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The Joint Dynamics of Equity Market Factors

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Abstract

The 4 equity market factors from Fama and French (1993) and Carhart (1997) are pervasive in academia and practice. However, not much is known about their joint distribution and dynamics. We find striking evidence of asymmetric tail dependence across the factors. While the linear factor correlations are small and even negative, the extreme correlations are large and positive, so that the linear correlations drastically overstate the benefits of diversification across the factors. We model the nonlinear factor dependence dynamics and explore their economic importance in a portfolio allocation experiment showing that significant economic value is earned when acknowledging nonlinear dependence.

I. Introduction

Establishing a manageable set of factors that capture a substantial share of the cross-sectional variation in equity returns is a core pursuit in empirical asset pricing. In their seminal contribution, Fama and French (1993) find that the cross section of stock returns is well explained by a simple linear 3-factor model comprised of a broad market premium, the spread between small and big market capitalization stocks, and the spread between value and growth stocks. In addition, Jegadeesh and Titman (1993) and Carhart (1997) point to the importance of a momentum factor in explaining observed stock returns. The momentum factor consists of the returns realized by buying a portfolio of stocks that have performed well during the past year and selling a portfolio of stocks that performed poorly during the same period. We refer to the 4 factors as market, size, value, and momentum.

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A number of other factors have been proposed. For example, Pastor and Stambaugh (2003) suggest a specific measure of market liquidity influencing the cross section of average returns, and Vassalou (2003) constructs a mimicking portfolio for news related to future gross domestic product growth. Ang, Hodrick, Xing, and Zhang (AHXZ) (2006) examine the role of aggregate and idiosyncratic volatility in expected returns, while Ang, Hodrick, Xing, and Zhang (2009) extend this analysis in an international context. Similarly, Fu (2009) finds a positive relation between expected return and conditional idiosyncratic risk. Following AHXZ (2006) and Engle and Lee (1999), Adrian and Rosenberg (2008) show that cross-sectional average returns are related to short-run and long-run volatility components in market returns. Other recent contributions include Boudoukh, Michaely, Richardson, and Roberts (2007), who explore the role of payouts (dividends and stock repurchases) as a contemporaneous risk factor and predictive variable for stock returns. While we focus on the standard 4 factors, it is important to note that our analysis is applicable to a much larger number of factors as well.

Beside their usefulness in cross-sectional asset pricing, factor models are used for risk management and portfolio optimization; see, for example, Chan, Karceski, and Lakonishok (1998), (1999) and Briner and Connor (2008). Excellent textbook treatments of standard approaches in portfolio management can be found in Lhabitant (2004), Connor, Goldberg, and Korajczyk (2010), and Brandt (2010).

Spread portfolios such as value, size, and momentum are popular not just because they help explain the cross-sectional variation in returns but also because they are nearly orthogonal to each other and to the market factor. Therefore, when used as regressors in a factor model, they lead to more precise loading estimates than would an alternative set of highly correlated factors. While having orthogonal factors is clearly beneficial, our main contribution is to show that focusing solely on linear dependence is perilous. We show that nonlinear factor dependence is important empirically, and when ignored it will lead to underestimation of extreme risks and suboptimal portfolio allocations.

The clear presence of nonlinear dependence necessitates a detailed investigation of the dynamics and the distributional properties of the factors. Consider, for example, portfolio optimization involving a large set of stocks. In such applications, a linear factor approach is desirable because it reduces significantly the dimension of the risk model. By estimating each asset's loadings on the set of factors, the covariance matrix of the assets can be expressed as the sum of the idiosyncratic risks and the quadratic form of the factor loadings and the factors' covariance matrix. However, if the joint distribution of the factors is not normal, then the distribution of the assets and the portfolio will not be normal either. Proper modeling of the joint factor distribution is essential. If either the marginal distribution of each factor or their joint distribution is not properly modeled, then the factor model will lead to erroneous conclusions regarding the portfolio return distribution. While a linear 4-factor model may offer a good description of the cross-sectional distribution of expected returns, a normally distributed 4-factor model does not offer a good description of the joint distribution of returns. We suggest instead dynamic nonnormal models that can accurately capture nonlinear factor dependence and thus portfolio tail risk.

Nonlinear risk in factor models is related to the literature on asymmetric correlation in equity returns. Building on Longin and Solnik (2001) and Ang and Bekaert (2002), Ang and Chen (2002) show that the correlation between domestic equity portfolios and the aggregate market is greater in down markets than in up markets. For example, a portfolio comprised of small, value, or loser stocks have greater correlation asymmetries. These findings are important in an asset-pricing context. Measuring asymmetric correlation as measured by downside beta, Ang, Chen, and Xing (2006) find a positive relation between downside risk and expected returns that is not explained by other traditional risk factors. In order to examine the importance of univariate and multivariate asymmetry on an optimal allocation between a small- and a large-capitalization portfolio, Patton (2004) uses a rotated Gumbel copula that is able to produce asymmetric correlation. Furthermore, Hong, Tu, and Zhou (2007) find that incorporating asymmetric dependence is important for portfolio selection for investors with disappointment aversion preferences.[1]

Our main contributions are as follows:

First, we extend the bivariate analysis in Ang and Chen (2002) to the 4 standard equity market factors. Modeling directly the 4 factors presents an interesting econometric challenge. We find striking evidence of nonlinear dependence for daily, weekly, and monthly returns that is much stronger than implied by the conventional linear correlation coefficients, which in most cases are close to 0. We focus on weekly returns and find that an asymmetric Student t copula is able to capture the factor asymmetry and dependence in a parsimonious way. Importantly, it can produce strong asymmetric tail dependence in virtually uncorrelated factors.

Second, we present evidence that the nonlinear dependence between the 4 factors has economic value for risk-averse investors who allocate capital using the factor model. The investor takes positions directly in the 4-factor portfolios and can take on leverage but is subject to margin requirements (MRs). With 27 years of weekly out-of-sample portfolio returns, we show that the annualized improvement in certainty equivalence (vs. a normal factor model) can reach 1.17% when using an asymmetric copula model instead of the multivariate normal distribution. The statistical significance of these results is verified by bootstrapping the difference in realized certainty equivalence.

Third, we show that our findings lead to different risk estimates from those based on the multivariate normality assumption. During the 2006–2010 period, the expected shortfall (ES) for an equal-weighted portfolios of the 4 factors is 20%–50% higher when using the asymmetric copula for risk management, and this difference is robust to allowing for time-varying correlations.

Applications of copulas in finance typically restrict attention to the two-dimensional case, which is motivated, for example, by a single-factor model.

[1] Other important contributions related to asymmetric dependence include Poon, Rockinger, and Tawn (2004), Tsafack (2009), Sancetta and Satchell (2007), Xu and Li (2009), Mazzotta (2008), Campbell, Koedijk, and Kofman (2002), Okimoto (2008), and Hatherley and Alcock (2007).

While many types of asymmetric copulas are not tractable in dimensions higher than two, ours is parsimonious and workable for many more than 4 factors. We focus on the standard 4-factor model in this paper because it is so widely used in academia and practice.

Our results show that the evidence for univariate as well as multivariate nonnormality is strong but also that the dependence across factors is dynamic. The joint nonnormality and intricate dynamics of the 4 standard factors came on display during the so-called quant meltdown of Aug. 2007. Khandani and Lo (2007), (2011) investigate the extent to which the meltdown was caused by equity hedge funds massively exiting certain strategies, thereby producing increased correlations between value, size, and momentum returns. Our model allows for dynamic correlations, which presents an additional source of risk. Asness, Moskowitz, and Pedersen (2013) find that returns to value strategies are positively correlated when applied to stocks, country equity indices, governments bonds, currencies, and commodities. Similarly, momentum returns are positively correlated across asset classes. Not surprisingly, these correlations are found to rise considerably during extreme market events. Once again, a properly specified factor model requires correlation dynamics, which we estimate and apply in a portfolio allocation context.

Our paper proceeds as follows: Various descriptive statistics, including threshold correlations of the factor returns, are reported in Section II, which also models dynamics in factor return mean and volatility. Section III introduces copula models that can capture nonlinear and dynamic dependence across factors. Section IV considers the economic importance of the nonlinear dependence from a portfolio allocation and risk management perspective. Section V presents reverse threshold correlations for weekly returns, threshold correlations for daily and monthly returns, and discussions of alternative copulas. Section VI concludes.

II. Factor Returns and Residuals

We study weekly equity factor returns observed from July 5, 1963, to Dec. 31, 2010. Market, size, and value factors are constructed as in Fama and French (1993), who use the median market capitalization to form size portfolios and use the lower and upper terciles of book-to-market ratios to construct the value factor.[2] The market factor is the value-weighted return on all NYSE, AMEX, and NASDAQ stocks less the 1-month T-bill rate. Every June, the median size of NYSE stocks is used to split the stocks into two size portfolios. The 30th and 70th percentiles of NYSE stocks' book-to-market ratios are used to sort stocks into three book-to-market portfolios. All portfolios are value weighted. The size factor is obtained by computing the spread between the average return of the three small-capitalization portfolios and the average return of the three large-capitalization portfolios. The value factor is the average return of the two value portfolios less the average return on the two growth portfolios.

Each month, the 30th and 70th percentiles of NYSE stocks' returns from months $t-12$ to $t-2$ are used to construct three prior-return-sorted portfolios

[2]We rely on the factor data available from Kenneth French's data library (http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html).

containing all stocks that have sufficient history. Stocks are also separated into two size portfolios using the NYSE median market capitalization. Value-weighted portfolios are used to construct the momentum factor as the difference between the average return on the two high-prior-return portfolios and the mean return on the two low-prior-return portfolios.

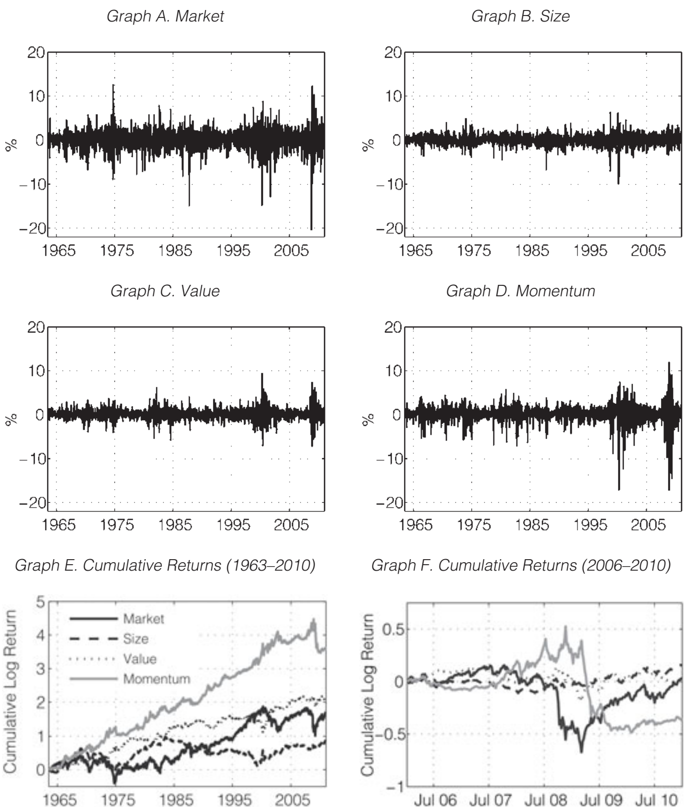
A. Factor Returns

Graphs A–D of Figure 1 plot the times series of returns for each factor, and summary statistics are provided in Table 1. Table 1 shows that all factors exhibit a high degree of volatility around the mean at the weekly frequency.

FIGURE 1

Time Series of Returns and Cumulative Returns

Graphs A–D of Figure 1 show the time series of weekly returns for each factor for the period July 5, 1963–Dec. 31, 2010. Graphs E–F show the cumulative log returns for each factor for the periods July 5, 1963–Dec. 31, 2010 (Graph E) and Jan. 5, 2006–Dec. 31, 2010 (Graph F).



The market, size, and momentum factor distributions are highly asymmetric, as evident by the large negative skewness in Table 1. The value factor distribution, on the other hand, is close to symmetric.

All 4-factor distributions have fat tails as evident by the large excess kurtosis estimates in Table 1. Figure 2 provides further evidence of the nonnormality

TABLE 1

Descriptive Statistics of Weekly Factor Returns (1963–2010)

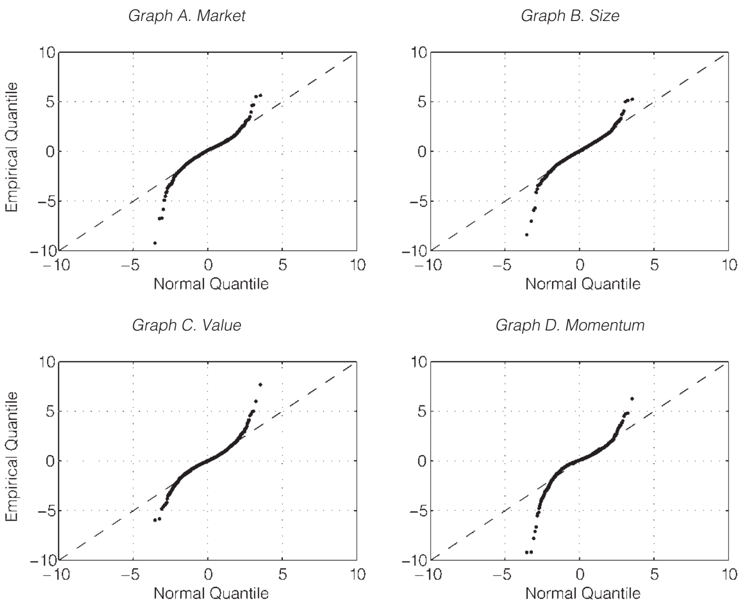
We report sample moments, autocorrelations, and cross correlations for weekly log returns of the 4 factors. The sample period is from July 5, 1963, to Dec. 31, 2010. Significant correlations are marked by * and ** denoting the 5% and 1% levels, respectively.

Sample Moments	Market	Size	Value	Momentum
Annualized mean	3.63%	1.81%	4.30%	7.55%
Annualized volatility	15.95%	8.54%	8.73%	13.56%
Skewness	–0.75	–0.44	0.18	–1.44
Excess kurtosis	7.01	5.04	5.16	12.38
Autocorrelations				
First-order	0.006	0.112**	0.113**	0.099**
Second-order	0.044	0.103**	0.084**	0.084*
Third-order	0.009	0.106**	0.067*	0.052
Cross Correlations				
Market	—	0.048*	–0.306**	–0.110**
Size	—	—	–0.128**	0.039
Value	—	—	—	–0.190**

FIGURE 2

Quantile-Quantile Plots for Returns from July 5, 1963, to Dec. 31, 2010

For each observation we scatter plot the empirical quantile on the vertical axis against the corresponding quantile from the standard normal distribution on the horizontal axis. If returns are normally distributed, then the data points will fall randomly around the 45° line, which is marked by dashes.



in weekly factor returns: The empirical factor quantiles are plotted against the quantiles from a normal distribution, so that deviations from the 45° line signal nonnormality. Figure 2 clearly shows that both tails are fat in all 4-factor returns. Part of the large excess kurtosis found in the factor return series is likely driven by volatility dynamics, which we therefore model in Section II.C.

As a measure of linear dependence, Table 1 reports the sample correlations across the factor returns. Note that the correlations are close to 0 or even negative.

The largest positive correlation is +0.05 between the market and size factors, while the largest negative correlation is -0.31 between the market and value factors. This near orthogonality is part of the reason for the widespread use of these factors in portfolio management.

Graphs E and F of Figure 1 provide a complementary picture of the relationship between the factors. Graph E depicts the cumulative log returns during the period 1963–2010. The long-term returns on momentum are quite striking. Note also that the size factor accumulated losses during the period 1995–2000, while the market rallied significantly. Graph F depicts the cumulative log returns on the factors since 2006 and shows how the momentum factor crashed in the early part of 2009 while the overall market was recovering. This apparent lack of dependence between the factors is of course interesting from a diversification perspective.

B. Factor Return Threshold Correlations

It is only in the case of the multivariate normal distribution that simple linear correlations fully characterize the dependence across returns. The strong evidence of nonnormality we have found in the individual factor returns suggests that the simple correlations reported in Table 1 could be concealing nonlinear dependencies across factors.

In order to explore dependence further, we rely on the threshold (or exceedance) correlations previously applied by Longin and Solnik (2001) and Ang and Bekaert (2002) to country indexes, by Ang and Chen (2002) to various equity portfolios, and by Patton (2004) to large- and small-capitalization portfolios.

Following Patton (2004), we define the threshold correlation $\bar{\rho}_{ij}(u)$ with respect to the quantiles of the empirical univariate distribution of factors i and j by

$$\bar{\rho}_{ij}(u) \quad = \quad \begin{cases} \operatorname{corr}(r_i, r_j \mid r_i < F_i^{-1}(u), r_j < F_j^{-1}(u)), & \text{when } u < 0.5, \\ \operatorname{corr}(r_i, r_j \mid r_i \geq F_i^{-1}(u), r_j \geq F_j^{-1}(u)), & \text{when } u \geq 0.5, \end{cases}$$

where u is a threshold between 0 and 1, and $F_i^{-1}(u)$ is the empirical quantile of the univariate distribution of r_i. Thus, the threshold correlation reports the linear correlation between two assets for the subset of observations lying in the bottom-left or top-right quadrant defined by the two univariate quantiles.

In the bivariate normal distribution, the threshold correlation approaches 0 as the threshold approaches 0 or 1. The empirical threshold correlations can therefore be used as a benchmark for the bivariate distribution of each pair of factor returns.

The left-hand panels of Figures 3 and 4 show the scatter plots of standardized weekly returns for the six possible pairs of factor returns. A remarkable feature of these returns is that even a factor pair with a relatively large negative correlation, such as market versus value, contains many outliers in the bottom-left and top-right quadrants of the scatter.

The empirical threshold correlations in the right-hand panels of Figures 3 and 4 are compared to the one implied by a bivariate normal distribution fitted on each pair of factors.[3] Threshold correlations are computed only for threshold values for which at least 20 pairs of returns are available. The differences between

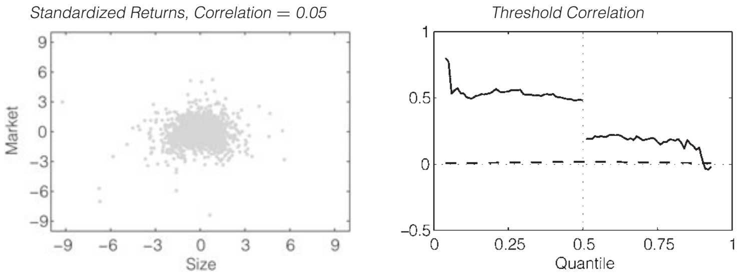
[3]The analytical expression for the exceedance correlation for a bivariate normal distribution can be found in the appendix of Ang and Chen (2002).

FIGURE 3

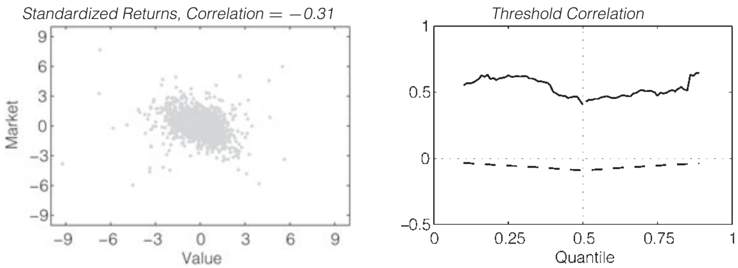
Scatter Plots and Threshold Correlations for Market versus Other Factors

Figure 3 presents scatter plots in the left-hand panels and threshold correlations in the right-hand panels between the market premium and the 3 other factors. Our sample consists of weekly returns from July 5, 1963, to Dec. 31, 2010. The linear correlations are provided in the titles of the left-hand panels. The continuous line in the right-hand panels represents the correlation when both variables are below (above) a threshold when this threshold is below (above) the median. The dashed line represents the threshold correlation function for a bivariate normal distribution using the linear correlation coefficient from the data.

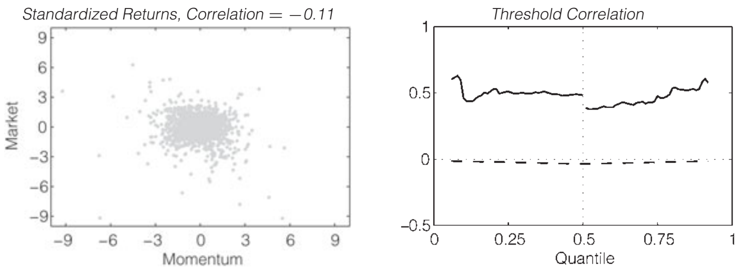
Graph A. Market versus Size



Graph B. Market versus Value



Graph C. Market versus Momentum



the empirical (solid lines) and normal (dashed lines) threshold correlations are striking. For example, while the unconditional correlation between the market and size factors indicates near independence under the bivariate normality assumption, the threshold correlation in the top-right panel of Figure 3 is positive and clearly larger below the median than above. Also, the market-value pair in the right-hand panel of Graph B in Figure 3 exhibits large and positive threshold correlations, while its unconditional correlation is slightly negative.

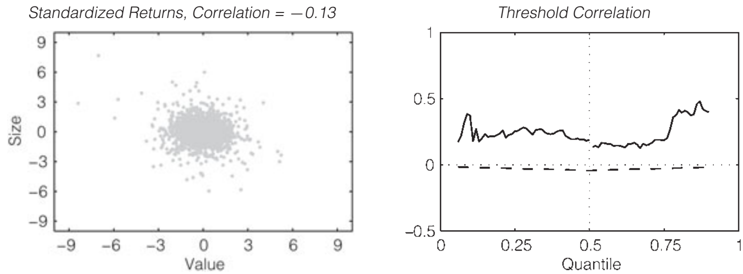
While the simple linear correlations are close to 0 and often negative, the threshold correlations in Figures 3 and 4 are virtually always positive and very

FIGURE 4

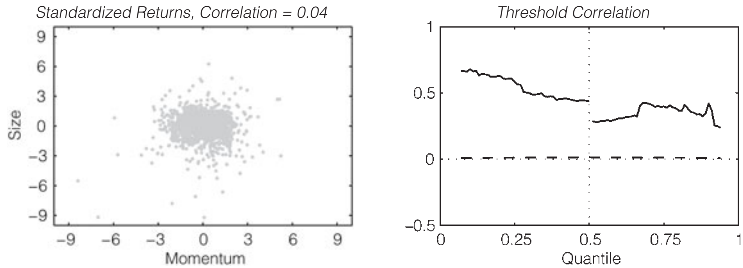
Scatter Plots and Threshold Correlations for Factors Other Than Market

Figure 4 presents scatter plots in the left-hand panels and threshold correlations in the right-hand panels between factors pairs not involving the market premium. Our sample consists of weekly returns from July 5, 1963, to Dec. 31, 2010. The linear correlations are provided in the titles of the left-hand panels. The continuous line in the right-hand panels represents the correlation when both variables are below (above) a threshold when this threshold is below (above) the median. The dashed line represents the threshold correlation function for a bivariate normal distribution using the linear correlation coefficient from the data.

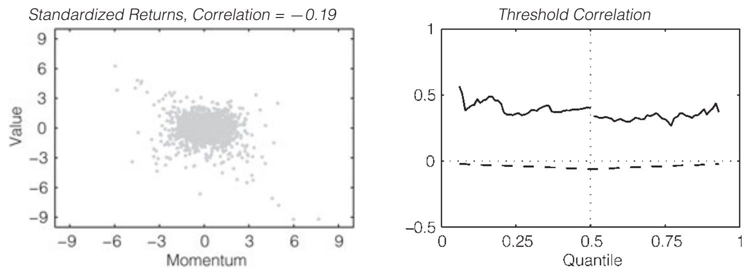
Graph A. Size versus Value



Graph B. Size versus Momentum



Graph C. Value versus Momentum



often large. In some cases the threshold correlations are even increasing as the thresholds get more extreme. This is evident, for example, in the right-hand panel of Graph B in Figure 4, where the threshold correlation for size versus momentum increases when the threshold decreases below the median. The implications for a fund manager holding a portfolio that is long small stocks and long momentum are serious: The simple correlation is low, suggesting that diversification is high, but when both value and momentum perform poorly, their correlation is in fact very high.

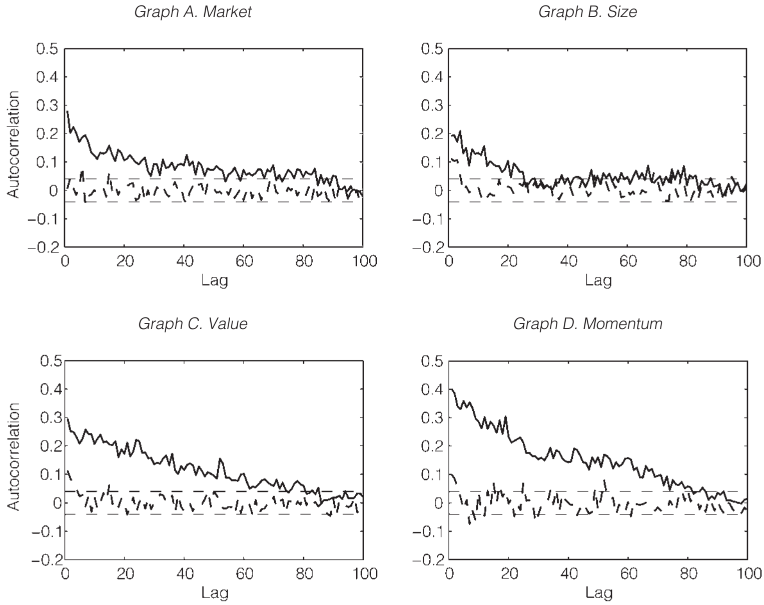
C. Factor Return Dynamics

Table 1 shows that the returns on the size, value, and momentum factors contain some serial correlation for the first three weekly lags. The dashed lines in Figure 5 show the empirical autocorrelation function for the 4 factors for lags of up to 100 weeks. The horizontal lines denote the 95% confidence bands around 0 and suggest that the short-lag autocorrelations are indeed significant for the size, value, and momentum factors. The *p*-values obtained from a Ljung-Box (L-B) test (not reported) suggest that serial correlation is marginally significant in the market factor also.

FIGURE 5

Autocorrelation Functions of Returns and Absolute Returns

Autocorrelation of weekly returns (dashed line) and absolute returns (solid line) from July 5, 1963, to Dec. 31, 2010. The horizontal dotted lines provide a 95% confidence interval around 0.



Financial assets typically display much stronger serial correlation in return magnitudes (measured by squares or absolutes) compared with the serial correlation in returns themselves. The solid lines in Figure 5 show that the weekly factor returns follow this pattern. All 4 factors display strong persistence in absolute returns.

We proceed by modeling the dynamics evident in Figure 5 using standard univariate autoregressive (AR)-generalized autoregressive conditional heteroskedasticity (GARCH) processes. We estimate the conditional mean using a simple autoregressive model of order 3 (AR(3)) specification,

$$r_{j,t} = \phi_{0,j} + \phi_{1,j} r_{j,t-1} + \phi_{2,j} r_{j,t-2} + \phi_{3,j} r_{j,t-3} + \sigma_{j,t}\epsilon_{j,t}, \tag{1}$$

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where $r_{j,t}$ is the return of factor j at time t. We hasten to add that the AR(3) specification is not meant to replace an economic model of expected returns; rather, it is needed to ensure consistent estimation of the second- and higher-order moments. Our analysis focuses on higher-order moments, and we do not attempt to explicitly model risk premia in the factor returns.

The conditional variance of daily returns is modeled using a GARCH dynamic of the form

$$\text{(2)} \qquad \sigma_{j,t}^2 \quad = \quad \omega_j + \beta_j \sigma_{j,t-1}^2 + \alpha_j \sigma_{j,t-1}^2 \left(\epsilon_{j,t-1} - \theta_j\right)^2 .$$

The θ_j parameter captures the so-called leverage effect, which appears when a negative innovation has a stronger impact on the conditional variance than a positive shock of the same magnitude. Several specifications have been proposed to incorporate the leverage effect. We rely on the nonlinear GARCH (NGARCH) model suggested by Engle and Ng (1993). Notice that as is typical in a GARCH model, $\sigma_{j,t}^2$ is observed at the end of day $t-1$, which makes the model very tractable and maximum likelihood estimation easy.

Based on previous studies, we expect the leverage parameter to be positive for the market factor. But as the 3 other factors contain both long and short equity positions, the expected sign of θ_j is much less clear for those.

Panel A of Table 2 presents the AR-GARCH estimates and diagnostics when ϵ is assumed to follow a normal distribution. As expected, the variance persistence

TABLE 2

AR-GARCH Models of Individual Factor Returns (1963–2010)

We report parameter estimates and model diagnostics for the AR-GARCH model with normal shocks (Panel A) and skewed t shocks (Panel B). Standard errors (in parentheses) are calculated from the outer product of the gradient at the optimum parameter values. The model estimated is $r_t = \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + \phi_3 r_{t-3} + \sigma_t \epsilon_t$, where $\sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \alpha \sigma_{t-1}^2 (\epsilon_{t-1} - \theta)^2$. Here, ω is fixed by variance targeting. The p-values for Ljung-Box (L-B) tests on the residuals and the absolute residuals are provided using 20 lags for both tests. The empirical skewness and excess kurtosis of the residuals are compared to the model-implied levels from the normal and asymmetric models.

	Market	Size	Value	Momentum
Panel A. Normal Distribution				
Parameter Estimates				
ϕ_0	6.30E–4	1.51E–4	5.41E–4	1.55E–3
	(3.34E–4)	(2.04E–4)	(1.83E–4)	(1.87E–4)
ϕ_1	0.030	0.086	0.137	0.107
	(0.020)	(0.021)	(0.019)	(0.019)
ϕ_2	0.052	0.108	0.037	0.007
	(0.021)	(0.021)	(0.022)	(0.020)
ϕ_3	0.022	0.073	0.069	–0.004
	(0.019)	(0.019)	(0.021)	(0.020)
β	0.752	0.849	0.862	0.831
	(0.018)	(0.017)	(0.011)	(0.011)
α	0.127	0.113	0.119	0.114
	(0.012)	(0.012)	(0.009)	(0.008)
θ	0.757	0.125	–0.059	–0.638
	(0.091)	(0.048)	(0.051)	(0.056)
ν	—	—	—	—
κ	—	—	—	—

(continued on next page)

TABLE 2 (continued)

AR-GARCH Models of Individual Factor Returns (1963–2010)

	Market	Size	Value	Momentum
Panel A. Normal Distribution (continued)				
Diagnostics				
Log-likelihood	6,291	7,715	7,862	7,099
Variance persistence	0.951	0.963	0.981	0.991
L-B(20) *p*-value	0.26	0.50	0.72	0.44
Absolute L-B(20) *p*-value	0.37	0.74	0.39	0.13
Empirical skewness	–0.56	–0.26	0.10	–0.56
Model skewness	0.00	0.00	0.00	0.00
Empirical excess kurtosis	1.61	1.42	1.08	1.51
Model excess kurtosis	0.00	0.00	0.00	0.00
Panel B. Skewed t Distribution				
Parameter Estimates				
ϕ_0	7.26E–4	1.41E–4	4.78E–4	1.58E–3
	(2.90E–4)	(2.03E–4)	(1.83E–4)	(2.22E–4)
ϕ_1	0.012	0.101	0.147	0.102
	(0.021)	(0.021)	(0.020)	(0.019)
ϕ_2	0.045	0.124	0.042	0.012
	(0.021)	(0.021)	(0.021)	(0.019)
ϕ_3	0.024	0.076	0.076	0.006
	(0.020)	(0.020)	(0.021)	(0.020)
β	0.794	0.849	0.877	0.830
	(0.020)	(0.022)	(0.015)	(0.016)
α	0.105	0.109	0.106	0.119
	(0.014)	(0.015)	(0.013)	(0.013)
θ	0.796	0.152	–0.076	–0.585
	(0.131)	(0.086)	(0.077)	(0.079)
ν	10.047	8.827	8.425	7.478
	(1.593)	(1.319)	(1.311)	(1.082)
κ	–0.221	–0.061	0.021	–0.161
	(0.027)	(0.029)	(0.029)	(0.029)
Diagnostics				
Log-likelihood	6,357	7,753	7,892	7,165
Variance persistence	0.965	0.961	0.983	0.989
L-B(20) *p*-value	0.22	0.49	0.79	0.46
Absolute L-B(20) *p*-value	0.65	0.74	0.40	0.18
Empirical skewness	–0.59	–0.26	0.11	–0.56
Model skewness	–0.52	–0.16	0.06	–0.46
Empirical excess kurtosis	1.73	1.48	1.14	1.53
Model excess kurtosis	1.26	1.27	1.36	1.96

implied by the model is close to 1. The leverage effect parameter θ_j is significantly positive for the market factor, as expected, but it is much smaller for size and insignificant for value. Note that the leverage effect is significantly negative for momentum: A positive return on the momentum factor increases momentum volatility more than a negative return of the same magnitude.

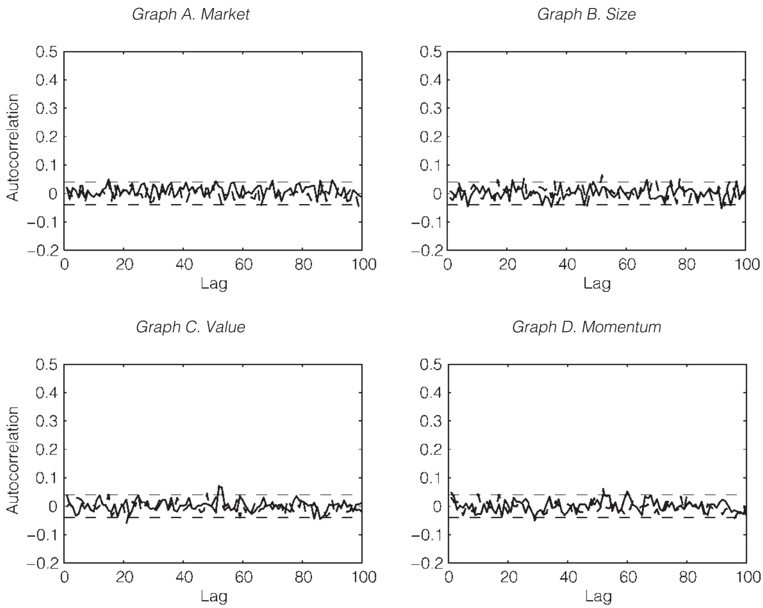
Figure 6 shows the autocorrelation functions for residuals and absolute residuals from the AR-GARCH model. Comparing the autocorrelation functions in Figure 6 with those found in Figure 5 strongly suggests that the AR-GARCH model has picked up the expected return and volatility dynamics in returns. This observation is confirmed by the *p*-values obtained from an L-B test on the residuals and absolute residuals as reported among the diagnostics in Table 2, which indicate that serial correlations have been removed.

The normal distribution assumption implies that the model-based skewness and excess kurtosis of ϵ is 0. The diagnostics in Table 2 show that the empirical skewness of ϵ is negative for market, size, and momentum and slightly positive

FIGURE 6

Autocorrelation Functions of AR-GARCH Residuals and Absolute Residuals

Autocorrelation of AR-GARCH residuals (dashed line) and absolute residuals (solid line) from July 5, 1963, to Dec. 31, 2010. The horizontal dotted lines provide a 95% confidence interval around 0.



for value. The asymmetric GARCH model has removed some of the skewness from the factor returns found in Table 1, but some still remains. Excess kurtosis is 0 in the normal distribution, but the empirical ϵ still contain positive excess kurtosis. The GARCH model has also removed much of the excess kurtosis found in Table 1, but some still remains.

The inability of the normal distribution to match skewness and kurtosis in the factor residuals leads us to consider the skewed t distribution of Hansen (1994). We denote the skewed t probability density function (PDF) of factor j by $f_j(\epsilon_{j,t};\kappa_j,\nu_j)$ and define it in Appendix A. The parameter κ_j is related to skewness, and ν_j is related to kurtosis. The distribution of returns will be dynamic due to the AR-GARCH model, and we can write

$$f_{j,t}\left(r_{j,t+1}\right) \quad = \quad \sigma_{j,t+1}^{-1} f_j\left(\epsilon_{j,t+1};\kappa_j,\nu_j\right).$$

Panel B of Table 2 presents the estimation results for the skewed t distribution. When comparing the residuals' skewness and kurtosis to those implied by a skewed t distribution using the estimated parameters, we see that a much better fit is obtained. The skewed t specification is also preferred to the normal AR-GARCH based on the likelihood values.

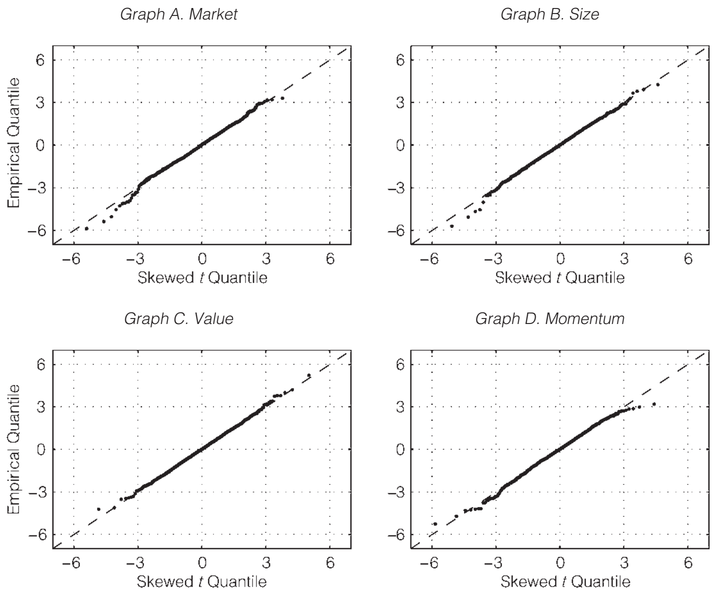
Figure 7 presents the quantile-quantile plots of the AR-GARCH residuals against the skewed t distribution. Figure 7 shows that the skewed AR-GARCH model delivers shocks that are very close in distribution to the assumed skewed

t distribution. Below, we will rely on the skewed t version of the AR-GARCH model when modeling factor dependence.

FIGURE 7

Quantile-Quantile Plots for the Skewed t AR-GARCH Residuals

For each observation we scatter plot the empirical quantile on the vertical axis against the corresponding quantile from the skewed t distribution on the horizontal axis. If the AR-GARCH residuals adhere to the skewed t distribution, then the data points will fall on the 45° line, which is marked by dashes. The parameters for the skewed t distribution are from Table 2.



III. Modeling Factor Dependence

In the analysis so far, we have found clear evidence of nonnormality in the marginal distributions as well as clear evidence of asymmetry in the threshold correlations. Together, these results strongly suggest nonnormality in the multivariate distribution of factor returns. Fortunately, copula models provide a powerful and flexible framework for linking nonnormal marginal distributions allowing for nonnormality in the multivariate distribution.

Patton (2006) builds on Sklar (1959) and shows that the joint conditional distribution of N factors, $f_t(r_{1,t+1},\ldots,r_{N,t+1})$, can be decomposed into the marginal distributions and a copula function as follows:

$$f_t(r_{1,t+1},\ldots,r_{N,t+1}) \quad = \quad c_t(\eta_{1,t+1},\ldots,\eta_{N,t+1})\prod_{j=1}^{N} f_{j,t}(r_{j,t+1}),$$

where $c_t(\eta_{1,t+1},\ldots,\eta_{N,t+1})$ is the conditional copula density function,

$$\eta_{j,t+1} \quad = \quad F_{j,t}(r_{j,t+1}) \quad \equiv \quad \int_{-\infty}^{r_{j,t+1}} f_{j,t}(r)dr$$

is the marginal probability for factor j, and $f_{j,t}(r_{j,t+1})$ is the univariate conditional density function from above.

While we have already modeled the univariate distributions, $f_{j,t}(r_{j,t+1})$, we now need to decide on an appropriate functional form for the copula function $c_t(F_{1,t}(r_{1,t+1}), \ldots, F_{N,t}(r_{N,t+1}))$. We first consider constant copula functions and then dynamic copulas.

A. Constant Copula Models

From the asymmetric threshold correlations obtained above, we know that an asymmetric copula function is required. Upon an extensive copula model selection study (detailed in Section V), we settle on a copula model built from the multivariate skewed t distribution in Demarta and McNeil (2005).

The multivariate skewed t distribution provides a parsimonious specification in which univariate and multivariate asymmetry are driven by an N-dimensional vector of parameters λ. In the skewed t copula, the univariate skewness is captured by the univariate distributions modeled above, and the vector λ only has to capture multivariate asymmetry. We denote the skewed t copula density function by $c(\eta_1, \ldots, \eta_N; \lambda, \nu_c, \Psi)$, where ν_c denotes the scalar degree-of-freedom parameter and Ψ denotes the copula correlation matrix. Further details on the skewed t copula function are provided in Appendix C.

The copula parameters are estimated by maximizing $\sum_{t=1}^{T} \ln c(\eta_{1,t}, \ldots, \eta_{N,t}; \lambda, \nu_c, \Psi)$. Standard errors are computed using Chen and Fan (2006).[4]

Panel A of Table 3 reports the estimates for three different constant copulas: the skewed t copula described above, the symmetric t copula special case where $\lambda_j \rightarrow 0$ for all 4 factors, and the normal copula special case where further $\nu_c \rightarrow \infty$. From the log-likelihood values, we see that moving from left to right, the greatest improvement in likelihood comes from using the symmetric t rather than normal copula, even though only one parameter is added in this case. The λs are generally significant, suggesting that the skewed t copula offers additional improvements in fit. We also compute the pseudo-likelihood ratio test of Chen and Fan (2006) to determine whether the skewed t copula is significantly better than the symmetric t copula, which is equivalent to testing for the null hypothesis that all λs are equal to 0. The test statistic has a standard normal distribution, and the value reported in the last line of Table 3 indicates that it rejects in favor of the skewed t copula.

B. Dynamic Copula Models

Following Christoffersen, Errunza, Jacobs, and Langlois (2012) and Jin (2009), we now allow the conditional copula correlation matrix of the normal, t, and skewed t copulas to evolve through time. We rely on the dynamic conditional

[4]When estimating the copula parameters, we use the empirical distribution of the residuals to construct $\eta_{j,t}$. This increases efficiency and ensures the validity of the Chen and Fan (2006) standard errors.

TABLE 3

Estimation Results for Factor Dependence Models (1963–2010)

Table 3 presents parameter estimates for the dependence models. All models are estimated by maximum likelihood. Standard errors (in parentheses) are computed using the methodology of Chen and Fan (2006). The last line presents the pseudo-likelihood ratio test statistics for the null hypothesis that the asymmetry parameters in the skewed *t* copula are all equal to 0. Here, * and ** indicate significance at 5% and 1% levels, respectively.

	Panel A. Constant Correlation			*Panel B. Dynamic Correlation*		
	Normal Copula	Symmetric *t* Copula	Skewed *t* Copula	Normal Copula	Symmetric *t* Copula	Skewed *t* Copula
Parameter Estimates						
ν_C		4.520	4.740		9.070	8.940
		(0.272)	(0.002)		(0.906)	(0.025)
λ_{MARKET}			−0.019			−0.055
			(0.010)			(0.003)
λ_{SIZE}			−0.069			−0.111
			(0.015)			(0.001)
λ_{VALUE}			0.036			0.030
			(0.031)			(0.001)
$\lambda_{\text{MOMENTUM}}$			−0.161			−0.115
			(0.044)			(0.003)
β_C				0.886	0.884	0.885
				(0.025)	(0.008)	(0.010)
α_C				0.089	0.073	0.092
				(0.020)	(0.005)	(0.008)
ρ(MARKET,SIZE)	−0.015	0.000	−0.002	0.013	0.039	0.037
ρ(MARKET,VALUE)	−0.356	−0.352	−0.349	−0.378	−0.379	−0.376
ρ(MARKET,MOMENTUM)	0.107	0.116	0.113	0.083	0.090	0.090
ρ(SIZE,VALUE)	−0.047	−0.046	−0.038	−0.087	−0.109	−0.107
ρ(SIZE,MOMENTUM)	0.013	0.023	−0.017	0.032	0.040	0.035
ρ(VALUE,MOMENTUM)	−0.077	−0.094	−0.080	−0.110	−0.139	−0.138
Model Properties						
Correlation persistence	0	0	0	0.975	0.957	0.977
Log-likelihood	188.4	433.7	448.8	1,052.1	1,152.1	1,161.3
No. of parameters	6	7	11	8	9	13
Pseudo-likelihood ratio test			2.551**			5.703**

correlation (DCC) model of Engle (2002), where the correlation matrix dynamic is generated via

$$Q_t = Q(1-\beta_c-\alpha_c)+\beta_c Q_{t-1}+\alpha_c z_{t-1}z_{t-1}^{\top}.$$

In Engle's (2002) dynamic linear correlation model, we have $z_{t-1}=\epsilon_{t-1}$ so that the correlation dynamics are updated using the vector of standardized returns. But in our copula application of the DCC model, $z_{j,t}$ instead denotes the standardized version of the fractile $F_c^{-1}(\eta_{j,t})$, where F_c^{-1} is the inverse univariate cumulative distribution function (CDF) from the specific copula.[5] The matrix Q is defined as the sample correlation of z_t.[6] The dynamic correlations are obtained using the following normalization of the elements of the matrix Q_t:

$$\rho_{ij,t} = [\Psi_t]_{i,j} = \frac{[Q_t]_{i,j}}{\sqrt{[Q_t]_{i,i}[Q_t]_{j,j}}}.$$

[5] See Appendix B for details of this standardization.

[6] We implement the modified DCC model in Aielli (2013), but the differences between this and Engle's (2002) original model are very small.

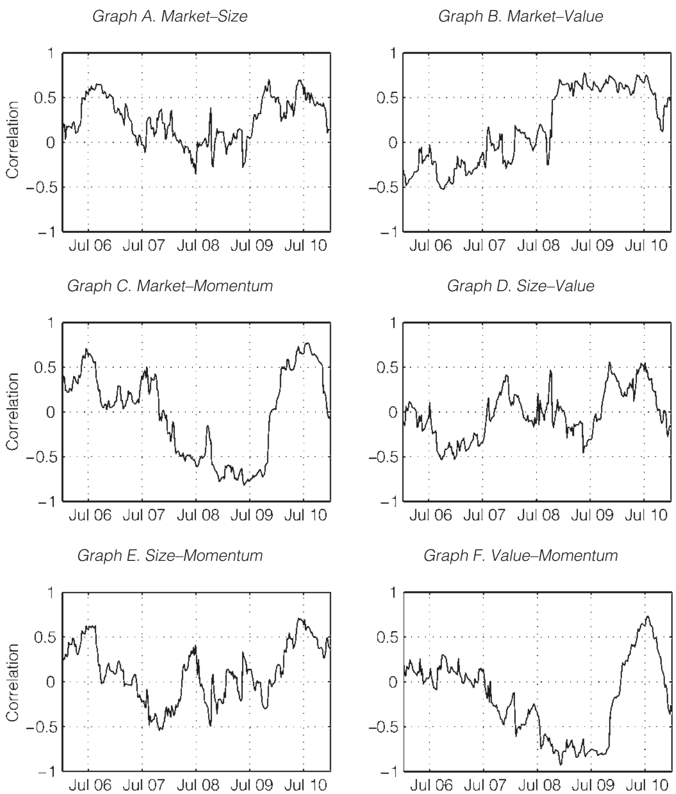
Panel B of Table 3 provides the estimates of the dynamic copula models. Note that the dynamic copula log-likelihoods in Panel B are significantly higher than their constant versions in Panel A. Also, the hypothesis that all λs are equal to 0 is again rejected, indicating that the skewed t copula is preferred to its symmetric version.

Figure 8 plots the elements of Ψ_t over time from the skewed t copula model. We restrict attention to the 2006–2010 period. The variation in correlation across time is striking. For instance, the conditional copula correlation between the market and momentum factors ranges from -0.5 to 0.5 during this period. Consider also the correlation between value and momentum, which increases from -0.8 to +0.7 during a very short period in late 2009 and early 2010. These rapid reversals in correlation show that standard risk management techniques based on constant correlations are misleading.

FIGURE 8

Dynamic Copula Correlations (2006–2010)

We report dynamic conditional copula correlation for each pair of factors from Jan. 2006 to Dec. 2010. The correlations are obtained by estimating the dynamic skewed t copula model on the factor return residuals from the AR-GARCH model. The entire 1963–2010 sample is used in estimation of the models.



Although not shown, the dynamic correlation patterns are quite similar across the copula models. Allowing for time-varying correlation in the factors appears to be crucial in properly capturing equity factor interdependence.

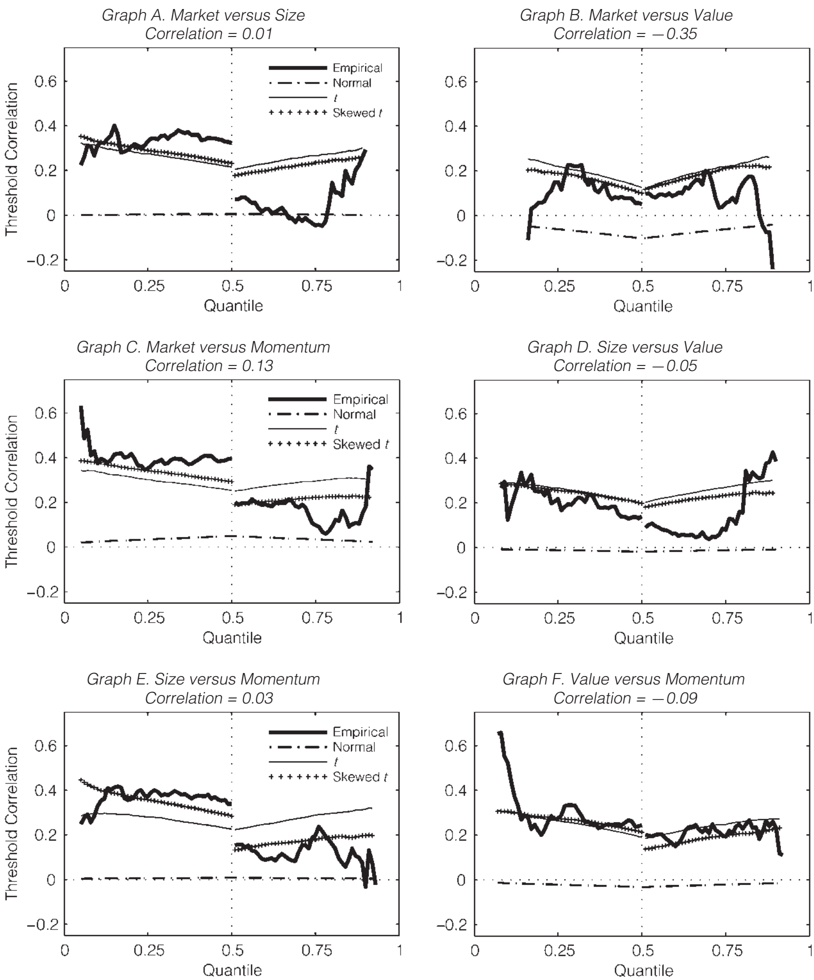
C. Copula Threshold Correlations

At this point it is natural to ask if the estimated copula models are able to capture the asymmetric threshold correlations found in Figures 3 and 4. Given the variance dynamics in factor returns, we need to assess the cross-sectional dependence in factor residuals, ϵ, rather than in the factor returns, r. The empirical threshold correlations of ϵ are shown in thick solid lines in Figure 9. When comparing the weekly threshold correlations for returns in Figures 3 and 4 with those for the return residuals in Figure 9, it appears that the univariate dynamic models

FIGURE 9

Threshold Correlations for Factor Residuals and Copula Models

We present threshold correlations computed on AR-GARCH residuals from July 5, 1963, to Dec. 31, 2010. The thick continuous line represents the empirical correlation. The linear sample correlations are provided in the titles for each pair of factors. The threshold correlation functions are computed for thresholds for which there are at least 20 data points available. We compare the empirical correlations to those implied by the normal copula and the constant *t* and skewed *t* copulas.



have removed some of the threshold correlation, but clearly much is still left. Just as the AR-GARCH models have removed some skewness and kurtosis from the univariate factor returns and make the factor residuals closer to normal than the factor returns, so too have they made the factor residuals closer to multivariate normal than were the original factor returns.

When comparing the empirical threshold correlations of ϵ in Figure 9 with the threshold correlations implied by the copula models, it appears that the skewed t copula (lines marked with "+") is able to produce the asymmetric threshold correlations required by the data. The additional flexibility introduced by the asymmetry parameters can be observed, especially for the following factor pairs: market versus momentum (middle-left panel) and size versus momentum (bottom-left panel).

Not surprisingly, the copula threshold correlations in Figure 9 do not match up perfectly with their empirical counterparts. Two remarks are in order in this regard. First, the empirical threshold correlations are estimated with uncertainty (especially in the extremes). Second, the copula models are estimated by maximizing the likelihoods and not by directly fitting the empirical threshold correlation patterns. The ultimate test of the models is in their economic relevance for portfolio allocation and risk management. This is the topic to which we now turn.

IV. Economic Implications

We find rather striking statistical evidence of nonlinear dependence between the market, size, value, and momentum factors, and we now examine if these findings are important in economic terms. In particular, we assess the economic cost of ignoring time-varying and nonlinear dependence between the factors when using the factors for portfolio allocation.

In order to address this issue, we consider expected constant relative risk aversion (CRRA) utility-maximizing investors. CRRA utility functions are widely used for studying portfolio choice in finance (see, e.g., Ait-Sahalia and Brandt (2001)), partly for their analytical tractability. But as CRRA functions are locally mean-variance preferences, they will most likely yield conservative (i.e., low) estimates of the economic cost of ignoring nonnormality in the factors. Hong et al. (2007) and Ang, Bekaert, and Liu (2005) use disappointment aversion preferences instead, because they are better suited to take into account asymmetric correlation, but they are less tractable analytically.

A. Portfolio Selection Framework

Consider an investor who directly takes positions in the 4-factor portfolios. As in Jagannathan and Ma (2003), we constrain the weights to be positive to prevent them from taking on extreme values. Note that this constraint does not prohibit short sales in our application, as 3 out of 4 factors involve short positions. To further restrict the set of the admissible portfolio that investors can choose from, we follow Pastor and Stambaugh (2000) and impose the MR that customers of U.S. broker-dealers face under the Federal Reserve's Regulation T. Regulation

T imposes an upper limit of 2 on the ratio of total position to capital corresponding to a 50% minimum margin.

Investing in the 4 factors can be viewed as a hedge fund employing quantitative equity strategies. While Regulation U states that Regulation T applies not only to broker-dealers' customers, but to any U.S. investors, there are several ways for hedge funds to circumvent the 50% limit. First, broker-dealers are granted looser restrictions for their own accounts, and so some hedge funds have registered as broker-dealers. Second, a joint back office operation can be established between a fund and its broker. Third, a fund managed in the United States can register offshore and limit its financing to offshore broker-dealers. Fourth, higher levels of leverage can be obtained by using over-the-counter derivatives such as total return swaps.[7] We therefore also consider investors who can lever themselves more than Regulation T allows. We impose an MR of either 20% or 50% of the fund's exposure, that is, we impose

$$w_{\text{MARKET}} + 2\left(w_{\text{SIZE}} + w_{\text{VALUE}} + w_{\text{MOMENTUM}}\right) \quad \leq \quad \frac{1}{\text{MR}},$$

where all weights are nonnegative, and the weights for the spread portfolios are multiplied by 2 as they involve both short and long positions.

We are now ready to describe the real-time implementation of the investment problem.

B. Implementing Real-Time Investing

We begin the real-time investment process by estimating the skewed t AR-GARCH model on each factor using the first 20 years of weekly returns spanning 1963–1983. We then estimate the constant and dynamic copula models in Table 3 on the 20 years of AR-GARCH residuals. In addition, we implement the multivariate standard normal distribution with constant and dynamic correlations as benchmarks. Note that both benchmark models allow for dynamic variances in the individual factors.

We re-estimate the models once a year using all the data available up to that point in time. While the parameter estimates are updated annually, the conditional factor means, variances, and correlations are updated weekly. In all models, we set the factor's expected return to the average return computed over the previous 2 years. This enables us to focus attention on the impact of higher moments on portfolio selection.

Once the conditional 1-week-ahead multivariate distribution, $f_t(r_{t+1})$, is constructed, we find the optimal portfolio weights by maximizing the 1-week expected CRRA utility

$$\text{(3)} \qquad \max_{w_t} \text{E}_t \left[U(1 + r_{f,t+1} + w_t^\top r_{t+1}) \right] = \int \frac{\left(1 + r_{f,t+1} + w_t^\top r_{t+1}\right)^{1-\gamma}}{(1-\gamma)} f_t(r_{t+1}) dr_{t+1},$$

[7]See McCrary (2002) for details on how hedge funds can create leverage.

where $r_{f,t+1}$ is the weekly return of the 1-month T-bill, and r_{t+1} is the vector of returns for the 4 factors. For simplicity, we ignore intertemporal hedging demands.

The integrals are solved by simulating 100,000 variates for the 4 factors from the multivariate conditional return distribution $f_t(r_{t+1})$. The ex post investment performance is computed from the first week of July 1983 until the end of Dec. 2010, thus producing 1,436 real-time or out-of-sample returns.

C. Investment Results

The real-time investment results are given in Table 4 for investors with an MR of 20%, and in Table 5 we report on MR = 50%. We consider three different levels of relative risk aversion, namely $\gamma = 3$ in Panel A, $\gamma = 7$ in Panel B, and $\gamma = 10$ in Panel C. Various standard statistics, including mean, volatility, skewness, and excess kurtosis, are computed for the ex post realized portfolio returns in Tables 4 and 5.

In order to compare the economic value of the different dependence models, we compute the certainty equivalent (CE) of the average realized utility computed

TABLE 4

Out-of-Sample Results for the Investor with MR of 20%

Table 4 presents out-of-sample results for the investor investing with 20% margin requirement (MR) in the 4 factors. The out-of-sample period is from July 1, 1983, to Dec. 31, 2010, for a total of 1,436 weekly returns. For each level of relative risk aversion, the performance of the three copulas is compared to the benchmark normal distribution. Panels A, B, and C show the results for relative risk aversion coefficients of 3, 7, and 10, respectively. We report the realized moments of the portfolio returns, the average turnover, as well as the certainty equivalent (CE). The annualized difference in CE is the difference between the CE for each model and the normal benchmark multiplied by 52. We also report bootstrap *p*-values testing the significance of the differences in CEs. We test each of the three alternative models against the normal benchmark.

	Constant Correlation Models				*Dynamic Correlation Models*			
	Normal Distribution	Normal Copula	Symmetric *t* Copula	Skewed *t* Copula	Normal Distribution	Normal Copula	Symmetric *t* Copula	Skewed *t* Copula
Panel A. $\gamma = 3$								
Annualized mean (%)	19.549	19.494	19.659	19.724	21.095	21.195	21.287	20.481
Annualized volatility (%)	32.208	31.857	31.958	31.730	31.123	30.584	30.715	28.473
Skewness	–1.248	–1.299	–1.222	–1.184	–0.894	–0.906	–0.896	–0.980
Excess kurtosis	8.472	8.778	7.917	7.438	4.972	4.670	4.589	5.178
Average turnover (%)	12.707	12.301	12.397	12.417	13.129	12.854	12.900	13.115
(CE – 1) (bp)	16.808	17.129	17.372	17.823	21.435	22.297	22.318	23.387
Annualized diff. in CE (%)	—	0.167	0.293	0.528	—	0.448	0.459	1.015
p-value	—	0.031	0.051	0.034	—	0.001	0.019	0.016
Panel B. $\gamma = 7$								
Annualized mean (%)	17.158	17.189	17.145	17.326	19.726	19.712	19.709	19.124
Annualized volatility (%)	20.201	19.997	19.855	19.547	19.005	18.649	18.701	17.605
Skewness	–0.968	–0.969	–0.936	–0.924	–0.745	–0.737	–0.740	–0.775
Excess kurtosis	4.017	3.973	3.734	3.599	2.724	2.576	2.581	2.561
Average turnover (%)	9.311	8.954	9.016	9.005	10.396	10.170	10.203	9.625
(CE – 1) (bp)	7.946	8.527	8.862	10.000	16.181	16.996	16.867	18.162
Annualized diff. in CE (%)	—	0.302	0.476	1.068	—	0.424	0.357	1.030
p-value	—	0.015	0.014	0.007	—	0.003	0.037	0.020
Panel C. $\gamma = 10$								
Annualized mean (%)	16.375	16.297	16.248	16.498	18.120	18.076	18.085	17.808
Annualized volatility (%)	16.315	16.143	15.993	15.754	15.526	15.235	15.284	14.517
Skewness	–0.816	–0.829	–0.805	–0.794	–0.715	–0.707	–0.718	–0.708
Excess kurtosis	2.953	2.913	2.786	2.735	2.291	2.209	2.227	2.199
Average turnover (%)	7.931	7.605	7.653	7.576	9.072	8.913	8.914	8.535
(CE – 1) (bp)	6.936	7.304	7.719	8.941	12.885	13.655	13.519	15.134
Annualized diff. in CE (%)	—	0.191	0.407	1.043	—	0.400	0.330	1.170
p-value	—	0.055	0.026	0.023	—	0.009	0.060	0.021

TABLE 5

Out-of-Sample Results for Investor with MR of 50%

Table 5 presents out-of-sample results for the investor investing with 50% margin requirement (MR) in the 4 factors. The out-of-sample period is from July 1, 1983, to Dec. 31, 2010, for a total of 1,436 weekly returns. For each level of relative risk aversion, the performance of the three copulas is compared to the benchmark normal distribution. Panels A, B, and C show the results for relative risk aversion coefficients of 3, 7, and 10, respectively. We report the realized moments of the portfolio returns, the average turnover, as well as the certainty equivalent (CE). The annualized difference in CE is the difference between the CE for each model and the normal benchmark multiplied by 52. We also report bootstrap *p*-values testing the significance of the differences in CEs. We test each of the three alternative models against the normal benchmark.

	Constant Correlation Models				*Dynamic Correlation Models*			
	Normal Distribution	Normal Copula	Symmetric *t* Copula	Skewed *t* Copula	Normal Distribution	Normal Copula	Symmetric *t* Copula	Skewed *t* Copula
Panel A. $\gamma = 3$								
Annualized mean (%)	11.759	11.875	11.866	11.948	11.949	12.091	12.064	11.749
Annualized volatility (%)	18.745	18.655	18.726	18.648	18.354	18.251	18.296	17.406
Skewness	–1.127	–1.123	–1.126	–1.154	–1.132	–1.133	–1.126	–1.256
Excess kurtosis	7.439	7.345	7.340	7.594	7.198	7.137	7.106	8.340
Average turnover (%)	4.903	4.827	4.810	4.812	4.684	4.613	4.584	5.480
(CE − 1) (bp)	15.729	16.021	15.949	16.162	16.384	16.733	16.648	16.658
Annualized diff. in CE (%)	—	0.151	0.114	0.225	—	0.181	0.137	0.142
p-value	—	0.050	0.103	0.012	—	0.040	0.119	0.269
Panel B. $\gamma = 7$								
Annualized mean (%)	10.807	10.799	10.854	10.885	11.357	11.421	11.440	11.383
Annualized volatility (%)	13.352	13.252	13.304	13.217	12.914	12.758	12.815	12.116
Skewness	–1.285	–1.343	–1.264	–1.217	–0.909	–0.924	–0.913	–0.824
Excess kurtosis	9.197	9.617	8.688	8.113	5.139	4.941	4.845	6.140
Average turnover (%)	5.154	5.011	5.045	5.055	5.281	5.205	5.212	5.562
(CE − 1) (bp)	9.908	10.031	10.093	10.323	11.869	12.234	12.186	13.156
Annualized diff. in CE (%)	—	0.064	0.096	0.216	—	0.190	0.165	0.670
p-value	—	0.101	0.113	0.046	—	0.017	0.096	0.032
Panel C. $\gamma = 10$								
Annualized mean (%)	10.239	10.239	10.244	10.259	11.272	11.327	11.342	11.383
Annualized volatility (%)	11.071	10.995	11.021	10.929	10.594	10.433	10.470	9.979
Skewness	–1.179	–1.215	–1.173	–1.151	–0.836	–0.846	–0.838	–0.517
Excess kurtosis	6.606	6.808	6.360	6.074	4.200	4.036	3.971	5.628
Average turnover (%)	4.677	4.512	4.572	4.567	4.967	4.838	4.856	5.055
(CE − 1) (bp)	8.390	8.527	8.515	8.753	11.555	11.974	11.936	13.040
Annualized diff. in CE (%)	—	0.071	0.065	0.189	—	0.218	0.198	0.772
p-value	—	0.091	0.216	0.061	—	0.003	0.042	0.043

using the $T = 1{,}436$ out-of-sample weekly returns. The CE for each model is computed as

$$\mathrm{CE} \quad = \quad U^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}\frac{(1+r_{p,t})^{1-\gamma}}{1-\gamma}\right) \quad = \quad \left(\frac{1}{T}\sum_{t=1}^{T}(1+r_{p,t})^{1-\gamma}\right)^{\frac{1}{1-\gamma}},$$

where the argument of the inverse utility function U^{-1} in the first equality is the realized average utility and where

$$r_{p,t} \quad = \quad r_{f,t} + w_{t-1}^{\top} r_t$$

are the out-of-sample portfolio returns.

We also report the difference in realized CE between each model and the multivariate normal benchmark model, and we annualize the measure for ease of presentation. As we normalize the initial wealth to be $1 in each period, the difference in CE measures between two models can be seen as the proportion of wealth the investor would be willing to forego to be indifferent ex post between the portfolio allocations from the alternative model and the benchmark model.

For example, an investor with an MR = 20% and a relative risk aversion of 7 realizes a gain of 1.068%/52 = 0.0205% or 2.05 basis points (bp) per week if she uses the constant skewed *t* copula instead of the multivariate normal model.

The CE results in Tables 4 and 5 are quite striking. First, all three copulas always improve upon the normal distribution. This is true both for the constant and the dynamic copulas. Second, the skewed *t* copula performs the best in all but one case across constant and dynamic models. Third, each dynamic dependence model always dominates its constant counterpart in terms of CE. This is true for the copula models as well as the normal distribution.

One may wonder whether richer models lead to better performance by generating more trading, and whether accounting for transaction costs would lower the realized return on those portfolios. We argue that this is not the case by reporting the turnover for each model, defined as the percentage change in weights averaged across time and factors,

$$\text{Average turnover (\%)} \quad = \quad \frac{100}{4T}\sum_{t=1}^{T}\sum_{i=1}^{4}\left|w_{i,t} - w_{i,t-1}\right|.$$

Average turnover values are around 4%–13%, depending on risk aversion. More importantly, they do not vary much within each of the panels, which suggests that improvements in realized utility across models are not driven by differences in turnover.

The improvements found in the alternative nonnormal models, especially the ones using a skewed *t* copula, are large in economic terms. The next section examines if these differences are statistically significant.

D. Significance of Results

In order to assess the statistical differences between the performances of the CRRA investors' portfolios, we use the method of Politis and Romano (1994) to bootstrap the difference in realized CE of each copula with respect to the multivariate normal benchmark model. This yields a distribution of differences in realized CEs, and we can infer whether the actual differences presented in Tables 4 and 5 are significantly larger than 0. We compute in each case the bootstrap *p*-value, which represents the proportion of bootstrapped differences that fall below 0. A small *p*-value indicates that the difference in CE realized by this specification is significant, while a value near 1 suggests that the specification is worse than the benchmark model. The bootstrap *p*-values are computed in each case using 100,000 bootstrap replications.

The *p*-values against the benchmark normal model are presented for each level of risk aversion, and for either the constant or the dynamic copula models. The results in Table 4 for MR = 20% are quite striking. The *p*-values for the skewed *t* copulas are smaller than 5% in all six cases. When leverage is large, the nonnormal risk models offer important economic benefits.

The analysis is repeated on the results in Table 5 for MR = 50%. In this case, the skewed *t* copulas have *p*-values smaller than 5% in four of six cases. When the margin requirement is larger and investors are able to take on less leverage, then careful risk management is still important.

E. Implications for Risk Management

The portfolio allocation experiment above is, of course, quite specific in nature. However, our findings of variance dynamics and dynamic nonlinear dependence across factors have important implications for risk measurement more generally. The broader risk management implications are relevant for investors investing in the 4 factors, as in the previous section, as well as for investors using the factors to model a wider set of assets.

To assess the broader implications of the models, we now investigate the effect of the different models on a generic portfolio risk measure, namely ES. The 1% ES is defined as the expected loss when the loss is in the 1% tail of the distribution,

$$\mathrm{ES}_{t+1}^{0.01} \quad = \quad -\mathrm{E}_t\left[r_{p,t+1} | r_{p,t+1} < F_{p,t}^{-1}(0.01)\right].$$

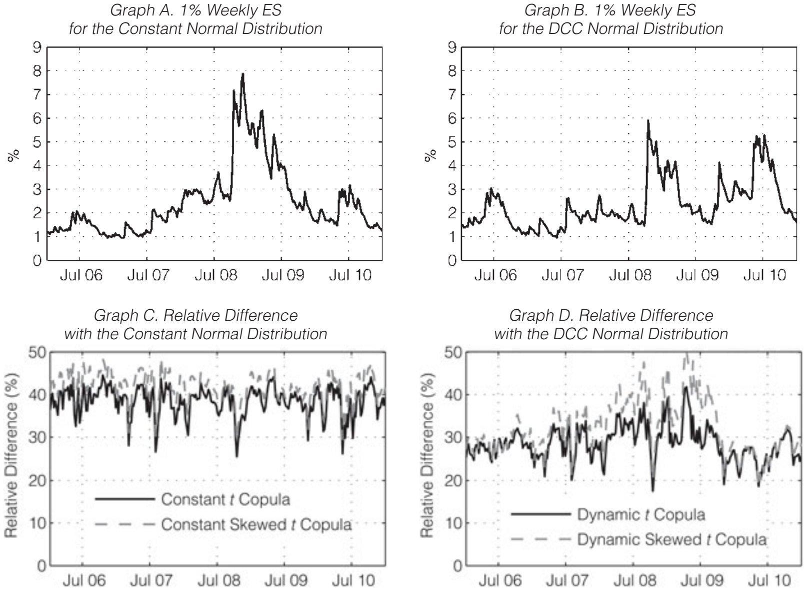
ES is preferable to the more conventional value-at-risk measure because ES emphasizes the magnitude of large losses (see, e.g., Basak and Shapiro (2001)).

Figure 10 presents the 1% weekly ES for an equal-weighted 4-factor portfolio rebalanced weekly during the dramatic period from Jan. 2006 through Dec. 2010. Graphs A and B show the ES from the benchmark constant correlation normal distribution (Graph A) and for the DCC normal distribution (Graph B). ES increases during market turmoil, for instance, during the financial crisis

FIGURE 10

ES for Equal-Weighted Portfolios of Factors

We report in Graphs A and B of Figure 10 the 1% weekly expected shortfall (ES) measure for an equal-weighted portfolio of the 4 factors from Jan. 2006 to Dec. 2010. Graph A presents the risk measure for the normal distribution with constant correlation, and Graph B for the DCC normal distribution. Graphs C and D show the relative difference in ES implied by either the *t* or the skewed *t* copulas. The expected returns and volatilities are from the AR-GARCH model in all cases.



of 2007–2008. Perhaps surprisingly, the ES based on dynamic correlation in Graph B is significantly lower during the fall 2008 compared to the one based on constant correlation in Graph A. This is a reflection of the lower dynamic correlation between momentum and the other factors during that period, as was evident in Figure 8.

Graphs C and D of Figure 10 report the relative difference in ES between the normal distribution models and the t and skewed t copulas. The difference between the constant skewed t copula and the normal distribution ranges from 20% to 50%. This difference is robust to allowing for time-varying correlations. We conclude that ignoring the multivariate nonnormality in equity factors leads to a large underestimation of portfolio risk.

V. Further Analysis

In this section, we present additional results of our analysis. First, we check that the deviations from normality are also present when short positions in the factors are considered. Then we verify the robustness of our results by examining the threshold correlation for daily and monthly returns. Finally, we discuss the copula specification search that has led us to favor the t and skewed t copula with DCC dynamics.

A. Reverse Threshold Correlations

We first examine the dependence structure in weekly factor returns from a different perspective. So far in the literature, threshold correlations have mainly been used to inspect the dependence in highly correlated equity portfolios, for which it was natural to look at bivariate returns falling in the bottom-left or top-right quadrants. For uncorrelated or even negatively correlated returns, it is relevant to look at the top-left or bottom-right quadrants as well. To this end, we define the reverse threshold correlation $\widetilde{\rho}_{ij}(u)$ as

$$\widetilde{\rho}_{ij}(u) \quad = \quad \begin{cases} \operatorname{corr}(r_i, r_j \mid r_i < F_i^{-1}(u), r_j > F_j^{-1}(1-u)), & \text{when } u < 0.5, \\ \operatorname{corr}(r_i, r_j \mid r_i \geq F_i^{-1}(u), r_j \leq F_j^{-1}(1-u)), & \text{when } u \geq 0.5. \end{cases}$$

Figure 11 reports the empirical reverse threshold correlation for weekly returns as well as the ones implied by the bivariate normal distribution. Again, the empirical threshold correlation patterns are markedly different from normality. We thus conclude that deviations from multivariate normality in factor returns are not limited to the cases when 2-factor returns are of the same sign.

B. Daily and Monthly Returns

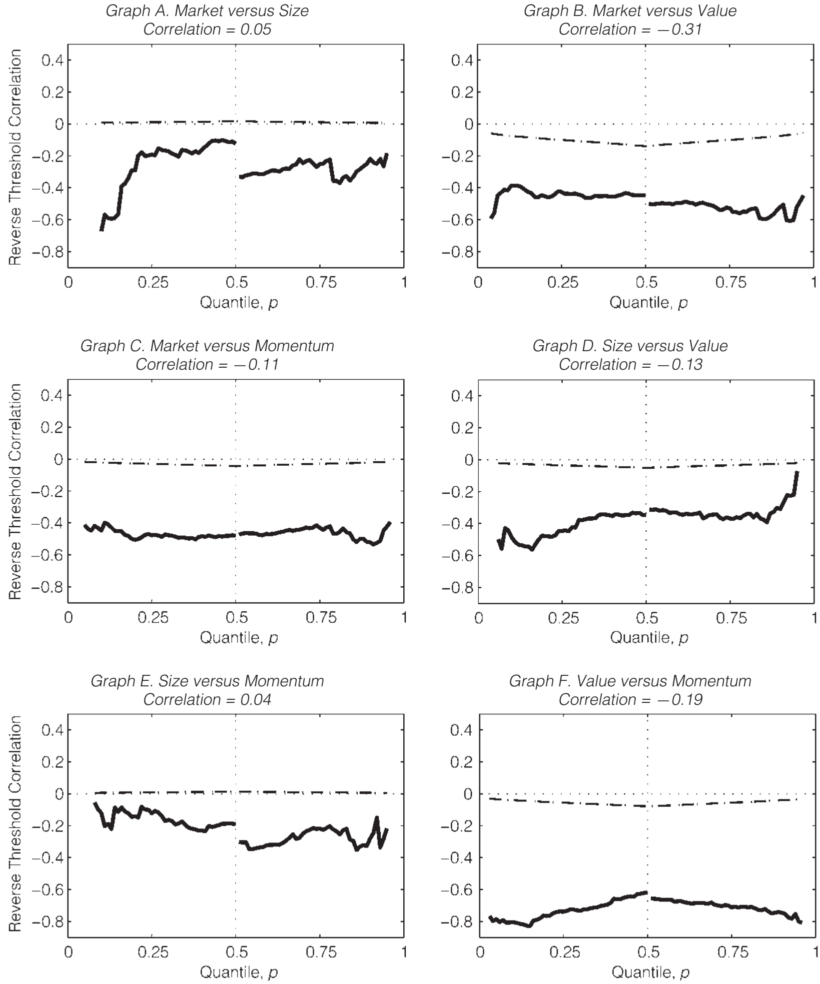
So far we have focused attention solely on weekly factor returns. Given the dynamics found in the variances and correlations of the weekly returns, the temporal aggregation of factor returns is not obvious, and we therefore briefly study factor returns at two other frequencies as well.

In particular, we examine the presence of nonlinear dependence between equity market factors on a daily and monthly basis. Table 6 presents the descriptive

FIGURE 11

Reverse Threshold Correlation on Weekly Returns

Figure 11 presents the reverse threshold correlations of weekly returns from July 5, 1963, to Dec. 31, 2010. The linear correlations are provided in the titles of the graphs. Below the median ($p = 0.5$) the solid line represents the correlation when the first variable is below its pth quantile and the second above its $(1 - p)$th quantile. Above the median ($p = 0.5$), the solid line represents the correlation when the first variable is above its pth quantile and the second below its $(1 - p)$th quantile. The dash-dot line represents the analytical reverse threshold correlation function for a bivariate normal distribution using the linear correlation coefficient from the data.



statistics for daily returns in Panel A and for monthly returns in Panel B for the period July 1963–Dec. 2010. Daily returns for the market, size, and momentum factors exhibit negative skewness, and all factors display significant excess kurtosis. Monthly returns for the market and momentum factors still display negative skewness, and the excess kurtosis for size and momentum are large, suggesting that the univariate nonnormality in factor returns is persistent as the investor horizon increases.

TABLE 6

Descriptive Statistics for Daily and Monthly Factor Returns

We report descriptive statistics for daily returns in Panel A of Table 6 and for monthly returns in Panel B from July 1963 to Dec. 2010. Significant correlations are marked by * and ** denoting 5% and 1% levels, respectively.

	Market	Size	Value	Momentum
Panel A. Daily Returns				
Annualized mean	5.32%	1.88%	4.82%	8.20%
Annualized volatility	15.62%	8.07%	7.78%	11.14%
Skewness	−0.53	−1.21	0.08	−1.06
Excess kurtosis	17.13	27.26	8.25	16.49
Autocorrelations				
First-order	0.069**	0.052**	0.165**	0.230**
Second-order	−0.033*	0.024*	0.034**	0.058**
Third-order	0.016	0.042**	0.025*	0.038**
Cross Correlations				
Market	—	−0.180**	−0.305**	−0.126**
Size	—	—	−0.049**	0.068**
Value	—	—	—	−0.191**
Panel B. Monthly Returns				
Annualized mean	5.35%	3.25%	4.80%	8.61%
Annualized volatility	15.71%	11.00%	10.19%	15.06%
Skewness	−0.56	0.53	−0.02	−1.43
Excess kurtosis	1.99	5.56	2.43	10.81
Autocorrelations				
First-order	0.090*	0.059	0.156**	0.063
Second-order	−0.037	0.039	0.037	−0.064
Third-order	0.023	−0.082	0.039	0.017
Cross Correlations				
Market	—	0.307**	−0.304**	−0.126**
Size	—	—	−0.235**	−0.003
Value	—	—	—	−0.160**

The correlation between the market and size factors varies from −0.18 to 0.05 to 0.31 when going from daily to weekly to monthly returns. However, the linear correlations between the other factors are remarkably stable across return horizons.

Figure 12 presents the threshold correlation for daily (continuous line) and monthly returns (dotted line). Not surprisingly, there are some differences between the patterns for weekly returns in Figures 3 and 4 and those in Figure 12, but threshold correlations for both daily and monthly returns remain markedly different from the ones implied by the normality assumption: Daily and monthly factor returns exhibit strong tail dependence, which is crucial for portfolio and risk management and which is not captured by the normal distribution. We thus conclude that the multivariate nonnormality in factor returns is persistent as the investor horizon increases.

C. Alternative Copula Functions

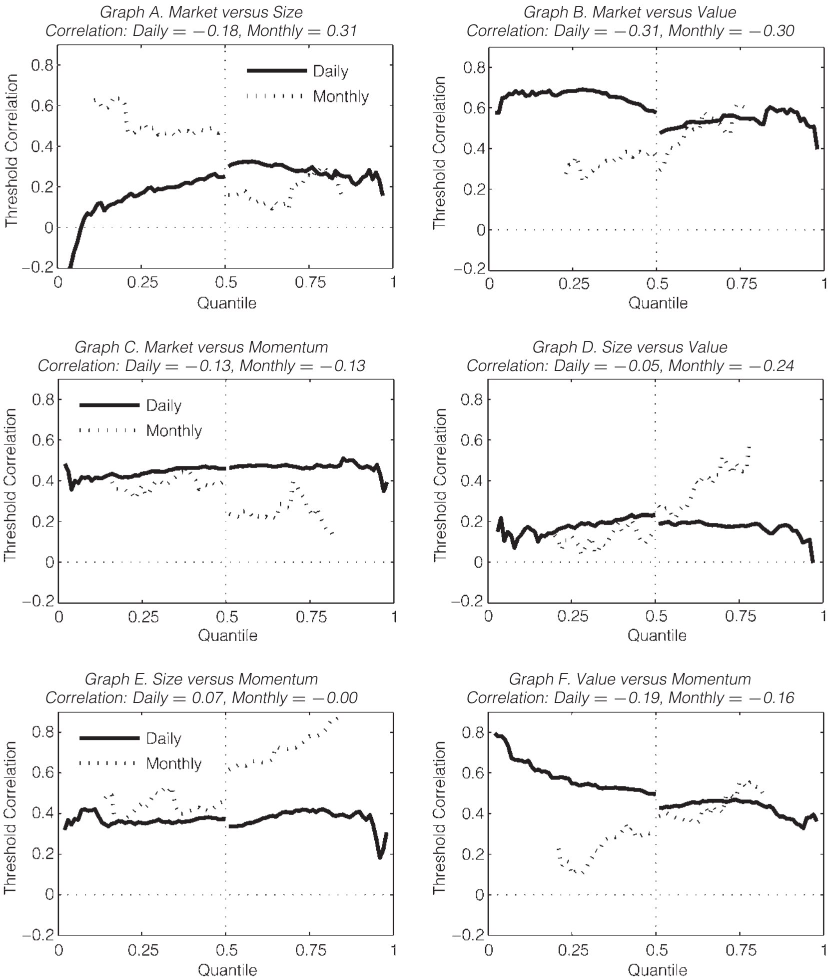
In order to fit the factor return data, we need copulas that can capture multivariate fat tails, which are often measured in terms of tail dependence. The lower tail dependence (LTD) and upper tail dependence (UTD) coefficients are defined, respectively, as

$$\begin{aligned} \text{LTD}\left(r_i, r_j\right) &= \lim_{u\to 0} \Pr\left[r_j \leq F_j^{-1}(u)\,|r_i \leq F_i^{-1}(u)\right], \\ \text{UTD}\left(r_i, r_j\right) &= \lim_{u\to 1} \Pr\left[r_j > F_j^{-1}(u)\,|r_i > F_i^{-1}(u)\right]. \end{aligned}$$

FIGURE 12

Threshold Correlation Functions on Daily and Monthly Returns

We show threshold correlation functions computed on daily returns (continuous line) and monthly returns (dotted line) from July 5, 1963, to Dec. 31, 2010. Returns are standardized by their unconditional mean and standard deviation. The lines represent the correlation when both returns are below (above) the threshold when the threshold is below (above) the median. The linear correlations are provided in the titles for each pair of factors. The threshold correlation functions are computed for thresholds for which at least 20 data points are available.



Tail dependence and threshold correlation are related concepts. Most importantly, a copula having zero tail dependence generates correlations approaching 0 as the threshold nears 0 or 1.

The normal copula has zero LTD and UTD, and the t copula has nonzero and symmetric LTD and UTD coefficients. The skewed t copula we use allows for nonzero tail dependence coefficients, which in turn differ between the upper and lower tails.

Before settling on the *t* and skewed *t* copulas as our favored alternative to the normal copula specification, we investigated several copulas from the Archimedean family detailed in Joe (1997) and Patton (2009). For example, we considered in detail the Clayton and Gumbel copulas. Along with the normal and *t* copulas, they are arguably some of the most often used copulas in the financial literature. The Clayton and Gumbel copulas have nonzero LTD and UTD, respectively. Moreover, these copulas are potentially able to produce asymmetric threshold correlation patterns. However, the Clayton or Gumbel is not capable of capturing asymmetric threshold correlations while keeping the linear correlation close to 0. Unfortunately this is the empirically relevant case for factor returns, as we saw above.

The Clayton and Gumbel copulas are defined with only one parameter, and the range of dependence they can generate is limited to positive levels. These asymmetric copulas have stronger dependence in the lower-left quadrant or in the upper-right quadrant. This means that they will produce few observations lying in the upper-left or lower-right quadrants. This limitation leads to very low levels of likelihood when we estimate the models on the factor return data.

We also consider rotated (survival) versions of these copulas as in Patton (2004) and mixtures with the normal copula as in Hong et al. (2007). However, the likelihood levels favored the *t* and skewed *t* copulas, and we rely on these models instead.

D. Alternative Copula Dynamics

Regime-switching models are arguably the main alternative to the DCC for modeling time-varying dependence we use in our analysis. For the use of regime-switching models, see among others Ang and Bekaert (2002), Pelletier (2006), and Garcia and Tsafack (2011). In these models, each regime has a different level of dependence, and the choice of regime in each period is governed by an unobservable Markov chain.

Asymmetric copulas such as the Clayton, Gumbel, or Joe-Clayton are difficult to generalize in higher dimension because they are defined with either one or two parameters. Chollete, Heinen, and Valdesogo (2009) recently proposed a regime-switching copula with two regimes in which one regime is characterized by a normal copula and the other by a canonical vine copula. Canonical vine copulas circumvent the dimensionality problem by assuming that the multivariate distribution can be decomposed into a hierarchy of bivariate functions.

We estimate a model in which one regime is specified as a normal copula and the other as a *t* copula, and another model in which both regimes are characterized by *t* copulas. Note that in these two models, much modeling flexibility is gained because the second regime's distribution is a series of *t* copulas, each having a different correlation and degree of freedom. Finally, we combine a normal copula with rotated Gumbel copulas. Such a specification is interesting because of its ability to capture asymmetry in dependence.

When estimating the three regime-switching models and comparing them with the DCC copulas in Table 3, we find that the DCC copulas provided a better fit and did so with fewer parameters. We therefore do not include the estimation results in the paper.[8]

VI. Conclusion

The large-scale nature of equity portfolio selection and risk management often requires a factor approach. The Fama-French (1993) and momentum factors are pervasive in cross-sectional asset pricing and are also increasingly used in portfolio allocation. We therefore study their dynamic and distributional properties in detail.

Our analysis shows that the conditional variance of all 4-factor returns is dynamic, persistent, and well captured by an asymmetric GARCH model. We also find that the skewed t distribution provides a good fit to the factor residuals.

There is strong evidence of nonlinear dependence across factors, which we model using the copula implied by a skewed version of the multivariate t distribution. This copula model is capable of generating the strongly asymmetric patterns in nonlinear dependence observed across factors, while preserving the relatively modest linear correlations found in the returns data.

We use the new copula models to investigate the economic importance of modeling the nonlinear and dynamic dependence between the factors. Using a real-time portfolio selection experiment, we find strong economic gains from modeling nonlinear factor dependence. The skewed t copula leads to higher realized investor utility than do other dependence models. Dynamic correlations offer large economic benefits as well. In a more generic risk management application, we show that the nonnormal factor model has important implications for portfolio risk measurement.

Several important challenges are left for future research. First, we only study the 4-factor model in this paper. Clearly, extending our analysis beyond the 4-factor model would be interesting. It would also be interesting to investigate which economic variables drive the level of factor variance, correlation, and asymmetry. This analysis could be conducted using the methodology of Engle and Rangel (2008), (2012).

Appendix A. Skewed *t* Distribution for Residuals

We first define the univariate skewed t distribution from Hansen (1994) as

$$f(\epsilon;\kappa,\nu) = \begin{cases} bc\left(1+\dfrac{1}{\nu-2}\left(\dfrac{b\epsilon+a}{1-\kappa}\right)^2\right)^{-\frac{\nu+1}{2}} & \text{if } \epsilon < -\dfrac{a}{b} \\ bc\left(1+\dfrac{1}{\nu-2}\left(\dfrac{b\epsilon+a}{1+\kappa}\right)^2\right)^{-\frac{\nu+1}{2}} & \text{if } \epsilon \geq -\dfrac{a}{b} \end{cases},$$

[8] All estimation results are available from the authors.

where

$$a = 4\kappa c\frac{\nu-2}{\nu-1}, \qquad b^2 = 1+3\kappa^2-a^2, \qquad c = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi(\nu-2)}\Gamma\left(\frac{\nu}{2}\right)}.$$

The skewed t distribution has zero mean, unit variance, and its skewness and kurtosis are

$$\begin{aligned} \mathrm{E}\left[\epsilon^3\right] &= \frac{m_3-2am_2+2a^3}{b^3}, \\ \mathrm{E}\left[\epsilon^4\right] &= \frac{m_3-4am_3+6a^2m_2-3a^4}{b^4}, \end{aligned}$$

where

$$\begin{aligned} m_2 &= 1+3\kappa^2, \\ m_3 &= 16c\kappa(1+\kappa^2)\frac{(\kappa-2)^2}{(\kappa-1)(\kappa-3)}, \text{ if } \kappa>3, \\ m_4 &= 3\frac{\kappa-2}{\kappa-4}(1+10\kappa^2+5\kappa^4), \text{ if } \kappa>4. \end{aligned}$$

The following sections contain the PDFs for the t copulas used.

Appendix B. t Copula

The CDF of the t copula with correlation matrix Ψ and scalar degree of freedom ν_c is given by

$$C^t_{\Psi,\nu_c}(\eta) = T_{\Psi,\nu_c}\left(T^{-1}_{\nu_c}(\eta_1),\ldots,T^{-1}_{\nu_c}(\eta_N)\right),$$

where $T_{\Psi,\nu_c}(\cdot)$ is the multivariate t CDF and $T^{-1}_{\nu_c}(\cdot)$ is the univariate t inverse CDF.

The PDF is

$$c^t_{\Psi,\nu_c}(\eta) = \frac{t_{\Psi,\nu_c}\left(T^{-1}_{\nu_c}(\eta_1),\ldots,T^{-1}_{\nu_c}(\eta_N)\right)}{\prod_{j=1}^N t_{\nu_c}\left(T^{-1}_{\nu_c}(\eta_j)\right)},$$

where $t_{\Psi,\nu_c}(\cdot)$ and $t_{\nu_c}(\cdot)$ are, respectively, the multivariate t PDF and the univariate t PDF.

When standardizing the fractiles $z_j = T^{-1}_{\nu_c}(\eta_j)$ used in the DCCs specification, we use the fact that the covariance of the fractiles is given by $(\nu_c/(\nu_c-2))\Psi$.

Appendix C. Skewed t Copula

The PDF of the skewed t copula defined from the asymmetric t distribution is given by

$$\begin{aligned} c^{st}_{\Psi,\nu_c,\lambda}(\eta) &= \\ & \frac{2^{\frac{(\nu_c-2)(N-1)}{2}} K_{\frac{\nu_c+N}{2}}\left(\sqrt{(\nu_c+z^\top\Psi^{-1}z)\,\lambda^\top\Psi^{-1}\lambda}\right)e^{z^\top\Psi^{-1}\lambda}}{\Gamma\left(\frac{\nu_c}{2}\right)^{1-N}|\Psi|^{\frac{1}{2}}\left(\sqrt{(\nu_c+z^\top\Psi^{-1}z)\,\lambda^\top\Psi^{-1}\lambda}\right)^{-\frac{\nu_c+N}{2}}\left(1+\frac{1}{\nu_c}z^\top\Psi^{-1}z\right)^{\frac{\nu_c+N}{2}}} \\ & \times\prod_{j=1}^N \frac{\left(\sqrt{(\nu_c+z_j^2)\,\lambda_j^2}\right)^{-\frac{\nu_c+1}{2}}\left(1+\frac{z_j^2}{\nu_c}\right)^{\frac{\nu_c+1}{2}}}{K_{\frac{\nu_c+1}{2}}\left(\sqrt{(\nu_c+z_j^2)\,\lambda_j^2}\right)e^{z_j\lambda_j}}, \end{aligned}$$

where $K(\cdot)$ is the modified Bessel function of the third kind. We define $z_j = \mathrm{ST}^{-1}_{\nu_c,\lambda_j}(\eta_j)$, where $\mathrm{ST}^{-1}_{\nu_c,\lambda_j}(\eta_j)$ is the skewed t univariate quantile function that is constructed via simulation.

When simulating, we rely on the stochastic representation of the skewed t distribution,

$$X = \sqrt{W}Y + \lambda W,$$

where W is an inverse gamma variable, $W \sim IG(\nu_c/2, \nu_c/2)$, Y is a vector of correlated normal variables, $Y \sim N(0, \Psi)$, and Y and W are independent. Now z_j is found from the empirical quantile function of a large number of simulated X_j values.

To standardize the z fractiles used in the DCC specification, note that the expected value is given by

$$\mathrm{E}[X] = \mathrm{E}(\mathrm{E}[X|W]) = \mathrm{E}[W]\lambda = \frac{\nu_c}{\nu_c - 2}\lambda,$$

and the covariances of the fractiles are given by

$$\begin{aligned}\mathrm{cov}(X) &= \mathrm{E}(\mathrm{var}[X|W]) + \mathrm{var}(\mathrm{E}[X|W]) \\ &= \frac{\nu_c}{\nu_c - 2}\Psi + \frac{2\nu_c^2 \lambda\lambda^\top}{(\nu_c - 2)^2(\nu_c - 4)}.\end{aligned}$$

Note that as $\lambda \to 0$ element-wise, we obtain the symmetric t copula, and if we further let $\nu_c \to \infty$, then we have the normal copula.

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