{aligned}$$

Figure 9.6 shows the threshold correlation between u_1 and u_2 for different values of the copula correlation ρ^*. Naturally, the normal copula threshold correlations look similar to the normal distribution threshold correlations in Figure 9.3.

Note that the threshold correlations are computed from the u_1 and u_2 probabilities and not from the z_1 and z_2 shocks, which was the case in Figures 9.1 through 9.5. The normal copula gives us flexibility by allowing the marginal distributions F_1 and F_2 to be flexible but the multivariate aspects of the normal distribution remains: The threshold correlations go to zero for extreme u_1 and u_2 observations, which is likely not desirable in a risk management model where extreme moves are often highly correlated across assets.

4.3 The t Copula

The normal copula is relatively convenient and much more flexible than the normal distribution but for many financial risk applications it does not allow for enough dependence between the tails of the distributions of the different assets. This was illustrated by the normal copula threshold correlations in Figure 9.6, which decay to zero for extreme tails.

Fortunately a copula model can be built from the t distribution as well. Consider first the bivariate case. The bivariate t copula CDF is defined by

$$G(u_1, u_2; \rho^*, d) = t_{(d,\rho^*)}\left(t^{-1}(u_1; d), t^{-1}(u_2; d)\right)$$

where $t_{(d,\rho^*)}(*)$ denotes the (not standardized) symmetric multivariate t distribution, and $t^{-1}(u; d)$ denotes the inverse CDF of the symmetric (not standardized) univariate t distribution, which we denoted $t_{u_1}^{-1}(d)$ in Chapter 6.

The corresponding bivariate t copula PDF is

$$\begin{aligned} g(u_1, u_2; \rho^*, d) &= \frac{t_{(d,\rho^*)}(t^{-1}(u_1; d), t^{-1}(u_2; d))}{f_{t(d)}\left(t^{-1}(u_1; d); d\right) f_{t(d)}\left(t^{-1}(u_2; d); d\right)} \\ &= \frac{\Gamma\left(\frac{d+2}{2}\right)}{\sqrt{1-\rho^2}\,\Gamma\left(\frac{d}{2}\right)} \left(\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)}\right)^2 \\ &\quad \times \frac{\left(1 + \frac{(t^{-1}(u_1; d))^2 + (t^{-1}(u_2; d))^2 - 2\rho t^{-1}(u_1; d) t^{-1}(u_1; d)}{d(1-\rho^2)}\right)^{-\frac{d+2}{2}}}{\left(1 + \frac{(t^{-1}(u_1; d))^2}{d}\right)^{-\frac{d+1}{2}} \left(1 + \frac{(t^{-1}(u_2; d))^2}{d}\right)^{-\frac{d+1}{2}}} \end{aligned}$$

In Figure 9.7 we plot the threshold correlation between u_1 and u_2 for different values of the copula correlation ρ^* and the tail fatness parameter d. Naturally, the t copula threshold correlations look similar to the t distribution threshold correlations in Figure 9.4 but different from the normal threshold correlations in Figure 9.6.

The t copula can generate large threshold correlations for extreme moves in the assets. Furthermore it allows for individual modeling of the marginal distributions, which allows for much flexibility in the resulting multivariate distribution.

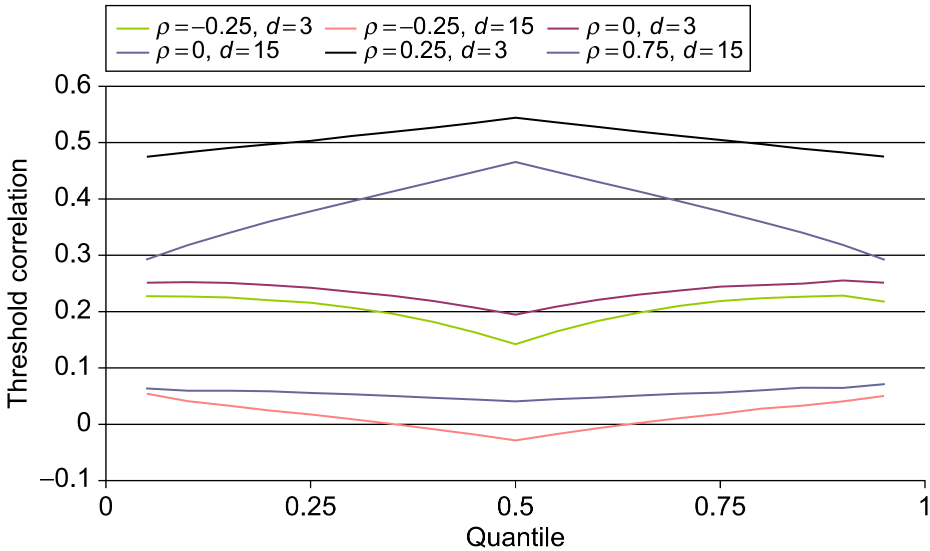
In the general case of n assets we have the t copula CDF

$$G(u_1, \ldots, u_n; \Upsilon^*, d) = t_{(d,\Upsilon^*)}\left(t^{-1}(u_1; d), \ldots, t^{-1}(u_n; d)\right)$$

and the t copula PDF

$$\begin{aligned} g(u_1, \ldots, u_n; \Upsilon^*, d) &= \frac{t_{(d,\Upsilon^*)}(t^{-1}(u_1; d), \ldots, t^{-1}(u_n; d))}{\prod_{i=1}^n t\left(t^{-1}(u_i; d); d\right)} \\ &= \frac{\Gamma\left(\frac{d+n}{2}\right)}{|\Upsilon^*|^{\frac{1}{2}}\,\Gamma\left(\frac{d}{2}\right)} \left(\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)}\right)^n \frac{\left(1 + \frac{1}{d} t^{-1}(u; d)' \Upsilon^{*-1} t^{-1}(u; d)\right)^{-\frac{d+n}{2}}}{\prod_{i=1}^n \left(1 + \frac{(t^{-1}(u_i; d))^2}{d}\right)^{-\frac{d+1}{2}}} \end{aligned}$$

Figure 9.7 Simulated threshold correlations from the symmetric t copula with various parameters.



Notes: We simulate a large number of realizations from the bivariate symmetric t copula. The figure shows the threshold correlations from the simulated data when using various values of the copula correlation and d parameter.

Notice that d is a scalar, which makes the t copula somewhat restrictive but also makes it implementable for many assets.

Maximum likelihood estimation can again be used to estimate the parameters d and Υ^* in the t copula. We need to maximize

$$\ln L_g = \sum_{t=1}^{T} \ln g(u_{1,t}, \ldots, u_{n,t})$$

defining again the copula shocks for asset i on day t as follows:

$$z_{i,t}^* = t^{-1}(u_{i,t}; d) = t^{-1}(F_i\left(z_{i,t}\right); d)$$

In large dimensions we need to target the copula correlation matrix, which can be done as before using

$$\Upsilon^* = \frac{1}{T}\sum_{t=1}^{T} z_t^* z_t^{*\prime}$$

With this matrix preestimated we will only be searching for the parameter d in the maximization of $\ln L_g$ earlier.

4.4 Other Copula Models

An asymmetric t copula can be developed from the asymmetric multivariate t distribution in the same way that we developed the symmetric t copula from the multivariate t distribution earlier.

Figure 9.8 shows the iso-probability or probability contour plots of the bivariate normal copula, the symmetric t copula, and the asymmetric (or skewed) t copula with positive or negative λ. Each line in the contour plot represents the combinations of z_1 and z_2 that correspond to an equal level of probability. The more extreme values of z_1 and z_2 in the outer contours therefore correspond to lower levels of probability. We have essentially taken the bivariate distribution, which is a 3D graph, and sliced it at different levels of probability. The probability levels for each ring are the same across the four panels in Figure 9.8.

Consider the bottom-left corner of each panel in Figure 9.8. This corresponds to extreme outcomes where both assets have a large negative shock. Notice that the symmetric t copula and particularly the asymmetric t copula with negative λ can accommodate the largest (negative) shocks on the outer contours. The two univariate distributions are assumed to be standard normal in Figure 9.8.

In large dimensions it may be necessary to restrict the asymmetry parameter λ to be the same across all or across subsets of the assets. But note that the asymmetric t copula still offers flexibility because we can use the univariate asymmetric t distribution in Chapter 6 to model the marginal distributions so that the λ in the asymmetric t copula only has to capture multivariate aspects of asymmetry. In the multivariate asymmetric t distribution the vector of λ parameters needs to capture asset-specific as well as multivariate asymmetries.

We have only considered normal and t copulas here. Other classes of copula functions exist as well. However, only a few copula functions are applicable in high dimensions; that is, when the number of assets, n, is large.

So far we have assumed that the copula correlation matrix, Υ^*, is constant across time. However, we can let the copula correlations be dynamic using the DCC approach in Chapter 7. We would now use the copula shocks $z_{i,t}^*$ as data input into the estimation of the dynamic copula correlations instead of the $z_{i,t}$ that were used in Chapter 7.

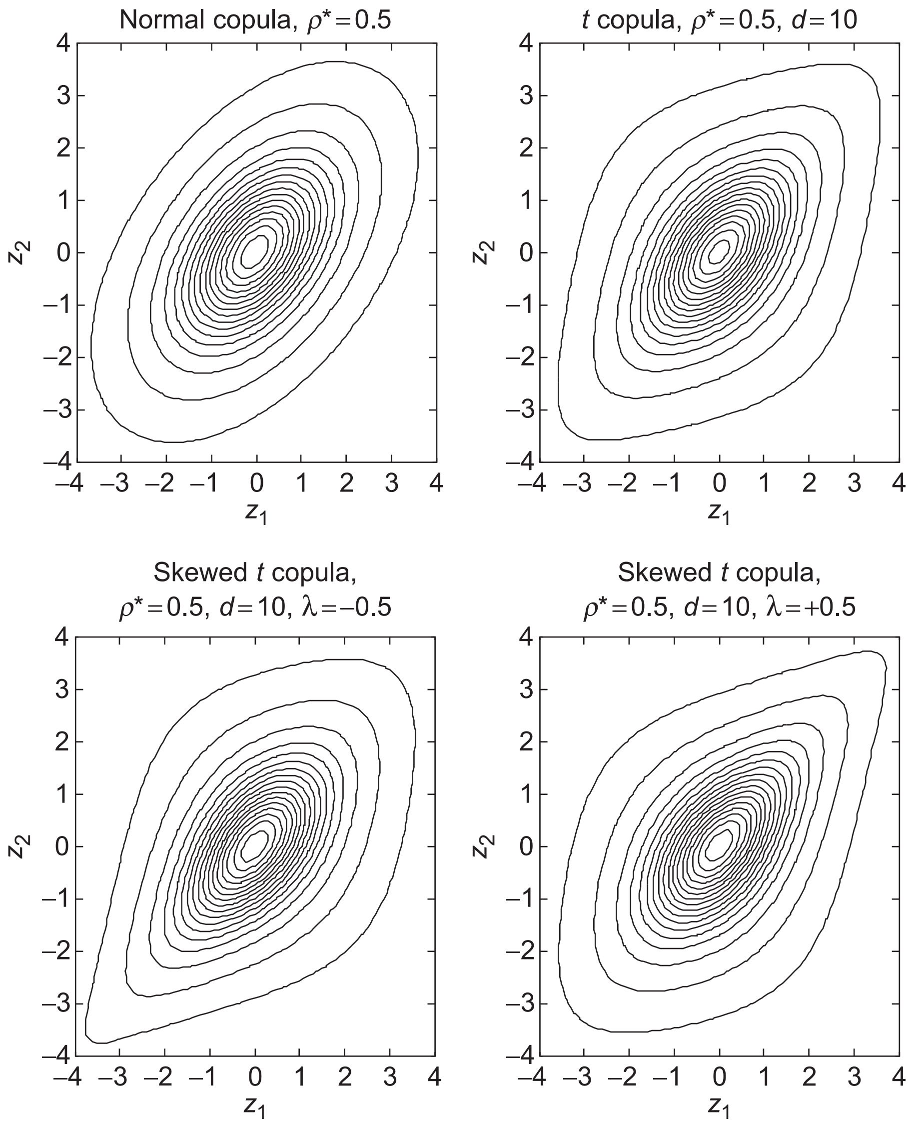
5 Risk Management Using Copula Models

5.1 Copula VaR and ES by Simulation

When we want to compute portfolio *VaR* and *ES* from copula models we need to rely on Monte Carlo simulation. Monte Carlo simulation essentially reverses the steps taken in model building. Recall that we have built the copula model from returns as follows:

- First, estimate a dynamic volatility model, $\sigma_{i,t}$ (Chapters 4 and 5), on each asset to get from observed return $R_{i,t}$ to shock $z_{i,t} = r_{i,t}/\sigma_{i,t}$.

Figure 9.8 Contour probability plots for the normal, symmetric t, and asymmetric skewed t copula.



Notes: We plot the contour probabilities for the normal, symmetric t, and asymmetric skewed t copulas. The marginal distributions are assumed to be standard normal. Each line on the figure corresponds to a particular probability level. The probability levels are held fixed across the four panels.

- Second, estimate a density model for each asset (Chapter 6) to get the probabilities $u_{i,t} = F_i(z_{i,t})$ for each asset.
- Third, estimate the parameters in the copula model using $\ln L_g = \sum_{t=1}^{T} \ln g(u_{1,t}, \ldots, u_{n,t})$.

When we simulate data from the copula model we need to reverse the steps taken in the estimation of the model. We get the algorithm:

- First, simulate the probabilities $(u_{1,t}, \ldots, u_{n,t})$ from the copula model.
- Second, create shocks from the copula probabilities using the marginal inverse CDFs $z_{i,t} = F_i^{-1}(u_{i,t})$ on each asset.
- Third, create returns from shocks using the dynamic volatility models, $r_{i,t} = \sigma_{i,t} z_{i,t}$ on each asset.

Once we have simulated MC vectors of returns from the model we can easily compute the simulated portfolio returns using a given portfolio allocation. The portfolio *VaR*, *ES*, and other measures can then be computed on the simulated portfolio returns in Chapter 8. For example, the 1% *VaR* will be the first percentile of all the simulated portfolio return paths.

5.2 Integrated Risk Management

Integrated risk management is concerned with the aggregation of risks across different business units within an organization. Each business unit may have its own risk model but the senior management needs to know the overall risk to the organization arising in the aggregate from the different units. In short, senior management needs a method for combining the marginal distributions of returns in each business unit.

In the simplest (but highly unrealistic) case, we can assume that the multivariate normal model gives a good description of the overall risk of the firm. If the correlations between all the units are one (also not realistic) then we get a very simple result. Consider first the bivariate case

$$\begin{aligned} VaR_{t+1}^{p} &= -\sqrt{w_{1,t}^2\sigma_{1,t}^2 + w_{2,t}^2\sigma_{2,t}^2 + 2w_{1,t}w_{2,t}\rho_{12,t}\sigma_{1,t}\sigma_{2,t}}\,\Phi_p^{-1} \\ &= -\sqrt{\left(w_{1,t}\sigma_{1,t} + w_{2,t}\sigma_{2,t}\right)^2}\,\Phi_p^{-1} \\ &= \left(w_{1,t}VaR_{1,t+1}^{p} + w_{2,t}VaR_{2,t+1}^{p}\right) \end{aligned}$$

where we have assumed the weights are positive. The total *VaR* is simply the (weighted) sum of the two individual business unit *VaR*s under these specific assumptions.

In the general case of n business units we similarly have

$$VaR_{t+1}^{p} = \sum_{i=1}^{n} w_{i,t} VaR_{i,t+1}^{p}$$

but again only when the returns are multivariate normal with correlation equal to one between all pairs of units.

In the more general case where the returns are not normally distributed with all correlations equal to one, we need to specify the multivariate distribution from the individual risk models. Copulas do exactly that and they are therefore very well suited for integrated risk management. But we do need to estimate the copula parameters and also need to rely on Monte Carlo simulation to compute organization wide *VaR*s and other risk measures. The methods in this and the previous chapter can be used for this purpose.

6 Summary

Multivariate risk models require assumptions about the multivariate distribution of return shocks. The multivariate normal distribution is by far the most convenient model but it does not allow for enough extreme dependence in most risk management applications. We can use the threshold correlation to measure extreme dependence in observed asset returns and in the available multivariate distributions. The multivariate symmetric t and in particular the asymmetric t distribution provides the larger threshold correlations that we need, but in high dimension the asymmetric t may be cumbersome to estimate. Copula models allow us to link together a wide range of marginal distributions. The normal and t copulas we have studied are fairly flexible and are applicable in high dimensions. Copulas are also well suited for integrated risk management where the risk models from individual business units must be linked together to provide a sensible aggregate measure of risk for the organization as a whole.

Further Resources

For powerful applications of threshold correlations in equity markets, see Longin and Solnik (2001), Ang and Chen (2002), and Okimoto (2008).

Sklar's theorem is proved in Sklar (1959). The multivariate symmetric and asymmetric t distributions are analyzed in Demarta and McNeil (2005), who also develop the t copula model. Jondeau and Rockinger (2006) develop the copula-GARCH approach advocated here.

Thorough treatments of copula models are provided in the books by Cherubini et al. (2004) and McNeil et al. (2005). Surveys focusing on risk management applications of copulas can be found in Embrechts et al. (2003, 2002), Fischer et al. (2009), and Patton (2009).

Model selection in the context of copulas is studied in Chen and Fan (2006) and Kole et al. (2007). Default correlation modeling using copulas is done in Li (2000).

Dynamic copula models have been developed in Patton (2004, 2006), Patton and Oh (2011), Chollete et al. (2009), Christoffersen et al. (2011), Christoffersen and Langlois (2011), and Creal et al. (2011). Hafner and Manner (2010) suggest a stochastic copula approach that requires simulation in estimation.

A framework for integrated risk management using copulas is developed in Rosenberg and Schuermann (2006). Copula models are also well suited for studying financial contagion as done in Rodriguez (2007).

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Empirical Exercises

Open the Chapter9Data.xlsx file from the web site.

1. Replicate the threshold correlations in Figures 9.1 and 9.2. Use a grid of thresholds from 0.15 to 0.85 in increments of 0.01.
2. Simulate 10,000 data points from a bivariate normal distribution to replicate the thresholds in Figure 9.3.
3. Estimate a normal copula model on the S&P 500 and 10-year bond return data. Assume that the marginal distributions have RiskMetrics volatility with symmetric t shocks. Estimate the d parameter for each asset first. Assume that the correlation across the two assets is constant.
4. Simulate 10,000 sets of returns from the model in exercise 3. Compute the 1% VaR and ES from the model.

The answers to these exercises can be found in the Chapter9Results.xlsx file on the companion site.

For more information see the companion site at
http://www.elsevierdirect.com/companions/9780123744487