Euler Method + Runge Kutta Methods + Newton-Raphson &

$$\frac{dx}{dt} = v \quad , \quad \frac{dv}{dt} = a$$

1st order linear approximation → Euler method

- $\Delta x = v\Delta t$
- $\Delta v = a\Delta t$

, in general $a = a(v, x)$

explicit: $\begin{cases} x_{i+1} = x_i + v_i \Delta t \\ v_{i+1} = v_i + a_i \Delta t \end{cases}$

Symplectic $\begin{cases} x_{i+1} = x_i + v_{i+1}\Delta t & \leftarrow \text{second} \\ v_{i+1} = v_i + a_i \Delta t & \leftarrow \text{first} \end{cases}$

implicit: $\begin{cases} x_{i+1} = x_i + v_{i+1} \Delta t \\ v_{i+1} = v_i + a_{i+1} \Delta t \end{cases}$

If $a(v,x)$ is linear, then the implicit method can be solved as a linear system of equations. (however most are non-linear ⇒ explicit used more often)

(set $-k/m = 1$)

Example: Harmonic Oscillator, $F = -kx = ma \Rightarrow a = -x$

explicit: $\begin{cases} x_{i+1} = x_i + v_i \Delta t \\ v_{i+1} = v_i - x_i \Delta t \end{cases}$

$$\begin{bmatrix} 1 & -\Delta t \\ \Delta t & 1 \end{bmatrix} \begin{pmatrix} x_{i+1} \\ v_{i+1} \end{pmatrix} = \begin{pmatrix} x_i \\ v_i \end{pmatrix}$$

⇑ Solve

implicit: $\begin{cases} x_{i+1} = x_i + v_{i+1} \Delta t \\ v_{i+1} = v_i - x_{i+1} \Delta t \end{cases} \Leftrightarrow \begin{matrix} x_{i+1} - v_{i+1}\Delta t = x_i \\ v_{i+1} + x_{i+1}\Delta t = v_i \end{matrix}$

$$\begin{pmatrix} x_{i+1} \\ v_{i+1} \end{pmatrix} = \frac{1}{\Delta t^2 + 1} \begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1 \end{bmatrix} \begin{pmatrix} x_i \\ v_i \end{pmatrix}$$

Euler Method for Pendulum

$$\frac{d\theta}{dt} = \frac{P_\theta}{ml} \; ; \; \frac{dP}{dt} = -\sin\theta$$

$$\boxed{\begin{array}{ll} \dot{x} = v & \dot{\theta} = \frac{P_\theta}{ml} \\ \dot{v} = a & \dot{P}_\theta = -mgl\sin\theta \end{array}}$$

$$\Delta\theta = P\Delta t$$

$$\Delta P = -\sin\theta\,\Delta t$$

Explicit

$$\theta_{i+1} = \theta_i + P_i\,\Delta t$$

$$P_{i+1} = P_i + (-\sin\theta_i)\,\Delta t$$

Implicit

$$\theta_{i+1} = \theta_i + P_{i+1}\,\Delta t \quad (1)$$

$$P_{i+1} = P_i + (-\sin\theta_{i+1})\,\Delta t \quad (2)$$

$$\Rightarrow \begin{cases} \theta_{i+1} - P_{i+1}\Delta t = \theta_i \\ P_{i+1} + \sin\theta_{i+1}\Delta t = P_i \end{cases}$$

$$\rightarrow \begin{bmatrix} 1 & -\Delta t \\ s & \end{bmatrix}$$

(2) → (1):

$$\theta_{i+1} = \theta_i + (P_i + \sin(\theta_{i+1})\Delta t)\Delta t$$

$$\boxed{0 = \theta_i - \theta_{i+1} + (P_i - \sin(\theta_{i+1})\Delta t)\Delta t}$$

Kør rootfinding for θ_{i+1}, start: θ_i

$$\boxed{\begin{array}{l} f(\theta_1) = \theta_0 - \theta_1 + (P_0 - \sin\theta_1 \cdot \Delta t)\Delta t \\ f'(\theta_1) = -1 - \cos\theta_1 \cdot \Delta t^2 \end{array}}$$

Symplectic

$$\theta_{i+1} = \theta_i + P_{i+1}\Delta t \longleftarrow \text{second}$$

$$P_{i+1} = P_i - \sin\theta_i\,\Delta t \longleftarrow \text{first}$$

General Runge-Kutta

$$y_{i+1} = y_i + \phi h$$

$\boxed{k_i = \text{slope estimate}}$

$$\phi(x_i, y_i, h) = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h \;,\; y_i + q_{11} k_1 h)$$

$$k_3 = f(x_i + p_2 h \;,\; y_i + q_{21} k_1 h + q_{22} k_2 h)$$

$$\vdots$$

$$k_n = f\left(x_i + p_{n-1} h \;,\; y_i + q_{n-1,1} k_1 + q_{n-1,2} k_2 h + \cdots + q_{n-1,n-1} k_{n-1} h\right)$$

2nd Order Runge-Kutta Methods

$$y_{i+1} = y_i + \phi h$$

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h = \text{Taylor Series Expansion}$$

where $k_1 = f(x_i, y_i)$

$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$

$\Downarrow$

$a_1 + a_2 = 1$

$a_2 p_1 = \frac{1}{2}$

$a_2 q_{11} = \frac{1}{2}$

3 eqns
4 unknowns : a_1, a_2, p_1, q_{11}

Newton-Raphson Root finding Method

$f(x) \approx f(x_0) + f'(x_0)(\underset{\downarrow\, x_{i+1}}{x} - \underset{\downarrow\, x_i}{x_0})$, set $f(x) = 0$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

x_i = initial guess

x_{i+1} = next guess