CHONGQING UNIVERSITY OF POSTS AND TELECOMMUNICATIONS

Assignment Report

TERM:	**Spring 2022**
Module:	**EE1613 Advanced Math (II)**
CLASS:	34092102
BRUNEL ID:	2161047
NAME:	Liu Xukang
TUTOR:	Yang Yingxue

April 2022

1^{st} Assignment of Advanced Math (II)

Abstract

1 Aim

To review what we have learned this semester. Further investigate and understand differential equation, vector space, partial derivates, plane equations, image of curves, etc.

2 Objectives

1) Review chapters 7 to 9. Briefly make a note to help me master the corresponding knowledge points.
2) Do some exercises our teacher give us. Then find the knowledge points which do not master skilled.
3) Make a conclusion. It includes important knowledge points in each chapter.

3 Theory

3.1 Chapter 7: Differential equation

1) First order differential equation with variables separable:

A first order differential equation$y' = F(x, y)$ is called a variable separable equation if the function F can be factored into the form:$F(x, y) = \frac{f(x)}{g(y)}$ and f(x), g(x) is continuous, g(y)≠0.

2) Homogeneous differential equation:

A first order differential equation $F(x, y, y') = 0$ is called homogeneous differential equation if it can be rewritten as: $\frac{dy}{dx} = \varphi(\frac{y}{x})$.

Let $u = \frac{y}{x}$

$$y = ux$$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

3) Linear Equation of First Order:

A differential equation of type

$$\frac{dy}{dx} + p(x)y = Q(x)$$

where P(x) and Q(x) are continuous functions of x, is called a linear differential equation of first order.

4) Bernoulli equation:

The equation

$$\frac{dy}{dx} + p(x)y = Q(x)y^n \ (n \neq 0,1)$$

is called Bernoulli equation.

5) Structure of solutions of linear differential equations:

$$y'' + P(x)y' + Q(x)y = 0. \qquad (1)$$

If $y_1(x)$ and $y_2(x)$ are two solutions of (1), then

$$y = C_1y_1(x) + C_2y_2(x)$$

Let $y^*(x)$is a particular solution of Second order nonhomogeneous linear equation, Y(x) is a general solution of corresponding homogeneous equations.

$$y'' + P(x)y' + Q(x)y = f(x)$$
$$y = Y(x) + y^*(x)$$

6) **Second Order Linear Homogeneous Differential Equations with Constant Coefficients:**

For each of the equation$y'' + py' + qy = 0$, we can write the so-called characteristic (auxiliary) equation:

$$r^2 + pr + q = 0$$

The general solution of the homogeneous differential equation depends on the roots of the characteristic quadratic equation.

If $D = p^2 - 4q > 0$

$$y(x) = C_1e^{r_1x} + C_2e^{r_2x}$$

If $D = p^2 - 4q = 0$

$$y(x) = (C_1x + C_2)e^{r_1x}$$

If $D = p^2 - 4q < 0$

$$y(x) = e^{\alpha x}[C_1cos\ (\beta x) + C_2sin\ (\beta x)]$$

7) **Second Order Linear Nonhomogeneous Differential Equations with Constant Coefficients:**

The nonhomogeneous differential equation of this type has the form

$$y'' + py' + qy = f(x)$$

Case1: $f(x) = P_m(x)e^{\lambda x}$

$$y^* = x^k R_m(x)e^{\lambda x}$$

Case2: $f(x) = e^{\lambda x}[P_m(x)\cos(\beta x) + Q_n(x)\sin(\beta x)]$

$$y^* = \lambda^k e^{\lambda x}\left[R_m^{(1)}(x)\cos\omega x + R_m^{(2)}(x)\sin\omega x\right]$$

3.2 Chapter 8: vector space

1) **Dot product:**

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

2) **Cross product:**

$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|sin\theta$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

3) **Mixed product:**

$$[\vec{a}\ \ \vec{b}\ \ \vec{c}] = (\vec{a}\times\vec{b})\cdot\vec{c}$$

4) **Distance from a point to a plane:**

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

The second method is

$$d = \frac{|\overrightarrow{P_1P_0}\times\vec{a}|}{|\vec{a}|}$$

5) **Point-normal form of the equation of a plane:**

Normal vector is $\vec{n} = (A, B, C)$ and passes through the point $M_0(x_0, y_0, z_0)$.

The plane equation is

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

6) **General form of the equation of a plane:**

$$Ax + By + Cz + D = 0$$

7) Symmetric form equations of a straight line:

The equation of the line L that passes through the point P_0 and is parallel to the vector $\vec{a} = (l, m, \mathrm{n})$ is

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$$

8) Parametric equations of a straight line:

$$\begin{cases} x = x_0 + lt, \\ y = y_0 + mt, \ (t \in R) \\ z = z_0 + nt. \end{cases}$$

3.3 Chapter 9: Multivariable function

1) Partial derivate:

$$\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$
$$\frac{\partial}{\partial y} f(x, y) = f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

2) Necessary conditions for differentiability:

Suppose that a function $z = f\ (x,\ y)$ is differentiable at a point (x_0, y_0), then

(a) f must be continuous at the point (x_0, y_0);
(b) both partial derivatives of function f at the point (x_0, y_0) exist and $A = f_x(x_0, y_0), B = f_y(x_0, y_0)$, that is

$$dz|_{(x_0, y_0)} = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy.$$

3) Sufficient condition for differentiability:

If the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ of a function $z = f(x, y)$, both exist and are continuous in a neighborhood of the point, (x_0, y_0), then the function f is differentiable at the point (x_0, y_0).

4) Partial derivatives of multivariable composite function:

Chain rule: If $z = f(u(x,\ y),\ v(x,\ y))$, then the following chain rule holds:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

c.

5) **Jacobi determinant:**

$$J = \frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix}$$

6) **The limit of vector-valued function:**

If $\boldsymbol{f}(t) = f_1(t)\boldsymbol{i} + f_2(t)\boldsymbol{j} + f_3(t)\boldsymbol{k}$, then the limit $\lim\limits_{t\to t_0} \boldsymbol{f}(t)$ exists if and only if all of the limits $\lim\limits_{t\to t_0} f_1(t)$, $\lim\limits_{t\to t_0} f_2(t)$, $\lim\limits_{t\to t_0} f_3(t)$ exist, and

$$\lim_{t\to t_0} \boldsymbol{f}(t) = \lim_{t\to t_0} f_1(t)\boldsymbol{i} + \lim_{t\to t_0} f_2(t)\boldsymbol{j} + \lim_{t\to t_0} f_3(t)\boldsymbol{k}$$

7) **The derivate of vector-valued function:**

If $\boldsymbol{f}(t) = f_1(t)\boldsymbol{i} + f_2(t)\boldsymbol{j} + f_3(t)\boldsymbol{k}$, then $\boldsymbol{f}(t)$ is differentiable if and only if $f_1(t)$, $f_2(t)$, $f_3(t)$ are differentiable and

$$\boldsymbol{f}'(t) = f'_1(t)\boldsymbol{i} + f'_2(t)\boldsymbol{j} + f'_3(t)\boldsymbol{k}$$

8) **Tangent plane equation:**

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$$

9) **Normal plane equation:**

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)} = \frac{y-y_0}{F_y(x_0,y_0,z_0)} = \frac{z-z_0}{F_z(x_0,y_0,z_0)}$$

10) **Formula for the directional derivative:**

$$\frac{\partial f}{\partial l}|_{(x_0,y_0)} = f_x(x_0,y_0)\cos\alpha + f_y(x_0,y_0)\cos\beta$$

11) **Gradient:**

$$\mathbf{grad}\, f(x_0,y_0) = f_x(x_0,y_0)\boldsymbol{i} + f_y(x_0,y_0)\boldsymbol{j}$$

4 Methods and Solutions

4.1 Task 1

(a) $\boldsymbol{x(y+2)y' = \ln x + 1}$

Solution:

$$(\mathrm{y}+2)\frac{dy}{dx} = \frac{\mathrm{lnx}+1}{\mathrm{x}}$$

$$(\mathrm{y}+2)\mathrm{dy} = \left(\frac{\mathrm{lnx}+1}{\mathrm{x}}\right)\mathrm{dx}$$

$$\int (y+2)\,dx = \int \frac{\ln x + 1}{x}dx$$

$$\frac{1}{2}y^2 + 2y = \frac{1}{2}(\ln x)^2 + \ln x + C$$

(b) $\boldsymbol{(x^2+xy)y' = y^2}$

Solution:

$$(x^2+xy)\frac{dy}{dx} = y^2$$

$$\frac{dy}{dx} = \frac{y^2}{x^2+xy} = \frac{\left(\frac{y}{x}\right)^2}{1+\left(\frac{y}{x}\right)}$$

Let $u = \frac{y}{x}$

$$\frac{dy}{dx} = x\frac{du}{dx} + u = \frac{\left(\frac{y}{x}\right)^2}{1+(\frac{y}{x})^2} = \frac{u^2}{1+u}$$

Then $\int \frac{1+u}{24^2-u}du = \int \frac{dx}{x}$

$$-\ln|u| + \frac{1}{2}\ln|2\mathrm{u}-1| = \ln|x| + C_1$$

$$\frac{\sqrt{2u-1}}{|u|} = |x|e^{C_1}$$

$$\frac{2\left(\frac{y}{x}\right)-1}{\left(\frac{y}{x}\right)^2} = x^2e^C$$

$$2y = x(y^2e^C+1)$$

(c) $\boldsymbol{x^2y' = -xy - 2}$

Solution:

$$\frac{dy}{dx} + \frac{1}{x}y = -\frac{2}{x^2}$$

By linear equation of first order

Let $p(y) = \frac{1}{x}$ $Q(x) = -\frac{2}{x^2}$

$y = e^{-\int p(x)\,dx}\left(\int Q(x)e^{\int p(x)dx}\,dx + C\right)$

$= e^{-\int \frac{1}{x}dx}\left(\int -\frac{2}{x^2}e^{\int \frac{1}{x}dx}\,dx + C\right)$

$= \frac{1}{|x|}(-2\ln|x| + C)$

(d) $\boldsymbol{y'' = \sqrt{1-(y')^2}}$

Solution:

Let $y' = p(x)$

$y'' = p' = \sqrt{1-p^2}$

$\frac{dp}{dx} = \sqrt{1-p^2}$

$\frac{dp}{\sqrt{1-p^2}} = dx$

$\int \frac{dp}{\sqrt{1-p^2}} = \int dx$

$\arcsin p = x + C_1$

$\frac{dy}{dx} = p = \sin(x + C_1)$

$\int dy = \int \sin(x + C_1)\,dx$

$y = -\cos(x + C_1) + C_2$

(e) $\boldsymbol{\sqrt{x}y'' = (y')^2}$

Solution:

Let $y' = p(x)$

$\sqrt{x}\frac{dp}{dx} = p^2$

$\int \frac{dp}{p^2} = \int \frac{dx}{\sqrt{x}}$

$\frac{1}{p} = -2\sqrt{x} + C_1$

$\frac{dy}{dx} = \frac{1}{-2\sqrt{x} + C_1}$

$\int dy = \int \frac{1}{-2\sqrt{x} + C_1}\,dx$

$y = -\sqrt{x} - \frac{1}{2}C_1 \ln\left|-2\sqrt{x} + C_1\right| + C_2$

(f) $\boldsymbol{y'' + y' - 6y = 26x}$

Solution:

The characteristic equation is

$r^2 + r - 6 = 0$

So $r_1 = 2 \quad r_2 = -3$

The general solution is

$Y = C_1 e^{2x} + C_2 e^{-3x}$

This is a second order linear homogenous differential equation with constant coefficients

Let $f(x) = 26x \quad \lambda = 0 \quad p_m(x) = 26x$

$\lambda = 0$ is not the root of $\lambda^2 + \lambda - 6 = 0$

Let $y^* = ax + b$ is a particular solution of the original equation

$a - 6(ax + b) = 26x$

$$\begin{cases} -6a = 26 \\ a - 6b = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a = -\dfrac{13}{3} \\ b = -\dfrac{13}{18} \end{cases}$$

$y^* = -\frac{13}{3}x - \frac{13}{18}$

So, the general solution of the original equation is

$y = C_1 e^{2x} + C_2 e^{-3x} - \frac{13}{3}x - \frac{13}{18}$

(g) $\boldsymbol{y'' - 5y' + 4y = e^x}$

Solution:

The characteristic equation is

$r^2 - 5r + 4 = 0$

So $r_1 = 1 \quad r_2 = 4$

The general solution is

$Y = C_1 e^x + C_2 e^{4x}$

This is a second order linear homogenous differential equation with constant coefficients

Let $f(x) = e^x \quad \lambda = 1 \quad p_m(x) = 1$

$\lambda = 1$ is the single root of $\lambda^2 - 5\lambda + 4 = 0$

Let $y^* = axe^x$

$(y^*)' = a(x+1)e^x \quad (y^*)'' = a(x+2)e^x$

$a(x+2)e^x - 5a(x+1)e^x + 4ae^x = e^x$

$\Rightarrow a = -\frac{1}{3}$

$y^* = -\frac{1}{3}xe^x$

So, the general solution of the original equation

$y = C_1e^x + C_2e^{4x} - \frac{1}{3}xe^x$

(h) $\boldsymbol{y'' - 7y' + 12y = 8\sin x + e^{4x}}$

Solution:

The characteristic equation is

$r^2 - 7\mathrm{r} + 12 = 0$

So $r_1 = 3 \quad r_2 = 4$

The general solution is

$Y = C_1e^{3x} + C_2e^{4x}$

This is a second order linear homogenous differential equation with constant coefficients

Let $f_1(x) = e^{4x} \quad (\lambda = 4, p_m(x) = 1)$

$\lambda = 4$ is the single root of $\lambda^2 - 7\lambda + 12 = 0$

So let $y_1 = xae^{4x}$

$(y_1)' = a(4x+1)e^{4x} \quad (y_1)'' = 8a(2x+1)e^{4x}$

$8a(2x+1)e^{4x} - 7a(4x+1)e^{4x} + 12xae^{4x} = e^{4x}$

$\Rightarrow a = 1$

$y_1 = xe^{4x}$

Let $f_2(x) = 8\sin x \quad (\lambda = 0, P_l(\pi) = 0, Q_n(x) = 8, \omega = 1)$

$\lambda = 0$ is not the root of $\lambda^2 - 7\lambda + 12 = 0$

So let $y_2 = \mathrm{a}\sin x + b\cos x$

$(y_2)' = a\cos x - b\sin x \quad (y_2)'' = -a\sin x - b\cos x$

$-a\sin x - b\cos x - 7(a\cos x - b\sin x) + 12(\mathrm{a}\sin x + b\cos x) = 8sinx$

$$\begin{cases} 11b - 7a = 0 \\ 11a + 7b = 8 \end{cases} \Rightarrow \begin{cases} a = \dfrac{44}{85} \\ b = \dfrac{28}{85} \end{cases}$$

So $y_2 = \frac{44}{85}\sin x + \frac{28}{85}\cos x$

So, the general solution of the original operation is

$y = C_1e^{2x} + C_2e^{-3x} - \frac{13}{3}x - \frac{13}{18}$

(i) $\boldsymbol{y' + \frac{y}{x} = y^2}$

Solution:

Both side of the equation are divided by y^2

$$y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = -1$$

Let $z = y^{-1}$

$$\frac{dz}{dx} - \frac{1}{x}z = -1$$

By linear equation of first order

Let $p_{(x)} = -\frac{1}{x}$ $Q(x) = -1$

$$z = e^{-\int -\frac{1}{x}dx}\left(\int -e^{\int -\frac{1}{x}dx}\,dx + C\right)$$

$$= -x\ln x + cx$$

So $y = \frac{1}{-x\ln x + Cx}$

4.2 Task 2

(a) Find an equation of the plane that passes through the points $\boldsymbol{P(1,3,2)}$, $\boldsymbol{Q(3,-1,6)}$ and $\boldsymbol{R(5,2,0)}$, and find the area of the triangle $\boldsymbol{\Delta PQR}$.

Solution:

$\overrightarrow{PQ} = (2,-4,4)$ $\overrightarrow{PR} = (4,-1,-2)$

The normal vector

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\vec{i} + 20\vec{j} + 14\vec{k}$$

So, the equation of the plane is

$$12(x-1) + 20(y-3) + 14(z-2) = 0$$

$$\Rightarrow 6x + 10y + 7z - 50 = 0$$

The area of the triangle

$$S = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \sqrt{185}$$

(b) Find the equation of the plane that passes through the point $(1,2,3)$ and contains the line $\boldsymbol{x = 3t,\ y = 1 + t,\ z = 2 - t}$.

Solution:

According to the parametric equation of the line

Let $t=0$ $P_1(0,1,2)$

Let $t=1$ $P_2(3,2,1)$

P_1, P_2 are on the line

Let Q is the point (1,2,3)

$\overrightarrow{QP_1}=(-1,-1,-1)$ $\overrightarrow{QP_2}=(2,0,-2)$

The normal vector

$$\vec{n}=\overrightarrow{QP_1}\times\overrightarrow{QP_2}=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ -1 & -1 & -1\\ 2 & 0 & -2\end{vmatrix}=2\vec{\imath}-4\vec{\jmath}+2\vec{k}$$

So, the equation of the plane is

$2(x-1)-4(y-2)+2(z-3)=0$

$\Rightarrow x-2y+z=0$

(c) Find the distance from the point $\boldsymbol{P(1,2,3)}$ to the plane $\boldsymbol{x+6y+4z=3}$ and find the distance from the point $\boldsymbol{Q(4,1,-2)}$ to the line $\boldsymbol{\frac{x-1}{1}=\frac{y-3}{-2}=\frac{z-4}{-3}}$.

Solution:

$$d=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}=\frac{|1\times1+2\times6+3\times4-3|}{\sqrt{1^2+6^2+4^2}}=\frac{22}{\sqrt{53}}$$

For the line $\frac{x-1}{1}=\frac{y-3}{-2}=\frac{z-4}{-3}$

The direction vector $\vec{s}=(1,-2,-3)$

So, the parametric equation form of straight line

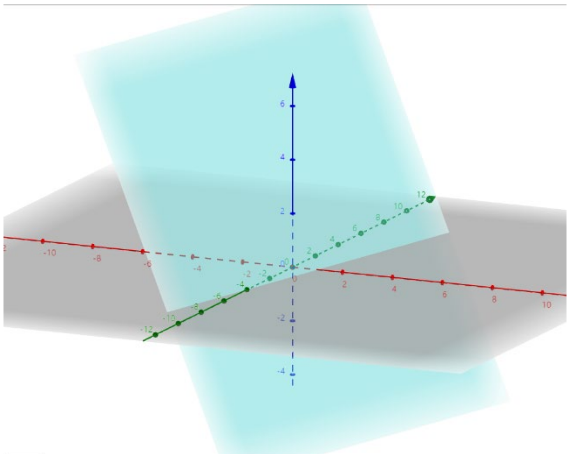
$$\begin{cases} x=t+1\\ y=-2t+3\\ z=-3t+4\end{cases}$$

Let $t=0$ $M(1,3,4)$ $\overrightarrow{QM}=(-3,2,6)$

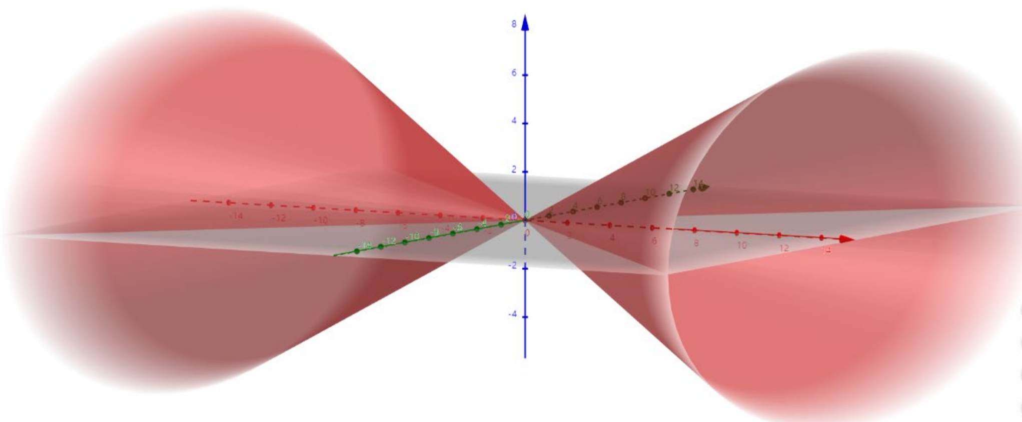
$$d=\frac{|\overrightarrow{QM}\times\vec{s}|}{|\vec{s}|}=\sqrt{\frac{61}{14}}$$

4.3 Task 3

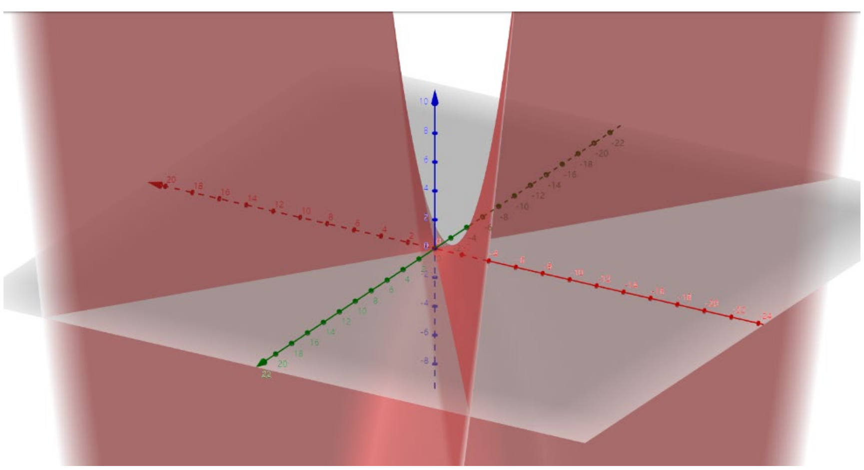
(a) $4x-y+2z=4$



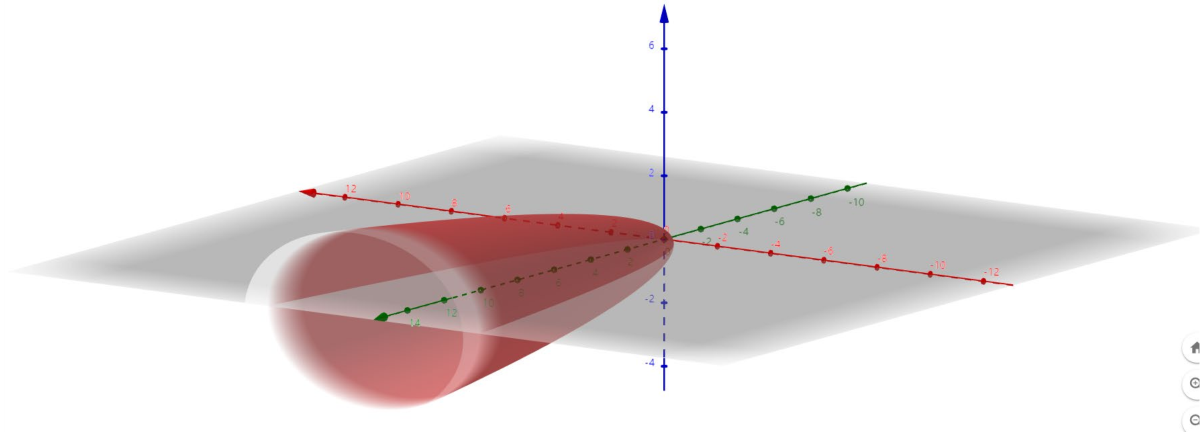
(b) $x^2 = y^2 + 4z^2$



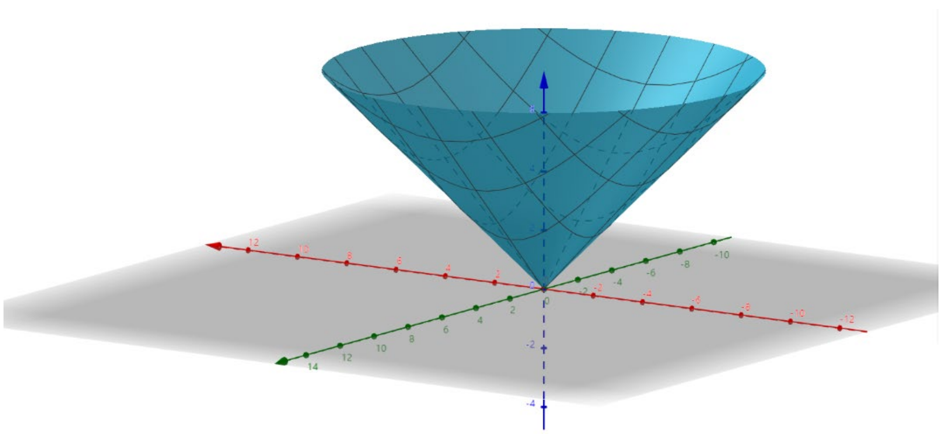
(c) $\frac{x^2}{2} - \frac{y^2}{4} = 2z$



(d) $y = x^2 + 2z^2$



(e) $z = \sqrt{x^2 + y^2}$



4.4 Task4

(a) If $f(x,y) = \frac{xy^2}{x^2+y^4}$, does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist? Explain your reason.

Solution:

When going to (0,0) along with y=x

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{xy^2}{x^2+y^4} = \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+x^4} = 0$$

When going to (0,0) along with $x = y^2$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{xy^2}{x^2+y^4} = \lim_{(x,y)\to(0,0)} = \frac{y^4}{2y^4} = \frac{1}{2}$$

The two limits are not equal

So, the $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(b) If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x\,\partial y}$

Solution:

$$\frac{\partial f}{\partial x} = 3x^2 + 2xy^3$$

$$\frac{\partial f}{\partial y} = 3x^2y^2 - 4y$$

$$\frac{\partial^2 f}{\partial x^2} = 6x + 2y^3$$

$$\frac{\partial^2 f}{\partial y^2} = 6x^2y - 4$$

$$\frac{\partial^2 f}{\partial x\, \partial y} = 6xy^2$$

5 Conclusion

5.1 Differential equation

(1) Differential equation with variables separable

If a first order differential equation can be denoted by

$$g(y)dy = f(x)dx$$

Then the original function is called **differential equation with variables separable**.

Example:

$y' = 2xy$

$3x^2 + 5x - y' = 0$

$y' = 10^{x+y}$

are differential equation with variables separable, and

$(x^2 + y^2)dx - xydy = 0$

$y' = \frac{x}{y} + \frac{y}{x}$

are not differential equation with variable separate, because they cannot separate dy and dx

(2) Humongous equation

If a first order equation can be reduced to

$$\frac{dy}{dx} = \varphi(\frac{y}{x}).$$

It is called homogeneous equation.

Example:

$$(xy - y^2)dx - (x^2 - 2xy)dy = 0$$

is a humongous equation.

Problem solving method:

Let $\frac{y}{x} = u$

$$\frac{dy}{dx} = u + \frac{xdu}{dx}$$

We obtain

$$u + \frac{xdu}{dx} = \varphi(u)$$

Both sides integral

$$\int \frac{du}{\varphi(u) - u} = \int \frac{dx}{x}$$

(3) Linear equation of first order

A differential equation of type

$$\frac{dy}{dx} + p(x)y = Q(x)$$

where P(x) and Q(x) are continuous functions of x, is called a linear differential equation of first order.

Problem solve method:

$$y = e^{-\int p(x)\,dx}\left(\int Q(x)e^{\int p(x)dx}\,dx + C\right)$$

(4) Bernoulli equation

The equation

$$\frac{dy}{dx} + p(x)y = Q(x)y^n \; (n \neq 0,1)$$

is called Bernoulli equation.

Example:

$$\frac{dy}{dx} + \frac{y}{x} = a(lnx)y^2$$

(5) Reducible higher order differential equations

Case 1: $\boldsymbol{y^{(n)} = f(x)}$

$$y^{(n-1)} = \int f(x)dx + C_1$$
$$y^{(n-2)} = \int \left[\int f(x)dx + C_1\right]dx + C_2$$
$$\dots\dots$$

Example:

$$y''' = e^{2x} - cosx$$

Case 2: $\boldsymbol{y'' = f(x, y')}$

Let $y' = p(x)$,

$$y'' = p' = \frac{dp}{dx} = f(x, p)$$
$$p(x) = \varphi(x, C_1)$$
$$\frac{dy}{dx} = p(x) = \varphi(x, C_1)$$
$$y(x) = \int \varphi(x, C_1) + C$$

Example:

$$(1 + x^2)y'' = 2xy'$$

Case 3: $\boldsymbol{y'' = f(y, y')}$

Let $y' = p(y)$

$$y'' = \frac{d(y')}{dx} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = \frac{dp}{dy}p$$
$$p\frac{dp}{dy} = f(y, p)$$
$$y' = p(y) = \varphi(y, C_1)$$
$$\int \frac{1}{\varphi(y, C_1)}dy = x + C_2$$

Example:

$$yy'' - y'^2 = 0$$

(6) Structure of solutions of linear differential equations:

$$y'' + P(x)y' + Q(x)y = 0. \qquad (1)$$

If $y_1(x)$ and $y_2(x)$ are two solutions of (1), then

$$y = C_1 y_1(x) + C_2 y_2(x)$$

Let $y^*(x)$is a particular solution of Second order nonhomogeneous linear equation, Y(x) is a general solution of corresponding homogeneous equations.

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y = Y(x) + y^*(x)$$

(7) Second Order Linear Homogeneous Differential Equations with Constant Coefficients:

For each of the equation$y'' + py' + qy = 0$, we can write the so-called characteristic (auxiliary) equation:

$$r^2 + pr + q = 0$$

The general solution of the homogeneous differential equation depends on the roots of the characteristic quadratic equation.

If $\boldsymbol{D = p^2 - 4q > 0}$

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

If $\boldsymbol{D = p^2 - 4q = 0}$

$$y(x) = (C_1 x + C_2)e^{r_1 x}$$

If $\boldsymbol{D = p^2 - 4q < 0}$

$$y(x) = e^{\alpha x}[C_1 cos\,(\beta x) + C_2 sin\,(\beta x)]$$

8) Second Order Linear Nonhomogeneous Differential Equations with Constant Coefficients:

The nonhomogeneous differential equation of this type has the form

$$y'' + py' + qy = f(x)$$

Case1: $\boldsymbol{f(x) = P_m(x)e^{\lambda x}}$

$$y^* = x^k R_m(x) e^{\lambda x}$$

Case2: $\boldsymbol{f(x) = e^{\lambda x}[P_m(x)\cos(\beta x) + Q_n(x)\sin(\beta x)]}$

$$y^* = \lambda^k e^{\lambda x}\left[R_m^{(1)}(x)\cos\omega x + R_m^{(2)}(x)\sin\omega x\right]$$

Example:

$$y'' + 5y' + 6y = xe^{2x}$$

The particular solution $y*$ of this equation is in case 1, and $(\lambda = 2, P_m(x) = x)$

$$y'' + y = xcos2x$$

The particular solution $y*$ of this equation is in case 2, and $(\lambda = 0, \omega = 2, P_l(x) = x, Q_n(x) = 0)$

5.2 Vector space

(1) Dot product

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$$

(2) Cross product

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| sin\theta$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Example:

$$\boldsymbol{a} = (2,1,-1) \quad \boldsymbol{b} = (1,-1,2)$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = 2 \times 1 - 1 \times 1 - 1 \times 2 = -1$$

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix} = \boldsymbol{i} - 5\boldsymbol{j} - 3\boldsymbol{k}$$

(3) Mixed product

$$[\vec{a} \ \ \vec{b} \ \ \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

(4) **Point-normal form of the equation of a plane**

Normal vector is $\vec{n} = (A, B, C)$ and passes through the point $M_0(x_0, y_0, z_0)$.

The plane equation is

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Example:

Through the point (2, -3,0) and take $\boldsymbol{n} = (1,-2,3)$ as a normal vector

The plane equation is

$$(x - 2) - 2(y + 3) + 3z = 0$$

(5) General form of the equation of a plane:

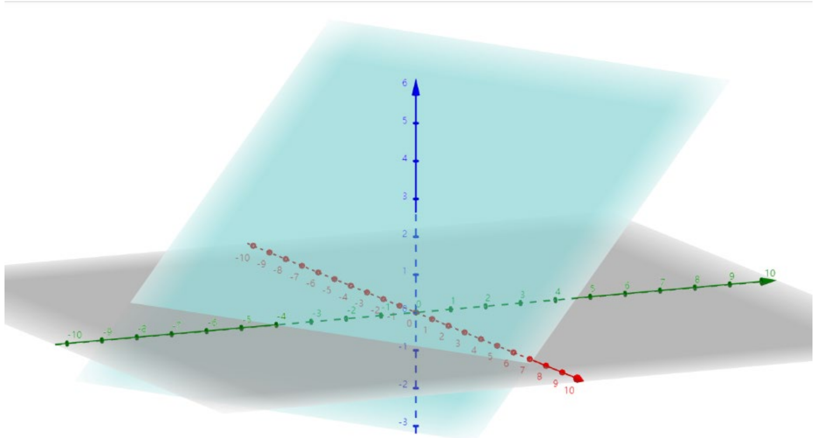
$$Ax + By + Cz + D = 0$$

Example:

For the example in (4), the general form of the equation of a plane is

$$x - 2y + 3z - 8 = 0$$

The image of the plane is



(6) Symmetric form equations of a straight line:

The equation of the line L that passes through the point P_0 and is parallel to the vector $\vec{a} = (l, m, \mathrm{n})$ is

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$$

(7) Parametric equations of a straight line

$$\begin{cases} x = x_0 + lt, \\ y = y_0 + mt, \ (t \in R) \\ z = z_0 + nt. \end{cases}$$

(8) Distance from a point to a plane

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

(9) Distance from a point to a line in space

$$d = \frac{\overrightarrow{PS} \times \boldsymbol{v}}{\boldsymbol{v}}$$

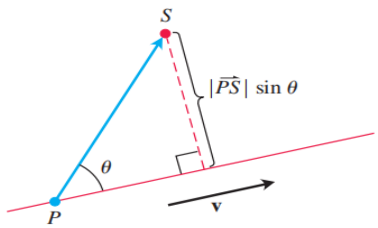


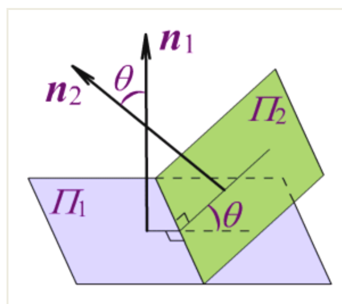
FIGURE 12.38 The distance from S to the line through P parallel to $\mathbf{v}$ is $|\overrightarrow{PS}|\sin\theta$, where θ is the angle between $\overrightarrow{PS}$ and $\mathbf{v}$.

(10) Angle between two planes

There are two normal vectors $\overrightarrow{n_1}, \overrightarrow{n_2}$.

$$\vec{n}_1 = (A_1, B_1, C_1), \vec{n}_2 = (A_2, B_2, C_2)$$

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|} = \frac{|A_1A_2 + B_1B_2 + C_1C_2|}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}$$



5.3 Multivariable function

(1) Methods on finding the double limit

- Rationalization
- Properties of limit
- Squeezing
- Using formulas
- Equivalent to replace
- Replace overall

(2) Method of Judging the double limit does not exist

- **It does not exist along some curve**

  Example:

$\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x-y}{x+y}$ It does not exist along $y=-x$

- **It is not the same along different curves**

  Example:

  $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x-y}{x+y}$ It is 1 along $y=0$, and is -1 along $x=0$.

- **Both two second limits exist, but they are not equal.**

  Example:

  $$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x-y}{x+y}$$

  $$\lim_{x\to 0}\lim_{y\to 0} \frac{x-y}{x+y} = \mathop{lim}_{x\to 0}\frac{x-0}{x+0} = 1,$$

  $$\lim_{y\to 0}\lim_{x\to 0} \frac{x-y}{x+y} = \mathop{lim}_{x\to 0}\frac{0-y}{0+y} = -1$$

(3) Partial derivate

$$\frac{\partial}{\partial x}f(x,y) = f_x(x,y) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$$
$$\frac{\partial}{\partial y}f(x,y) = f_y(x,y) = \mathop{lim}_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$$

(4) Necessary conditions for differentiability:

Suppose that a function $z=f\,(x,\ y)$ is differentiable at a point (x_0,y_0), then

(a) f must be continuous at the point (x_0,y_0);
(b) both partial derivatives of function f at the point (x_0,y_0) exist and $A=f_x(x_0,y_0), B=f_y(x_0,y_0)$, that is

$$dz|_{(x_0,y_0)} = f_x(x_0,y_0)dx + f_y(x_0,y_0)dy$$

Example:

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0\end{cases}$$

First, we known that f is continuous at (0, 0).

Second, from

$$f(0+\Delta x,0)-f(0,0)=0, f(0,0+\Delta y)-f(0,0)=0,$$

we have

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=0,$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0,0+\Delta y) - f(0,0)}{\Delta y} = 0.$$

Both the partial differentials of f exist.

If f is differentiable at (0,0), then we have

$$\Delta f = f_x(0,0)\Delta x + f_y(0,0)\Delta y + o(\rho) = o(\rho).$$

$$\Rightarrow \lim_{\rho \to 0} \frac{\Delta f}{\rho} = \lim_{\rho \to 0} \frac{o(\rho)}{\rho} = 0.$$

But in fact, we have

$$\Delta f = f(0+\Delta x, 0+\Delta y) - f(0,0) = \frac{\Delta x \Delta y}{\sqrt{\Delta x^2 + \Delta y^2}}$$

If we consider that $P'(\Delta x, \Delta y)$ is going to (0,0) along with $y = x$, then

$$\lim_{\rho \to 0} \frac{\Delta f}{\rho} = \lim_{\rho \to 0} \frac{\Delta x \Delta y}{\Delta x^2 + \Delta y^2} \overset{\Delta y = \Delta x}{=} \lim_{\Delta x \to 0} \frac{\Delta x^2}{\Delta x^2 + \Delta x^2} = \frac{1}{2} \neq 0.$$

Actually, $\lim_{\rho \to 0} \frac{\Delta x \Delta y}{\Delta x^2 + \Delta y^2}$ does not exist.

Thus, the function f is not differentiable at the point $(0,\ 0)$.

(5) Sufficient condition for differentiability:

If the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ of a function $z = f(x,y)$, both exist and are continuous in a neighborhood of the point, (x_0, y_0), then the function f is differentiable at the point (x_0, y_0).

(6) Partial derivatives of multivariable composite function

Chain rule:

If $z = f(u(x,\ y),\ v(x,\ y))$, then the following chain rule holds:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

Case 1:

$$z = f(u,v), u = \varphi(t), v = \Psi(t)$$

$$\frac{dz}{dx} = \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt}$$

Case 2:

$$z = f(u,v) = f[\varphi(x,y), \Psi(x,y)]$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

Case 3:

$$z=f[\varphi(x,y),\Psi(y)], u=\varphi(x,y), v=\Psi(y)$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

(7) Invariance of the (first order) total differential form

Let $\mathrm{z}=\mathrm{f}(\mathrm{u},\mathrm{v}), \mathrm{u}=\mathrm{u}(\mathrm{x},\mathrm{y}), \mathrm{v}=\mathrm{v}(\mathrm{x},\mathrm{y})$, if we regard u,v as directly variables, we have

$$dz=\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv$$

If we regard u, v as intermediate variables, we find

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$$

(8) Implicit Differentiation (one function)

- **Existence of an implicit function:**

Suppose that the function $F(x,\ y)$ of two variables has the properties:

(a)$F(x_0,y_0)=0$;

(b) Both partial derivatives of the function F are continuous in a neighborhood of the point (x_0,y_0);

(c) $F_y(x_0,y_0)\neq 0$.

Then

(a) There exists one and only one function $y=f(x)$ determined by $F(x,y)=0$ in the $U(x_0,y_0)$, such that $y_0=f(x_0)$ and $F[x,f(x)]\equiv 0$;

(b) the function $y=f\ (x)$ has a continuous derivative in $U(x_0,y_0)$, And

$$\frac{dy}{dx}=-\frac{F_x}{F_y}$$

- **Existence of multivariable implicit function:**

If $F(x,y,z)$ has continuous partial derivatives in some neighborhood of (x_0,y_0,z_0), and $F(x_0,y_0,z_0)=0, F_z(x_0,y_0,z_0)\neq 0$, then $F(x,y,z)=0$ determines a unique continuous function $z=f(x,y)$ in the neighborhood of (x_0,y_0,z_0) whose partial derivatives are continuous and $z_0=f(x_0,y_0)$. Moreover

$$\Rightarrow F_x+F_z\cdot\frac{\partial z}{\partial x}=0\Rightarrow\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}$$

Similarly, we have

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

(9) Implicit Differentiation (more than one equation)

Suppose that the system of two equations of functions

$$\begin{cases} F(x, y, u, v) = 0, \\ G(x, y, u, v) = 0. \end{cases}$$

has determined two functions of two variables with continuous partial derivatives

$$u = u(x, y), v = v(x, y)$$

So that

$$\begin{cases} F(x, y, u(x, y), v(x, y)) \equiv 0, \\ G(x, y, u(x, y), v(x, y)) \equiv 0. \end{cases}$$

To get $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ we differentiate both sides of the two identities with respect to x

$$\begin{cases} F(x, y, u(x, y), v(x, y)) \equiv 0, \\ G(x, y, u(x, y), v(x, y)) \equiv 0. \end{cases}$$

$$\Rightarrow \begin{cases} F_x + F_u \cdot \dfrac{\partial u}{\partial x} + F_v \cdot \dfrac{\partial v}{\partial x} = 0, \\ G_x + G_u \cdot \dfrac{\partial u}{\partial x} + G_v \cdot \dfrac{\partial v}{\partial x} = 0. \end{cases}$$

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -F_x & F_v \\ -G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}$$

$$\frac{\partial v}{\partial x} = \frac{\begin{vmatrix} F_u & -F_x \\ G_u & -G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}$$

(10) The limit of vector-valued function

If $\boldsymbol{f}(t) = f_1(t)\boldsymbol{i} + f_2(t)\boldsymbol{j} + f_3(t)\boldsymbol{k}$, then the limit $\lim\limits_{t \to t_0} \boldsymbol{f}(t)$ exists if and only if all of the limits $\lim\limits_{t \to t_0} f_1(t)$, $\lim\limits_{t \to t_0} f_2(t)$, $\lim\limits_{t \to t_0} f_3(t)$ exist, and

$$\lim_{t \to t_0} \boldsymbol{f}(t) = \lim_{t \to t_0} f_1(t)\boldsymbol{i} + \lim_{t \to t_0} f_2(t)\boldsymbol{j} + \lim_{t \to t_0} f_3(t)\boldsymbol{k}$$

(11) The derivate of vector-valued function:

If $\boldsymbol{f}(t) = f_1(t)\boldsymbol{i} + f_2(t)\boldsymbol{j} + f_3(t)\boldsymbol{k}$, then $\boldsymbol{f}(t)$ is differentiable if and only if $f_1(t)$, $f_2(t)$, $f_3(t)$ are differentiable and

$$\boldsymbol{f}'(t) = f'_1(t)\boldsymbol{i} + f'_2(t)\boldsymbol{j} + f'_3(t)\boldsymbol{k}$$

(12) Tangent line and normal plane to a space curve

- **Parametric equations of curves**

$$\begin{cases} x = \varphi(t) \\ y = \Psi(t) \\ z = \omega(t) \end{cases}$$

- **Tangent equation**

$$\frac{x - x_0}{\varphi'(t_0)} = \frac{y - y_0}{\Psi'(t_0)} = \frac{z - z_0}{\omega'(t_0)}$$

- **Normal line equation**

$$\varphi'(t_0)(x - x_0) + \Psi'(t_0)(y - y_0) + \omega'(t_0)(z - z_0) = 0$$

(13) Tangent planes and normal lines of surfaces

- **Tangent plane equation**

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

- **Normal line equation**

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

(14) Directional derivatives

Suppose that the function $z = f(x, y)$ is differentiable at the point (x_0, y_0), then the directional derivative in any direction $\boldsymbol{l}$ at the point (x_0, y_0) exists and

$$\frac{\partial f}{\partial l}|_{(x_0, y_0)} = f_x(x_0, y_0)\cos\alpha + f_y(x_0, y_0)\cos\beta,$$

where $\boldsymbol{e}_l = (\cos\alpha, \cos\beta)$ is a unit vector in the direction $\boldsymbol{l}$, α, β are the direction angles of $\boldsymbol{l}$.

(15) Gradient

Denoted by

$$\mathbf{grad}\, f(x_0, y_0) \text{ or } \nabla f(x_0, y_0)$$

$$\mathbf{grad}\, f(x_0, y_0) = f_x(x_0, y_0)\boldsymbol{i} + f_y(x_0, y_0)\boldsymbol{j}$$

(16) Necessary condition for an extreme value

If (x_0, y_0) is an extreme point of $z = f(x, y)$, and $f_x(x_0, y_0), f_y(x_0, y_0)$ exist, then$f_x(x_0, y_0) = 0, f_y(x_0, y_0) = 0$.

(17) Sufficient condition for an extreme value

$$f_x(x_0, y_0) = 0,\ f_y(x_0, y_0) = 0$$

$P_0(x_0, y_0)$ is a stationary point of f. Let

$$f_{xx}(x_0, y_0) = A, f_{xy}(x_0, y_0) = B, f_{yy}(x_0, y_0) = C$$

- If $AC - B^2 > 0$ and $A > 0$, then $P_0(x_0, y_0)$ is minimum value.
- If $AC - B^2 > 0$ and $A < 0$, then $P_0(x_0, y_0)$ is maximum value.
- If $AC - B^2 < 0$, $P_0(x_0, y_0)$ is not an extreme point, this point is a saddle point.
- If $AC - B^2 = 0$, that is we cannot determine whether$P_0(x_0, y_0)$ is an extreme value of f or not.

(18) **Method of Lagrange multipliers**

The objective function

$$z = f(x, y)$$

In the condition

$$\varphi(x, y) = 0$$

$$L(x, y, \lambda) = f(x, y) + \lambda\varphi(x, y)$$

Set its partial derivatives equal to zero

$$\begin{cases} L_x(x_0, y_0, \lambda_0) = f_x(x_0, y_0) + \lambda_0\varphi_x(x_0, y_0) = 0, \\ L_y(x_0, y_0, \lambda_0) = f_y(x_0, y_0) + \lambda_0\varphi_y(x_0, y_0) = 0, \\ L_\lambda(x_0, y_0, \lambda_0) = \varphi(x_0, y_0) = 0. \end{cases}$$

Solving the equations for the roots (x_0, y_0, λ_0).

Determine whether point (x_0, y_0) is the required extreme point or not. This method is called the method of Lagrange multipliers.