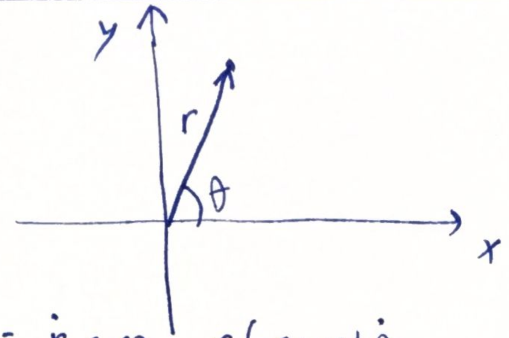
(around biggest body, not CM)

Hamiltonian Mechanics : Kepler problem (polar coordinates)

$q_1 = r$ 0

$q_2 = \theta$



$\vec{F} = -G\frac{Mm}{r^2}\hat{r}$

$V(r) = -\int_{\infty}^{r} -G\frac{M\cancel{m}}{r'^2}\hat{r}\,d\vec{r'} = \frac{-W}{m}$

$= +GM\left[-\frac{1}{r'}\right]_{\infty}^{r}$

$\Rightarrow V(r) = -\frac{GM}{r} = -\frac{\mu}{r}$

$\boxed{\mu \equiv GM}$

$x = r\cos\theta \quad ; \quad \dot{x} = \dot{r}\cos\theta + r(-\sin\theta)\cdot\dot{\theta}$

$y = r\sin\theta \quad ; \quad \dot{y} = \dot{r}\sin\theta + r\cos\theta\cdot\dot{\theta}$

$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$

$= \frac{m}{2}\left[(\dot{r}\cos\theta - r\sin\theta\cdot\dot{\theta})^2 + (\dot{r}\sin\theta + r\cos\theta\cdot\dot{\theta})^2\right]$

$= \frac{m}{2}\left[\dot{r}^2\cos^2\theta + r^2\sin^2\theta\cdot\dot{\theta}^2 + \dot{r}^2\sin^2\theta + r^2\cos^2\theta\cdot\dot{\theta}^2 - 2r\dot{r}\cos\theta\sin\theta\cdot\dot{\theta} + 2r\dot{r}\cos\theta\sin\theta\dot{\theta}\right]$

$= \frac{m}{2}\left[\dot{r}^2 + r^2\dot{\theta}^2\right]$

$V = -\frac{GM}{r} = -\frac{\mu}{r}$, $L = T - V \Rightarrow L = \frac{m}{2}\left(\dot{r}^2 + (r\dot{\theta})^2\right] + \frac{\mu}{r}$

Generalized momentae

1. $p_r = \frac{\partial L}{\dot{r}} = m\dot{r}$, $p_\theta = \frac{\partial L}{\partial\dot{\theta}} = mr^2\dot{\theta}$

Transform for $\dot{q}_i$'s

2. $\dot{r} = \frac{p_r}{m}$; $\dot{\theta} = \frac{p_\theta}{mr^2}$

$\boxed{H = \sum_{i=1}^{n} p_i\dot{q}_i - L}$

Hamiltonian

3. $H = p_r\cdot\frac{p_r}{m} + p_\theta\cdot\frac{p_\theta}{mr^2} - \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\mu}{r}$

$\Leftrightarrow H = \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} - \frac{m}{2}\left(\frac{p_r^2}{m^2} + r^2\frac{p_\theta^2}{m^2r^4}\right) + \frac{\mu}{r}$

$\Leftrightarrow H = \boxed{\frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{\mu}{r}}$

Hamilton's equations of motion

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$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = -\frac{\mu}{r^2}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$$