

A few Concepts from Analytical Mechanics

LECTURE NOTE

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This note summarizes a few important concepts relevant to the dynamical systems that arise in analytical mechanics. We begin with Lagrangian mechanics, and proceed to a brief discussion of Hamiltonian mechanics.

1. GENERALISED COORDINATES

Given a system of point masses or solid bodies moving in \mathbb{R}^3 , the first important step in the mathematical analysis is to establish so called 'generalised coordinates', or 'degrees of freedom' for the system. These are the (smallest number of) coordinates (or geometrical quantities interpreted as coordinates) needed to specify uniquely the positional state of the system. For instance, a pendulum, even though it is moving in a plane, needs only one coordinate (the angle) to specify the configuration. Two point masses at each end of a rigid rod (a dumbbell) moving in a plane requires three coordinates to specify its configuration: two to locate one point on the rod, and one number giving the angle of the rod relative to a reference line in the plane.

Although the number n of generalised coordinates is unique for a given system; there is often a choice between several kinds of coordinates; for instance, position in a plane may be specified either by cartesian or by polar coordinates. As we shall see later, a good choice of coordinates can sometimes be useful in the further process of solving the equations of motion.

EXERCISE 1. Establish generalised coordinates for

- (1) A dumbbell in \mathbb{R}^3
- (2) A point mass moving on a helix in \mathbb{R}^3 .
- (3) Two masses in \mathbb{R}^2 connected with a spring.
- (4) N point masses moving in \mathbb{R}^3 .

▽

In general we denote the chosen generalised coordinates by $q_1 \dots q_n$. We count the number n of generalised coordinates and say that the mechanical system has n degrees of freedom.

For a system of N point masses moving in \mathbb{R}^3 and forming a system with n degrees of freedom we will have $3N$ equations

$$\begin{aligned} x_1 &= x_1(q_1 \dots q_n) \\ &\vdots \\ z_N &= z_N(q_1 \dots q_n) \end{aligned}$$

- 1) $3+2 = 5$ (3 for point on dumbbell, 2 angles for orientation)
- 2) 1 (height \rightarrow angle)
- 3) 4 (separation distance + angle) cm position)
- 4) $3 \cdot N$

relating the cartesian coordinates $x_1, y_1, z_1 \dots x_N, y_N, z_N$ to the generalised coordinates $q_1 \dots q_n$.

EXERCISE. For the systems in the Exercise 1, and with the generalised coordinates you have chosen, write down the transformations

$$\begin{aligned}x_1 &= x_1(q_1 \dots q_n), \\y_1 &= y_1(q_1 \dots q_n), \\z_1 &= z_1(q_1 \dots q_n) \\&\vdots \\z_N &= z_N(q_1 \dots q_n)\end{aligned}$$

explicitly. ▽

2. KINETIC ENERGY

Kinetic energy is energy associated with motion. For a single point mass m moving with velocity $\mathbf{v} = \dot{\mathbf{x}}$ in \mathbb{R}^3 the kinetic energy is given by $T = \frac{1}{2}m\mathbf{v} \cdot \mathbf{v} = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$.

From the expressions for the cartesian coordinates in terms of the generalised coordinates, one can now by differentiation (remember the 'chain rule'!) find expressions for

$$\begin{aligned}\dot{x}_1 &= \dot{x}_1(q_1 \dots q_n, \dot{q}_1 \dots \dot{q}_n), \\&\vdots \\ \dot{z}_N &= \dot{z}_N(q_1 \dots q_n, \dot{q}_1 \dots \dot{q}_n)\end{aligned}$$

so that the kinetic energy T can be written down as a function $T = T(q_1 \dots q_n, \dot{q}_1 \dots \dot{q}_n)$. The quantities $\dot{q}_1 \dots \dot{q}_N$ are called the *generalised velocities*.

EXAMPLE The pendulum has $x = \ell \sin \theta$ and $y = \ell \cos \theta$, and consequently $T = \frac{1}{2}m(\ell \dot{\theta})^2$. △

3. POTENTIAL ENERGY

Physics comes in when we specify potential energy as a function of the generalised coordinates (and possibly generalised velocities): $V = V(q_1 \dots q_n, \dot{q}_1 \dots \dot{q}_n)$. Well known examples are: potential energy for a point mass near the surface of the Earth ($V = mgz$ where z is the height above the surface of the Earth), potential energy of a spring ($V = \frac{1}{2}kX^2$ where X is the extension of the spring), potential energy of two gravitationally interacting masses, m_1 and m_2 ($V = -G \frac{m_1 m_2}{|\mathbf{r}_2 - \mathbf{r}_1|}$).

4. POSITION-VELOCITY SPACE

Notice that the space in which Lagrangian mechanics unfolds is the $2n$ -dimensional space with coordinates $q_1 \dots q_n, \dot{q}_1 \dots \dot{q}_n$. This space is known as the *position-velocity space*

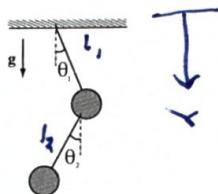
(sometimes also called the 'phase space', but we shall reserve that name for a different space, see the section on Hamiltonian mechanics).

5. THE LAGRANGIAN

The Lagrangian is on position-velocity space a function given as the difference between the kinetic and the potential functions:

$$L = L(q_1 \dots q_n, \dot{q}_1 \dots \dot{q}_n, t) \equiv T(q_1 \dots q_n, \dot{q}_1 \dots \dot{q}_n, t) - V(q_1 \dots q_n, \dot{q}_1 \dots \dot{q}_n, t)$$

EXAMPLE For the double pendulum with (point) masses m_1 and m_2 in a gravitational field



the angles θ_1 and θ_2 specify the configuration of the system. Thus, the set (θ_1, θ_2) constitutes our q 's. The coordinate transformation from ^{top} rectangular coordinates are given by (verify this!)

$$\begin{aligned} x_1 &= l_1 \sin \theta_1 \\ y_1 &= l_1(1 - \cos \theta_1) + l_2 \\ x_2 &= l_1 \sin \theta_1 + l_2 \sin \theta_2 \\ y_2 &= l_1(1 - \cos \theta_1) + l_2(1 - \cos \theta_2) \end{aligned}$$

Using the chain rule, one can now find the velocities, \dot{x}_1 , \dot{y}_1 , \dot{x}_2 , and \dot{y}_2 as functions of $q_1 \dots q_n$ and $\dot{q}_1 \dots \dot{q}_n$. Finding this transformation constitutes the necessary *kinematics*. The *physics* is introduced when we identify the kinetic and potential energy functions. In terms of rectangular coordinates, the kinetic energy is here $T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)$, and the potential energy is here $V = m_1gy_1 + m_2gy_2$ where g is the magnitude of the acceleration of gravity. Δ

EXERCISE. Show that the Lagrangian for the double pendulum with generalised coordinates (θ_1, θ_2) is given by

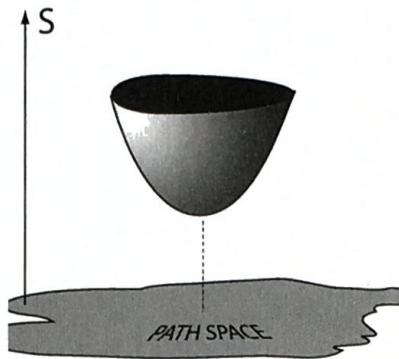
$$\begin{aligned} L = \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2) \\ - m_1g(l_1(1 - \cos \theta_1) + l_2) - m_2g(l_1(1 - \cos \theta_1) + l_2(1 - \cos \theta_2)) \end{aligned}$$

∇

6. THE EULER-LAGRANGE EQUATION

The Euler-Lagrange equations is a system of n ordinary differential equations, one for each degree of freedom q_i . They can be derived from the following variational principle:

On the space of all differentiable paths leading from some initial point $(q(0), \dot{q}(0), 0)$ to some final point $(q(t), \dot{q}(t), t)$, find the path for which the functional $S = \int_0^t L(q(t), \dot{q}(t), t) dt$ (the action) is an extremum. The derivation can be found in many textbooks on mechanics or on variational principles. This mechanical principle is called the *principle of least action*.



In index notation the Euler-Lagrange equations are:

$$\boxed{\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \quad i = 1, \dots, n.}$$

EXAMPLE Single unconstrained particle moving in a potential. For this simple case, the Lagrangian is $L = \frac{1}{2}m\ddot{\mathbf{r}}^2 - V(\mathbf{r})$. The Euler-Lagrange equations become $m\ddot{\mathbf{r}} = -\mathbf{grad}V(\mathbf{r})$, which we recognise as Newton's second law. We thus see that the Euler-Lagrange equations are identical to Newton's second law in the situations where the forces are simple. When more complicated forces act, in particular forces of constraint, the Euler-Lagrange equations still have solutions in accordance with Newton's second law, but the Euler-Lagrange equations employ more natural coordinates and are thus simpler to write down and (hopefully) solve. Δ

EXAMPLE Bead on a rotating wire. Consider a bead of mass m forced to move on a straight frictionless wire which is rotating with angular velocity ω about an axis through one endpoint of the wire and perpendicular to the wire.

The kinetic energy of the bead is:

$$T = \frac{m}{2}(\dot{r}^2 + r^2\omega^2)$$

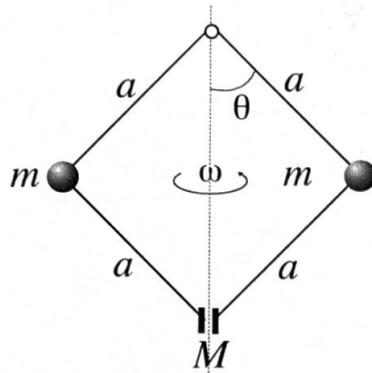
There is no potential energy.

Consequently, the r -Euler-Lagrange equation is : $\ddot{r} = r\omega^2$

△

EXAMPLE The Governor. Consider the system illustrated below (early design for steam engine power control):

$\omega = \rho g / c y b$
Vint/hustish



The two masses m rotate about the symmetry axis around in a circle perpendicular to the plane of the paper with a constant (impressed angular velocity ω). The arms all have length a , and are fixed at the top to the symmetry axis, at the bottom to a mass M which, as the arms flex, is able to move up or down the symmetry axis.

If we denote the angle between the top arms and the axis by θ , the kinetic and potential energies are given by

$$\begin{aligned} T(\theta, \dot{\theta}) &= 2 \frac{m}{2} \left(a^2 \dot{\theta}^2 + \omega^2 a^2 \sin^2 \theta \right) + \frac{M}{4} (4a^2 \sin^2(\theta) \dot{\theta}^2) \\ V(\theta, \dot{\theta}) &= -2(m+M)ga \cos \theta \end{aligned}$$

△

EXERCISE. Write the Euler-Lagrange equation for The Governor. Plot the phase plane portrait for various values of the physical parameters. ▽

7. CONSERVATION LAWS IN LAGRANGIAN DYNAMICS

Theorem. Suppose the Lagrangian does not depend explicitly on time, $\frac{\partial L}{\partial t} = 0$. Then the position-velocity space function

$$E(q, \dot{q}) \equiv \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = \text{constant}$$

if

is a constant of the motion: $E(q(t), \dot{q}(t)) = E(q(0), \dot{q}(0))$ along a solution to the Euler-Lagrange equations.

Proof: Consider the total time derivative of the Lagrangian, and use that along a solution curve we have the Euler-Lagrange equations satisfied: ?

$$\begin{aligned}\frac{d}{dt}L &= \sum_i \left(\frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right) \\ &= \sum_i \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right) \\ &= \sum_i \frac{d}{dt} \left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right)\end{aligned}$$

so that

$$\frac{d}{dt} \left(\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \right) = 0$$

□

Remark. If the kinetic energy is a (homogeneous) quadratic function of the generalised velocities, i.e., if $T = \sum_{i,j} a_{i,j} \dot{q}_i \dot{q}_j$, then $\sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T$, and thus, in this case, we have

$$E = T + V \quad , \text{i.e., } E \text{ becomes the total mechanical energy}$$

Question: For The Governor, is the total mechanical energy conserved? [Ans: no].

why not?

Definition. A *cyclic* coordinate is a coordinate q_k which does not occur in the Lagrangian, i.e., $\frac{\partial L}{\partial q_k} = 0$

Exercise. For a particle in a uniform gravitational field in \mathbb{R}^3 , determine which coordinates are cyclic. x and y ? ▽

Theorem If a coordinate q_k is cyclic, then the quantity $\frac{\partial L}{\partial \dot{q}_k}$ is a constant of the motion. ▽

Exercise. Prove this theorem. Hint: Use the Euler-Lagrange equations. ▽

EXAMPLE The Kepler Problem. Consider a point mass moving in \mathbb{R}^3 and subject to a potential which depends only on the distance to the center of the coordinate system:

$$V = V(r) = -\frac{\mu}{r}$$

Clearly, spherical polar coordinates are the coordinates most suited to reflect this form of the potential. Using the form of the kinetic energy derived in a previous exercise, we have that

$$L = \frac{m}{2} \left(\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\phi})^2 \right) + \frac{\mu}{r}$$

The angle ϕ is cyclic. Consequently, $r^2 \sin\theta^2 \dot{\phi}$ is a conserved quantity; let us denote it by M . The Lagrangian therefore has the form

$$L = \frac{m}{2} \left(\dot{r}^2 + (r\dot{\theta})^2 + 2M\dot{\phi} \right) + \frac{\mu}{r}$$

$\bar{r} \times \vec{\dot{r}}$

↑
2Ṁφ

$$= 2r^2 \sin^2\theta \dot{\phi}^2 = 2 \cdot \text{original} ?$$

$$\cdot \frac{m}{r} K_J = \frac{m^3}{r} K_S$$

Now we see that the coordinate θ is also 'secretly' cyclic. As a consequence, the quantity $mr^2\dot{\theta}$ is also conserved.

The two conserved numbers are the θ and ϕ components of the vector

$$\underline{\ell} \equiv m\mathbf{r} \times \dot{\mathbf{r}}$$

Since the r -component of this vector will always be zero, we conclude that all three components of the vector $\underline{\ell}$ are constant in time, and that the vector thus is constant vector.

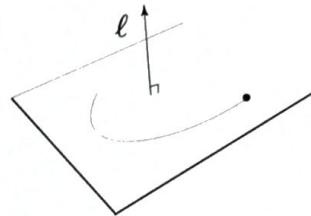
This vector is called the angular momentum vector, and since its constancy is a consequence only on the geometrical nature of the potential, we note the following theorem:

Theorem. For all *central force problems* (i.e., problems where the potential depends only on the radial coordinate r), the angular momentum vector $\underline{\ell} = m\mathbf{r} \times \dot{\mathbf{r}}$ is constant in time. □

Corollary. For central force problems, the motion takes place in a plane inside R^3 . □

Proof: If the vector \mathbf{r} were to leave the geometrical plane spanned by the initial vectors \mathbf{r} and $\dot{\mathbf{r}}$ it would cause $\underline{\ell}$ to change, in violation of the theorem. □

The corollary therefore tells us that rotational symmetry reduces the number of degrees of freedom to two, and that we should consider the motion to be taking place inside a plane (the plane to which $\underline{\ell}$ is normal). Obviously the plane contains the coordinate origo.



We therefore consider planar polar coordinates (r, θ) in this plane as a new set of coordinates. In these coordinates, (r, θ) , the Lagrangian is:

$$L(r, \dot{r}, \theta, \dot{\theta}) = \frac{m}{2}(\dot{r}^2 + (r\dot{\theta})^2) + \frac{\mu}{r}$$

Note, that θ still is cyclic, so that $mr^2\dot{\theta} = \text{constant} = \ell$. This is Kepler's Area Law. As a consequence, there is now only one Euler-Lagrange equation:

$$m\ddot{r} - \frac{\ell^2}{mr^3} = -\frac{\mu}{r^2}$$

This looks like 1-D motion in the potential (the 'effective potential')

$$V^{eff} = -\frac{\mu}{r} + \frac{\ell^2}{2mr^2}$$

2-body problem

EL-equation

&
 V_{eff}

We have to calculate this

For the motion the energy $E = \frac{r^2}{2} + V^{eff}(r)$ is conserved.

8. GENERALIZED MOMENTA

The transition from the Lagrangian description to the Hamiltonian description is effected first by changing dynamical variables from the set $(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n)$ to the set $(q_1, \dots, q_n, p_1, \dots, p_n)$. The n new variables p_i are called the (generalised) momenta. The transformation is effected in the following way. First, one forms the generalized momenta p_i for $i = 1 \dots n$, as functions of the q_i 's and the \dot{q}_i 's:

$$p_i(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}_i} \quad i = 1, \dots, n$$

Next, one uses these equations to express the \dot{q}_i 's as functions of the p_i 's and the q_i 's, so that we obtain the transformations

$$(2) \quad \dot{q}_i = \dot{q}_i(q_1 \dots q_n, p_1 \dots p_n, t) \quad i = 1, \dots, n.$$

$$L = \frac{1}{2}m\dot{r}^2 - V(r)$$

$$P = \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

9. THE HAMILTONIAN

Using (??) one forms the expression

$$H(q_1 \dots q_n, p_1 \dots p_n, t) = \sum_{i=1}^n p_i \dot{q}_i - L(q, \dot{q}, t)$$

$$H = P\dot{q} - L(q, \dot{q}, t)$$

$$= m\dot{r}^2 - \frac{1}{2}m\dot{r}^2 + V(r)$$

$$= \frac{m\dot{r}^2}{2} + V(r)$$

$$= \frac{P^2}{2m} + V(r)$$

where, on the right hand side, each \dot{q}_i is expressed as $\dot{q}_i = \dot{q}_i(q_1 \dots q_n, p_1 \dots p_n)$.

The function $H(q_1 \dots q_n, p_1 \dots p_n)$ is the *Hamiltonian* for the system.

EXAMPLE Particle in potential V . Since the Lagrangian is $L = \frac{1}{2}m\dot{r}^2 - V(r)$, the momenta are given by $P = \dot{r}$ and H becomes $H(r, P) = \frac{P^2}{2m} + V(r)$. Δ

$$m \quad \text{Particle in Pot. } V(r)$$

10. PHASE SPACE

The Lagrangian L is a real valued function on the $2n$ -dimensional space with coordinates $(q_1 \dots q_n, \dot{q}_1 \dots \dot{q}_n)$.

In contrast to this, the Hamiltonian H is a function on the space $(q_1 \dots q_n, p_1 \dots p_n)$. This space is called the *phase space* for the system.

Remark. From the point of view of differential geometry, if the configuration space has the structure of a *differential manifold*, kinetic energy is a *Riemannian structure* on this manifold, and the position-velocity space is the *tangent bundle* of this manifold. The Lagrangian is a function from the tangent bundle to \mathbb{R} . The phase space is the *co-tangent bundle*, and the Hamiltonian is a function from the tangent bundle to \mathbb{R} . An extremely powerful and elegant mathematical machinery for doing classical mechanics evolves from these concepts. Developing this formalism will, however, take us too far astray from the

** co-tangent!*

mainstream of 01248. A classic introductory text to this subject is [?]. A more physical standard text is [?].



11. HAMILTON'S EQUATIONS OF MOTION

why time?

Suppose we consider the total time differential of the phase space function H :

$$dH = \sum_i \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt$$



then, from the definition of H ,

$$dH = \sum_i \dot{q}_i dp_i + p_i d\dot{q}_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial L}{\partial t} dt$$

Comparing terms (and using the definition of p_i), we then have, in addition to the identity $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$ the equations

$$\begin{aligned}\dot{q}_i &= \frac{\partial H}{\partial p_i} & i = 1, \dots, n \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} & i = 1, \dots, n\end{aligned}$$

These equations of motion are $2n$ first-order ordinary differential equations (Hamilton's equations). Note the sign in the \dot{p}_i equations!

EXAMPLE Particle in \mathbb{R}^3 in conservative potential. In this case, $H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r})$ and Hamilton's equations become

$$\begin{aligned}\dot{\mathbf{r}} &= \frac{\mathbf{p}}{m} \\ \dot{\mathbf{p}} &= -\mathbf{grad} V(\mathbf{r})\end{aligned}$$

which are (the first-order form of) Newton's equations of motion. Δ

EXAMPLE The Pendulum. Here, $H = \frac{p_\theta^2}{2m\ell^2} - mgl \cos \theta$, and Hamilton's equations become

$$\begin{aligned}\dot{\theta} &= \frac{p_\theta}{m\ell^2} \\ \dot{p}_\theta &= -mgl \sin \theta\end{aligned}$$

(D-pendul) ; Landau Δ)

her:
 $q \leftrightarrow \theta$
 $p \leftrightarrow p_\theta$

12. CONSERVED QUANTITIES

The change of any quantity $J = J(q_1 \dots q_n, p_1 \dots p_n, t)$ as the p_i 's and the q_i 's evolve along a solution curve $(\mathbf{p}(t), \mathbf{q}(t))$ governed by Hamilton's equations, is given by

$$\frac{d}{dt} J = \sum_{i=1}^n \frac{\partial J}{\partial p_i} \dot{p}_i + \frac{\partial J}{\partial q_i} \dot{q}_i + \frac{\partial J}{\partial t} = \sum_{i=1}^n -\frac{\partial J}{\partial p_i} \left(\frac{\partial H}{\partial q_i} \right) + \frac{\partial J}{\partial q_i} \frac{\partial H}{\partial p_i} + \frac{\partial J}{\partial t}$$

Note that, for the Hamiltonian function itself,

$$\frac{d}{dt}H = \sum_{i=1}^n -\frac{\partial H}{\partial p_i}\left(\frac{\partial H}{\partial q_i}\right) + \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

*autonomous
= time-independent*

Thus, if the Hamiltonian is not explicitly time dependent (not being explicitly time dependent is sometimes called autonomous), we have that $dH/dt = 0$, meaning that the initial value of the Hamiltonian is conserved along the $(\mathbf{p}(t), \mathbf{q}(t))$ flow.

For $n = 1$, phase space is 2-dimensional, and if the Hamiltonian is autonomous, the solution curves coincide with the level curves of the Hamiltonian. Thus, in this special case, we do not even need to solve differential equations to construct the phase space portrait; plotting level curves gives the same picture. Of course, just looking at the level curves one will get no information about the dynamics, i.e., about time evolution along these curves; for this information one will have to integrate Hamilton's equations.

The Hamiltonian analog to the Lagrangian result on cyclic coordinates is given by the following

Theorem. If a variable, either some q_k or some p_j is missing from the Hamiltonian, the conjugate variable (either p_k or q_j respectively) is a constant of the motion.

Proof. The k th or the j th of Hamilton's equations provides the conservation law. \square

Consider now the situation where either *all* the generalised coordinates (or *all* the generalised momenta) are missing in the Hamiltonian. Then we are able to characterise completely and analytically the solution.

Theorem. If all the variables q_k are cyclic, the entire system of equations can be integrated.

Proof. From the previous theorem, all of the momenta p_k are constant. Consequently, the remaining first-order equations for the q_k all have the form $\dot{q} = \omega_k$ where $\omega_k = \frac{\partial H}{\partial p_k}$ are constants (since H is a function of the p_k s only, and they are all constant). These first-order equations have the solutions $q_k(t) = \omega_k t + q_k(0)$ and the system is solved. \square

We see that solving of Hamilton's equations of motion becomes a trivial task if coordinates can be found such that all the q_k s are cyclic. We can therefore shift the focus of our efforts from integration to *coordinate transformations*, as long as we are certain that the form of the dynamical equations of motion retain their Hamiltonian form when expressed in terms of the 'new' variables.

Coordinate transformations of the phase plane that preserve the form of the equations of motion come in four basic varieties, characterised by their so called generating function, F . We will specify the procedure for 1-D, with one 'old' coordinate q , and one 'old' momentum p . This set is to be transformed to a set of 'new' variables, (Q, P) . We thus

seek two equations of type

$$(3) \quad \begin{aligned} Q &= Q(q, p) \\ P &= P(q, p) \end{aligned}$$

A generating function is a function of one 'old' and one 'new' variable. The four basic varieties are then

$$\begin{aligned} F_1 &= F_1(q, Q) \\ F_2 &= F_2(q, P) \\ F_3 &= F_3(Q, p) \\ F_4 &= F_4(P, p) \end{aligned}$$

The transformation rules that accompany these four types of generating functions are given as follows. For each generating function, two functional equations can be found by partial differentiation of the generating function. By suitable inversion one can then extract transformation equations of the form (??).

The rules are

$$(4) \quad \frac{\partial F_1(q, Q)}{\partial q} = p, \quad \frac{\partial F_1(q, Q)}{\partial Q} = -P$$

$$(5) \quad \frac{\partial F_2(q, P)}{\partial q} = p, \quad \frac{\partial F_2(q, P)}{\partial P} = Q$$

$$(6) \quad \frac{\partial F_3(Q, p)}{\partial p} = -q, \quad \frac{\partial F_3(Q, p)}{\partial Q} = -P$$

$$(7) \quad \frac{\partial F_4(P, p)}{\partial p} = -q, \quad \frac{\partial F_4(P, p)}{\partial P} = Q$$

EXAMPLE All transformation schemes should be able to describe the identity transformations. In this case, the appropriate generating function is of type F_2 and is simply: $F_2(q, P) = qP$. Using the above formulas, one will arrive at $Q = q$ and $P = p$. Δ

13. CANONICAL PERTURBATION THEORY

It is often the case that the Hamiltonian in a given problem can be split into a part where the coordinates are cyclic and a small part where they are not. This sets the stage for the following approach. We seek coordinate transformations that will systematically remove the coordinate dependence in the full Hamiltonian.

In the most general case, one will have to pursue at each order of ϵ a new coordinate transformation which removes to this order of ϵ the coordinate dependence. We shall not here go into the general procedure (or its pitfalls). Only the simplest case, first order perturbation theory, will be covered. We shall for notational convenience restrict ourselves to the 1-D case, where, as we have seen, all autonomous Hamiltonians are integrable. The higher dimensional version is a straightforward generalisation.

Suppose one has a Hamiltonian which expressed in the phase space coordinates (θ, J) has the form

$$H = H^{(0)}(J) + \epsilon H^{(1)}(\theta, J)$$

We seek a coordinate transformation from the set (θ, J) to a new set of coordinates $(\tilde{\theta}, \tilde{J})$, which to first order in ϵ will remove the θ dependence in the term $H^{(1)}$, without, of course, re-introducing a $\tilde{\theta}$ dependence into the term $H^{(0)}$. Thus, in order to stay close to the identity transformation $F_2 = \theta \tilde{J}$, we seek an F_2 transformation of form

$$F_2 = \theta \tilde{J} + \epsilon F_2^1(\theta, \tilde{J})$$

which, to first order in ϵ will make $H^{(1)}$ independent of $\tilde{\theta}$. According to the transformation rule (??) we have the relations

$$\begin{aligned} J &= \tilde{J} + \epsilon \frac{\partial F_2^1(\theta, \tilde{J})}{\partial \theta} \\ \tilde{\theta} &= \theta + \epsilon \frac{\partial F_2^1(\theta, \tilde{J})}{\partial \tilde{J}} \end{aligned}$$

which, *to first order in ϵ* , can be written

$$\begin{aligned} J &= \tilde{J} + \epsilon \frac{\partial F_2^1(\tilde{\theta}, \tilde{J})}{\partial \tilde{\theta}} \\ \theta &= \tilde{\theta} - \epsilon \frac{\partial F_2^1(\tilde{\theta}, \tilde{J})}{\partial \tilde{J}} \end{aligned}$$

Using these relations in the expression for H , we will get an expression of form

$$\begin{aligned} (8) \quad \tilde{H}(\tilde{\theta}, \tilde{J}) &= H^{(0)}(\tilde{J}) + \frac{\partial H^{(0)}(\tilde{J})}{\partial \tilde{J}} \epsilon \frac{\partial F_2^1(\tilde{\theta}, \tilde{J})}{\partial \tilde{\theta}} + \epsilon H^{(1)}(\tilde{\theta}, \tilde{J}) \\ &= H^{(0)}(\tilde{J}) + \omega \epsilon \frac{\partial F_2^1(\tilde{\theta}, \tilde{J})}{\partial \tilde{\theta}} + \epsilon H^{(1)}(\tilde{\theta}, \tilde{J}) \end{aligned}$$

where we have Taylor expanded for J in $H^{(0)}$, and in the term $H^{(1)}$ simply substituted $\theta = \tilde{\theta}$, and $J = \tilde{J}$ since this term is already to first order in ϵ .

We wish to chose F_2^1 in such a way that there is no $\tilde{\theta}$ dependence on the right hand side of (??). The term $\epsilon H^{(1)}(\tilde{\theta}, \tilde{J})$ will in general be a sum of a $\tilde{\theta}$ *independent* part $\langle H^{(1)} \rangle$, which, if it cannot be directly separated from $H^{(1)}$ can be found as a $\tilde{\theta}$ -average of $H^{(1)}$, and a $\tilde{\theta}$ -*dependent* term, which is simply the rest, $H^{(1)} - \langle H^{(1)} \rangle$.

Thus, we demand that

$$\omega \frac{\partial F_2^1}{\partial \tilde{\theta}} = -(H^{(1)} - \langle H^{(1)} \rangle)(\tilde{\theta}, \tilde{J})$$

or

$$F_2^1(\tilde{\theta}, \tilde{J}) = -\frac{1}{\omega} \int (H^{(1)} - \langle H^{(1)} \rangle)(\tilde{\theta}, \tilde{J}) d\theta$$

This will make, to first order in ϵ , the new Hamiltonian function equal to

$$\tilde{H}(\tilde{J}) = H^{(0)}(\tilde{J}) + \epsilon \langle H^{(1)} \rangle(\tilde{J})$$

which is what we wanted.

EXAMPLE For a Hamiltonian $H(\theta, J) = J^2 + \epsilon(aJ + bJ \sin \theta)$, find the first-order frequency.

Solution. From first order canonical perturbation theory, we see that $\epsilon < H^{(1)} > (\tilde{J}) = \epsilon a \tilde{J}$, so the Hamiltonian will be $\tilde{H}(\tilde{J}) = \tilde{J}^2 + \epsilon a \tilde{J}$, and the frequency ω consequently $\omega = \frac{\partial \tilde{H}}{\partial \tilde{J}} = 2\tilde{J} + \epsilon a$.

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