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Multivariate distributions with generalized inverse Gaussian marginals, and associated Poisson mixtures

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ABSTRACT

Several types of multivariate extensions of the inverse Gaussian (IG) distribution and the reciprocal inverse Gaussian (RIG) distribution are proposed. Some of these types are obtained as random-additive-effect models by means of well-known convolution properties of the IG and RIG distributions, and they have one-dimensional IG or RIG marginals. They are used to define a flexible class of multivariate Poisson mixtures.

RÉSUMÉ

Plusieurs généralisations multidimensionnelles des lois gaussienne inverse (IG) et gaussienne inverse réciproque (RIG) sont proposées. Certaines sont obtenues sous forme de modèles à effets aléatoires additifs au moyen de propriétés de la convolution des lois IG et RIG; elles ont des marginales unidimensionnelles identiques, IG ou RIG. Elles sont utilisées afin d'obtenir une riche classe de mélanges de lois de Poisson.

1. INTRODUCTION

The many interesting and useful properties of the generalized inverse Gaussian distribution, and in particular of the inverse Gaussian distribution and the reciprocal inverse Gaussian distribution (i.e., the distribution of the reciprocal of an inverse Gaussian random variable), have led to a search for multivariate versions of these distributions. Various proposals for multivariate extensions of the inverse Gaussian have been put forward by Wasan (1968), Al-Hussaini and Abd-El-Hakim (1981), Barndorff-Nielsen (1983), Barndorff-Nielsen and Blæsild (1983, 1988), Iyengar (1985), and Kocherlakota (1986). However, none of the more tractable among these have all one-dimensional marginals following the inverse Gaussian law. In this paper we construct several types of multivariate — pure or reciprocal or generalized — inverse Gaussian distributions whose one-dimensional marginals are all of the same type. Furthermore we shall employ some of these to discuss multivariate versions of the one-dimensional generalized inverse Gaussian-Poisson (or Sichel) distribution. These include several types of multivariate negative binomial distributions.

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In Section 2 we recapitulate various properties of the generalized inverse Gaussian distributions that are needed for the following discussion. Multivariate — pure or reciprocal or generalized — inverse Gaussian distributions, with one-dimensional marginals of the same type and having a full positive orthant as support, are defined in Section 4. These distributions are all of the random-additive-effects type, and they rely on certain convolution properties of the generalized inverse Gaussian distributions. Section 3 is a preliminary to Section 4 and consists of a brief discussion of random-additive-effects models. In Section 5 the distributions introduced in Section 4 are employed as mixing distributions for Poisson variates. The resulting classes of multivariate discrete distributions comprise previous definitions of a multivariate negative binomial distribution and of a generalized inverse Gaussian-Poisson distribution as special cases. The final Section 6 contains a discussion of another type of multivariate generalized inverse Gaussian distributions which have generalized inverse Gaussian marginals and various other pleasing properties, but also have the drawback that their support is only part of the positive orthant: more specifically, a hyperbolic region.

The questions of inference under the models considered are not discussed here. However, we wish to draw attention to a succesful analysis by Andersen (1987) of a large data set concerning illness among Danish smiths. In this analysis Andersen proposed a new type of two-dimensional negative-binomial distribution, mentioned in Section 5 below, that was required because the classical definition of a bivariate negative binomial distribution, due to Arbous and Kerrich (1951), was incapable of modelling the correlation structure in the data.

2. REVIEW OF SOME BASIC PROPERTIES OF THE GENERALIZED INVERSE GAUSSIAN DISTRIBUTIONS

Most of the distributions considered in this paper are particular instances of the *generalized inverse Gaussian distribution* GIG (λ, χ, ψ) or are related to this class of distributions. As a preliminary to the discussion in the following sections we review here some of its basic properties. For a comprehensive exposition on the generalized inverse Gaussian distributions see Jørgensen (1982); see also Barndorff-Nielsen (1978). None of these refer, however, to some interesting early work of Etienne Halphen* (1941), which was followed up by Le Cam and Morlat (1949) and Morlat (1956), and in which what is now known as the generalized inverse Gaussian distribution is introduced and applied to problems in hydrology. In view of this work, the generalized inverse Gaussian distributions could appropriately be termed the Halphen distributions. We are indebted to Michel Maurin for drawing our attention to the papers by Halphen, LeCam, and Morlat.

The GIG(λ, χ, ψ) distribution has probability density function

$$\frac{(\psi/\chi)^{\lambda/2}}{2K_\lambda(\sqrt{\chi\psi})} x^{\lambda-1} e^{-\frac{1}{2}(\chi x^{-1} + \psi x)}, \qquad x > 0, \tag{2.1}$$

where K_λ denotes the modified Bessel function of the third kind and with index λ. The domain of variation of (λ, χ, ψ) is given by

$$\begin{array}{lll} \chi \geq 0, & \psi > 0 & \text{if } \lambda > 0, \\ \chi > 0, & \psi > 0 & \text{if } \lambda = 0, \\ \chi > 0, & \psi \geq 0 & \text{if } \lambda < 0. \end{array}$$

*Published under the name of Daniel Dugué.

In case $\chi = 0$ or $\psi = 0$, the norming constant in (2.1) is determined via the asymptotic relation

$$K_\lambda(x) \sim \Gamma(\lambda)2^{\lambda-1}x^{-\lambda} \quad \text{as} \quad x \downarrow 0 \qquad (\lambda > 0)$$

and the formula

$$K_\lambda(x) = K_{-\lambda}(x)$$

Denoting the law of a random variable x by $\mathcal{L}(x)$, we have

$$\mathcal{L}(x) = \mathrm{GIG}(\lambda, \chi, \psi) \quad \Leftrightarrow \quad \mathcal{L}(x^{-1}) = \mathrm{GIG}(-\lambda, \psi, \chi) \tag{2.2}$$

and, for every $c > 0$,

$$\mathcal{L}(x) = \mathrm{GIG}(\lambda, \chi, \psi) \quad \Rightarrow \quad \mathcal{L}(cx) = \mathrm{GIG}(\lambda, c\chi, c^{-1}\psi). \tag{2.3}$$

The *inverse Gaussian distribution* $\mathrm{IG}(\chi, \psi)$ has probability density function

$$\frac{\sqrt{\chi}}{\sqrt{2\pi}} e^{\sqrt{\chi\psi}} x^{-\frac{3}{2}} e^{-\frac{1}{2}(\chi x^{-1} + \psi x)}, \tag{2.4}$$

and in view of the well-known formula

$$K_{-\frac{1}{2}}(x) = K_{\frac{1}{2}}(x) = \sqrt{\pi/2}\, x^{-\frac{1}{2}} e^{-x},$$

one has that $\mathrm{IG}(\chi, \psi) = \mathrm{GIG}(-\frac{1}{2}, \chi, \psi)$. If $\mathcal{L}(x) = \mathrm{IG}(\chi, \psi)$ then $\mathcal{L}(x^{-1}) = \mathrm{GIG}(\frac{1}{2}, \psi, \chi)$. For this reason the distribution $\mathrm{GIG}(\frac{1}{2}, \chi, \psi)$, whose density function is

$$\frac{\sqrt{\psi}}{\sqrt{2\pi}} e^{\sqrt{\chi\psi}} x^{-\frac{1}{2}} e^{-\frac{1}{2}(\chi x^{-1} + \psi x)}, \tag{2.5}$$

is referred to as the *reciprocal inverse Gaussian distribution* $\mathrm{RIG}(\chi, \psi)$.

For $\lambda > 0$ and $\chi = 0$ (2.1) becomes

$$\frac{(\psi/2)^\lambda}{\Gamma(\lambda)} x^{\lambda-1} e^{-\frac{1}{2}\psi x}, \tag{2.6}$$

i.e., $\mathrm{GIG}(\lambda, 0, \psi)$ is the *gamma distribution* $\Gamma(\lambda, \psi/2)$.

If $\mathcal{L}(x) = \mathrm{GIG}(\lambda, \chi, \psi)$, it follows from (2.1) that the moment generating function (or Laplace transform) of X is

$$M_{\mathrm{GIG}(\lambda,\chi,\psi)}(s) = \mathcal{E}\{e^{sx}\} = \left(1 - \frac{2s}{\psi}\right)^{-\lambda/2} \frac{K_\lambda(\sqrt{\chi(\psi - 2s)})}{K_\lambda(\sqrt{\chi\psi})}. \tag{2.7}$$

Similarly, (2.4)–(2.6) imply that

$$M_{\mathrm{IG}(\chi,\psi)}(s) = e^{\sqrt{\chi\psi} - \sqrt{\chi(\psi-2s)}}, \tag{2.8}$$

$$M_{\mathrm{RIG}(\chi,\psi)}(s) = \left(1 - \frac{2s}{\psi}\right)^{-\frac{1}{2}} e^{\sqrt{\chi\psi} - \sqrt{\chi(\psi-2s)}}, \tag{2.9}$$

and

$$M_{\Gamma(\lambda,\psi/2)}(s) = \left(1 - \frac{2s}{\psi}\right)^{-\lambda}. \tag{2.10}$$

Using the formulae (2.7)–(2.10) or arguing directly, the following known convolution properties of the generalized inverse Gaussian distributions are easily verified:

PROPERTY 1. $\mathrm{IG}(\chi,\psi) * \mathrm{IG}(\chi',\psi) = \mathrm{IG}\big((\sqrt{\chi}+\sqrt{\chi'})^2,\psi\big)$.

PROPERTY 2. $\mathrm{IG}(\chi,\psi) * \mathrm{RIG}(\chi',\psi) = \mathrm{RIG}\big((\sqrt{\chi}+\sqrt{\chi'})^2,\psi\big)$.

PROPERTY 3. $\mathrm{GIG}(-\lambda,\chi,\psi) * \Gamma(\lambda,\psi/2) = \mathrm{GIG}(\lambda,\chi,\psi)$ *for every* $\lambda > 0$.

3. MODELS OF RANDOM-ADDITIVE-EFFECTS TYPE

Suppose $z_1,\dots,z_m$ are random variables of the form

$$z_i = \alpha_i u + v_i, \tag{3.1}$$

where u and $v_1,\dots,v_m$ are independent random variables and $\alpha_1,\dots,\alpha_m$ are parameters, i.e., the family of joint distributions of $z_1,\dots,z_m$ constitutes a model of random-additive-effects. This type of construct is primarily of interest when u and $v_1,\dots,v_m$ follow parametric models that are closed under scale transformations and are such that the convolution of an arbitrary member of the distributions for u with an arbitrary member of the distributions for the v_i's has a simple form.

Well-known examples of random-additive-effects models are, besides the normal variance-component models, a form of multivariate gamma distribution and a form of multivariate Poisson distribution. See, for instance, Mardia (1970). In the next section we shall use the convolution properties 1–3 stated at the end of the previous section to define multivariate — pure or reciprocal or generalized — inverse Gaussian distributions having one-dimensional marginals of the same kind.

First, however, we note a few elementary facts about random-additive-effects models as defined here. Letting $s = (s_1,\dots,s_m)$, $z = (z_1,\dots,z_m)$, and $s\cdot z = s_1z_1 + \cdots + s_mz_m$, it follows from (3.1) that the random vector z has moment generating function

$$\begin{aligned} M_z(s) &= \mathcal{E}\{e^{s\cdot z}\} \\ &= \mathcal{E}\{e^{(\alpha\cdot s)u + s\cdot v}\} \\ &= M_u(\alpha\cdot s)\prod_{i=1}^{m} M_{v_i}(s_i). \end{aligned} \tag{3.2}$$

Similarly,

$$M_{z_i}(s_i) = M_u(\alpha_i s_i) M_{v_i}(s_i). \tag{3.3}$$

In the random additive effects models we consider in the next section, we assume that the variables u, v_i, and z_i are all positive. Clearly, this implies that the parameters α_i should be nonnegative, and furthermore that in general it is not possible to give an explicit expression for the joint density of $z = (z_1,\dots,z_m)$. However, in the applications of those models, discussed in Section 5, it is the explicit knowledge of the cumulant generating function of z rather than the density function of z that is of importance.

4. MULTIVARIATE IG, RIG, AND GIG DISTRIBUTIONS OF RANDOM-ADDITIVE-EFFECTS TYPE

4.1. Multivariate IG distribution.

If in (3.1) one takes $u \sim \mathrm{IG}(\chi,\psi)$ and $v_i \sim \mathrm{IG}(\chi_i,\psi/\alpha_i)$, then the distribution of $(z_1,\dots,z_m)$ has IG marginals. (Here and in the following we use $\sim$ to mean "distributed as".) Specifically, it follows from (2.3) and Property 1 that the marginal distribution of z_i is

$$z_i \sim \mathrm{IG}\big((\sqrt{\chi\alpha_i} + \sqrt{\chi_i})^2,\ \psi/\alpha_i\big).$$

The formulae (2.8) and (3.2) imply that the moment generating function of z is

$$M_z(s) = e^{\sqrt{\chi\psi}-\sqrt{\chi(\psi-2\alpha\cdot s)}} \prod_{i=1}^{m} e^{\sqrt{\chi_i\psi/\alpha_i}-\sqrt{\chi_i(\psi/\alpha_i-2s_i)}}.$$

Moreover, it follows from (2.2) that the marginal distributions of $z^{-1} = (z_1^{-1}, \ldots, z_m^{-1})$ are

$$z_i^{-1} \sim \mathrm{RIG}\big(\psi/\alpha_i,\ (\sqrt{\chi\alpha_i} + \sqrt{\chi_i})^2\big).$$

4.2. Multivariate RIG distribution.

Equation (2.3) and Property 2 imply that if $u \sim \mathrm{IG}(\chi, \psi)$ and $v_i \sim \mathrm{RIG}(\chi_i, \psi/\alpha_i)$, then the distribution of $(z_1, \ldots, z_m)$ has RIG marginals; more precisely,

$$z_i \sim \mathrm{RIG}\big((\sqrt{\chi\alpha_i} + \sqrt{\chi_i})^2,\ \psi/\alpha_i\big).$$

From (2.8), (2.9), and (3.2) the moment generating function of z is seen to be

$$M_z(s) = e^{\sqrt{\chi\psi}-\sqrt{\chi(\psi-2\alpha\cdot s)}} \prod_{i=1}^{m} \left(1 - \frac{2\alpha_i s_i}{\psi}\right)^{-\frac{1}{2}} e^{\sqrt{\chi_i\psi/\alpha_i}-\sqrt{\chi_i(\psi/\alpha_i-2s_i)}}.$$

The marginal distributions of $z^{-1} = (z_1^{-1}, \ldots, z_m^{-1})$ are

$$z_i^{-1} \sim \mathrm{IG}\big(\psi/\alpha_i,\ (\sqrt{\chi\alpha_i} + \sqrt{\chi_i})^2\big).$$

4.3. Multivariate GIG distributions.

If $u \sim \Gamma(\lambda, \psi/2)$ and $v_i \sim \mathrm{GIG}(-\lambda, \chi_i, \psi/\alpha_i)$, it follows from (2.3) and Property 3 that the marginal distribution of z_i is

$$z_i \sim \mathrm{GIG}(\lambda, \chi_i, \psi/\alpha_i).$$

In particular, if $\lambda = \frac{1}{2}$, the distribution of $(z_1, \ldots, z_m)$ has RIG marginals.

Using (2.7), (2.10), and (3.2), the moment generating function of z in the general case becomes

$$M_z(s) = \left(1 - \frac{2(\alpha\cdot s)}{\psi}\right)^{-\lambda} \prod_{i=1}^{m} \left(1 - \frac{2\alpha_i s_i}{\psi}\right)^{\lambda/2} \frac{K_\lambda\big(\sqrt{\chi_i(\psi/\alpha_i - 2s_i)}\big)}{K_\lambda(\sqrt{\chi_i\psi/\alpha_i})}.$$

For $\lambda = \frac{1}{2}$ this takes the simple form

$$M_z(s) = \left(1 - \frac{2(\alpha\cdot s)}{\psi}\right)^{-\frac{1}{2}} \prod_{i=1}^{m} e^{\sqrt{\chi_i\psi/\alpha_i}-\sqrt{\chi_i(\psi/\alpha_i-2s_i)}}.$$

Finally, note that (2.2) implies that the marginal distributions of $z^{-1} = (z_1^{-1}, \ldots, z_m^{-1})$ are

$$z_i^{-1} \sim \mathrm{GIG}(-\lambda, \psi/\alpha_i, \chi_i).$$

5. MULTIVARIATE VERSIONS OF THE GENERALIZED INVERSE GAUSSIAN-POISSON DISTRIBUTION

Sichel (1971) introduced the distribution on the nonnegative integers which is obtained by mixing the Poisson distribution with the generalized inverse Gaussian distribution, i.e., by treating the parameter of the Poisson distribution as a random variable having a GIG distribution. The resulting distribution is known as the Sichel distribution or the generalized inverse Gaussian-Poisson distribution. It has been found to be useful in a variety of contexts, in particular in linguistics; see Sichel (1971, 1973a,b, 1974, 1975, 1982a, 1985, 1986), and cf. also Sichel (1982b), Atkinson and Yeh (1982), and Dean, Lawless, and Willmot (1989). A multivariate extension has recently been proposed by Stein, Zucchini, and Juritz (1987).

Here another type of multivariate extension will be discussed. The construction we shall consider is analogous to that of a two-dimensional negative binomial distribution due to Andersen (1987), who made a successful application to a set of data concerning cases of illness among a group of smiths.

Let $y_1, \dots, y_m$ and $z_1, \dots, z_m$ be random variables and suppose that conditionally on $z_1, \dots, z_m$ the variables $y_1, \dots, y_m$ are independent and Poisson-distributed with means $z_1, \dots, z_m$ respectively. Suppose moreover that the joint distribution of $z_1, \dots, z_m$ is of the additive-effects type discussed in Section 3, that is, $z_1, = \alpha_1 u + v_1, \dots, z_m = \alpha_m u + v_m$, where $u, v_1, \dots, v_m$ are independent random variables and $\alpha_1, \dots, \alpha_m$ are parameters. The joint distribution of $y_1, \dots, y_m$ then has a generating function of the form

$$\begin{aligned} G(t_1, \dots, t_m) &= \mathcal{E}\{t_1^{y_1} \cdots t_m^{y_m}\} \\ &= M_u\left(\sum \alpha_i(t_i - 1)\right) \prod_{i=1}^{m} M_{v_i}(t_i - 1). \end{aligned} \tag{5.1}$$

The point probabilities of $(y_1, \dots, y_m)$ can be determined by power-series expansion of the expression on the right-hand side. We note that

$$\mathcal{E} y_i = \alpha_i \mathcal{E} u + \mathcal{E} v_i$$

and

$$Cov(y_i, y_j) = \begin{cases} \alpha_i^2 \, \mathcal{V}ar \; u + \mathcal{V}ar \; v_i + \alpha_i \mathcal{E} u + \mathcal{E} v_i & \text{if } i = j, \\ \alpha_i \alpha_j \, \mathcal{V}ar \; u & \text{if } i \neq j. \end{cases}$$

We now discuss a number of special cases of this construction.

EXAMPLE 5.1. If $u \sim \Gamma(\lambda, \psi/2)$ and the v_i's are all 0, we have the multivariate version of the two-dimensional negative-binomial distribution introduced by Arbous and Kerrich (1951). An application of the symmetric version of this distribution, i.e., the distribution obtained by assuming that $\alpha_1 = \alpha_2$, may be found in Arbous and Sichel (1954).

EXAMPLE 5.2. If $u \sim \mathrm{GIG}(\lambda, \chi, \psi)$ and the v_i's are all 0, the multivariate extension of the GIG-Poisson distribution considered by Stein, Zucchini, and Juritz (1987) results. The one-dimensional versions of the IG-Poisson distribution and of the GIG-Poisson were introduced by Holla (1966) and Sichel (1971), respectively.

The condition $v_i = 0$ in Examples 5.1 and 5.2 is equivalent to proportionality of the conditional means of y_i given z_i (or of y_i given u), $i = 1, \dots, m$. In the following three examples we assume that the conditional mean of y_i is of the form (3.1) with v_i nondegenerate.

EXAMPLE 5.3. If $u \sim \Gamma(\kappa, 1)$ and $v_i \sim \Gamma(\kappa_i, \beta_i^{-1})$, it follows from (5.1) that the generating function of $(y_1, \ldots, y_m)$ is

$$G(t_1, \ldots, t_m) = \left\{1 - \sum \alpha_i(t_i - 1)\right\}^{-\kappa} \prod_{i=1}^{m} \{1 - \beta_i(t_i - 1)\}^{-\kappa_i}. \tag{5.2}$$

Since the generating function of the negative-binomial distribution $b^-(\kappa, p)$ with probability function

$$b^-(x; \kappa, p) = \binom{\kappa + x - 1}{x} p^x (1-p)^\kappa$$

is

$$G(t) = \left(1 - \frac{p}{1-p}(t-1)\right)^{-\kappa},$$

it is seen that the marginal distribution of y_i is negative binomial in the following three cases:

(1) $\alpha_i = 0$. In this case the y's are independent and $y_i \sim b^-\big(\kappa_i, \beta_i/(1+\beta_i)\big)$.
(2) $\beta_i = 0$. This case corresponds to that considered in Example 5.1. One has

$$y_i \sim b^-\left(\kappa, \frac{\alpha_i}{1+\alpha_i}\right)$$

and furthermore

$$\sum y_i \sim b^-\left(\kappa, \frac{\sum \alpha_i}{1 + \sum \alpha_i}\right).$$

(3) $\alpha_i = \beta_i$. Note that in this case (3.1) may be written as

$$z_i = \alpha_i(u + w_i),$$

where $u \sim \Gamma(\kappa, 1)$ and $w_i \sim \Gamma(\kappa_i, 1)$. It follows from (5.2) that

$$y_i \sim b^-\left(\kappa + \kappa_i, \frac{\alpha_i}{1+\alpha_i}\right).$$

The bivariate version of the distribution in (3) was introduced by Andersen (1987) in the abovementioned investigation of cases of illness among a group of smiths. Some properties of this distribution have also been considered by Stein and Juritz (1987); see their Section 2.1.

EXAMPLE 5.4. If $u \sim \mathrm{IG}(\chi, \psi)$ and $v_i \sim \mathrm{RIG}(\chi_i, \psi/\alpha_i)$, then each of the random variables y_i follows an IG-Poisson distribution. The set of parameters $z = (z_1, \ldots, z_m)$ of the Poisson distributions has the multivariate IG distribution discussed in Section 4.1. Using (5.1), the generating function of $(y_1, \ldots, y_m)$ is seen to be

$$G(t_1, \ldots, t_m) = e^{\sqrt{\chi\psi} - \sqrt{\chi\{\psi - 2\sum \alpha_i(t_i - 1)\}}} \prod_{i=1}^{m} e^{\sqrt{\chi_i\psi/\alpha_i} - \sqrt{\chi_i\{\psi/\alpha_i - 2(t_i - 1)\}}}.$$

The two-dimensional version of this distribution has been introduced in Section 2.2 of Stein and Juritz (1987).

EXAMPLE 5.5. If $u \sim \mathrm{IG}(\chi, \psi)$ and $v_i \sim \mathrm{RIG}(\chi_i, \psi/\alpha_i)$, then each of the random variables y_i follows a RIG-Poisson distribution where $z_i \sim \left((\sqrt{\chi\alpha_i} + \sqrt{\chi_i})^2, \psi/\alpha_i\right)$, as seen from Section 4.2. In this case (5.1) takes the form

$$G(t_1, \ldots, t_m) = e^{\sqrt{\chi\psi} - \sqrt{\chi\{\psi - 2\sum \alpha_i(t_i - 1)\}}} \prod_{i=1}^{m} \left(1 - \frac{2\alpha_i(t_i - 1)}{\psi}\right)^{-\frac{1}{2}} \times e^{\sqrt{\chi_i\psi/\alpha_i} - \sqrt{\chi_i\{\psi/\alpha_i - 2(t_i - 1)\}}}.$$

EXAMPLE 5.6. If $u \sim \Gamma(\lambda, \psi/2)$ and $v_i \sim \mathrm{GIG}(-\lambda, \chi_i, \psi/\alpha_i)$, then each of the random variables y_i follows a GIG-Poisson distribution, where, according to Section 4.3, the parameter of the Poisson distribution z_i is distributed as $\mathrm{GIG}(\lambda, \chi_i, \psi/\alpha_i)$. The generating function of $(y_1, \ldots, y_m)$ is

$$G(t_1, \ldots, t_m) = \left(1 - 2\frac{\sum \alpha_i(t_i - 1)}{\psi}\right)^{-\lambda} \times \prod_{i=1}^{m} \left(1 - \frac{2\alpha_i(t_i - 1)}{\psi}\right)^{\lambda/2} \frac{K_\lambda\left(\sqrt{\chi_i\{\psi/\alpha_i - 2(t_i - 1)\}}\right)}{K_\lambda(\sqrt{\chi_i\psi/\alpha_i})}.$$

We conclude this section by indicating the point probability function of the bivariate compound Poisson distributions considered in the examples in this section.

For fixed value of λ and χ the $\mathrm{GIG}(\lambda, \chi, \psi)$ distribution may be considered as an exponential family of the form

$$a(\lambda, \chi, \psi)b(x, \chi)x^{\lambda-1}e^{-\frac{1}{2}\psi x}. \tag{5.3}$$

If x is distributed according to (5.3), one obtains that

$$\mathcal{E}\{x^k e^{-\alpha x}\} = \frac{a(\lambda, \chi, \psi)}{a(\lambda + k, \chi, \psi + 2\alpha)}. \tag{5.4}$$

Assuming that given (z_1, z_2) the variates y_1 and y_2 are independent and Poisson-distributed with means z_1 and z_2, respectively, and that

$$z_i = \alpha_i u + v_i, \qquad i = 1, 2, \tag{5.5}$$

where u, v_1, v_2 are independent and where each of these variates has a distribution of the form (5.3), one has that

$$\begin{aligned}
p(y_1, y_2) &= \int\int\int e^{-z_1 - z_2} \frac{z_1^{y_1}}{y_1!} \frac{z_2^{y_2}}{y_2!} f(u)f(v_1)f(v_2)\, du\, dv_1\, dv_2 \\
&= \frac{1}{y_1!} \frac{1}{y_2!} \sum_{i_1=0}^{y_1} \sum_{i_2=0}^{y_2} \binom{y_1}{i_1} \binom{y_2}{i_2} \alpha_1^{i_1} \alpha_2^{i_2} \\
&\quad \times \int\int\int u^{i_1+i_2} v_1^{y_1-i_1} v_2^{y_2-i_2} \\
&\quad \times e^{-(\alpha_1+\alpha_2)u - v_1 - v_2} f(u)f(v_1)f(v_2)\, du\, dv_1\, dv_2 \\
&= \frac{1}{y_1!} \frac{1}{y_2!} \sum_{i_1=0}^{y_1} \sum_{i_2=0}^{y_2} \binom{y_1}{i_1} \binom{y_2}{i_2} \alpha_1^{i_1} \alpha_2^{i_2} \\
&\quad \times \mathcal{E}\{u^{i_1+i_2} e^{-(\alpha_1+\alpha_2)u}\} \mathcal{E}\{v_1^{y_1-i_1} e^{-v_1}\} \mathcal{E}\{v_2^{y_2-i_2} e^{-v_2}\}.
\end{aligned} \tag{5.6}$$

Consequently, the point probability function of the bivariate compound Poisson distribution, defined using (5.5), is obtainable from (5.4) and (5.6). As an example one finds that the point probability function of the bivariate negative-binomial distribution considered in (3) of Example 5.3 is

$$p(y_1, y_2) = \frac{\alpha_1^{y_1}\alpha_2^{y_2}}{y_1!y_2!}\sum_{i_1=0}^{y_1}\sum_{i_2=0}^{y_2}\binom{y_1}{i_1}\binom{y_2}{i_2}\kappa^{\langle i_1+i_2\rangle}(1+\alpha_1+\alpha_2)^{-(\kappa+i_1+i_2)}$$
$$\times\, \kappa_1^{\langle y_1-i_1\rangle}(1+\alpha_1)^{-(\kappa_1+y_1-i_1)}\kappa_2^{\langle y_2-i_2\rangle}(1+\alpha_2)^{-(\kappa_2+y_2-i_2)}, \quad (5.7)$$

where $\kappa^{\langle i_1+i_2\rangle} = \Gamma(\kappa+i_1+i_2)/\Gamma(\kappa) = \kappa(\kappa+1)\cdots(\kappa+i_1+i_2-1)$.

The point probability function of the marginal distribution of y_i, $i = 1, 2$, as well as those of the other multivariate compound Poisson distributions considered in the examples in this section, may be found in a similar way.

6. DISTRIBUTIONS WITH GIG MARGINALS DEFINED ON HYPERBOLIC SUBSETS OF R_+^m

We first consider the case $m = 2$. Suppose $\lambda > 0$. From (2.2) and Property 3 it follows that if

$$\mathcal{L}(u, v) = \mathrm{GIG}(-\lambda, \chi, \psi) * \Gamma(\lambda, \psi/2)$$

and if

$$x = (x_1, x_2) = (u^{-1}, u+v),$$

then $x_1 \sim \mathrm{GIG}(\lambda, \psi, \chi)$ and $x_2 \sim \mathrm{GIG}(\lambda, \chi, \psi)$. Furthermore, setting $y = (y_1, y_2) = (x_1^{-1}, x_2^{-1})$, Equation (2.2) implies that $y_1 \sim \mathrm{GIG}(-\lambda, \chi, \psi)$ and $y_2 \sim \mathrm{GIG}(-\lambda, \psi, \chi)$. The density functions of the distributions of x and y are easily seen to be, respectively,

$$\frac{(\chi\psi)^{\lambda/2}}{2^{\lambda+1}\Gamma(\lambda)K_\lambda(\sqrt{\chi\psi})}(x_1x_2-1)^{\lambda-1}e^{-\frac{1}{2}(\chi x_1+\psi x_2)}, \qquad x \in S_x, \quad (6.1)$$

with support $S_x = \{(x_1, x_2) : x_1, x_2 > 0,\ x_1x_2 > 1\}$, and

$$\frac{(\chi\psi)^{\lambda/2}}{2^{\lambda+1}\Gamma(\lambda)K_\lambda(\sqrt{\chi\psi})}(y_1y_2)^{-\lambda-1}(1-y_1y_2)^{\lambda-1}e^{-\frac{1}{2}(\chi y_1^{-1}+\psi y_2^{-1})}, \qquad y \in S_y, \quad (6.2)$$

with support $S_y = \{(y_1, y_2) : y_1, y_2 > 0,\ y_1y_2 < 1\}$. We denote the distributions in (6.1) and (6.2) by $\mathrm{GIG}_2(\lambda, \chi, \psi)$ and $\mathrm{GIG}_2(-\lambda, \chi, \psi)$, respectively.

We do not here intend to study the general class of GIG_2 distributions in detail, but restrict ourselves to mentioning some of its nice properties.

For fixed value of λ ($\neq 0$) the set of distributions $\{\mathrm{GIG}_2(\lambda, \chi, \psi) : \chi > 0,\ \psi > 0\}$ is a regular exponential family of order 2. Furthermore, if $\lambda > 0$, the family is a linear exponential family with moment generating function

$$M_{\mathrm{GIG}_2(\lambda,\chi,\psi)}(s) = \left(1-\frac{2s_1}{\chi}\right)^{-\lambda/2}\left(1-\frac{2s_2}{\psi}\right)^{-\lambda/2}\frac{K_\lambda\big(\sqrt{(\chi-2s_1)(\psi-2s_2)}\big)}{K_\lambda(\sqrt{\chi\psi})}.$$

By construction the marginal distributions of the GIG_2 distribution are GIG. Moreover, if $x \sim \mathrm{GIG}_2(\lambda, \chi, \psi)$ with $\lambda > 0$, then the conditional distribution of x_1 given x_2 is a translated gamma distribution. More precisely, one has

$$(x_1 - x_2^{-1})|x_2 \sim \Gamma(\lambda, \chi/2),$$

and similarly,

$$(x_2 - x_1^{-1})|x_1 \sim \Gamma(\lambda, \psi/2).$$

Setting $\lambda = \frac{1}{2}$ in (6.1) and (6.2), we obtain the distributions $\mathrm{RIG}_2(\chi, \psi)$ and $\mathrm{IG}_2(\chi, \psi)$ with densities, respectively,

$$\frac{\sqrt{\chi\psi}}{2\pi} e^{\sqrt{\chi\psi}} (x_1 x_2 - 1)^{-\frac{1}{2}} e^{-\frac{1}{2}(\chi x_1 + \psi x_2)}, \qquad x \in S_x, \tag{6.3}$$

and

$$\frac{\sqrt{\chi\psi}}{2\pi} e^{\sqrt{\chi\psi}} (y_1 y_2)^{-\frac{3}{2}} (1 - y_1 y_2)^{-\frac{1}{2}} e^{-\frac{1}{2}(\chi y_1^{-1} + \psi y_2^{-1})}, \qquad y \in S_y. \tag{6.4}$$

The two-dimensional reciprocal inverse Gaussian distribution (6.3) has the following simple moment generating function:

$$M_{\mathrm{RIG}_2(\chi,\psi)}(s) = \frac{e^{\sqrt{\chi\psi}(1 - \sqrt{1-2s_1/\chi}\sqrt{1-2s_2/\psi})}}{\sqrt{1-2s_1/\chi}\sqrt{1-2s_2/\psi}}.$$

We note in passing that the distribution (6.3) is reminiscent of a distribution considered in Example 2 of Barndorff-Nielsen (1980), the roles of parameters and random variables being, as it were, reversed between the two distributions. The distribution referred to has support R_+^2 and probability density function

$$\frac{(\chi\psi - 1)^{\nu}}{2^{\nu+1}\Gamma(\nu)} I_{\nu-1}(\sqrt{x_1 x_2})(x_1 x_2)^{\frac{1}{2}(\nu - 1)} e^{-\frac{1}{2}(\chi x_1 + \psi x_2)},$$

where $\nu > 0$ is considered as fixed, while the parameters χ and ψ satisfy $\chi > 0$, $\psi > 0$, and $\chi\psi > 1$. Each of these two distributions may serve as a rather tractable prior for the other, particularly in the case $\nu = \frac{1}{2}$.

The GIG_2 distributions may be generalized to higher dimensions by repeating the argument leading to the GIG_2 distribution. To be specific, let $u, v_1, v_2, \ldots$ and $w_1, w_2, \ldots$ be independent random variables such that $u \sim \mathrm{GIG}(-\lambda, \chi, \psi)$ with $\lambda > 0$, $v_i \sim \Gamma(\lambda, \psi/2)$, and $w_i \sim \Gamma(\lambda, \chi/2)$. Setting

$$\begin{aligned} x_1 &= u^{-1}, \\ x_{2i} &= x_{2i-1}^{-1} + v_i, \qquad i = 1, 2, \ldots, \\ x_{2i+1} &= x_{2i}^{-1} + w_i, \qquad i = 1, 2, \ldots, \end{aligned}$$

we denote the distribution of $x = (x_1, \ldots, x_m)$, the first m of these variables, by $\mathrm{GIG}_m(\lambda, \chi, \psi)$. Furthermore, if $x \sim \mathrm{GIG}_m(\lambda, \chi, \psi)$, we denote the distribution of $y = x^{-1} = (x_1^{-1}, \ldots, x_m^{-1})$ by $\mathrm{GIG}_m(-\lambda, \chi, \psi)$. The supports of these distributions are, respectively,

$$S_x = \{x : x_i > 0,\ x_i x_{i+1} > 1,\ i = 1, \ldots, m-1\}$$

and

$$S_y = \{y : y_i > 0,\ y_i y_{i+1} < 1,\ i = 1, \ldots, m-1\}.$$

It follows from the formula (2.2) and the convolution property 3 of Section 2 that if $x \sim \mathrm{GIG}_m(\lambda, \chi, \psi)$ with $\lambda > 0$, then the distribution of the marginals x_i is given by

$$x_i \sim \begin{cases} \mathrm{GIG}(\lambda, \psi, \chi) & \text{if } i \text{ is odd,} \\ \mathrm{GIG}(\lambda, \chi, \psi) & \text{if } i \text{ is even.} \end{cases}$$

Similarly, if $y \sim \mathrm{GIG}_m(-\lambda, \chi, \psi)$ with $\lambda > 0$, one has

$$y_i \sim \begin{cases} \mathrm{GIG}(-\lambda, \chi, \psi) & \text{if } i \text{ is odd,} \\ \mathrm{GIG}(-\lambda, \psi, \chi) & \text{if } i \text{ is even.} \end{cases}$$

It is possible to write down the densities of the GIG_m distributions and show some properties similar to those of the GIG_2 distribution, but here we will not discuss these distributions further.

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