APPENDIX

Gravitational Potential of a Sphere

E

Figure E.1 shows a point mass m with Cartesian coordinates (x, y, z) as well a system of N point masses m_1, m_2, m_3,… , m_N. The ith one of these particles has mass m_i and coordinates (x_i, y_i, z_i). The total mass of the N particles is M,

$$M = \sum_{i=1}^{N} m_i \tag{E.1}$$

The position vector drawn from m_i to m is $\mathbf{r}_i$ and the unit vector in the direction of $\mathbf{r}_i$ is

$$\hat{\mathbf{u}}_i = \frac{\mathbf{r}_i}{r_i}$$

The gravitational force exerted on m by m_i is opposite in direction to $\mathbf{r}_i$, and is given by

$$\mathbf{F}_i = -\frac{Gmm_i}{r_i^2}\hat{\mathbf{u}}_i = -\frac{Gmm_i}{r_i^3}\mathbf{r}_i$$

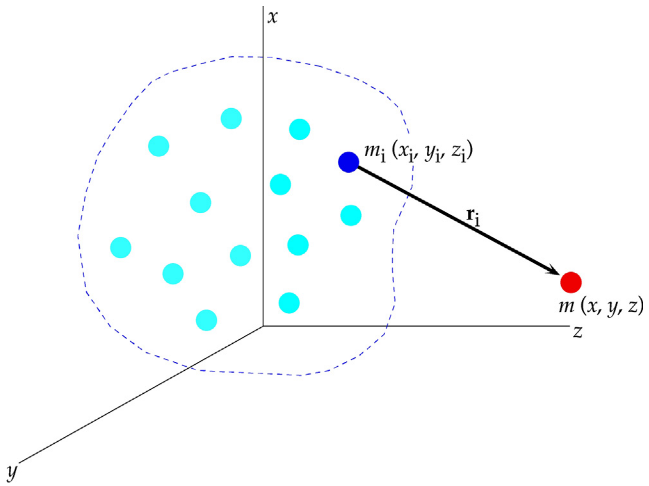

FIGURE E.1

A System of Point Masses and a Neighboring Test Mass *m*.

The potential energy of this force is

$$V_i = -G\frac{mm_i}{r_i} \tag{E.2}$$

The total gravitational potential energy of the system due to the gravitational attraction of all of the N particles is

$$V = \sum_{i=1}^{N} V_i \tag{E.3}$$

Therefore, the total force of gravity $\mathbf{F}$ on the mass m is

$$\mathbf{F} = -\nabla V = -\left(\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right) \tag{E.4}$$

Consider the solid sphere of mass M and radius R_0 illustrated in Figure E.2. Instead of a discrete system as above, we have a continuum with mass density ρ. Each "particle" is a differential element $\mathrm{d}M = \rho \mathrm{d}v$ of the total mass M. Equation (E.1) becomes

$$M = \iiint\limits_{v} \rho \mathrm{d}v \tag{E.5}$$

where dv is the volume element and v is the total volume of the sphere. In this case, Eqn (E.2) becomes

$$\mathrm{d}V = -G\frac{m\mathrm{d}M}{r} = -Gm\frac{\rho \mathrm{d}v}{r}$$

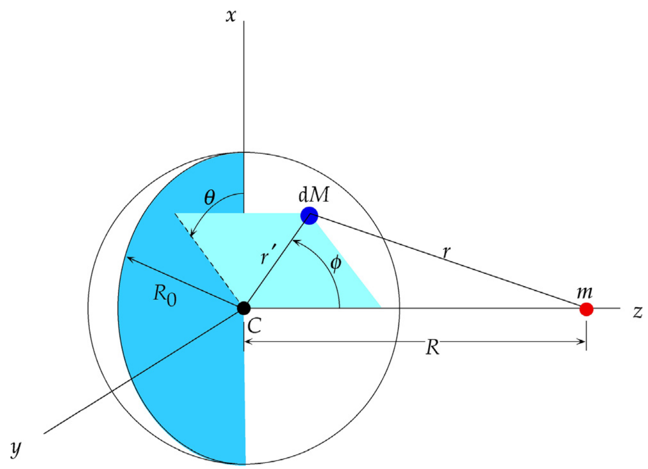

FIGURE E.2

Sphere with a Spherically Symmetric Mass Distribution.

where r is the distance from the differential mass $\mathrm{d}M$ to the finite point mass m. Equation (E.3) is replaced by

$$V = -Gm \iiint_v \frac{\rho \mathrm{d}v}{r} \tag{E.6}$$

Let the mass of the sphere have a spherically symmetric distribution, which means that the mass density ρ depends only on r', the distance from the center C of the sphere. An element of mass $\mathrm{d}M$ has spherical coordinates (r', θ, ϕ), where the angle θ is measured in the xy plane of a Cartesian coordinate system with origin at C, as shown in Figure E.2. In spherical coordinates the volume element is

$$\mathrm{d}v = r'^2 \sin\phi \mathrm{d}\phi \mathrm{d}r' \mathrm{d}\theta \tag{E.7}$$

Therefore Eqn (E.5) becomes

$$M = \int_{\theta=0}^{2\pi} \int_{r'=0}^{R_0} \int_{\phi=0}^{\pi} \rho r'^2 \sin\phi \mathrm{d}\phi \mathrm{d}r' \mathrm{d}\theta = \left(\int_0^{2\pi} \mathrm{d}\theta \right) \left(\int_0^{\pi} \sin\phi \mathrm{d}\phi \right) \left(\int_0^{R_0} \rho r'^2 \mathrm{d}r' \right)$$
$$= (2\pi)(2) \left(\int_0^{R_0} \rho r'^2 \mathrm{d}r' \right)$$

so that the mass of the sphere is given by

$$M = 4\pi \int_{r'=0}^{R_0} \rho r'^2 \mathrm{d}r' \tag{E.8}$$

Substituting Eqn (E.7) into Eqn (E.6) yields

$$V = -Gm \int_{\theta=0}^{2\pi} \int_{r'=0}^{R_0} \int_{\phi=0}^{\pi} \frac{\rho r'^2 \sin\phi \mathrm{d}\phi \mathrm{d}r' \mathrm{d}\theta}{r} = -2\pi Gm \left[\int_0^{R_0} \left(\int_0^{\pi} \frac{\sin\phi \mathrm{d}\phi}{r} \right) \rho r'^2 \mathrm{d}r' \right] \tag{E.9}$$

The distance r is found by using the law of cosines,

$$r = \left(R^2 + r'^2 - 2r'R\cos\phi \right)^{\frac{1}{2}}$$

where R is the distance from the center of the sphere to the mass m. Differentiating this equation with respect to ϕ, holding r' constant, yields

$$\frac{\mathrm{d}r}{\mathrm{d}\phi} = \frac{1}{2}\left(R^2 + r'^2 - 2r'R\cos\phi \right)^{-\frac{1}{2}} (2r'R\sin\phi \mathrm{d}\phi) = \frac{r'R\sin\phi}{r}$$

so that

$$\sin\phi \mathrm{d}\phi = \frac{r\mathrm{d}r}{r'R}$$

It follows that

$$\int_{\phi=0}^{\pi} \frac{\sin\phi \mathrm{d}\phi}{r} = \frac{1}{r'R} \int_{R-r'}^{R+r'} \mathrm{d}r = \frac{2}{R}$$

Substituting this result along with Eqn (E.8) into Eqn (E.9) yields

$$V = -\frac{GMm}{R} \tag{E.10}$$

We conclude that the gravitational potential energy, and hence (from Eqn (E.4)) the gravitational force, of a sphere with a spherically symmetric mass distribution M is the same as that of a point mass M located at the center of the sphere.