CHAPTER

Orbits in Three Dimensions

4

CHAPTER OUTLINE

4.1 Introduction

The discussion of orbital mechanics up to now has been confined to two dimensions, that is, to the plane of the orbits themselves. This chapter explores the means of describing orbits in three-dimensional space, which, of course, is the setting for real missions and orbital maneuvers. Our focus will be on the orbits of earth satellites, but the applications are to any two-body trajectories, including interplanetary missions to be discussed in Chapter 8.

We begin with a discussion of the ancient concept of the celestial sphere and the use of right ascension (RA) and declination (Dec) to define the location of stars, planets, and other celestial objects on the sphere. This leads to the establishment of the inertial geocentric equatorial frame of reference and the concept of state vector. The six components of this vector give the instantaneous position and velocity of an object relative to the inertial frame and define the characteristics of the orbit. Following this discussion is a presentation of the six classical orbital elements, which also uniquely define the shape and orientation of an orbit and the location of a body on it. We then show how to transform the state vector into orbital elements, and vice versa, taking advantage of the perifocal frame introduced in Chapter 2.

Orbital Mechanics for Engineering Students. http://dx.doi.org/10.1016/B978-0-08-097747-8.00004-9

We go on to summarize two of the major perturbations of earth orbits due to the earth's nonspherical shape. These perturbations are exploited to place satellites in sun-synchronous and Molniya orbits.

The chapter concludes with a discussion of ground tracks and how to compute them.

4.2 Geocentric right ascension–declination frame

The coordinate system used to describe earth orbits in three dimensions is defined in terms of earth's equatorial plane, the ecliptic plane, and the earth's axis of rotation. The ecliptic is the plane of the earth's orbit around the sun, as illustrated in Figure 4.1. The earth's axis of rotation, which passes through the north and south poles, is not perpendicular to the ecliptic. It is tilted away by an angle known as the obliquity of the ecliptic, ε. For the earth, ε is approximately 23.4°. Therefore, the earth's equatorial plane and the ecliptic intersect along a line, which is known as the vernal equinox line. On the calendar, "vernal equinox" is the first day of spring in the northern hemisphere, when the noontime sun crosses the equator from south to north. The position of the sun at that instant defines the location of a point in the sky called the vernal equinox, for which the symbol γ is used. On the day of the vernal equinox, the number of hours of daylight and darkness are equal, hence the word equinox. The other equinox occurs precisely one-half year later, when the sun crosses back over the equator from north to south, thereby defining the first day of autumn. The vernal equinox lies today in the constellation Pisces, which is visible in the night sky during the fall. The direction of the vernal equinox line is from the earth toward γ, as shown in Figure 4.1.

For many practical purposes, the vernal equinox line may be considered fixed in space. However, it actually rotates slowly because the earth's tilted spin axis precesses westward around the normal to the ecliptic at the rate of about 1.4° per century. This slow precession is due primarily to the action of the sun and the moon on the nonspherical distribution of mass within the earth. Due to the centrifugal force of rotation about its own axis, the earth bulges very slightly outward at its equator. This effect is shown

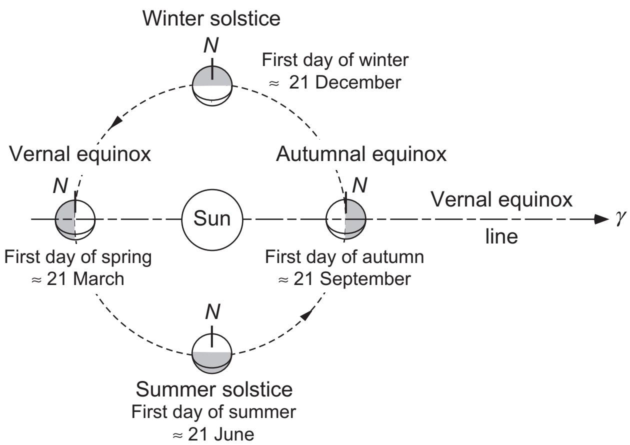


FIGURE 4.1

The earth's orbit around the sun, viewed from above the ecliptic plane, showing the change of seasons in the northern hemisphere.

highly exaggerated in Figure 4.2. One of the bulging sides is closer to the sun than the other, so the force of the sun's gravity $\mathbf{f}_1$ on its mass is slightly larger than the force $\mathbf{f}_2$ on opposite the side, farthest from the sun. The forces $\mathbf{f}_1$ and $\mathbf{f}_2$, along with the dominant force $\mathbf{F}$ on the spherical mass, comprise the total force of the sun on the earth, holding it in its solar orbit. Taken together, $\mathbf{f}_1$ and $\mathbf{f}_2$ produce a net clockwise moment (a vector into the page) about the center of the earth. That moment would rotate the earth's equator into alignment with the ecliptic if it were not for the fact that the earth has an angular momentum directed along its south-to-north polar axis due to its spin around that axis at an angular velocity ω_E of about 360° per day. The effect of the moment is to rotate the angular momentum vector in the direction of the moment (into the page). The result is that the spin axis is forced to precess in a counterclockwise direction around the normal to the ecliptic, sweeping out a cone as illustrated in the figure. The moon exerts a torque on the earth for the same reason, and the combined effect of the sun and the moon is a precession of the spin axis, and hence γ, with a period of about 26,000 years. The moon's action also superimposes a small nutation on the precession. This causes the obliquity ε to vary with a maximum amplitude of 0.0025° over a period of 18.6 years.

About 4000 years ago, when the first recorded astronomical observations were being made, γ was located in the constellation Aries, the ram. The Greek letter γ is a descendant of the ancient Babylonian symbol resembling the head of a ram.

To the human eye, objects in the night sky appear as points on a celestial sphere surrounding the earth, as illustrated in Figure 4.3. The north and south poles of this fixed sphere correspond to those of the earth rotating within it. Coordinates of latitude and longitude are used to locate points on the celestial sphere in much the same way as on the surface of the earth. The projection of the earth's equatorial plane outward onto the celestial sphere defines the celestial equator. The vernal equinox γ, which lies on the celestial equator, is the origin for measurement of longitude, which in astronomical parlance is called right ascension. Right ascension (RA or α) is measured along the celestial equator in degrees east from the vernal equinox (astronomers measure right ascension in hours instead of degrees, where 24 hours equals 360 degrees). Latitude on the celestial sphere is called declination. Declination (Dec or δ) is measured along a meridian in degrees, positive to the north of the equator and negative to the south. Figure 4.4 is a sky chart showing how the heavenly grid appears from a given

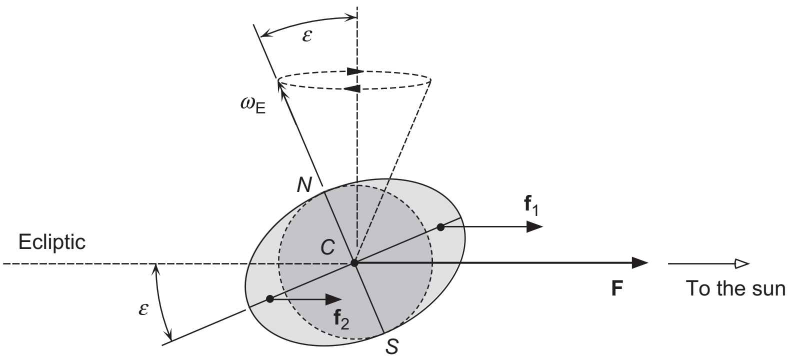


FIGURE 4.2

Secondary (perturbing) gravitational forces on the earth.

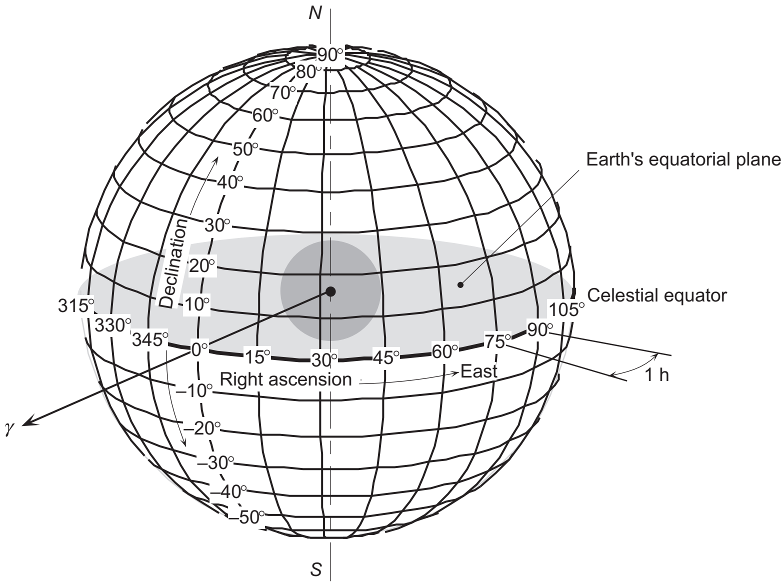


FIGURE 4.3

The celestial sphere, with grid lines of right ascension and declination.

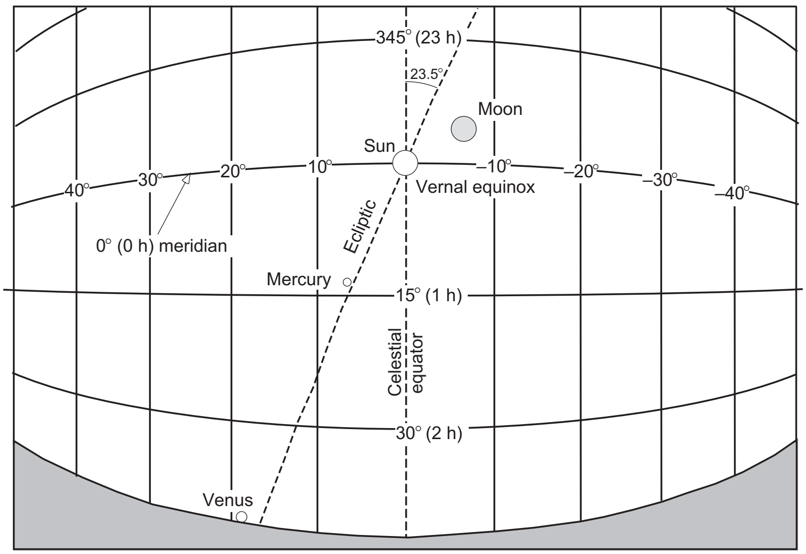


FIGURE 4.4

A view of the sky above the eastern horizon from 0° longitude on the equator at 9 AM local time, March 20, 2004 (precession epoch AD 2000).

Table 4.1 Venus and Moon Ephemeris for 0 h Universal Time (Precession Epoch: 2000 AD)

	Venus		Moon	
Date	**RA**	**Dec**	**RA**	**Dec**
January 1, 2004	21 h 05.0 min	−18° 36′	1 h 44.9 min	+8° 47′
February 1, 2004	23 h 28.0 min	−04° 30′	4 h 37.0 min	+24° 11′
March 1, 2004	01 h 30.0 min	+10° 26′	6 h 04.0 min	+08° 32′
April 1, 2004	03 h 37.6 min	+22° 51′	9 h 18.7 min	+21° 08′
May 1, 2004	05 h 20.3 min	+27° 44′	11 h 28.8 min	+07° 53′
June 1, 2004	05 h 25.9 min	+24° 43′	14 h 31.3 min	−14° 48′
July 1, 2004	04 h 34.5 min	+17° 48′	17 h 09.0 min	−26° 08′
August 1, 2004	05 h 37.4 min	+19° 04′	21 h 05.9 min	−21° 49′
September 1, 2004	07 h 40.9 min	+19° 16′	00 h 17.0 min	−00° 56′
October 1, 2004	09 h 56.5 min	+12° 42′	02 h 20.9 min	+14° 35′
November 1, 2004	12 h 15.8 min	+00° 01′	05 h 26.7 min	+27° 18′
December 1, 2004	14 h 34.3 min	−13° 21′	07 h 50.3 min	+26° 14′
January 1, 2005	17 h 12.9 min	−22° 15′	10 h 49.4 min	+11° 39′

point on the earth. Notice that the sun is located at the intersection of the equatorial and ecliptic planes, so this must be the first day of spring.

Stars are so far away from the earth that their positions relative to each other appear stationary on the celestial sphere. Planets, comets, satellites, etc., move on the fixed backdrop of the stars. A table of the coordinates of celestial bodies as a function of time is called an ephemeris, for example, the *Astronomical Almanac* (US Naval Observatory, 2013). Table 4.1 is an abbreviated ephemeris for the moon and for Venus. An ephemeris depends on the location of the vernal equinox at a given time or epoch, for we know that even the positions of the stars relative to the equinox change slowly with time. For example, Table 4.2 shows the celestial coordinates of the star Regulus at five epochs since AD 1700. Currently, the position of the vernal equinox in the year 2000 is used to define the standard grid of the celestial sphere. In 2025, the position will be updated to that of the year 2050, and so on at 25-year intervals. Since observations are made relative to the actual orientation of the earth, these measurements must be transformed into the standardized celestial frame of reference. As Table 4.2 suggests, the adjustments will be small if the current epoch is within 25 years of the standard precession epoch.

Table 4.2 Variation of the Coordinates of the Star Regulus Due to Precession of the Equinox

Precession Epoch	RA	Dec
1700 AD	9 h 52.2 min (148.05°)	+13° 25′
1800 AD	9 h 57.6 min (149.40°)	+12° 56′
1900 AD	10 h 3.0 min (150.75°)	+12° 27′
1950 AD	10 h 5.7 min (151.42°)	+12° 13′
2000 AD	10 h 8.4 min (152.10°)	+11° 58′

4.3 State vector and the geocentric equatorial frame

At any given time, the state vector of a satellite comprises its velocity $\mathbf{v}$ and orbital acceleration $\mathbf{a}$. Orbital mechanics is concerned with specifying or predicting state vectors over intervals of time. From Chapter 2, we know that the equation governing the state vector of a satellite traveling around the earth is, under the familiar assumptions,

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \tag{4.1}$$

$\mathbf{r}$ is the position vector of the satellite relative to the center of the earth. The components of $\mathbf{r}$ and, especially, those of its time derivatives $\dot{\mathbf{r}} = \mathbf{v}$ and $\ddot{\mathbf{r}} = \mathbf{a}$, must be measured in a nonrotating frame attached to the earth. A commonly used nonrotating right-handed Cartesian coordinate system is the geocentric equatorial frame shown in Figure 4.5. The X-axis points in the vernal equinox direction. The XY plane is the earth's equatorial plane, and the Z-axis coincides with the earth's axis of rotation and points northward. The unit vectors $\hat{\mathbf{I}}$, $\hat{\mathbf{J}}$, and $\hat{\mathbf{K}}$ form a right-handed triad. The nonrotating geocentric equatorial frame serves as an inertial frame for the two-body earth satellite problem, as embodied in Eqn (4.1). It is not truly an inertial frame, however, since the center of the earth is always accelerating toward a third body, the sun (to say nothing of the moon), a fact that we ignore in the two-body formulation.

In the geocentric equatorial frame, the state vector is given in component form by

$$\mathbf{r} = X\hat{\mathbf{I}} + Y\hat{\mathbf{J}} + Z\hat{\mathbf{K}} \tag{4.2}$$

$$\mathbf{v} = v_X\hat{\mathbf{I}} + v_Y\hat{\mathbf{J}} + v_Z\hat{\mathbf{K}} \tag{4.3}$$

If r is the magnitude of the position vector, then

$$\mathbf{r} = r\hat{\mathbf{u}}_r \tag{4.4}$$

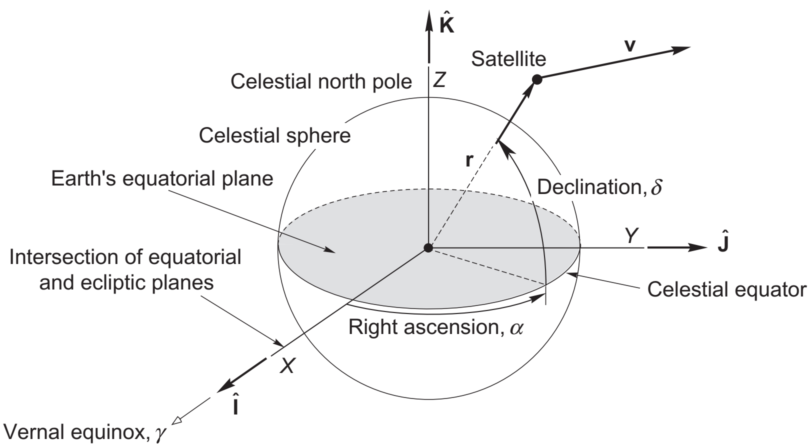


FIGURE 4.5

The geocentric equatorial frame.

Figure 4.5 shows that the components of $\hat{\mathbf{u}}_r$ (the direction cosines l, m, and n of $\hat{\mathbf{u}}_r$) are found in terms of the RA α and Dec δ as follows:

$$\hat{\mathbf{u}}_r = l\hat{\mathbf{I}} + m\hat{\mathbf{J}} + n\hat{\mathbf{K}} = \cos\delta\cos\alpha\hat{\mathbf{I}} + \cos\delta\sin\alpha\hat{\mathbf{J}} + \sin\delta\hat{\mathbf{K}} \tag{4.5}$$

From this we see that the Dec is obtained as $\delta = \sin^{-1}n$. There is no quadrant ambiguity since, by definition, the Dec lies between $-90°$ and $+90°$, which is precisely the range of the principal values of the arcsine function. It follows that $\cos\delta$ cannot be negative. Equation (4.5) also reveals that $l = \cos\delta\cos\alpha$. Hence, we find the RA from $\alpha = \cos^{-1}(l/\cos\delta)$, which yields two values of α between 0° and 360°. To determine the correct quadrant for α, we check the sign of the direction cosine $m = \cos\delta\sin\alpha$. Since $\cos\delta$ cannot be negative, the sign of m is the same as the sign of $\sin\alpha$. If $\sin\alpha > 0$, then α lies in the range 0°–180°, whereas $\sin\alpha < 0$ means that α lies between 180° and 360°.

ALGORITHM 4.1

Given the position vector $\mathbf{r} = X\hat{\mathbf{I}} + Y\hat{\mathbf{J}} + Z\hat{\mathbf{K}}$, calculate the RA α and Dec δ. This procedure is implemented in MATLAB as *ra_and_dec_from_r.m*, which appears in Appendix D.17.

1. Calculate the magnitude of $\mathbf{r}$: $r = \sqrt{X^2 + Y^2 + Z^2}$
2. Calculate the direction cosines of $\mathbf{r}$:
$$l = \frac{X}{r} \quad m = \frac{Y}{r} \quad n = \frac{Z}{r}$$
3. Calculate the Dec:
$$\delta = \sin^{-1}n$$
4. Calculate the RA:
$$\alpha = \begin{cases} \cos^{-1}\left(\frac{l}{\cos\delta}\right) & (m > 0) \\ 360° - \cos^{-1}\left(\frac{l}{\cos\delta}\right) & (m \le 0) \end{cases}$$

Although the position vector furnishes the RA and Dec, the RA and Dec alone do not furnish $\mathbf{r}$. For that we need the distance r in order to obtain the position vector from Eqn (4.4).

EXAMPLE 4.1

If the position vector of the International Space Station in the geocentric equatorial frame is

$$\mathbf{r} = -5368\hat{\mathbf{I}} - 1784\hat{\mathbf{J}} + 3691\hat{\mathbf{K}}\,(\text{km})$$

what are its RA and Dec?

Solution

We employ Algorithm 4.1.

Step 1:

$$r = \sqrt{(-5368)^2 + (-1784)^2 + 3691^2} = 6754 \text{ km}$$

Step 2:

$$l = \frac{-5368}{6754} = -0.7947 \quad m = \frac{-1784}{6754} = -0.2642 \quad n = \frac{3691}{6754} = 0.5462$$

Step 3:

$$\delta = \sin^{-1} 0.5464 = \boxed{33.12^\circ}$$

Step 4:

Since the direction cosine m is negative,

$$\alpha = 360^\circ - \cos^{-1}\left(\frac{l}{\cos\delta}\right) = 360^\circ - \cos^{-1}\left(\frac{-0.7947}{\cos 33.12^\circ}\right) = 360^\circ - 161.6^\circ = \boxed{198.4^\circ}$$

If we are provided the state vector ($\mathbf{r}_0$,$\mathbf{v}_0$) at a given instant, then we can determine the state vector at any other time in terms of the initial vector by means of the expressions

$$\begin{aligned} \mathbf{r} &= f\mathbf{r}_0 + g\mathbf{v}_0 \\ \mathbf{v} &= \dot{f}\mathbf{r}_0 + \dot{g}\mathbf{v}_0 \end{aligned} \tag{4.6}$$

where the Lagrange coefficients f and g and their time derivatives are given in Eqn (3.69). Specifying the total of six components of $\mathbf{r}_0$ and $\mathbf{v}_0$ therefore completely determines the size, shape, and orientation of the orbit.

EXAMPLE 4.2

At time t_0, the state vector of an earth satellite is

$$\mathbf{r}_0 = 1600\hat{\mathbf{I}} + 5310\hat{\mathbf{J}} + 3800\hat{\mathbf{K}} \text{ (km)} \tag{a}$$

$$\mathbf{v}_0 = -7.350\hat{\mathbf{I}} + 0.4600\hat{\mathbf{J}} + 2.470\hat{\mathbf{K}} \text{ (km/s)} \tag{b}$$

Determine the position and velocity 3200 s later and plot the orbit in three dimensions.

Solution

We will use the universal variable formulation and Algorithm 3.4, which was illustrated in detail in Example 3.7. Therefore, only the results of each step are presented here.

Step 1:

(α here is not to be confused with the right ascension.)

$\alpha = 1.4613 \times 10^{-4}$ km^{-1}. Since this is positive, the orbit is an ellipse.

Step 2:

$$\chi = 294.42 \text{ km}^{\frac{1}{2}}$$

Step 3:

$$f = -0.94843 \text{ and } g = -354.89\ \mathrm{s}^{-1}$$

Step 4:

$$\boxed{\mathbf{r} = 1090.9\hat{\mathbf{I}} - 5199.4\hat{\mathbf{J}} - 4480.6\hat{\mathbf{K}}\ (\mathrm{km})} \Rightarrow r = 6949.8\ \mathrm{km}$$

Step 5:

$$\dot{f} = 0.00045324\ \mathrm{s}^{-1},\ \ \dot{g} = -0.88479$$

Step 6:

$$\boxed{\mathbf{v} = 7.2284\hat{\mathbf{I}} + 1.9997\hat{\mathbf{J}} - 0.46311\ \hat{\mathbf{K}}\ (\mathrm{km/s})}$$

To plot the elliptical orbit, we observe that one complete revolution means a change in the eccentric anomaly E of 2π radians. According to Eqn (3.57), the corresponding change in the universal anomaly is

$$\chi = \sqrt{a}E = \sqrt{\frac{1}{\alpha}}E = \sqrt{\frac{1}{0.00014613}} \cdot 2\pi = 519.77\ \mathrm{km}^{\frac{1}{2}}$$

Letting χ vary from 0 to 519.77 in small increments, we employ the Lagrange coefficient formulation (Eqn (3.67) plus (3.69a) and (3.69b)) to compute

$$\mathbf{r} = \left[1 - \frac{\chi^2}{r_0}C\left(\alpha\,\chi^2\right)\right]\mathbf{r}_0 + \left[\Delta t - \frac{1}{\sqrt{\mu}}\chi^3 S\left(\alpha\,\chi^2\right)\right]\mathbf{v}_0$$

where Δt for a given value of χ is given by Eqn (3.49). Using a computer to plot the points obtained in this fashion yields Figure 4.6, which also shows the state vectors at t_0 and $t_0 + 3200$ s.

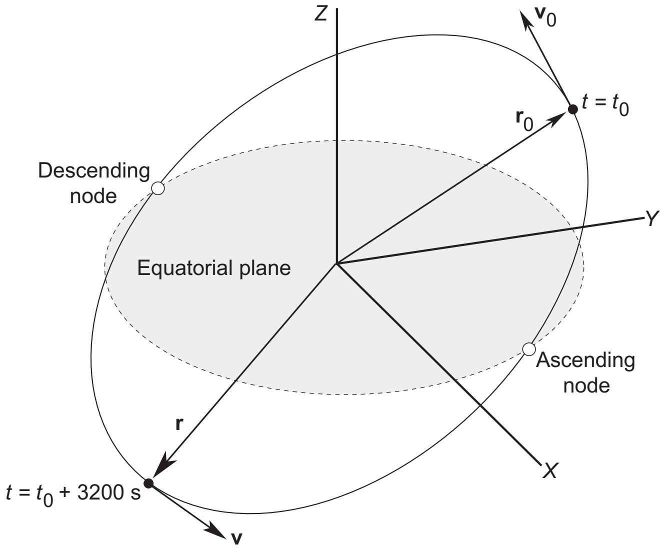


FIGURE 4.6

The orbit corresponding to the initial conditions given in Eqns (a) and (b) of Example 4.2.

The previous example illustrates the fact that the six quantities or orbital elements comprising the state vector **r** and **v** completely determine the orbit. Other elements may be chosen. The classical orbital elements are introduced and related to the state vector in the next section.

4.4 Orbital elements and the state vector

To define an orbit in the plane requires two parameters: eccentricity and angular momentum. Other parameters, such as the semimajor axis, the specific energy, and (for an ellipse) the period are obtained from these two. To locate a point on the orbit requires a third parameter, the true anomaly, which leads us to the time since perigee. Describing the orientation of an orbit in three dimensions requires three additional parameters, called the Euler angles, which are illustrated in Figure 4.7.

First, we locate the intersection of the orbital plane with the equatorial (XY) plane. This line is called the node line. The point on the node line where the orbit passes above the equatorial plane from below it is called the ascending node. The node line vector **N** extends outward from the origin through the ascending node. At the other end of the node line, where the orbit dives below the equatorial plane, is the descending node. The angle between the positive X-axis and the node line is the first Euler angle Ω, the RA of the ascending node. Recall from Section 4.2 that right ascension is a positive number lying between 0° and 360°.

The dihedral angle between the orbital plane and the equatorial plane is the inclination i, measured according to the right-hand rule, that is, counterclockwise around the node line vector from the equator to the orbit. The inclination is also the angle between the positive Z-axis and the normal to the plane of the orbit. The two equivalent means of measuring i are indicated in Figure 4.7. Recall from Chapter 2 that the angular momentum vector **h** is normal to the plane of the orbit. Therefore, the inclination i is the angle between the positive Z-axis and **h**. The inclination is a positive number between 0° and 180°.

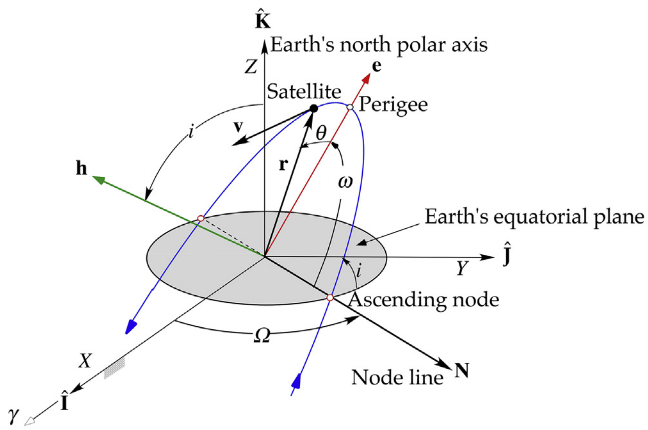


FIGURE 4.7

Geocentric equatorial frame and the orbital elements.

It remains to locate the perigee of the orbit. Recall that perigee lies at the intersection of the eccentricity vector **e** with the orbital path. The third Euler angle ω, the argument of perigee, is the angle between the node line vector **N** and the eccentricity vector **e**, measured in the plane of the orbit. The argument of perigee is a positive number between 0° and 360°.

In summary, the six orbital elements are

h: specific angular momentum.
i: inclination.
Ω: right ascension of the ascending node.
e: eccentricity.
ω: argument of perigee.
θ: true anomaly.

The angular momentum h and true anomaly θ are frequently replaced by the semimajor axis a and the mean anomaly M, respectively.

Given the position **r** and velocity **v** of a spacecraft in the geocentric equatorial frame, how do we obtain the orbital elements? The step-by-step procedure is outlined next in Algorithm 4.2. Note that each step incorporates results obtained in the previous steps. Several steps require resolving the quadrant ambiguity that arises in calculating the arccosine (recall Figure 3.4).

ALGORITHM 4.2

Obtain orbital elements from the state vector. A MATLAB version of this procedure appears in Appendix D.18. Applying this algorithm to orbits around other planets or the sun amounts to defining the frame of reference and substituting the appropriate gravitational parameter μ.

1. Calculate the distance:

$$r = \sqrt{\mathbf{r}\cdot\mathbf{r}} = \sqrt{X^2 + Y^2 + Z^2}$$

2. Calculate the speed:

$$v = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{v_X^2 + v_Y^2 + v_Z^2}$$

3. Calculate the radial velocity:

$$v_r = \mathbf{r}\cdot\mathbf{v}/r = (Xv_X + Yv_Y + Zv_Z)/r.$$

Note that if $v_r > 0$, the satellite is flying away from perigee. If $v_r < 0$, it is flying toward perigee.

4. Calculate the specific angular momentum:

$$\mathbf{h} = \mathbf{r}\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ X & Y & Z \\ v_X & v_Y & v_Z \end{vmatrix}$$

5. Calculate the magnitude of the specific angular momentum:

$$h = \sqrt{\mathbf{h}\cdot\mathbf{h}}$$

the first orbital element.

6. Calculate the inclination:

$$i = \cos^{-1}\left(\frac{h_Z}{h}\right) \tag{4.7}$$

This is the second orbital element. Recall that i must lie between 0° and 180°, which is precisely the range (principle values) of the arccosine function. Hence, there is no quadrant ambiguity to contend with here. If $90° < i \le 180°$, the angular momentum **h** points in a southerly direction. In that case, the orbit is retrograde, which means that the motion of the satellite around the earth is opposite to earth's rotation.

7. Calculate:

$$\mathbf{N} = \hat{\mathbf{K}} \times \mathbf{h} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0 & 0 & 1 \\ h_X & h_Y & h_Z \end{vmatrix} \tag{4.8}$$

This vector defines the node line.

8. Calculate the magnitude of **N**:

$$N = \sqrt{\mathbf{N}\cdot\mathbf{N}}$$

9. Calculate the right ascension of the ascending node:

$$\Omega = \cos^{-1}(N_X/N)$$

the third orbital element. If $(N_x/N) > 0$, then Ω lies in either the first or fourth quadrant. If $(N_x/N) < 0$, then Ω lies in either the second or third quadrant. To place Ω in the proper quadrant, observe that the ascending node lies on the positive side of the vertical XZ plane ($0 \le \Omega < 180°$) if $N_Y > 0$. On the other hand, the ascending node lies on the negative side of the XZ plane ($180° \le \Omega < 360°$) if $N_Y < 0$. Therefore, $N_Y > 0$ implies that $0 < \Omega < 180°$, whereas $N_Y < 0$ implies that $180° < \Omega < 360°$. In summary,

$$\Omega = \begin{cases} \cos^{-1}\left(\dfrac{N_X}{N}\right) & (N_Y \ge 0) \\ 360° - \cos^{-1}\left(\dfrac{N_X}{N}\right) & (N_Y < 0) \end{cases} \tag{4.9}$$

10. Calculate the eccentricity vector. Starting with Eqn (2.40):

$$\mathbf{e} = \frac{1}{\mu}\left[\mathbf{v}\times\mathbf{h} - \mu\frac{\mathbf{r}}{r}\right] = \frac{1}{\mu}\left[\mathbf{v}\times(\mathbf{r}\times\mathbf{v}) - \mu\frac{\mathbf{r}}{r}\right] = \frac{1}{\mu}\left[\overbrace{\mathbf{r}v^2 - \mathbf{v}(\mathbf{r}\cdot\mathbf{v})}^{\text{bac}-\text{cab rule}} - \mu\frac{\mathbf{r}}{r}\right]$$

so that

$$\mathbf{e} = \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\mathbf{r} - r\,v_r\mathbf{v}\right] \tag{4.10}$$

11. Calculate the eccentricity:

$$e = \sqrt{\mathbf{e}\cdot\mathbf{e}}$$

the fourth orbital element. Substituting Eqn (4.10) leads to a form depending only on the scalars obtained thus far,

$$e = \sqrt{1 + \frac{h^2}{\mu^2}\left(v^2 - \frac{2\mu}{r}\right)} \tag{4.11}$$

12. Calculate the argument of perigee:

$$\omega = \cos^{-1}\left(\frac{\mathbf{N}}{N}\cdot\frac{\mathbf{e}}{e}\right)$$

the fifth orbital element. If $\mathbf{N}\cdot\mathbf{e} > 0$, then ω lies in either the first or fourth quadrant. If $\mathbf{N}\cdot\mathbf{e} < 0$, then ω lies in either the second or third quadrant. To place ω in the proper quadrant, observe that perigee lies above the equatorial plane ($0 \le \omega < 180°$) if $\mathbf{e}$ points up (in the positive Z direction) and that perigee lies below the plane ($180° \le \omega < 360°$) if $\mathbf{e}$ points down. Therefore, $e_Z \ge 0$ implies that $0 < \omega < 180°$, whereas $e_Z < 0$ implies that $180° < \omega < 360°$. To summarize,

$$\omega = \begin{cases} \cos^{-1}\left(\dfrac{\mathbf{N}\cdot\mathbf{e}}{Ne}\right) & (e_Z \ge 0) \\[2ex] 360° - \cos^{-1}\left(\dfrac{\mathbf{N}\cdot\mathbf{e}}{Ne}\right) & (e_Z < 0) \end{cases} \tag{4.12}$$

13. Calculate the true anomaly:

$$\theta = \cos^{-1}\left(\frac{\mathbf{e}}{e}\cdot\frac{\mathbf{r}}{r}\right)$$

the sixth and final orbital element. If $\mathbf{e}\cdot\mathbf{r} > 0$, then θ lies in the first or fourth quadrant. If $\mathbf{e}\cdot\mathbf{r} < 0$, then θ lies in the second or third quadrant. To place θ in the proper quadrant, note that if the satellite is flying away from perigee ($\mathbf{r}\cdot\mathbf{v} \ge 0$), then $0 \le \theta < 180°$, whereas if the satellite is flying toward perigee ($\mathbf{r}\cdot\mathbf{v} < 0$), then $180° \le \theta < 360°$. Therefore, using the results of Step 3 above

$$\theta = \begin{cases} \cos^{-1}\left(\dfrac{\mathbf{e}}{e}\cdot\dfrac{\mathbf{r}}{r}\right) & (v_r \ge 0) \\[2ex] 360° - \cos^{-1}\left(\dfrac{\mathbf{e}}{e}\cdot\dfrac{\mathbf{r}}{r}\right) & (v_r < 0) \end{cases} \tag{4.13a}$$

Substituting Eqn (4.10) yields an alternative form of this expression,

$$\theta = \begin{cases} \cos^{-1}\left[\dfrac{1}{e}\left(\dfrac{h^2}{\mu r} - 1\right)\right] & (v_r \ge 0) \\[2ex] 360° - \cos^{-1}\left[\frac{1}{e}\left(\dfrac{h^2}{\mu r} - 1\right)\right] & (v_r < 0) \end{cases} \tag{4.13b}$$

The procedure described above for calculating the orbital elements is not unique.

EXAMPLE 4.3

Given the state vector,

$$\mathbf{r} = -6045\hat{\mathbf{I}} - 3490\hat{\mathbf{J}} + 2500\hat{\mathbf{K}}\ (\text{km})$$
$$\mathbf{v} = -3.457\hat{\mathbf{I}} + 6.618\hat{\mathbf{J}} + 2.533\hat{\mathbf{K}}\ (\text{km/s})$$

find the orbital elements h, i, Ω, e, ω, and θ using Algorithm 4.2.

Solution

Step 1:

$$r = \sqrt{\mathbf{r}\cdot\mathbf{r}} = \sqrt{(-6045)^2 + (-3490)^2 + 2500^2} = 7414\ \text{km} \tag{a}$$

Step 2:

$$v = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{(-3.457)^2 + 6.618^2 + 2.533^2} = 7.884\ \text{km/s} \tag{b}$$

Step 3:

$$v_r = \frac{\mathbf{v}\cdot\mathbf{r}}{r} = \frac{(-3.457)\cdot(-6045) + 6.618\cdot(-3490) + 2.533\cdot 2500}{7414} = 0.5575\ \text{km/s} \tag{c}$$

Since $v_r > 0$, the satellite is flying away from perigee.
Step 4:

$$\mathbf{h} = \mathbf{r}\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ -6045 & -3490 & 2500 \\ -3.457 & 6.618 & 2.533 \end{vmatrix} = -25{,}380\hat{\mathbf{I}} + 6670\hat{\mathbf{J}} - 52{,}070\hat{\mathbf{K}}\ \left(\text{km}^2/\text{s}\right) \tag{d}$$

Step 5:

$$h = \sqrt{\mathbf{h}\cdot\mathbf{h}} = \sqrt{(-25{,}380)^2 + 6670^2 + (-52{,}070)^2} = \boxed{58{,}310\ \text{km}^2/\text{s}} \tag{e}$$

Step 6:

$$i = \cos^{-1}\frac{h_Z}{h} = \cos^{-1}\left(\frac{-52{,}070}{58{,}310}\right) = \boxed{153.2^\circ} \tag{f}$$

Since i is greater than 90°, this is a retrograde orbit.
Step 7:

$$\mathbf{N} = \hat{\mathbf{K}}\times\mathbf{h} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0 & 0 & 1 \\ -25{,}380 & 6670 & -52{,}070 \end{vmatrix} = -6670\hat{\mathbf{I}} - 25{,}380\hat{\mathbf{J}}\ \left(\text{km}^2/\text{s}\right) \tag{g}$$

Step 8:

$$N = \sqrt{\mathbf{N}\cdot\mathbf{N}} = \sqrt{(-6670)^2 + (-25{,}380)^2} = 26{,}250\ \text{km}^2/\text{s} \tag{h}$$

Step 9:

$$\Omega = \cos^{-1}\frac{N_X}{N} = \cos^{-1}\left(\frac{-6670}{26{,}250}\right) = 104.7^\circ \text{ or } 255.3^\circ$$

From Eqn (g) we know that $N_Y < 0$; therefore, Ω must lie in the third quadrant,

$$\boxed{\Omega = 255.3^\circ} \tag{i}$$

Step 10:

$$\begin{aligned}
\mathbf{e} &= \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\mathbf{r} - r v_r \mathbf{v}\right] \\
&= \frac{1}{398{,}600}\left[\left(7.884^2 - \frac{398{,}600}{7414}\right)\left(-6045\hat{\mathbf{I}} - 3490\hat{\mathbf{J}} + 2500\hat{\mathbf{K}}\right)\right. \\
&\qquad \left. -(7414)(0.5575)\left(-3.457\hat{\mathbf{I}} + 6.618\hat{\mathbf{J}} + 2.533\hat{\mathbf{K}}\right)\right] \\
\mathbf{e} &= -0.09160\hat{\mathbf{I}} - 0.1422\hat{\mathbf{J}} + 0.02644\hat{\mathbf{K}}
\end{aligned} \tag{j}$$

Step 11:

$$e = \sqrt{\mathbf{e}\cdot\mathbf{e}} = \sqrt{(-0.09160)^2 + (-0.1422)^2 + (0.02644)^2} = \boxed{0.1712} \tag{k}$$

Clearly, the orbit is an ellipse.
Step 12:

$$\begin{aligned}
\omega &= \cos^{-1}\frac{\mathbf{N}\cdot\mathbf{e}}{Ne} = \cos^{-1}\left[\frac{(-6670)(-0.09160) + (-25{,}380)(-0.1422) + (0)(0.02644)}{(26{,}250)(0.1712)}\right] \\
&= 20.07^\circ \text{ or } 339.9^\circ
\end{aligned}$$

ω lies in the first quadrant if $e_Z > 0$, which is true in this case, as we see from Eqn (j). Therefore,

$$\boxed{\omega = 20.07^\circ} \tag{l}$$

Step 13:

$$\begin{aligned}
\theta &= \cos^{-1}\left(\frac{\mathbf{e}\cdot\mathbf{r}}{er}\right) = \cos^{-1}\left[\frac{(-0.09160)(-6045) + (-0.1422)\cdot(-3490) + (0.02644)(2500)}{(0.1712)(7414)}\right] \\
&= 28.45^\circ \text{ or } 331.6^\circ
\end{aligned}$$

From Eqn (c) we know that $v_r > 0$, which means $0 \le \theta < 180^\circ$. Therefore,

$$\boxed{\theta = 28.45^\circ}$$

Having found the orbital elements, we can go on to compute other parameters. The perigee and apogee radii are

$$r_\text{p} = \frac{h^2}{\mu}\frac{1}{1 + e\cos(0)} = \frac{58{,}310^2}{398{,}600}\frac{1}{1 + 0.1712} = 7284 \text{ km}$$

$$r_\text{a} = \frac{h^2}{\mu}\frac{1}{1 + e\cos(180^\circ)} = \frac{58{,}310^2}{398{,}600}\frac{1}{1 - 0.1712} = 10{,}290 \text{ km}$$

From these it follows that the semimajor axis of the ellipse is

$$a = \frac{1}{2}\left(r_\text{p} + r_\text{a}\right) = 8788 \text{ km}$$

This leads to the period,

$$T = \frac{2\pi}{\sqrt{\mu}}a^{\frac{3}{2}} = 2.278 \text{ h}$$

The orbit is illustrated in Figure 4.8.

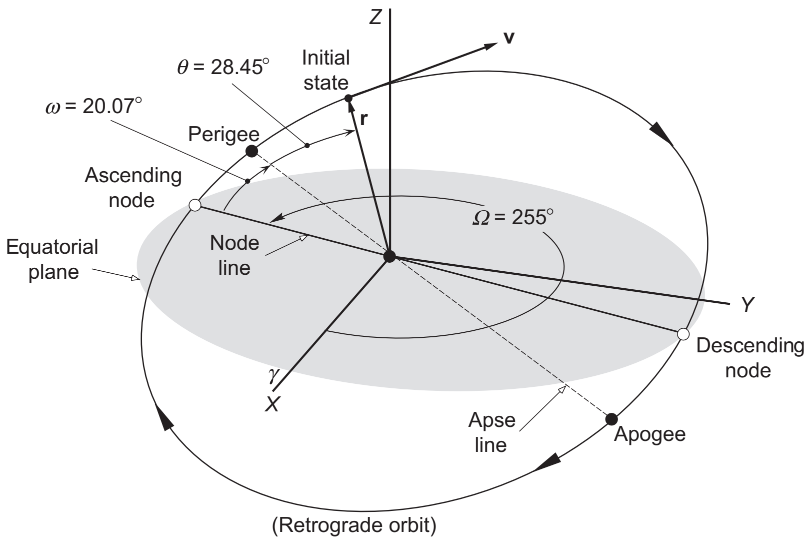


FIGURE 4.8

A plot of the orbit identified in Example 4.3.

We have seen how to obtain the orbital elements from the state vector. To arrive at the state vector, given the orbital elements, requires performing coordinate transformations, which are discussed in the next section.

4.5 Coordinate transformation

The Cartesian coordinate system was introduced in Section 1.2. Figure 4.9 shows two such coordinate systems: the unprimed system with axes xyz, and the primed system with axes $x'y'z'$. The orthogonal unit basis vectors for the unprimed system are $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$. The fact they are unit vectors means

$$\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1 \tag{4.14}$$

Since they are orthogonal,

$$\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{i}}\cdot\hat{\mathbf{k}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = 0 \tag{4.15}$$

The orthonormal basis vectors $\hat{\mathbf{i}}'$, $\hat{\mathbf{j}}'$, and $\hat{\mathbf{k}}'$ of the primed system share these same properties. That is,

$$\hat{\mathbf{i}}'\cdot\hat{\mathbf{i}}' = \hat{\mathbf{j}}'\cdot\hat{\mathbf{j}}' = \hat{\mathbf{k}}'\cdot\hat{\mathbf{k}}' = 1 \tag{4.16}$$

and

$$\hat{\mathbf{i}}'\cdot\hat{\mathbf{j}}' = \hat{\mathbf{i}}'\cdot\hat{\mathbf{k}}' = \hat{\mathbf{j}}'\cdot\hat{\mathbf{k}}' = 0 \tag{4.17}$$

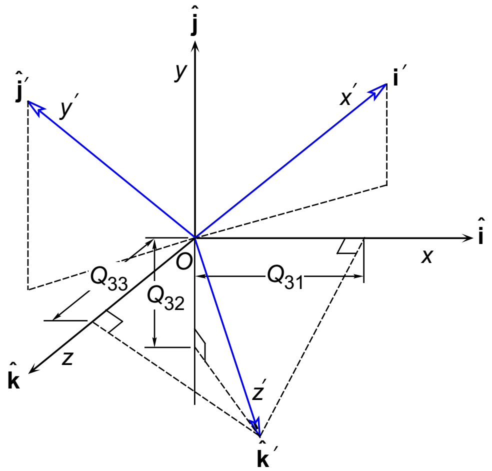


FIGURE 4.9

Two sets of Cartesian reference axes, *xyz* and *x′y′z′*.

We can express the unit vectors of the primed system in terms of their components in the unprimed system as follows:

$$
\begin{aligned}
\hat{\mathbf{i}}' &= Q_{11}\hat{\mathbf{i}} + Q_{12}\hat{\mathbf{j}} + Q_{13}\hat{\mathbf{k}} \\
\hat{\mathbf{j}}' &= Q_{21}\hat{\mathbf{i}} + Q_{22}\hat{\mathbf{j}} + Q_{23}\hat{\mathbf{k}} \\
\hat{\mathbf{k}}' &= Q_{31}\hat{\mathbf{i}} + Q_{32}\hat{\mathbf{j}} + Q_{33}\hat{\mathbf{k}}
\end{aligned}
\tag{4.18}
$$

The Q's in these expressions are just the direction cosines of $\hat{\mathbf{i}}'$, $\hat{\mathbf{j}}'$, and $\hat{\mathbf{k}}'$. Figure 4.9 illustrates the components of $\hat{\mathbf{k}}'$, which are, of course, the projections of $\hat{\mathbf{k}}'$ onto the x-, y-, and z-axes. The unprimed unit vectors may be resolved into components along the primed system to obtain a set of equations similar to Eqn (4.18).

$$
\begin{aligned}
\hat{\mathbf{i}} &= Q'_{11}\hat{\mathbf{i}}' + Q'_{12}\hat{\mathbf{j}}' + Q'_{13}\hat{\mathbf{k}}' \\
\hat{\mathbf{j}} &= Q'_{21}\hat{\mathbf{i}}' + Q'_{22}\hat{\mathbf{j}}' + Q'_{23}\hat{\mathbf{k}}' \\
\hat{\mathbf{k}} &= Q'_{31}\hat{\mathbf{i}}' + Q'_{32}\hat{\mathbf{j}}' + Q'_{33}\hat{\mathbf{k}}'
\end{aligned}
\tag{4.19}
$$

However, $\hat{\mathbf{i}}'\cdot\hat{\mathbf{i}} = \hat{\mathbf{i}}\cdot\hat{\mathbf{i}}'$, so that, from Eqns (4.18) and (4.19), we find $Q_{11} = Q'_{11}$. Likewise, $\hat{\mathbf{i}}'\cdot\hat{\mathbf{j}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{i}}'$, which, according to Eqns (4.18) and (4.19), means $Q_{12} = Q'_{21}$. Proceeding in this fashion, it is clear that the direction cosines in Eqn (4.18) may be expressed in terms of those in Eqn (4.19). That is, Eqn (4.19) may be written

$$
\begin{aligned}
\hat{\mathbf{i}} &= Q_{11}\hat{\mathbf{i}}' + Q_{21}\hat{\mathbf{j}}' + Q_{31}\hat{\mathbf{k}}' \\
\hat{\mathbf{j}} &= Q_{12}\hat{\mathbf{i}}' + Q_{22}\hat{\mathbf{j}}' + Q_{32}\hat{\mathbf{k}}' \\
\hat{\mathbf{k}} &= Q_{13}\hat{\mathbf{i}}' + Q_{23}\hat{\mathbf{j}}' + Q_{33}\hat{\mathbf{k}}'
\end{aligned}
\tag{4.20}
$$

Substituting Eqn (4.20) into Eqn (4.14) and making use of Eqns (4.16) and (4.17), we get the three relations

$$\begin{aligned} \hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = 1 \quad &\Rightarrow \quad Q_{11}^2 + Q_{21}^2 + Q_{31}^2 = 1 \\ \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = 1 \quad &\Rightarrow \quad Q_{12}^2 + Q_{22}^2 + Q_{32}^2 = 1 \\ \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1 \quad &\Rightarrow \quad Q_{13}^2 + Q_{23}^2 + Q_{33}^2 = 1 \end{aligned} \tag{4.21}$$

Substituting Eqn (4.20) into Eqn (4.15) and, again, making use of Eqns (4.16) and (4.17), we obtain the three equations

$$\begin{aligned} \hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = 0 \quad &\Rightarrow \quad Q_{11}Q_{12} + Q_{21}Q_{22} + Q_{31}Q_{32} = 0 \\ \hat{\mathbf{i}}\cdot\hat{\mathbf{k}} = 0 \quad &\Rightarrow \quad Q_{11}Q_{13} + Q_{21}Q_{23} + Q_{31}Q_{33} = 0 \\ \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = 0 \quad &\Rightarrow \quad Q_{12}Q_{13} + Q_{22}Q_{23} + Q_{32}Q_{33} = 0 \end{aligned} \tag{4.22}$$

Let $[\mathbf{Q}]$ represent the matrix of direction cosines of $\hat{\mathbf{i}}'$, $\hat{\mathbf{j}}'$, and $\hat{\mathbf{k}}'$ relative to $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$, as given by Eqn (4.18). $[\mathbf{Q}]$ is referred to as the direction cosine matrix (DCM).

$$[\mathbf{Q}] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{i}}'\cdot\hat{\mathbf{i}} & \hat{\mathbf{i}}'\cdot\hat{\mathbf{j}} & \hat{\mathbf{i}}'\cdot\hat{\mathbf{k}} \\ \hat{\mathbf{j}}'\cdot\hat{\mathbf{i}} & \hat{\mathbf{j}}'\cdot\hat{\mathbf{j}} & \hat{\mathbf{j}}'\cdot\hat{\mathbf{k}} \\ \hat{\mathbf{k}}'\cdot\hat{\mathbf{i}} & \hat{\mathbf{k}}'\cdot\hat{\mathbf{j}} & \hat{\mathbf{k}}'\cdot\hat{\mathbf{k}} \end{bmatrix} \tag{4.23}$$

The transpose of the matrix $[\mathbf{Q}]$, denoted $[\mathbf{Q}]^T$, is obtained by interchanging the rows and columns of $[\mathbf{Q}]$. Thus,

$$[\mathbf{Q}]^T = \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{i}}\cdot\hat{\mathbf{i}}' & \hat{\mathbf{i}}\cdot\hat{\mathbf{j}}' & \hat{\mathbf{i}}\cdot\hat{\mathbf{k}}' \\ \hat{\mathbf{j}}\cdot\hat{\mathbf{i}}' & \hat{\mathbf{j}}\cdot\hat{\mathbf{j}}' & \hat{\mathbf{j}}\cdot\hat{\mathbf{k}}' \\ \hat{\mathbf{k}}\cdot\hat{\mathbf{i}}' & \hat{\mathbf{k}}\cdot\hat{\mathbf{j}}' & \hat{\mathbf{k}}\cdot\hat{\mathbf{k}}' \end{bmatrix} \tag{4.24}$$

Forming the product $[\mathbf{Q}]^T[\mathbf{Q}]$, we get

$$[\mathbf{Q}]^T[\mathbf{Q}] = \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}$$

$$= \begin{bmatrix} Q_{11}^2 + Q_{21}^2 + Q_{31}^2 & Q_{11}Q_{12} + Q_{21}Q_{22} + Q_{31}Q_{32} & Q_{11}Q_{13} + Q_{21}Q_{23} + Q_{31}Q_{33} \\ Q_{12}Q_{11} + Q_{22}Q_{21} + Q_{32}Q_{31} & Q_{12}^2 + Q_{22}^2 + Q_{32}^2 & Q_{12}Q_{13} + Q_{22}Q_{23} + Q_{32}Q_{33} \\ Q_{13}Q_{11} + Q_{23}Q_{21} + Q_{33}Q_{31} & Q_{13}Q_{12} + Q_{23}Q_{22} + Q_{33}Q_{32} & Q_{13}^2 + Q_{23}^2 + Q_{33}^2 \end{bmatrix}$$

From this we obtain, with the aid of Eqns (4.21) and (4.22),

$$[\mathbf{Q}]^T[\mathbf{Q}] = [\mathbf{1}] \tag{4.25}$$

where

$$[\mathbf{1}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[**1**] stands for the identity matrix or unit matrix.

In a similar fashion, we can substitute Eqn (4.18) into Eqns (4.16) and (4.17) and make use of Eqns (4.14) and (4.15) to finally obtain

$$[\mathbf{Q}][\mathbf{Q}]^T = [1] \tag{4.26}$$

Since [**Q**] satisfies Eqns (4.25) and (4.26), it is called an orthogonal matrix.

Let **v** be a vector. It can be expressed in terms of its components along the unprimed system

$$\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$$

or along the primed system

$$\mathbf{v} = v'_x\hat{\mathbf{i}}' + v'_y\hat{\mathbf{j}}' + v'_z\hat{\mathbf{k}}'$$

These two expressions for **v** are equivalent (**v** = **v**) since a vector is independent of the coordinate system used to describe it. Thus,

$$v'_x\hat{\mathbf{i}}' + v'_y\hat{\mathbf{j}}' + v'_z\hat{\mathbf{k}}' = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}} \tag{4.27}$$

Substituting Eqn (4.20) into the right-hand side of Eqn (4.27) yields

$$\begin{aligned} v'_x\hat{\mathbf{i}}' + v'_y\hat{\mathbf{j}}' + v'_z\hat{\mathbf{k}}' = {} & v_x\left(Q_{11}\hat{\mathbf{i}}' + Q_{21}\hat{\mathbf{j}}' + Q_{31}\hat{\mathbf{k}}'\right) \\ & + v_y\left(Q_{12}\hat{\mathbf{i}}' + Q_{22}\hat{\mathbf{j}}' + Q_{32}\hat{\mathbf{k}}'\right) + v_z\left(Q_{13}\hat{\mathbf{i}}' + Q_{23}\hat{\mathbf{j}}' + Q_{33}\hat{\mathbf{k}}'\right) \end{aligned}$$

On collecting terms on the right, we get

$$\begin{aligned} v'_x\hat{\mathbf{i}}' + v'_y\hat{\mathbf{j}}' + v'_z\hat{\mathbf{k}}' = {} & (Q_{11}v_x + Q_{12}v_y + Q_{13}v_z)\hat{\mathbf{i}}' \\ & + (Q_{21}v_x + Q_{22}v_y + Q_{23}v_z)\hat{\mathbf{j}}' + (Q_{31}v_x + Q_{32}v_y + Q_{33}v_z)\hat{\mathbf{k}}' \end{aligned}$$

Equating the components of like unit vectors on each side of the equals sign yields

$$\begin{aligned} v'_x &= Q_{11}v_x + Q_{12}v_y + Q_{13}v_z \\ v'_y &= Q_{21}v_x + Q_{22}v_y + Q_{23}v_z \\ v'_z &= Q_{31}v_x + Q_{32}v_y + Q_{33}v_z \end{aligned} \tag{4.28}$$

In matrix notation, this may be written

$$\{\mathbf{v}'\} = [\mathbf{Q}]\{\mathbf{v}\} \tag{4.29}$$

where

$$\{\mathbf{v}'\} = \begin{Bmatrix} v'_x \\ v'_y \\ v'_z \end{Bmatrix} \quad \{\mathbf{v}\} = \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} \tag{4.30}$$

and $[\mathbf{Q}]$ is given by Eqn (4.23). Equation (4.28) (or Eqn (4.29)) shows how to transform the components of the vector $\mathbf{v}$ in the unprimed system into its components in the primed system. The inverse transformation, from primed to unprimed, is found by multiplying Eqn (4.29) throughout by $[\mathbf{Q}]^T$:

$$[\mathbf{Q}]^T\{\mathbf{v}'\} = [\mathbf{Q}]^T[\mathbf{Q}]\{\mathbf{v}\}$$

But, according to Eqn (4.25), $[\mathbf{Q}][\mathbf{Q}]^T = [\mathbf{1}]$, so that

$$[\mathbf{Q}]^T\{\mathbf{v}'\} = [1]\{\mathbf{v}\}$$

Since $[\mathbf{1}]\{\mathbf{v}\} = \{\mathbf{v}\}$, we obtain

$$\{\mathbf{v}\} = [\mathbf{Q}]^T\{\mathbf{v}'\} \tag{4.31}$$

Therefore, to go from the unprimed system to the primed system we use $[\mathbf{Q}]$, and in the reverse direction—from primed to unprimed—we use $[\mathbf{Q}]^T$.

EXAMPLE 4.4

In Figure 4.10, the x'-axis is defined by the line segment $O'P$. The $x'y'$ plane is defined by the intersecting line segments $O'P$ and $O'Q$. The z'-axis is normal to the plane of $O'P$ and $O'Q$ and obtained by rotating $O'P$ toward $O'Q$ and using the right-hand rule.

(a) Find the transformation matrix $[\mathbf{Q}]$.
(b) If $\{\mathbf{v}\} = \lfloor 2 \quad 4 \quad 6 \rfloor^T$, find $\{\mathbf{v}'\}$.
(c) If $\{\mathbf{v}'\} = \lfloor 2 \quad 4 \quad 6 \rfloor^T$, find $\{\mathbf{v}\}$.

Solution

(a) Resolve the directed line segments $\overrightarrow{O'P}$ and $\overrightarrow{O'Q}$ into components along the unprimed system:

$$\overrightarrow{O'P} = (-5-3)\hat{\mathbf{i}} + (5-1)\hat{\mathbf{j}} + (4-2)\hat{\mathbf{k}} = -8\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$
$$\overrightarrow{O'Q} = (-6-3)\hat{\mathbf{i}} + (3-1)\hat{\mathbf{j}} + (5-2)\hat{\mathbf{k}} = -9\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

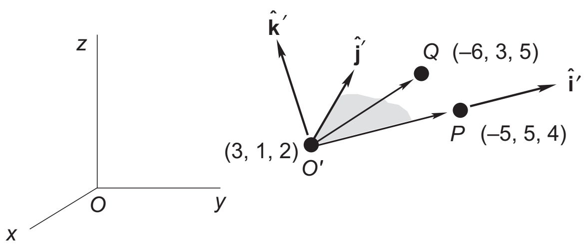


FIGURE 4.10

Defining a unit triad from the coordinates of three noncollinear points, O', P and Q.

Taking the cross product of $\overrightarrow{O'P}$ into $\overrightarrow{O'Q}$ yields a vector $\mathbf{Z}'$, which lies in the direction of the desired positive z'-axis:

$$\mathbf{Z}' = \overrightarrow{O'P} \times \overrightarrow{O'Q} = 8\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 20\hat{\mathbf{k}}$$

Taking the cross product of $\mathbf{Z}'$ into $\overrightarrow{O'P}$ then yields a vector $\mathbf{Y}'$, which points in the positive y' direction:

$$\mathbf{Y}' = \mathbf{Z}' \times \overrightarrow{O'P} = -68\hat{\mathbf{i}} - 176\hat{\mathbf{j}} + 80\hat{\mathbf{k}}$$

Normalizing the vectors $\overrightarrow{O'P}$, $\mathbf{Y}'$ and $\mathbf{Z}'$ produces the $\hat{\mathbf{i}}'$, $\hat{\mathbf{j}}'$, and $\hat{\mathbf{k}}'$ unit vectors, respectively. Thus

$$\hat{\mathbf{i}}' = \frac{\overrightarrow{O'P}}{\left\|\overrightarrow{O'P}\right\|} = -0.8729\hat{\mathbf{i}} + 0.4364\hat{\mathbf{j}} + 0.2182\hat{\mathbf{k}}$$

$$\hat{\mathbf{j}}' = \frac{\mathbf{Y}'}{\|\mathbf{Y}'\|} = -0.3318\hat{\mathbf{i}} - 0.8588\hat{\mathbf{j}} + 0.3904\hat{\mathbf{k}}$$

and

$$\hat{\mathbf{k}}' = \frac{\mathbf{Z}'}{\|\mathbf{Z}'\|} = 0.3578\hat{\mathbf{i}} + 0.2683\hat{\mathbf{j}} + 0.8944\hat{\mathbf{k}}$$

The components of $\hat{\mathbf{i}}'$, $\hat{\mathbf{j}}'$, and $\hat{\mathbf{k}}'$ are the rows of the DCM $[\mathbf{Q}]$. Thus,

$$[\mathbf{Q}] = \begin{bmatrix} -0.8729 & 0.4364 & 0.2182 \\ -0.3318 & -0.8588 & 0.3904 \\ 0.3578 & 0.2683 & 0.8944 \end{bmatrix}$$

(b)

$$\{\mathbf{v}'\} = [\mathbf{Q}]\{\mathbf{v}\} = \begin{bmatrix} -0.8729 & 0.4364 & 0.2182 \\ -0.3318 & -0.8588 & 0.3904 \\ 0.3578 & 0.2683 & 0.8944 \end{bmatrix} \begin{Bmatrix} 2 \\ 4 \\ 6 \end{Bmatrix} = \boxed{\begin{Bmatrix} 1.309 \\ -1.756 \\ 7.155 \end{Bmatrix}}$$

(c)

$$\{\mathbf{v}\} = [\mathbf{Q}]^T\{\mathbf{v}'\} = \begin{bmatrix} -0.8729 & -0.3318 & 0.3578 \\ 0.4364 & -0.8588 & 0.2683 \\ 0.2182 & 0.3904 & 0.8944 \end{bmatrix} \begin{Bmatrix} 2 \\ 4 \\ 6 \end{Bmatrix} = \boxed{\begin{Bmatrix} -0.9263 \\ -0.9523 \\ 7.364 \end{Bmatrix}}$$

Let us consider the special case in which the coordinate transformation involves a rotation about only one of the coordinate axes, as shown in Figure 4.11. If the rotation is about the x-axis, then according to Eqns (4.18) and (4.23),

$$\begin{aligned}
\hat{\mathbf{i}}' &= \hat{\mathbf{i}} \\
\hat{\mathbf{j}}' &= \left(\hat{\mathbf{j}}' \cdot \hat{\mathbf{i}}\right)\hat{\mathbf{i}} + \left(\hat{\mathbf{j}}' \cdot \hat{\mathbf{j}}\right)\hat{\mathbf{j}} + \left(\hat{\mathbf{j}}' \cdot \hat{\mathbf{k}}\right)\hat{\mathbf{k}} = \cos\phi\hat{\mathbf{j}} + \cos(90 - \phi)\hat{\mathbf{k}} = \cos\phi\hat{\mathbf{j}} + \sin(\phi)\hat{\mathbf{k}} \\
\hat{\mathbf{k}}' &= \left(\hat{\mathbf{k}}' \cdot \hat{\mathbf{j}}\right)\hat{\mathbf{j}} + \left(\hat{\mathbf{k}}' \cdot \hat{\mathbf{k}}\right)\hat{\mathbf{k}} = \cos(90^\circ + \phi)\hat{\mathbf{j}} + \cos\phi\hat{\mathbf{k}} = -\sin\phi\hat{\mathbf{j}} + \cos\phi\hat{\mathbf{k}}
\end{aligned}$$

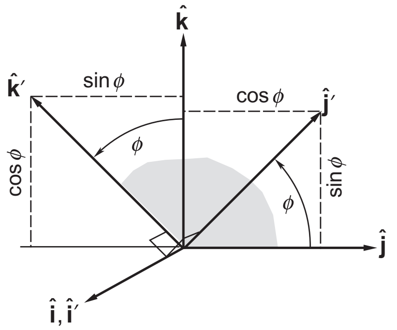


FIGURE 4.11

Rotation about the *x*-axis.

or

$$\begin{Bmatrix} \hat{\mathbf{i}}' \\ \hat{\mathbf{j}}' \\ \hat{\mathbf{k}}' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{Bmatrix}$$

The transformation from the xyz coordinate system to the $xy'z'$ system having a common x-axis is given by the direct cosine matrix on the right. Since this is a rotation through the angle ϕ about the x-axis, we denote this matrix by $[\mathbf{R}_1(\phi)]$, in which the subscript 1 stands for axis 1 (the x-axis). Thus,

$$[\mathbf{R}_1(\phi)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \tag{4.32}$$

If the rotation is about the y-axis, as shown in Figure 4.12, then Eqn (4.18) yields

$$\begin{aligned} \hat{\mathbf{i}}' &= \left(\hat{\mathbf{i}}' \cdot \hat{\mathbf{i}}\right)\hat{\mathbf{i}} + \left(\hat{\mathbf{i}}' \cdot \hat{\mathbf{k}}\right)\hat{\mathbf{k}} = \cos\phi\hat{\mathbf{i}} + \cos(\phi + 90^\circ)\hat{\mathbf{k}} = \cos\phi\hat{\mathbf{i}} - \sin\phi\hat{\mathbf{k}} \\ \hat{\mathbf{j}}' &= \hat{\mathbf{j}} \\ \hat{\mathbf{k}}' &= \left(\hat{\mathbf{k}}' \cdot \hat{\mathbf{i}}\right)\hat{\mathbf{i}} + \left(\hat{\mathbf{k}}' \cdot \hat{\mathbf{k}}\right)\hat{\mathbf{k}} = \cos(90^\circ - \phi)\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{k}} = \sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{k}} \end{aligned}$$

or, more compactly,

$$\begin{Bmatrix} \hat{\mathbf{i}}' \\ \hat{\mathbf{j}}' \\ \hat{\mathbf{k}}' \end{Bmatrix} = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{Bmatrix}$$

We represent this transformation between two Cartesian coordinate systems having a common y-axis (axis 2) as $[\mathbf{R}_2(\phi)]$. Therefore,

$$[\mathbf{R}_2(\phi)] = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \tag{4.33}$$

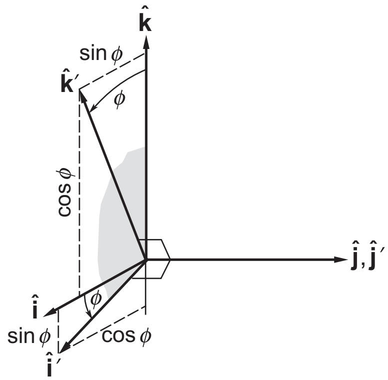


FIGURE 4.12

Rotation about the *y*-axis.

Finally, if the rotation is about the *z*-axis, as shown in Figure 4.13, then we have from Eqn (4.18) that

$$\begin{aligned}
\hat{\mathbf{i}}' &= \left(\hat{\mathbf{i}}' \cdot \hat{\mathbf{i}}\right)\hat{\mathbf{i}} + \left(\hat{\mathbf{i}}' \cdot \hat{\mathbf{j}}\right)\hat{\mathbf{j}} = \cos\phi\hat{\mathbf{i}} + \cos(90^\circ - \phi)\hat{\mathbf{j}} = \cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}} \\
\hat{\mathbf{j}}' &= \left(\hat{\mathbf{j}}' \cdot \hat{\mathbf{i}}\right)\hat{\mathbf{i}} + \left(\hat{\mathbf{j}}' \cdot \hat{\mathbf{j}}\right)\hat{\mathbf{j}} = \cos(90^\circ + \phi)\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}} = -\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}} \\
\hat{\mathbf{k}}' &= \hat{\mathbf{k}}
\end{aligned}$$

or

$$\begin{Bmatrix} \hat{\mathbf{i}}' \\ \hat{\mathbf{j}}' \\ \hat{\mathbf{k}}' \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{Bmatrix}$$

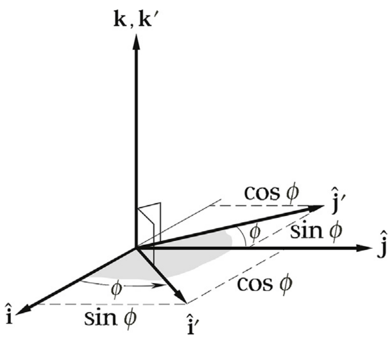


FIGURE 4.13

Rotation about the *z*-axis.

In this case, the rotation is around axis 3, the z-axis, so

$$[\mathbf{R}_3(\phi)] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.34}$$

The single transformation between the xyz and $x'y'z'$ Cartesian coordinate frames can be viewed as a sequence of three coordinate transformations, starting with xyz:

$$xyz \overbrace{\rightarrow}^{\alpha} x_1y_1z_1 \overbrace{\rightarrow}^{\beta} x_2y_2z_2 \overbrace{\rightarrow}^{\gamma} x'y'z'$$

Each coordinate system is obtained from the previous one by means of a rotation about one of the axes of the previous frame. Two successive rotations cannot be about the same axis. The first rotation angle is α, the second one is β, and the final one is γ. In specific applications, the Greek letters that are traditionally used to represent the three rotations are not α, β, and γ. For those new to the subject, however, it might initially be easier to remember that the first, second, and third rotation angles are represented by the first, second, and third letters of the Greek alphabet ($\alpha\beta\gamma$). Each one of the three transformations has the direct cosine matrix $[\mathbf{R}_i(\phi)]$, where $i = 1$, 2, or 3 and $\phi = \alpha$, β, or γ. The sequence of three such elementary rotations relating two different Cartesian frames of reference is called a Euler angle sequence. Each of the 12 possible Euler angle sequences has a direct cosine matrix $[\mathbf{Q}]$, which is the product of three elementary rotation matrices. The six symmetric Euler sequences are those that begin and end with rotation about the same axis:

$$\begin{matrix} [\mathbf{R}_1(\gamma)][\mathbf{R}_2(\beta)][\mathbf{R}_1(\alpha)] & [\mathbf{R}_1(\gamma)][\mathbf{R}_3(\beta)][\mathbf{R}_1(\alpha)] \\ [\mathbf{R}_2(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_2(\alpha)] & [\mathbf{R}_2(\gamma)][\mathbf{R}_3(\beta)][\mathbf{R}_2(\alpha)] \\ [\mathbf{R}_3(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_3(\alpha)] & [\mathbf{R}_3(\gamma)][\mathbf{R}_2(\beta)][\mathbf{R}_3(\alpha)] \end{matrix} \tag{4.35}$$

The asymmetric Euler sequences involve rotations about all three axes:

$$\begin{matrix} [\mathbf{R}_1(\gamma)][\mathbf{R}_2(\beta)][\mathbf{R}_3(\alpha)] & [\mathbf{R}_1(\gamma)][\mathbf{R}_3(\beta)][\mathbf{R}_2(\alpha)] \\ [\mathbf{R}_2(\gamma)][\mathbf{R}_3(\beta)][\mathbf{R}_1(\alpha)] & [\mathbf{R}_2(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_3(\alpha)] \\ [\mathbf{R}_3(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_2(\alpha)] & [\mathbf{R}_3(\gamma)][\mathbf{R}_2(\beta)][\mathbf{R}_1(\alpha)] \end{matrix} \tag{4.36}$$

One of the symmetric sequences that has frequent application in space mechanics is the "classical" Euler angle sequence,

$$[\mathbf{Q}] = [\mathbf{R}_3(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_3(\alpha)] \quad (0 \le \alpha < 360^\circ \quad 0 \le \beta \le 180^\circ \quad 0 \le \gamma < 360^\circ) \tag{4.37}$$

which is illustrated in Figure 4.14.

The first rotation is around the z-axis, through the angle α. It rotates the x and y axes into the x_1 and y_1 directions. Viewed down the z-axis, this rotation appears as shown in the insert at the top of the figure. The direct cosine matrix associated with this rotation is $[\mathbf{R}_3(\alpha)]$. The subscript means that the rotation is around the current (3) direction, which was the z-axis (and is now the z_1-axis).

The second rotation, represented by $[\mathbf{R}_1(\beta)]$, is around the x_1-axis through the angle β required to rotate the z_1-axis into the z_2-axis, which coincides with the target z'-axis. Simultaneously, y_1 rotates

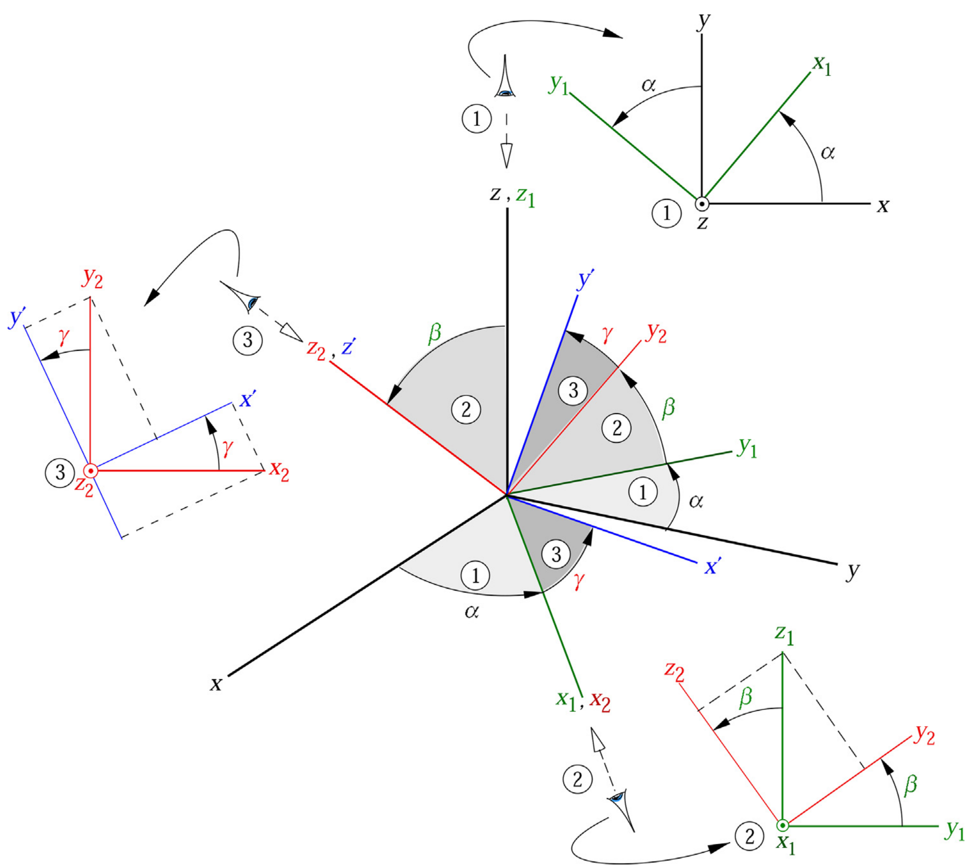


FIGURE 4.14

Classical Euler sequence of three rotations transforming *xyz* into *x′y′z′*. The "eye" viewing down an axis sees the illustrated rotation about that axis.

into y_2. The insert in the lower right of Figure 4.14 shows how this rotation appears when viewed from the x_1 direction.

$[\mathbf{R}_3(\gamma)]$ represents the third and final rotation, which rotates the x_2-axis (formerly the x_1-axis) and the y_2-axis through the angle γ around the z'-axis so that they become aligned with the target x'- and y'-axes, respectively. This rotation appears from the z'-direction as shown in the insert on the left of Figure 4.14.

Applying the transformation in Eqn (4.37) to the xyz components $\{\mathbf{b}\}_x$ of the vector $\mathbf{b} = b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}$ yields the components of the same vector in the $x'y'z'$ frame

$$\{\mathbf{b}\}_{x'} = [\mathbf{Q}]\{\mathbf{b}\}_x \quad \left(\mathbf{b} = b_{x'}\hat{\mathbf{i}}' + b_{y'}\hat{\mathbf{j}}' + b'_z\hat{\mathbf{k}}'\right)$$

That is

$$[\mathbf{Q}]\{\mathbf{b}\}_x = [\mathbf{R}_3(\gamma)][\mathbf{R}_1(\beta)]\overbrace{[\mathbf{R}_3(\alpha)]\{\mathbf{b}\}_x}^{\{\mathbf{b}\}_{x_1}} = [\mathbf{R}_3(\gamma)]\overbrace{[\mathbf{R}_1(\beta)]\{\mathbf{b}\}_{x_1}}^{\{\mathbf{b}\}_{x_2}} = \overbrace{[\mathbf{R}_3(\gamma)]\{\mathbf{b}\}_{x_2}}^{\{\mathbf{b}\}_{x'}}$$

The column vector $\{\mathbf{b}\}_{x_1}$ contains the components of the vector $\mathbf{b}$ $(\mathbf{b} = b_{x_1}\hat{\mathbf{i}}_1 + b_{y_1}\hat{\mathbf{j}}_1 + b_{z_1}\hat{\mathbf{k}}_1)$ in the first intermediate frame $x_1y_1z_1$. The column vector $\{\mathbf{b}\}_{x_2}$ contains the components of the vector $\mathbf{b}$ $(\mathbf{b} = b_{x_2}\hat{\mathbf{i}}_2 + b_{y_2}\hat{\mathbf{j}}_2 + b_{z_2}\hat{\mathbf{k}}_2)$ in the second intermediate frame $x_2y_2z_2$. Finally, the column vector $\{\mathbf{b}\}_{x'}$ contains the components in the target $x'y'z'$ frame.

Substituting Eqns (4.32)–(4.34) into Eqn (4.37) yields the direct cosine matrix of the classical Euler sequence $[\mathbf{R}_3(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_3(\alpha)]$,

$$[\mathbf{Q}] = \begin{bmatrix} -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \cos\alpha\cos\beta\sin\gamma + \sin\alpha\cos\gamma & \sin\beta\sin\gamma \\ -\sin\alpha\cos\beta\cos\gamma - \cos\alpha\sin\gamma & \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & \sin\beta\cos\gamma \\ \sin\alpha\sin\beta & -\cos\alpha\sin\beta & \cos\beta \end{bmatrix} \tag{4.38}$$

From this we can see that, given a direct cosine matrix $[\mathbf{Q}]$, the angles of the classical Euler sequence may be found as follows:

$$\tan\alpha = \frac{Q_{31}}{-Q_{32}} \quad \cos\beta = Q_{33} \quad \tan\gamma = \frac{Q_{13}}{Q_{23}} \quad \text{Classical Euler angle sequence} \tag{4.39}$$

We see that $\beta = \cos^{-1}Q_{33}$. There is no quadrant uncertainty because the principal values of the arccosine function coincide with the range of the angle β given in Eqn (4.37) (0°–180°). Finding α and γ involves computing the inverse tangent (arctan), whose principal values lie in the range −90° to +90°, whereas the range of both α and γ is 0°–360°. Placing $\tan^{-1}(y/x)$ in the correct quadrant is accomplished by taking into considering the signs of x and y. The MATLAB function *atan2d_0_360.m* in Appendix D.19 does just that.

ALGORITHM 4.3

Given the direct cosine matrix

$$[\mathbf{Q}] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}$$

find the angles $\alpha\beta\gamma$ of the classical Euler rotation sequence. This algorithm is implemented by the MATLAB function *dcm_to_euler.m* in Appendix D.20.

1. $\alpha = \tan^{-1}\left(\dfrac{Q_{31}}{-Q_{32}}\right) \quad (0 \le \alpha < 360°)$
2. $\beta = \cos^{-1}Q_{33} \quad (0 \le \beta \le 180°)$
3. $\gamma = \tan^{-1}\dfrac{Q_{13}}{Q_{23}} \quad (0 \le \gamma < 360°)$

EXAMPLE 4.5

If the DCM for the transformation from xyz to $x'y'z'$ is

$$[\mathbf{Q}] = \begin{bmatrix} 0.64050 & 0.75319 & -0.15038 \\ 0.76736 & -0.63531 & 0.086824 \\ -0.030154 & -0.17101 & -0.98481 \end{bmatrix}$$

find the angles α, β, and γ of the classical Euler sequence.

Solution

Use Algorithm 4.3.

Step 1:

$$\alpha = \tan^{-1}\left(\frac{Q_{31}}{-Q_{32}}\right) = \tan^{-1}\left(\frac{-0.030154}{-[-0.17101]}\right)$$

Since the numerator is negative and the denominator is positive, α must lie in the fourth quadrant. Thus

$$\tan^{-1}\left(\frac{-0.030154}{-[-0.17101]}\right) = \tan^{-1}(-0.17633) = -10^\circ \Rightarrow \boxed{\alpha = 350^\circ}$$

Step 2:

$$\beta = \cos^{-1} Q_{33} = \cos^{-1}(-0.98481) = \boxed{170.0^\circ}$$

Step 3:

$$\gamma = \tan^{-1}\frac{Q_{13}}{Q_{23}} = \tan^{-1}\left(\frac{-0.15038}{0.086824}\right)$$

The numerator is negative and the denominator is positive, so γ lies in the fourth quadrant,

$$\tan^{-1}\left(\frac{-0.15038}{0.086824}\right) = \tan^{-1}(-0.17320) = -60^\circ \Rightarrow \boxed{\gamma = 300^\circ}$$

Another commonly used set of Euler angles for rotating xyz into alignment with $x'y'z'$ is the asymmetric "yaw, pitch, and roll" sequence found in Eqn (4.36),

$$[\mathbf{Q}] = [\mathbf{R}_1(\gamma)][\mathbf{R}_2(\beta)][\mathbf{R}_3(\alpha)] \quad (0 \le \alpha < 360^\circ \quad -90^\circ < \beta < 90^\circ \quad 0 \le \gamma < 360^\circ) \tag{4.40}$$

It is illustrated in Figure 4.15.

The first rotation $[\mathbf{R}_3(\alpha)]$ is about the z-axis through the angle α. It carries the y-axis into the y_1-axis normal to the plane of z and x' while rotating the x-axis into x_1. This rotation appears as shown in the insert at the top right of Figure 4.15. The second rotation $[\mathbf{R}_2(\beta)]$, shown in the auxiliary view at the bottom right of the figure, is a pitch around y_1 through the angle β. This carries the x_1-axis into x_2, lined up with the target x' direction, and rotates the original z-axis (now z_1) into z_2. The final rotation $[\mathbf{R}_1(\gamma)]$ is a roll through the angle γ around the x_2-axis so as to carry y_2 (originally y_1) and z_2 into alignment with the target y'- and z'-axes.

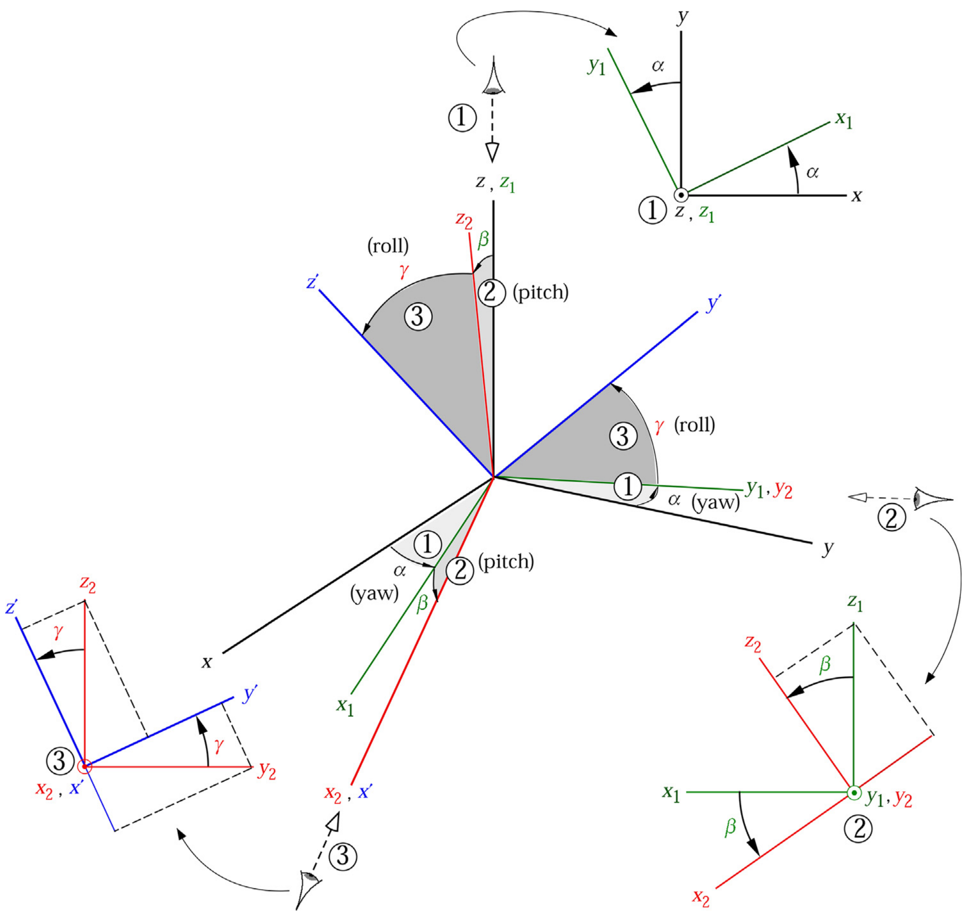


FIGURE 4.15

Yaw, pitch and roll sequence transforming *xyz* into $x'y'z'$.

Substituting Eqns (4.32)–(4.34) into Eqn (4.40) yields the DCM for the yaw, pitch, and roll sequence,

$$[\mathbf{Q}] = \begin{bmatrix} \cos\alpha\cos\beta & \sin\alpha\cos\beta & -\sin\beta \\ \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \cos\beta\sin\gamma \\ \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma & \cos\beta\cos\gamma \end{bmatrix} \quad (4.41)$$

From this it is apparent that

$$\tan\alpha = \frac{Q_{12}}{Q_{11}} \qquad \sin\beta = -Q_{13}$$

$$\tan\gamma = \frac{Q_{23}}{Q_{33}} \qquad \text{Yaw, pitch, and roll sequence} \quad (4.42)$$

For β we simply compute $\sin^{-1}(-Q_{13})$. There is no quadrant uncertainty because the principal values of the arcsine function coincide with the range of the pitch angle ($-90° < \beta < 90°$). Finding α and γ involves computing the inverse tangent, so we must once again be careful to place the results of these calculations in the range 0°–360°. As pointed out previously, the MATLAB function *atan2d_0_360.m* in Appendix D.19 takes care of that.

ALGORITHM 4.4

Given the direct cosine matrix

$$[\mathbf{Q}] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}$$

Find the angles $\alpha\beta\gamma$ of the yaw, pitch, and roll sequence. This algorithm is implemented by the MATLAB function *dcm_to_ypr.m* in Appendix D.21.

1. $\alpha = \tan^{-1}\dfrac{Q_{12}}{Q_{11}} \quad (0 \le \alpha < 360°)$
2. $\beta = \sin^{-1}(-Q_{13}) \quad (-90° < \beta < 90°)$
3. $\gamma = \tan^{-1}\dfrac{Q_{23}}{Q_{33}} \quad (0 \le \gamma < 360°)$

EXAMPLE 4.6

If the direct cosine matrix for the transformation from *xyz* to $x'y'z'$ is the same as it was in Example 4.5,

$$[\mathbf{Q}] = \begin{bmatrix} 0.64050 & 0.75319 & -0.15038 \\ 0.76736 & -0.63531 & 0.086824 \\ -0.030154 & -0.17101 & -0.98481 \end{bmatrix}$$

find the angles α, β, and γ of the yaw, pitch, and roll sequence.

Solution

Use Algorithm 4.4.

Step 1:

$$\alpha = \tan^{-1}\frac{Q_{12}}{Q_{11}} = \tan^{-1}\left(\frac{0.75319}{0.64050}\right)$$

Since both the numerator and the denominator are positive, α must lie in the first quadrant. Thus,

$$\tan^{-1}\left(\frac{0.75319}{0.64050}\right) = \tan^{-1}1.1759 = \boxed{49.62°}$$

Step 2:

$$\beta = \sin^{-1}(-Q_{13}) = \sin^{-1}[-(-0.15038)] = \sin^{-1}(0.15038) = \boxed{8.649°}$$

Step 3:

$$\gamma = \tan^{-1}\frac{Q_{23}}{Q_{33}} = \tan^{-1}\left(\frac{0.086824}{-0.98481}\right)$$

The numerator is positive and the denominator is negative, so γ lies in the second quadrant,

$$\tan^{-1}\left(\frac{0.086824}{-0.98481}\right) = \tan^{-1}(-0.088163) = -5.0383^\circ \Rightarrow \boxed{\gamma = 174.96^\circ}$$

4.6 Transformation between geocentric equatorial and perifocal frames

The perifocal frame of reference for a given orbit was introduced in Section 2.10. Figure 4.16 illustrates the relationship between the perifocal and geocentric equatorial frames. Since the orbit lies in the $\bar{x}\,\bar{y}$ plane, the components of the state vector of a body relative to its perifocal reference are, according to Eqns (2.119) and (2.125),

$$\mathbf{r} = \bar{x}\hat{\mathbf{p}} + \bar{y}\hat{\mathbf{q}} = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta}(\cos\theta\hat{\mathbf{p}} + \sin\theta\hat{\mathbf{q}}) \tag{4.43}$$

$$\mathbf{v} = \dot{\bar{x}}\hat{\mathbf{p}} + \dot{\bar{y}}\hat{\mathbf{q}} = \frac{\mu}{h}[-\sin\theta\hat{\mathbf{p}} + (e + \cos\theta)\hat{\mathbf{q}}] \tag{4.44}$$

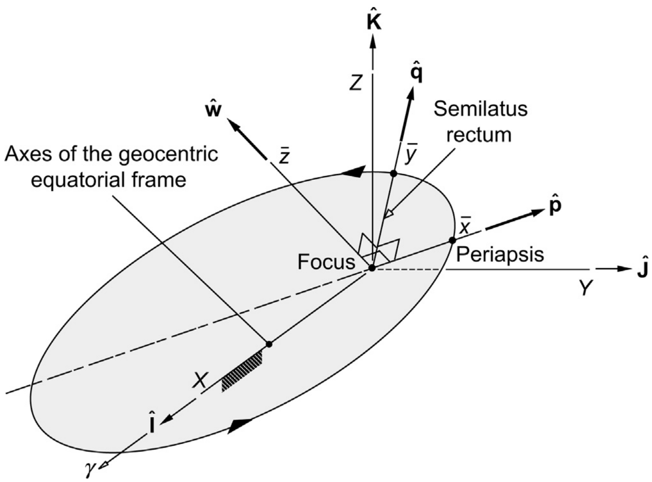


FIGURE 4.16

Perifocal ($\bar{x}\,\bar{y}\,\bar{z}$) and geocentric equatorial (XYZ) frames.

In matrix notation, these may be written

$$\{\mathbf{r}\}_{\bar{x}} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\begin{Bmatrix}\cos\theta \\ \sin\theta \\ 0\end{Bmatrix} \tag{4.45}$$

$$\{\mathbf{v}\}_{\bar{x}} = \frac{\mu}{h}\begin{Bmatrix}-\sin\theta \\ e+\cos\theta \\ 0\end{Bmatrix} \tag{4.46}$$

The subscript $\bar{x}$ is shorthand for "the $\bar{x}\,\bar{y}\,\bar{z}$ coordinate system" and is used to indicate that the components of these vectors are given in the perifocal frame, as opposed to, say, the geocentric equatorial frame (Eqns (4.2) and (4.3)).

The transformation from the geocentric equatorial frame into the perifocal frame may be accomplished by the classical Euler angle sequence $[\mathbf{R}_3(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_3(\alpha)]$ in Eqn (4.37). Refer to Figure 4.7. In this case, the first rotation angle is Ω, the right ascension of the ascending node. The second rotation is i, the orbital inclination angle, and the third rotation angle is ω, the argument of perigee. Ω is measured around the Z-axis of the geocentric equatorial frame, i is measured around the node line, and ω is measured around the $\bar{z}$-axis of the perifocal frame. Therefore, the direct cosine matrix $[\mathbf{Q}]_{X\bar{x}}$ of the transformation from XYZ to $\bar{x}\,\bar{y}\,\bar{z}$ is

$$[\mathbf{Q}]_{X\bar{x}} = [\mathbf{R}_3(\omega)][\mathbf{R}_1(i)][\mathbf{R}_3(\Omega)] \tag{4.47}$$

From Eqn (4.38) we get

$$[\mathbf{Q}]_{X\bar{x}} = \begin{bmatrix} -\sin\Omega\cos i\sin\omega + \cos\Omega\cos\omega & \cos\Omega\cos i\sin\omega + \sin\Omega\cos\omega & \sin i\sin\omega \\ -\sin\Omega\cos i\cos\omega - \cos\Omega\sin\omega & \cos\Omega\cos i\cos\omega - \sin\Omega\sin\omega & \sin i\cos\omega \\ \sin\Omega\sin i & -\cos\Omega\sin i & \cos i \end{bmatrix} \tag{4.48}$$

Remember that this is an orthogonal matrix, which means that the inverse transformation $[\mathbf{Q}]_{\bar{x}X}$, from $\bar{x}\,\bar{y}\,\bar{z}$ to XYZ, is given by $[\mathbf{Q}]_{\bar{x}X} = \left([\mathbf{Q}]_{X\bar{x}}\right)^T$, or

$$[\mathbf{Q}]_{\bar{x}X} = \begin{bmatrix} -\sin\Omega\cos i\sin\omega + \cos\Omega\cos\omega & -\sin\Omega\cos i\cos\omega - \cos\Omega\sin\omega & \sin\Omega\sin i \\ \cos\Omega\cos i\sin\omega + \sin\Omega\cos\omega & \cos\Omega\cos i\cos\omega - \sin\Omega\sin\omega & -\cos\Omega\sin i \\ \sin i\sin\omega & \sin i\cos\omega & \cos i \end{bmatrix} \tag{4.49}$$

If the components of the state vector are given in the geocentric equatorial frame,

$$\{\mathbf{r}\}_X = \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} \quad \{\mathbf{v}\}_X = \begin{Bmatrix} v_X \\ v_Y \\ v_Z \end{Bmatrix}$$

then the components in the perifocal frame are found by carrying out the matrix multiplications

$$\{\mathbf{r}\}_{\bar{x}} = \begin{Bmatrix} \bar{x} \\ \bar{y} \\ 0 \end{Bmatrix} = [\mathbf{Q}]_{X\bar{x}}\{\mathbf{r}\}_X \quad \{\mathbf{v}\}_{\bar{x}} = \begin{Bmatrix} \dot{\bar{x}} \\ \dot{\bar{y}} \\ 0 \end{Bmatrix} = [\mathbf{Q}]_{X\bar{x}}\{\mathbf{v}\}_X \tag{4.50}$$

Likewise, the transformation from perifocal to geocentric equatorial components is

$$\{\mathbf{r}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{r}\}_{\bar{x}} \quad \{\mathbf{v}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{v}\}_{\bar{x}} \tag{4.51}$$

ALGORITHM 4.5

Given the orbital elements h, e, i, Ω, ω, and θ, compute the state vectors **r** and **v** in the geocentric equatorial frame of reference. A MATLAB implementation of this procedure is listed in Appendix D.22. This algorithm can be applied to orbits around other planets or the sun.

1. Calculate position vector $\{\mathbf{r}\}_{\bar{x}}$ in perifocal coordinates using Eqn (4.45).
2. Calculate velocity vector $\{\mathbf{v}\}_{\bar{x}}$ in perifocal coordinates using Eqn (4.46).
3. Calculate the matrix $[\mathbf{Q}]_{\bar{x}X}$ of the transformation from perifocal to geocentric equatorial coordinates using Eqn (4.49).
4. Transform $\{\mathbf{r}\}_{\bar{x}}$ and $\{\mathbf{v}\}_{\bar{x}}$ into the geocentric frame by means of Eqn (4.51).

EXAMPLE 4.7

For a given earth orbit, the elements are $h = 80{,}000\ \mathrm{km^2/s}$, $e = 1.4$, $i = 30°$, $\Omega = 40°$, $\omega = 60°$, and $\theta = 30°$. Using Algorithm 4.5, find the state vectors **r** and **v** in the geocentric equatorial frame.

Solution

Step 1:

$$\{\mathbf{r}\}_{\bar{x}} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\begin{Bmatrix}\cos\theta\\ \sin\theta\\ 0\end{Bmatrix} = \frac{80{,}000^2}{398{,}600}\frac{1}{1+1.4\cos 30°}\begin{Bmatrix}\cos 30°\\ \sin 30°\\ 0\end{Bmatrix} = \begin{Bmatrix}6285.0\\ 3628.6\\ 0\end{Bmatrix}\ \mathrm{km}$$

Step 2:

$$\{\mathbf{v}\}_{\bar{x}} = \frac{\mu}{h}\begin{Bmatrix}-\sin\theta\\ e+\cos\theta\\ 0\end{Bmatrix} = \frac{398{,}600}{80{,}000}\begin{Bmatrix}-\sin 30°\\ 1.4+\cos 30°\\ 0\end{Bmatrix} = \begin{Bmatrix}-2.4913\\ 11.290\\ 0\end{Bmatrix}\ \mathrm{km/s}$$

Step 3:

$$[\mathbf{Q}]_{X\bar{x}} = \begin{bmatrix}\cos\omega & \sin\omega & 0\\ -\sin\omega & \cos\omega & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos i & \sin i\\ 0 & -\sin i & \cos i\end{bmatrix}\begin{bmatrix}\cos\Omega & \sin\Omega & 0\\ -\sin\Omega & \cos\Omega & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}\cos 60° & \sin 60° & 0\\ -\sin 60° & \cos 60° & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos 30° & \sin 30°\\ 0 & -\sin 30° & \cos 30°\end{bmatrix}\begin{bmatrix}\cos 40° & \sin 40° & 0\\ -\sin 40° & \cos 40° & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}-0.099068 & 0.89593 & 0.43301\\ -0.94175 & -0.22496 & 0.25\\ 0.32139 & -0.38302 & 0.86603\end{bmatrix}$$

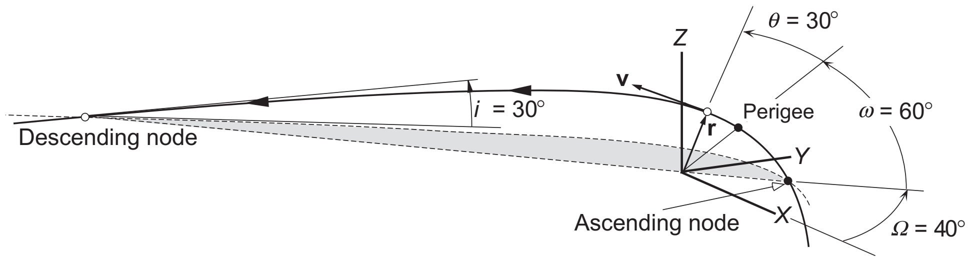


FIGURE 4.17

A portion of the hyperbolic trajectory of Example 4.7.

This is the direct cosine matrix for $XYZ \rightarrow \bar{x}\,\bar{y}\,\bar{z}$. The transformation matrix for $\bar{x}\,\bar{y}\,\bar{z} \rightarrow XYZ$ is the transpose,

$$[\mathbf{Q}]_{\bar{x}X} = \begin{bmatrix} -0.099068 & -0.94175 & 0.32139 \\ 0.89593 & -0.22496 & -0.38302 \\ 0.43301 & 0.25 & 0.86603 \end{bmatrix}$$

Step 4:
The geocentric equatorial position vector is

$$\{\mathbf{r}\}_X = [\mathbf{Q}]_{\bar{x}X}\,\{\mathbf{r}\}_{\bar{x}} = \begin{bmatrix} -0.099068 & -0.94175 & 0.32139 \\ 0.89593 & -0.22496 & -0.38302 \\ 0.43301 & 0.25 & 0.86603 \end{bmatrix} \begin{Bmatrix} 6285.0 \\ 3628.6 \\ 0 \end{Bmatrix} = \boxed{\begin{Bmatrix} -4040 \\ 4815 \\ 3629 \end{Bmatrix} (\text{km})} \qquad \text{(a)}$$

whereas the geocentric equatorial velocity vector is

$$\{\mathbf{v}\}_X = [\mathbf{Q}]_{\bar{x}X}\,\{\mathbf{v}\}_{\bar{x}} = \begin{bmatrix} -0.099068 & -0.94175 & 0.32139 \\ 0.89593 & -0.22496 & -0.38302 \\ 0.43301 & 0.25 & 0.86603 \end{bmatrix} \begin{Bmatrix} -2.4913 \\ 11.290 \\ 0 \end{Bmatrix} = \boxed{\begin{Bmatrix} -10.39 \\ -4.772 \\ 1.744 \end{Bmatrix} (\text{km/s})}$$

The state vectors **r** and **v** are shown in Figure 4.17. By holding all the orbital parameters except the true anomaly fixed and allowing θ to take on a range of values, we generate a sequence of position vectors $\mathbf{r}_{\bar{x}}$ from Eqn (4.37). Each of these is projected into the geocentric equatorial frame as in Eqn (a), using repeatedly the same transformation matrix $[\mathbf{Q}]_{\bar{x}X}$. By connecting the end points of all the position vectors $\mathbf{r}_X$, we trace out the trajectory illustrated in Figure 4.17.

4.7 Effects of the earth's oblateness

The earth, like all planets with comparable or higher rotational rates, bulges out at the equator because of centrifugal force. The earth's equatorial radius is 21 km (13 miles) larger than the polar radius. This flattening at the poles is called oblateness, which is defined as follows:

$$\text{Oblateness} = \frac{\text{Equatorial radius} - \text{Polar radius}}{\text{Equatorial radius}}$$

The earth is an oblate spheroid, lacking the perfect symmetry of a sphere (a basketball can be made an oblate spheroid by sitting on it). This lack of symmetry means that the force of gravity on an orbiting body is not directed toward the center of the earth. Although the gravitational field of a perfectly spherical planet depends only on the distance from its center, oblateness causes a variation also with latitude, that is, the angular distance from the equator (or pole). This is called a zonal variation. The dimensionless parameter that quantifies the major effects of oblateness on orbits is J_2, the second zonal harmonic. J_2 is not a universal constant. Each planet has its own value, as illustrated in Table 4.3, which lists variations of J_2 as well as oblateness.

Oblateness causes the right ascension Ω and the argument of periapsis ω to vary significantly with time.

In Chapter 12, we will show that the average rates of change of these two angles are

$$\dot{\Omega} = -\left[\frac{3}{2}\frac{\sqrt{\mu}J_2R^2}{\left(1-e^2\right)^2 a^{\frac{7}{2}}}\right]\cos i \tag{4.52}$$

and

$$\dot{\omega} = -\left[\frac{3}{2}\frac{\sqrt{\mu}J_2R^2}{\left(1-e^2\right)^2 a^{\frac{7}{2}}}\right]\left(\frac{5}{2}\sin^2 i - 2\right) \tag{4.53}$$

R and μ are the radius and gravitational parameter of the planet, respectively; a and e are the semimajor axis and eccentricity of the orbit, respectively; and i is the orbit's inclination.

In Eqn (4.52), observe that if $0 \le i < 90°$, then $\dot{\Omega} < 0$. That is, for prograde orbits, the node line drifts westward. Since the RA of the node continuously decreases, this phenomenon is called regression of the nodes. If $90° < i \le 180°$, we see that $\dot{\Omega} > 0$. The node line of retrograde orbits therefore advances eastward. For polar orbits ($i = 90°$), the node line is stationary.

In Eqn (4.53), we see that if $0° \le i < 63.4°$ or $116.6° < i \le 180°$, then $\dot{\omega}$ is positive, which means the perigee advances in the direction of the motion of the satellite (hence, the name "advance of perigee" for this phenomenon). If $63.4° < i \le 116.6°$, the perigee regresses, moving opposite to the direction of motion. $i = 63.4°$ and $i = 116.6°$ are the critical inclinations at which the apse line does not move.

Table 4.3 Oblateness and Second Zonal Harmonics

Planet	Oblateness	J_2
Mercury	0.000	60×10^{-6}
Venus	0.000	4.458×10^{-6}
Earth	0.003353	1.08263×10^{-3}
Mars	0.00648	1.96045×10^{-3}
Jupiter	0.06487	14.736×10^{-3}
Saturn	0.09796	16.298×10^{-3}
Uranus	0.02293	3.34343×10^{-3}
Neptune	0.01708	3.411×10^{-3}
(Moon)	0.0012	202.7×10^{-6}

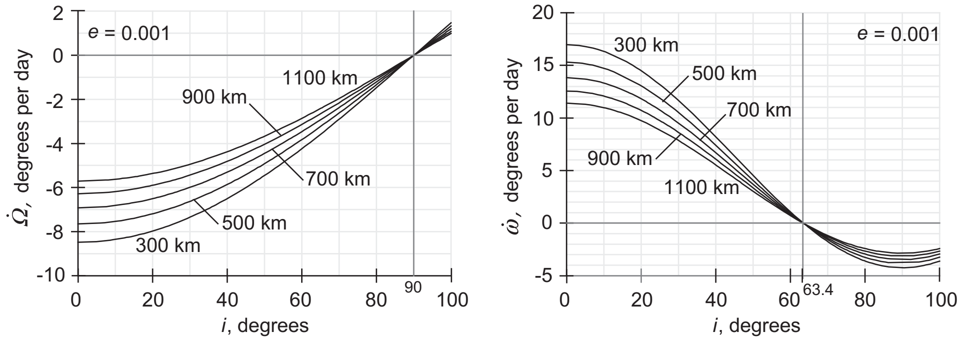


FIGURE 4.18

Regression of the node and advance of perigee for nearly circular orbits of altitudes 300–1100 km.

Observe that the coefficient of the trigonometric terms in Eqns (4.52) and (4.53) are identical, so that

$$\dot{\omega} = \dot{\Omega}\frac{(5/2)\sin^2 i - 2}{\cos i} \tag{4.54}$$

Figure 4.18 is a plot of Eqns 4.52 and 4.53 for several circular low earth orbits. The effect of oblateness on both $\dot{\Omega}$ and $\dot{\omega}$ is greatest at low inclinations, for which the orbit is near the equatorial bulge for longer portions of each revolution. The effect decreases with increasing semimajor axis because the satellite becomes further from the bulge and its gravitational influence. Obviously, $\dot{\Omega} = \dot{\omega} = 0$ if $J_2 = 0$ (no equatorial bulge).

It turns out (Chapter 12) that the J_2 effect produces zero time-averaged variations of the inclination, eccentricity, angular momentum, and semimajor axis.

EXAMPLE 4.8

A spacecraft is in a 280 km by 400 km orbit with an inclination of 51.43°. Find the rates of node regression and perigee advance.

Solution

The perigee and apogee radii are

$$r_p = 6378 + 280 = 6658 \text{ km} \quad r_a = 6378 + 400 = 6778 \text{ km}$$

Therefore, the eccentricity and semimajor axis are

$$e = \frac{r_a - r_p}{r_a + r_p} = 0.008931$$

$$a = \frac{1}{2}(r_a + r_p) = 6718 \text{ km}$$

From Eqn (4.52), we obtain the rate of node line regression.

$$\dot{\Omega} = -\left[\frac{3}{2}\frac{\sqrt{398{,}600} \times 0.0010826 \times 6378^2}{(1 - 0.008931 2^2)^2 \times 6718^{7/2}}\right]\cos 51.43° = -1.0465 \times 10^{-6} \text{ rad/s}$$

or

$$\boxed{\dot{\Omega} = 5.181° \text{ per day to the west}}$$

From Eqn (4.54),

$$\dot{\omega} = -1.0465 \times 10^{-6} \cdot \left(\frac{5}{2}\sin^2 51.43° - 2\right) = +7.9193 \times 10^{-7}\ \text{rad/s}$$

or

$$\boxed{\dot{\omega} = 3.920°\ \text{per day in the flight direction}}$$

The effect of orbit inclination on node regression and advance of perigee is taken advantage of for two very important types of orbits. Sun-synchronous orbits are those whose orbital plane makes a constant angle α with the radial from the sun, as illustrated in Figure 4.19. For that to occur, the orbital plane must rotate in inertial space with the angular velocity of the earth in its orbit around the sun, which is 360° per 365.26 days, or 0.9856° per day. With the orbital plane precessing eastward at this rate, the ascending node will lie at a fixed local time. In the illustration, it happens to be 3 PM. During every orbit, the satellite sees any given swath of the planet under nearly the same conditions of daylight or darkness day after day. The satellite also has a constant perspective on the sun. Sun-synchronous satellites, like the NOAA Polar-orbiting Operational Environmental Satellites and those of the Defense Meteorological Satellite Program, are used for global weather coverage, while Landsat and the French SPOT series are intended for high-resolution earth observation.

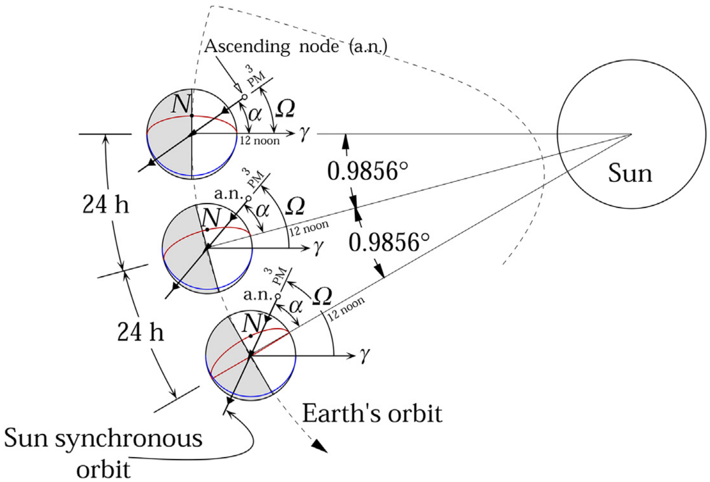


FIGURE 4.19

Sun-synchronous orbit.

EXAMPLE 4.9

A satellite is to be launched into a sun-synchronous circular orbit with period of 100 min. Determine the required altitude and inclination of its orbit.

Solution

We find the altitude z from the period relation for a circular orbit, Eqn (2.64):

$$T = \frac{2\pi}{\sqrt{\mu}}(R_E + z)^{\frac{3}{2}} \Rightarrow 100 \times 60 = \frac{2\pi}{\sqrt{398{,}600}}(6378 + z)^{\frac{3}{2}} \Rightarrow \boxed{z = 758.63\ \text{km}}$$

For a sun-synchronous orbit, the ascending node must advance at the rate

$$\dot{\Omega} = \frac{2\pi \text{ rad}}{365.26 \times 24 \times 3600 \text{ s}} = 1.991 \times 10^{-7} \text{ rad/s}$$

Substituting this and the altitude into Eqn (4.47), we obtain,

$$1.991 \times 10^{-7} = -\left[\frac{3}{2}\frac{\sqrt{398{,}600} \times 0.00108263 \times 6378^2}{(1-0^2)^2(6378+758.63)^{\frac{7}{2}}}\right]\cos i \Rightarrow \cos i = -0.14658$$

Thus, the inclination of the orbit is

$$\boxed{i = \cos^{-1}(-0.14658) = 98.43^\circ}$$

This illustrates the fact that sun-synchronous orbits are very nearly polar orbits ($i = 90^\circ$).

If a satellite is launched into an orbit with an inclination of 63.4° (prograde) or 116.6° (retrograde), then Eqn (4.53) shows that the apse line will remain stationary. The Russian space program made this a key element in the design of the system of Molniya ("lightning") communications satellites. All the Russian launch sites are above 45° latitude, the northernmost, Plesetsk, being located at 62.8°N. As we shall see in Chapter 6, launching a satellite into a geostationary orbit would involve a costly plane change maneuver. Furthermore, recall from Example 2.4 that a geostationary satellite cannot view effectively the far northern latitudes into which Russian territory extends.

The Molniya telecommunications satellites are launched from Plesetsk into 63° inclination orbits having a period of 12 h. From Eqn (2.83), we conclude that the major axis of these orbits is 53,000 km long. Perigee (typically 500 km altitude) lies in the southern hemisphere, while apogee is at an altitude of 40,000 km (25,000 miles) above the northern latitudes, farther out than the geostationary satellites.

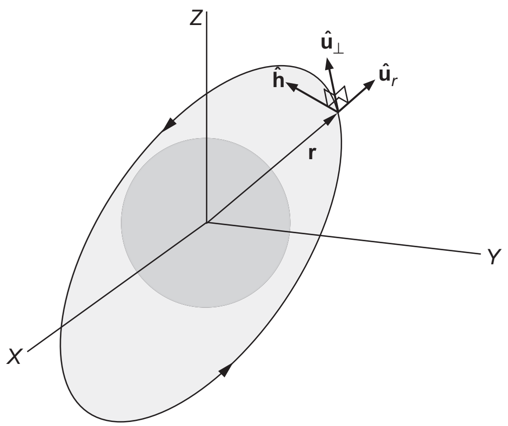


FIGURE 4.20

A typical Molniya orbit (to scale).

Figure 4.20 illustrates a typical Molniya orbit. A Molniya "constellation" consists of eight satellites in planes separated by 45°. Each satellite is above 30° north latitude for over 8 h, coasting toward and away from apogee.

EXAMPLE 4.10

Determine the perigee and apogee for an earth satellite whose orbit satisfies all the following conditions: it is sun-synchronous, its argument of perigee is constant, and its period is 3 h.

Solution

The period determines the semimajor axis,

$$T = \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}} \Rightarrow 3 \cdot 3600 = \frac{2\pi}{\sqrt{398{,}600}} a^{\frac{3}{2}} \Rightarrow a = 10{,}560 \text{ km}$$

For the apse line to be stationary, we know from Eqn (4.53) that $i = 64.435°$ or $i = 116.57°$. However, an inclination of less than 90° causes a westward regression of the node, whereas a sun-synchronous orbit requires an eastward advance, which $i = 116.57°$ provides. Substituting this, the semimajor axis, and the $\dot{\Omega}$ in radians per second for a sun-synchronous orbit (cf. Example 4.9) into Eqn (4.52), we get

$$1.991 \times 10^{-7} = -\frac{3}{2} \frac{\sqrt{398{,}600} \times 0.0010826 \times 6378^2}{\left(1 - e^2\right)^2 \times 10{,}560^{7/2}} \cos 116.57° \Rightarrow e = 0.3466$$

Now we can find the angular momentum from the period expression (Eqn (2.82))

$$T = \frac{2\pi}{\mu^2}\left(\frac{h}{\sqrt{1-e^2}}\right)^3 \Rightarrow 3 \cdot 3600 = \frac{2\pi}{398{,}600^2}\left(\frac{h}{\sqrt{1-0.34655^2}}\right)^3 \Rightarrow h = 60{,}850 \text{ km}^2/\text{s}$$

Finally, to obtain the perigee and apogee radii, we use the orbit formula:

$$z_p + 6378 = \frac{h^2}{\mu}\frac{1}{1+e} = \frac{60{,}860^2}{398{,}600}\frac{1}{1+0.34655} \Rightarrow \boxed{z_p = 522.6 \text{ km}}$$

$$z_a + 6378 = \frac{h^2}{\mu}\frac{1}{1-e} \Rightarrow \boxed{z_a = 7842 \text{ km}}$$

EXAMPLE 4.11

Given the following state vector of a satellite in geocentric equatorial coordinates,

$$\mathbf{r} = -3670\hat{\mathbf{I}} - 3870\hat{\mathbf{J}} + 4400\hat{\mathbf{K}} \text{ km}$$
$$\mathbf{v} = 4.7\hat{\mathbf{I}} - 7.4\hat{\mathbf{J}} + 1\hat{\mathbf{K}} \text{ km/s}$$

find the state vector after 4 days (96 h) of coasting flight, assuming that there are no perturbations other than the influence of the earth's oblateness on Ω and ω.

Solution

Four days is a long enough time interval that we need to take into consideration not only the change in true anomaly but also the regression of the ascending node and the advance of perigee. First, we must determine the orbital elements at the initial time using Algorithm 4.2, which yields

$$
\begin{aligned}
h &= 58{,}930\ \text{km}^2/\text{s} \\
i &= 39.687^\circ \\
e &= 0.42607 \ \ \text{(The orbit is an ellipse)} \\
\Omega_0 &= 130.32^\circ \\
\omega_0 &= 42.373^\circ \\
\theta_0 &= 52.404^\circ
\end{aligned}
$$

We use Eqn (2.71) to determine the semimajor axis,

$$a = \frac{h^2}{\mu}\frac{1}{1-e^2} = \frac{58{,}930^2}{398{,}600}\frac{1}{1-0.4261^2} = 10{,}640\ \text{km}$$

so that, according to Eqn (2.83), the period is

$$T = \frac{2\pi}{\sqrt{\mu}}a^{\frac{3}{2}} = 10{,}928\ \text{s}$$

From this we obtain the mean motion

$$n = \frac{2\pi}{T} = 0.00057495\ \text{rad/s}$$

The initial value E_0 of eccentric anomaly is found from the true anomaly θ_0 using Eqn (3.13a),

$$\tan\frac{E_0}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta_0}{2} = \sqrt{\frac{1-0.42607}{1+0.42607}}\tan\frac{52.404^\circ}{2} \Rightarrow E_0 = 0.60520\ \text{rad}$$

With E_0, we use Kepler's equation to calculate the time t_0 since perigee at the initial epoch,

$$nt_0 = E_0 - e\sin E_0 \Rightarrow 0.00057495t_0 = 0.60520 - 0.42607\sin 0.60520 \Rightarrow t_1 = 631.00\ \text{s}$$

Now we advance the time to t_f, that of the final epoch, given as 96 h later. That is, $\Delta t = 345{,}600$ s, so that

$$t_f = t_1 + \Delta t = 631.00 + 345{,}600 = 346{,}230\ \text{s}$$

The number of periods n_P since passing perigee in the first orbit is

$$n_P = \frac{t_f}{T} = \frac{346{,}230}{10{,}928} = 31.682$$

From this we see that the final epoch occurs in the 32nd orbit, whereas t_0 was in orbit 1. Time since passing perigee in the 32nd orbit, which we will denote t_{32}, is

$$t_{32} = (31.682 - 31)T \Rightarrow t_{32} = 7455.7\ \text{s}$$

The mean anomaly corresponding to that time in the 32nd orbit is

$$M_{32} = nt_{32} = 0.00057495 \times 7455.7 = 4.2866\ \text{rad}$$

Kepler's equation yields the eccentric anomaly:

$$
\begin{aligned}
&E_{32} - e\sin E_{32} = M_{32} \\
&E_{32} - 0.42607\sin E_{32} = 4.2866 \\
&\therefore\ E_{32} = 3.9721\ \text{rad} \quad \text{(Algorithm 3.1)}
\end{aligned}
$$

The true anomaly follows in the usual way,

$$\tan\frac{\theta_{32}}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E_{32}}{2} \Rightarrow \theta_{32} = 211.25^\circ$$

At this point, we use the newly found true anomaly to calculate the state vector of the satellite in perifocal coordinates. Thus, from Eqn (4.43)

$$\mathbf{r} = r\cos\theta_{32}\hat{\mathbf{p}} + r\sin\theta_{32}\hat{\mathbf{q}} = -11{,}714\hat{\mathbf{p}} - 7108.8\hat{\mathbf{q}}\ (\text{km})$$

or, in matrix notation,

$$\{\mathbf{r}\}_{\bar{x}} = \begin{Bmatrix} -11{,}714 \\ -7108.8 \\ 0 \end{Bmatrix} (\text{km})$$

Likewise, from Eqn (4.43),

$$\mathbf{v} = -\frac{\mu}{h}\sin\theta_{32}\hat{\mathbf{p}} + \frac{\mu}{h}(e + \cos\theta_{32})\hat{\mathbf{q}} = 3.509\ \ 3\hat{\mathbf{p}} - 2.9007\hat{\mathbf{q}}\ (\text{km/s})$$

or

$$\{\mathbf{v}\}_{\bar{x}} = \begin{Bmatrix} 3.5093 \\ -2.9007 \\ 0 \end{Bmatrix} (\text{km/s})$$

Before we can project **r** and **v** into the geocentric equatorial frame, we must update the right ascension of the node and the argument of perigee. The regression rate of the ascending node is

$$\begin{aligned}\dot{\Omega} &= -\left[\frac{3}{2}\frac{\sqrt{\mu}J_2R^2}{\left(1-e^2\right)^2 a^{\frac{7}{2}}}\right]\cos i = -\frac{3}{2}\frac{\sqrt{398{,}600}\times 0.00108263\times 6378^2}{\left(1-0.42607^2\right)^2\times 10{,}644^{\frac{7}{2}}}\cos 39.69^\circ \\ &= -3.8514\times 10^{-7}\ \text{rad/s}\end{aligned}$$

or

$$\dot{\Omega} = -2.2067\times 10^{-5}\ ^\circ/\text{s}$$

Therefore, right ascension at epoch in the 32nd orbit is

$$\Omega_{32} = \Omega_0 + \dot{\Omega}\Delta t = 130.32 + \left(-2.2067\times 10^{-5}\right)\times 345{,}600 = 122.70^\circ$$

Likewise, the perigee advance rate is

$$\dot{\omega} = -\left[\frac{3}{2}\frac{\sqrt{\mu}J_2R^2}{\left(1-e^2\right)^2 a^{\frac{7}{2}}}\right]\left(\frac{5}{2}\sin^2 i - 2\right) = 4.9072\times 10^{-7}\ \text{rad/s} = 2.8116\times 10^{-5}\ ^\circ/\text{s}$$

which means the argument of perigee at epoch in the 32nd orbit is

$$\omega_{32} = \omega_0 + \omega\Delta t = 42.373 + 2.8116\times 10^{-5}\times 345{,}600 = 52.090^\circ$$

Substituting the updated values of Ω and ω, together with the inclination i, into Eqn (4.47) yields the updated transformation matrix from geocentric equatorial to the perifocal frame,

$$\begin{aligned}[\mathbf{Q}]_{X\bar{x}} &= \begin{bmatrix} \cos\omega_{32} & \sin\omega_{32} & 0 \\ -\sin\omega_{32} & \cos\omega_{32} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{bmatrix}\begin{bmatrix} \cos\Omega_{32} & \sin\Omega_{32} & 0 \\ -\sin\Omega_{32} & \cos\Omega_{32} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos 52.09^\circ & \sin 52.09^\circ & 0 \\ -\sin 52.09^\circ & \cos 52.09^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 39.687^\circ & \sin 39.687^\circ \\ 0 & -\sin 39.687^\circ & \cos 39.687^\circ \end{bmatrix}\begin{bmatrix} \cos 122.70^\circ & \sin 122.70^\circ & 0 \\ -\sin 122.70^\circ & \cos 122.70^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix}\end{aligned}$$

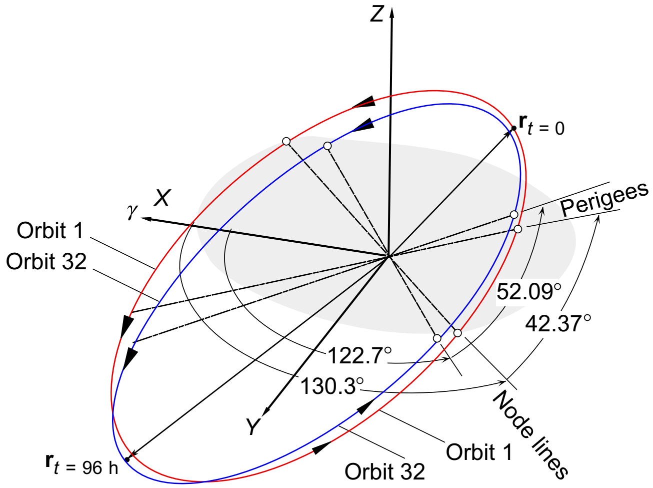


FIGURE 4.21

The initial and final position vectors.

or

$$[\mathbf{Q}]_{X\bar{x}} = \begin{bmatrix} -0.84285 & 0.18910 & 0.50383 \\ 0.028276 & -0.91937 & 0.39237 \\ 0.53741 & 0.34495 & 0.76955 \end{bmatrix}$$

For the inverse transformation, from perifocal to geocentric equatorial, we need the transpose of this matrix,

$$[\mathbf{Q}]_{\bar{x}X} = \begin{bmatrix} -0.84285 & 0.18910 & 0.50383 \\ 0.028276 & -0.91937 & 0.39237 \\ 0.53741 & 0.34495 & 0.76955 \end{bmatrix}^T = \begin{bmatrix} -0.84285 & 0.028276 & 0.53741 \\ 0.18910 & -0.91937 & 0.34495 \\ 0.50383 & 0.39237 & 0.76955 \end{bmatrix}$$

Thus, according to Eqn (4.51), the final state vector in the geocentric equatorial frame is

$$\{\mathbf{r}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{r}\}_{\bar{x}} = \begin{bmatrix} -0.84285 & 0.028276 & 0.53741 \\ 0.18910 & -0.91937 & 0.34495 \\ 0.50383 & 0.39237 & 0.76955 \end{bmatrix}\begin{Bmatrix} -11,714 \\ -7108.8 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 9672 \\ 4320 \\ -8691 \end{Bmatrix}(\text{km})$$

$$\{\mathbf{v}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{v}\}_{\bar{x}} = \begin{bmatrix} -0.84285 & 0.028276 & 0.53741 \\ 0.18910 & -0.91937 & 0.34495 \\ 0.50383 & 0.39237 & 0.76955 \end{bmatrix}\begin{Bmatrix} 3.5093 \\ -2.9007 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -3.040 \\ 3.330 \\ 0.6299 \end{Bmatrix}(\text{km/s})$$

or, in vector notation,

$$\boxed{\mathbf{r} = 9672\hat{\mathbf{I}} + 4320\hat{\mathbf{J}} - 8691\hat{\mathbf{K}}\ (\text{km}) \quad \mathbf{v} = -3.040\hat{\mathbf{I}} + 3.330\hat{\mathbf{J}} + 0.6299\hat{\mathbf{K}}\ (\text{km/s})}$$

The two orbits are plotted in Figure 4.21.

4.8 Ground tracks

The projection of a satellite's orbit onto the earth's surface is called its ground track. At a given instant, one can imagine a radial line drawn outward from the center of the earth to the satellite. Where this line pierces the earth's spherical surface is a point on the ground track. We locate this point by giving its latitude and longitude relative to the earth. As the satellite moves around the earth, the trace of these points is its ground track.

Because the satellite reaches a maximum and minimum latitude ("amplitude") during each orbit while passing over the equator twice, on a Mercator projection, the ground track of a satellite in low earth orbit often resembles a sine curve. If the earth did not rotate, there would be just one sinusoidlike track, traced repeatedly as the satellite orbits the earth. However, the earth rotates eastward beneath the satellite orbit at 15.04°/h, so the ground track advances westward at that rate. Figure 4.22 shows about two and a half orbits of a satellite, with the beginning and end of this portion of the ground track labeled. The distance between two successive crossings of the equator is measured to be 23.2°, which is the amount of earth rotation in one orbit of the spacecraft. Therefore, the ground track reveals that the period of the satellite is

$$T = \frac{23.2°}{15.04°/\mathrm{h}} = 1.54\ \mathrm{h} = 92.6\ \mathrm{min}$$

This is a typical low earth orbital period.

Given a satellite's position vector $\mathbf{r}$, we can use Algorithm 4.1 to find its right ascension and declination relative to the geocentric equatorial XYZ frame, which is fixed in space. The earth rotates at an angular velocity ω_E relative to this system. Let us attach an $x'y'z'$ Cartesian coordinate system to the earth with its origin located at the earth's center, as illustrated in Figure 1.18. The $x'y'$-axes lie in the equatorial plane and the z′-axis points north. (In Figure 1.18 the x'-axis is directed toward the prime meridian, which passes through Greenwich, England.) The XYZ and $x'y'z'$ differ only by the angle θ between the stationary X-axis and the rotating x'-axis. If the X- and x'-axes line up at time t_0, then at any time t thereafter the angle θ will be given by $\omega_E(t-t_0)$. The transformation from XYZ to $x'y'z'$ is represented by the elementary rotation matrix (recall Eqn (4.34)),

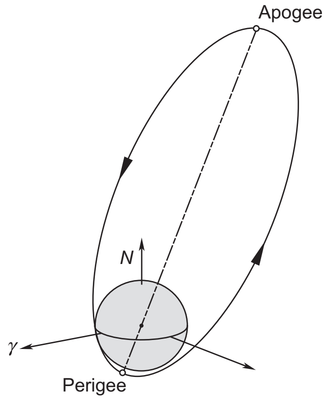


FIGURE 4.22

Ground track of a satellite.

$$[\mathbf{R}_3(\theta)] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \theta = \omega_E(t - t_0) \tag{4.55}$$

Thus, if the components of the position vector $\mathbf{r}$ in the inertial XYZ frame are given by $\{\mathbf{r}\}_x$, its components $\{\mathbf{r}\}_{x'}$ in the rotating, earth-fixed $x'y'z'$ frame are

$$\{\mathbf{r}\}_{x'} = [\mathbf{R}_3(\theta)]\{\mathbf{r}\}_X \tag{4.56}$$

Knowing $\{\mathbf{r}\}_{x'}$, we use Algorithm 4.1 to determine the right ascension (longitude east of x') and declination (latitude) in the earth-fixed system. These points are usually plotted on a rectangular Mercator projection of the earth's surface, as in Figure 4.22.

ALGORITHM 4.6

Given the initial orbital elements (h, e, a, T i, ω_0, Ω_0, and θ_0) of a satellite relative to the geocentric equatorial frame, compute the RA α and Dec δ relative to the rotating earth after a time interval Δt. This algorithm is implemented in MATLAB as the script *ground_track.m* in Appendix D.23.

1. Compute $\dot{\Omega}$ and $\dot{\omega}$ from Eqns (4.52) and (4.53).
2. Calculate the initial time t_0 (time since perigee passage):
   a. Find the eccentric anomaly E_0 from Eqn (3.13b).
   b. Find the mean anomaly M_0 from Eqn (3.14).
   c. Find t_0 from Eqn (3.15).
3. At time $t = t_0 + \Delta t$, calculate α and δ.
   a. Calculate the true anomaly:
      i. Find M from Eqn (3.8).
      ii. Find E from Eqn (3.14) using Algorithm 3.1.
      iii. Find θ from Eqn (3.13a).
   b. Update Ω and ω: $\Omega = \Omega_0 + \dot{\Omega}\Delta t \quad \omega = \omega_0 + \dot{\omega}\Delta t$
   c. Find $\{\mathbf{r}\}_x$ using Algorithm 4.5.
   d. Find $\{\mathbf{r}\}_{x'}$ using Eqns 4.55 and 4.56.
   e. Use Algorithm 4.1 to compute α and δ from $\{\mathbf{r}\}_{x'}$.
4. Repeat Step 3 for additional times ($t = t_0 + 2\Delta t, t = t_0 + 3\Delta t$, etc.).

EXAMPLE 4.12

An earth satellite has the following orbital parameters:

$r_p = 6700$ km	Perigee
$r_a = 10\,000$ km	Apogee
$\theta_0 = 230°$	True anomaly
$\Omega_0 = 270°$	Right ascension of the ascending node
$i_0 = 60°$	Inclination
$\omega_0 = 45°$	Argument of perigee

Calculate the right ascension (longitude east of x') and declination (latitude) relative to the rotating earth 45 min later.

Solution

First, we compute the semimajor axis a, eccentricity e, the angular momentum h, the semimajor axis a, and the period T. For the semimajor axis, we recall that

$$a = \frac{r_p + r_a}{2} = \frac{6700 + 10,000}{2} = 8350 \text{ km}$$

From Eqn (2.84) we get

$$e = \frac{r_a - r_p}{r_a + r_p} = \frac{10,000 - 6700}{10,000 + 6700} = 0.19760$$

Equation (2.50) yields

$$h = \sqrt{\mu r_p(1+e)} = \sqrt{398{,}600 \times 6700 \times (1 + 0.19760)} = 56{,}554 \text{ km}^2/\text{s}$$

Finally, we obtain the period from Eqn (2.83),

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2} = \frac{2\pi}{\sqrt{398{,}600}} 8350^{3/2} = 7593.5 \text{ s}$$

Now we can proceed with Algorithm 4.6.

Step 1:

$$\dot{\Omega} = -\left[\frac{3}{2}\frac{\sqrt{\mu}J_2 R_{\text{earth}}^2}{(1-e^2)a^{7/2}}\right]\cos i = -\left[\frac{3}{2}\frac{\sqrt{398{,}600} \times 0.0010836 \times 6378^2}{(1 - 0.19760^2)8350^{7/2}}\right]\cos 60^\circ = -2.3394 \times 10^{-7}\ ^\circ/\text{s}$$

$$\dot{\omega} = \dot{\Omega}\frac{5/2\sin^2 i - 2}{\cos i} = -2.3394 \times 10^{-5}\left(\frac{5/2\sin^2 60^\circ - 2}{\cos 60^\circ}\right) = 5.8484 \times 10^{-6}\ ^\circ/\text{s}$$

Step 2:

a.

$$E = 2\tan^{-1}\left(\tan\frac{\theta}{2}\sqrt{\frac{1-e}{1+e}}\right) = 2\tan^{-1}\left(\tan\frac{230^\circ}{2}\sqrt{\frac{1-0.19760}{1+0.19760}}\right) = -2.1059 \text{ rad}$$

b.

$$M = E - e\sin E = -2.1059 - 0.19760\sin(-2.1059) = -1.9360 \text{ rad}$$

c.

$$t_0 = \frac{M}{2\pi}T = \frac{-1.9360}{2\pi}\cdot 7593.5 = -2339.7 \text{ s} \quad (2339.7 \text{ s until perigee})$$

Step 3: $t = t_0 + 45 \text{ min} = -2339.7 + 45 \times 60 = 360.33 \text{ s} \quad (360.33 \text{ s after perigee})$

a.

$$M = 2\pi\frac{t}{T} = 2\pi\frac{360.33}{7593.5} = 0.29815 \text{ rad}$$

$$E - 0.19760\sin E = 0.29815 \quad \xrightarrow{\text{Algorithm 3.1}} \quad E = 0.36952 \text{ rad}$$

$$\theta = 2\tan^{-1}\left(\tan\frac{E}{2}\sqrt{\frac{1+e}{1-e}}\right) = 2\tan^{-1}\left(\tan\frac{0.36952}{2}\sqrt{\frac{1+0.19760}{1-0.19760}}\right) = 25.723^\circ$$

b.

$$\Omega = \Omega_0 + \dot{\Omega}\Delta t = 270^\circ + \left(-2.3394 \times 10^{-5}\ ^\circ/\mathrm{s}\right)(2700\ \mathrm{s}) = 269.94$$

$$\omega = \omega_0 + \dot{\omega}\Delta t = 45^\circ + \left(5.8484 \times 10^{-6}\ ^\circ/\mathrm{s}\right)(2700\ \mathrm{s}) = 45.016^\circ$$

c.

$$\{\mathbf{r}\}_X \overset{\text{Algorithm 4.5}}{\widehat{=}} \begin{Bmatrix} 3212.6 \\ -2250.5 \\ 5568.6 \end{Bmatrix} \text{(km)}$$

d.

$$\theta = \omega_E \Delta t = \frac{360^\circ\left(1 + \frac{1}{365.26}\right)}{24 \times 3600\mathrm{s}} \times 2700\ \mathrm{s} = 11.281^\circ$$

$$[\mathbf{R}_3(\theta)] = \begin{bmatrix} \cos 11.281^\circ & \sin 11.281^\circ & 0 \\ -\sin 11.281^\circ & \cos 11.281^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.98068 & 0.19562 & 0 \\ -0.19562 & 0.98068 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\{\mathbf{r}\}_{X'} = [\mathbf{R}_3(\theta)]\{\mathbf{r}\}_X = \begin{bmatrix} 0.98068 & 0.19562 & 0 \\ -0.19562 & 0.98068 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 3212.6 \\ -2250.5 \\ 5568.6 \end{Bmatrix} = \begin{Bmatrix} 2710.3 \\ -2835.4 \\ 5568.6 \end{Bmatrix} \text{(km)}$$

e.

$$\mathbf{r} = 2710.3\hat{\mathbf{I}}' - 2835.4\hat{\mathbf{J}}' + 5568.6\hat{\mathbf{K}}' \overset{\text{Algorithm 4.1}}{\widehat{\Rightarrow}} \boxed{\alpha = 313.7^\circ \quad \delta = 54.84^\circ}$$

The script *ground_track.m* in Appendix D.23 can be used to plot ground tracks. For the data of Example 4.12, the ground track for 3.25 periods appears in Figure 4.23. The ground track for one orbit of a Molniya satellite is featured more elegantly in Figure 4.24.

PROBLEMS

Section 4.3

4.1 For each of the following geocentric equatorial position vectors (in kilometers) find the RA and Dec.

$$\mathbf{r} = -3000\hat{\mathbf{I}} - 6000\hat{\mathbf{J}} - 9000\hat{\mathbf{K}} \tag{a}$$

$$\mathbf{r} = -3000\hat{\mathbf{I}} - 6000\hat{\mathbf{J}} - 9000\hat{\mathbf{K}} \tag{b}$$

$$\mathbf{r} = -9000\hat{\mathbf{I}} - 3000\hat{\mathbf{J}} + 6000\hat{\mathbf{K}} \tag{c}$$

$$\mathbf{r} = 6000\hat{\mathbf{I}} - 9000\hat{\mathbf{J}} - 3000\hat{\mathbf{K}} \tag{d}$$

{Partial Ans.: (b) $\alpha = 243.4^\circ$, $\delta = -53.30^\circ$}

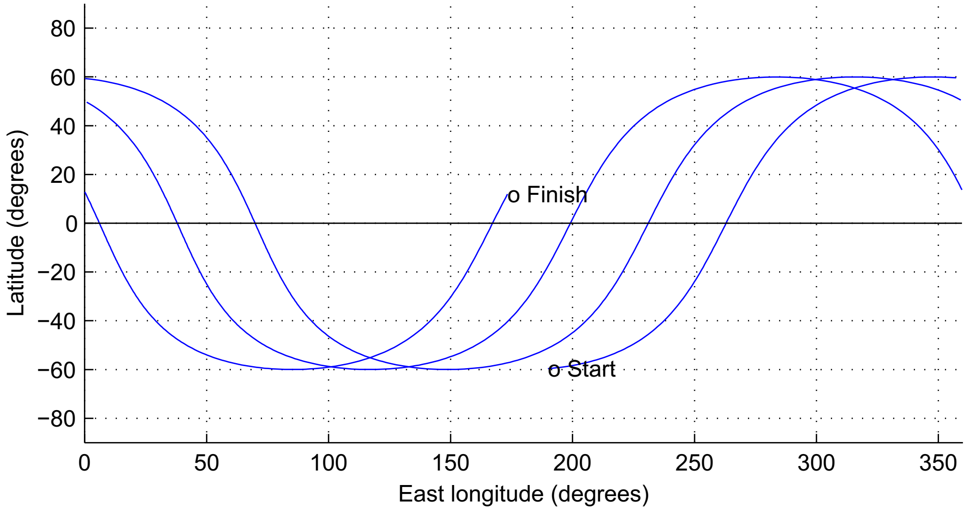


FIGURE 4.23

Ground track for 3.25 orbits of the satellite in Example 4.6.

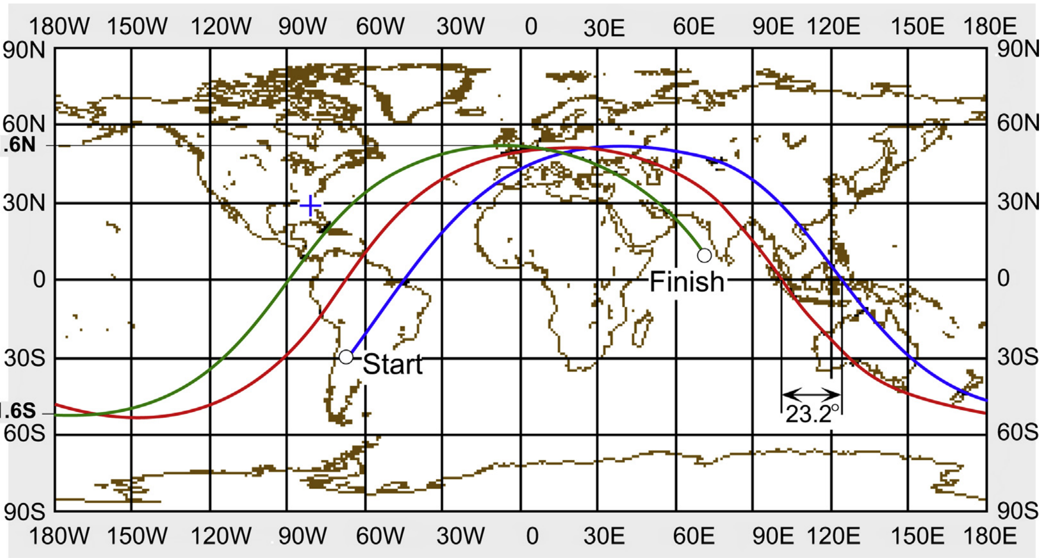


FIGURE 4.24

Ground track for two orbits of a Molniya satellite with a 12-h period. Tick marks are 1 h apart.

4.2 At a given instant, a spacecraft is 500 km above the earth, with an RA of 300° and Dec of −20° relative to the geocentric equatorial frame. Its velocity is 10 km/s directly north, normal to the equatorial plane. Find its RA and Dec 30 min later.
{Ans.: $\alpha = 120°$, $\delta = -29.98°$}

Section 4.4

4.3 Find the orbital elements of a geocentric satellite whose inertial position and velocity vectors in a geocentric equatorial frame are

$$\mathbf{r} = 2500\hat{\mathbf{I}} + 16{,}000\hat{\mathbf{J}} + 4000\hat{\mathbf{K}}(\text{km})$$

$$\mathbf{v} = -3\hat{\mathbf{I}} - \hat{\mathbf{J}} + 5\hat{\mathbf{K}}(\text{km/s})$$

{Ans.: $e = 0.4658$, $h = 98{,}623\ \text{km}^2/\text{s}$, $i = 62.52°$, $\Omega = 73.74°$, $\omega = 22.08°$, $\theta = 353.6°$}

4.4 At a given instant, the position **r** and velocity **v** of a satellite in the geocentric equatorial frame are $\mathbf{r} = -13{,}000\hat{\mathbf{K}}(\text{km})$ and $\mathbf{v} = 4\hat{\mathbf{i}} + 5\hat{\mathbf{J}} + 6\hat{\mathbf{K}}(\text{km/s})$. Find the orbital elements.
{Ans.: $h = 83{,}240\ \text{km}^2/\text{s}$, $e = 1.298$, $\Omega = 51.34°$, $\omega = 344.9°$, $\theta = 285.1°$, $i = 90°$}

4.5 At time t_0 (relative to perigee passage) the position **r** and velocity **v** of a satellite in the geocentric equatorial frame are $\mathbf{r} = 6500\hat{\mathbf{I}} - 7500\hat{\mathbf{J}} - 2500\hat{\mathbf{K}}(\text{km})$ and $\mathbf{v} = 4\hat{\mathbf{I}} + 3\hat{\mathbf{J}} - 3\hat{\mathbf{K}}\ \ (\text{km/s})$. Find the orbital elements.
{Ans.: $h = 58{,}656\ \text{km}^2/\text{s}$, $e = 0.2226$, $\Omega = 107.6°$, $\omega = 107.6°$, $\theta = 134.7°$, $i = 32.44°$}

4.6 Given that, with respect to the geocentric equatorial frame, $\mathbf{r} = -6000\hat{\mathbf{I}} - 1000\hat{\mathbf{J}} - 5000\hat{\mathbf{K}}$ (km), $\mathbf{v} = 6\hat{\mathbf{I}} - 7\hat{\mathbf{J}} - 2\hat{\mathbf{K}}$ (km/s), and the eccentricity vector is $\mathbf{e} = -0.4\hat{\mathbf{I}} - 0.5\hat{\mathbf{J}} - 0.6\hat{\mathbf{K}}$, calculate the true anomaly θ of the earth-orbiting satellite.
{Ans.: 328.6°}

4.7 Given that, relative to the geocentric equatorial frame, $\mathbf{r} = -6600\hat{\mathbf{I}} - 1300\hat{\mathbf{J}} - 5200\hat{\mathbf{K}}$ (km), the eccentricity vector is $\mathbf{e} = -0.4\hat{\mathbf{I}} - 0.5\hat{\mathbf{J}} - 0.6\hat{\mathbf{K}}$, and the satellite is flying toward perigee, calculate the inclination of the orbit.
{Ans.: 43.3°}

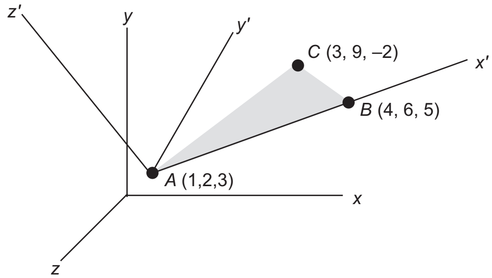
Section 4.5

4.8 The right-handed, primed *xyz* system is defined by the three points *A*, *B*, and *C*. The $x'y'$ plane is defined by the plane *ABC*. The x'-axis runs from *A* through *B*. The z'-axis is defined by the cross product of $\overrightarrow{AB}$ into $\overrightarrow{AC}$, so that the $+y'$-axis lies on the same side of the x'-axis as point C.

a) Find the DCM [**Q**] relating the two coordinate bases.

b) If the components of a vector **v** in the primed system are $\lfloor 2 \quad -1 \quad 3 \rfloor^T$, find the components of **v** in the unprimed system.

{Partial Ans.: (b) $\lfloor -1.307 \quad 2.390 \quad 2.565 \rfloor^T$}



4.9 The unit vectors in a *uvw* Cartesian coordinate frame have the following components in the *xyz* frame:

$$\hat{\mathbf{u}} = 0.26726\hat{\mathbf{i}} + 0.53452\hat{\mathbf{j}} + 0.80178\hat{\mathbf{k}}$$

$$\hat{\mathbf{v}} = -0.44376\hat{\mathbf{i}} + 0.80684\hat{\mathbf{j}} - 0.38997\hat{\mathbf{k}}$$

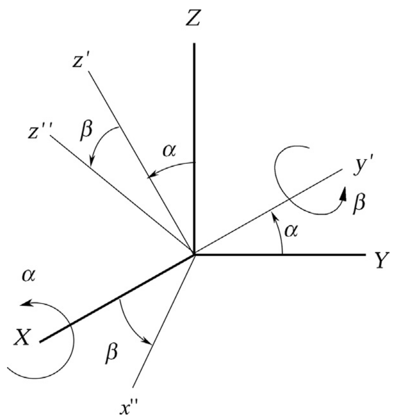
$$\hat{\mathbf{w}} = -0.85536\hat{\mathbf{i}} - 0.25158\hat{\mathbf{j}} + 0.45284\hat{\mathbf{k}}$$

If, in the *xyz* frame $\mathbf{V} = -50\hat{\mathbf{i}} + 100\hat{\mathbf{j}} + 75\hat{\mathbf{k}}$, find the components of the vector $\mathbf{V}$ in the *uvw* frame.

{Ans.: $\mathbf{V} = 100.2\hat{\mathbf{u}} + 73.62\hat{\mathbf{v}} + 51.57\hat{\mathbf{w}}$}

4.10 Calculate the DCM [**Q**] for the sequence of two rotations: $\alpha = 40°$ about the positive *X*-axis, followed by $\beta = 25°$ about the positive y'-axis. The result is that the *XYZ* axes are rotated into the $x''y''z''$ axes.

{Partial Ans.: $Q_{11} = 0.9063°$ $\quad Q_{12} = 0.2716°$ $\quad Q_{13} = -0.3237$}



4.11 For the DCM

$$[\mathbf{Q}] = \begin{bmatrix} 0.086824 & -0.77768 & 0.62264 \\ -0.49240 & -0.57682 & -0.65178 \\ 0.86603 & -0.25000 & -0.43301 \end{bmatrix}$$

calculate:

(a) the classical Euler angle sequence and

(b) the yaw, pitch, and roll angle sequence.

{Ans.: (a) $\alpha = 73.90°$, $\beta = 115.7°$, $\gamma = 136.31°$
(b) $\alpha = 276.37°$, $\beta = -38.51°$, $\gamma = 236.40°$}

4.12 What yaw, pitch, and roll sequence yields the same DCM as the classical Euler sequence $\alpha = 350°$, $\beta = 170°$, $\gamma = 300°$?
{Ans.: $\alpha = 49.62°$, $\beta = 8.649°$, $\gamma = 175.0°$}

4.13 What classical Euler angle sequence yields the same DCM as the yaw, pitch, and roll sequence $\alpha = 300°$, $\beta = -80°$, $\gamma = 30°$?
{Ans.: $\alpha = 240.4°$, $\beta = 81.35°$, $\gamma = 84.96°$}

Section 4.6

4.14 At time t_0 (relative to perigee passage), the position $\mathbf{r}$ and velocity $\mathbf{v}$ of a satellite in the geocentric equatorial frame are

$$\mathbf{r} = -5000\hat{\mathbf{I}} - 8000\hat{\mathbf{J}} - 2100\hat{\mathbf{K}} \text{ (km)}$$

$$\mathbf{v} = -4\hat{\mathbf{I}} + 3.5\hat{\mathbf{J}} - 3\hat{\mathbf{K}} \text{ (km/s)}$$

Find $\mathbf{r}$ and $\mathbf{v}$ at time $t_0 + 50$ min.
{Ans.: $\mathbf{r} = -1717\hat{\mathbf{I}} + 7604\hat{\mathbf{J}} - 2101\hat{\mathbf{K}}$ (km); $\mathbf{v} = 6.075\hat{\mathbf{I}} + 1.925\hat{\mathbf{J}} + 3.591\hat{\mathbf{K}}$(km/s)}

4.15 At time t_0 (relative to perigee passage), a spacecraft has the following orbital parameters: $e = 1.5$; perigee altitude = 300 km; $i = 35°$; $\Omega = 130°$; and $\omega = 115°$. Calculate $\mathbf{r}$ and $\mathbf{v}$ at perigee relative to (a) the perifocal reference frame and (b) the geocentric equatorial frame.
{Ans.: (a) $\mathbf{r} = 6678\hat{\mathbf{p}}$ (km), $\mathbf{v} = 12.22\hat{\mathbf{q}}$(km/s);
(b) $\mathbf{r} = -1984\hat{\mathbf{I}} - 5348\hat{\mathbf{J}} + 3471\hat{\mathbf{K}}$(km), $\mathbf{v} = 10.36\hat{\mathbf{I}} - 5.763\hat{\mathbf{J}} - 2.961\hat{\mathbf{K}}$ (km/s)}

4.16 For the spacecraft of Problem 4.15, calculate $\mathbf{r}$ and $\mathbf{v}$ at 2 h past perigee relative to (a) the perifocal reference frame and (b) the geocentric equatorial frame.
{Ans.: (a) $\mathbf{r} = -25{,}010\hat{\mathbf{p}} + 48{,}090\hat{\mathbf{q}}$ (km), $\mathbf{v} = -4.335\hat{\mathbf{p}} + 5.075\hat{\mathbf{q}}$ (km/s);
(b) $\mathbf{r} = 48{,}200\hat{\mathbf{I}} - 2658\hat{\mathbf{J}} - 24{,}660\hat{\mathbf{K}}$ (km), $\mathbf{v} = 5.590\hat{\mathbf{I}} + 1.078\hat{\mathbf{J}} - 3.484\hat{\mathbf{K}}$ (km/s)}

4.17 Calculate $\mathbf{r}$ and $\mathbf{v}$ for the satellite in Problem 4.15 at time $t_0 + 50$ min.
{Ans.: $\mathbf{r} = 23{,}047\hat{\mathbf{I}} - 6972.4\hat{\mathbf{J}} - 9219.6\hat{\mathbf{K}}$ (km), $\mathbf{v} = 6.6563\hat{\mathbf{I}} + 0.88638\hat{\mathbf{J}} - 3.9680\hat{\mathbf{K}}$ (km/s)}

4.18 For a spacecraft, the following orbital parameters are given: $e = 1.2$; perigee altitude = 200 km; $i = 50°$; $\Omega = 75°$; and $\omega = 80°$. Calculate $\mathbf{r}$ and $\mathbf{v}$ at perigee relative to (a) the perifocal reference frame and (b) the geocentric equatorial frame.
{Ans. (a) $\mathbf{r} = 6578\hat{\mathbf{p}}$ (km), $\mathbf{v} = 11.55\hat{\mathbf{q}}$ (km/s);
(b) $\mathbf{r} = -3726\hat{\mathbf{I}} + 2181\hat{\mathbf{J}} + 4962\hat{\mathbf{K}}$ (km), $\mathbf{v} = -4.188\hat{\mathbf{I}} - 10.65\hat{\mathbf{J}} + 1.536\hat{\mathbf{K}}$ (km/s)}

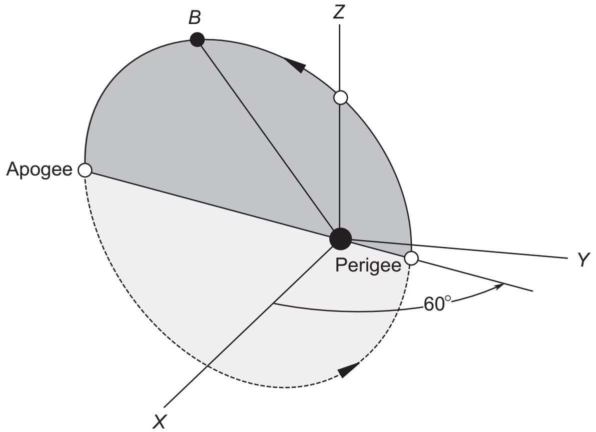
4.19 For the spacecraft of Problem 4.18, calculate $\mathbf{r}$ and $\mathbf{v}$ at 2 h past perigee relative to (a) the perifocal reference frame and (b) the geocentric equatorial frame.
{Ans.: (a) $\mathbf{r} = -26{,}340\hat{\mathbf{p}} + 37{,}810\hat{\mathbf{q}}$ (km), $\mathbf{v} = -4.306\hat{\mathbf{p}} + 3.298\hat{\mathbf{q}}$ (km/s);
(b) $\mathbf{r} = 1207\hat{\mathbf{I}} - 43{,}600\hat{\mathbf{J}} - 14{,}840\hat{\mathbf{K}}$ (km), $\mathbf{v} = 1.243\hat{\mathbf{I}} - 4.4700\hat{\mathbf{J}} - 2.810\hat{\mathbf{K}}$ (km/s)}

4.20 Given that $e = 0.7$, $h = 75{,}000$ km^2/s, and $\theta = 25°$, calculate the components of velocity in the geocentric equatorial frame if

$$[\mathbf{Q}]_{X\bar{x}} = \begin{bmatrix} -.83204 & -.13114 & 0.53899 \\ 0.02741 & -.98019 & -.19617 \\ 0.55403 & -.14845 & 0.81915 \end{bmatrix}$$

{Ans.: $\mathbf{v} = 2.103\hat{\mathbf{I}} - 8.073\hat{\mathbf{J}} - 2.885\hat{\mathbf{K}}$ (km/s)}

4.21 The apse line of the elliptical orbit lies in the XY plane of the geocentric equatorial frame, whose Z-axis lies in the plane of the orbit. At B (for which $\theta = 140°$), the perifocal velocity vector is $\{\mathbf{v}\}_{\bar{x}} = \lfloor -3.208 \quad -0.8288 \quad 0 \rfloor^T$ (km/s). Calculate the geocentric–equatorial components of the velocity at B.
{Ans.: $\{\mathbf{v}\}_X = \lfloor -1.604 \quad -2.778 \quad -0.8288 \rfloor^T$ (km/s)}



4.22 A satellite in earth orbit has the following orbital parameters: $a = 7016$ km, $e = 0.05$, $i = 45°$, $\Omega = 0°$, $\omega = 20°$, and $\theta = 10°$. Find the position vector in the geocentric equatorial frame.
{Ans.: $\mathbf{r} = 5776.4\hat{\mathbf{I}} + 2358.2\hat{\mathbf{J}} + 2358.2\hat{\mathbf{K}}$(km)}

Section 4.7

4.23 Calculate the orbital inclination required to place an earth satellite in a 300 km by 600 km sun-synchronous orbit.
{Ans.: 97.21°}.

4.24 A satellite in a circular, sun-synchronous low earth orbit passes over the same point on the equator once each day, at 12 o′clock noon. Calculate the inclination, altitude, and period of the orbit.
{Ans.: This problem has more than one solution.}

4.25 The orbit of a satellite around an unspecified planet has an inclination of 45°, and its perigee advances at the rate of 6° per day. At what rate does the node line regress?
{Ans.: $\dot{\Omega} = 5.656$ °/day}

4.26 At a given time, the position and velocity of an earth satellite in the geocentric equatorial frame are $\mathbf{r} = -2429.1\hat{\mathbf{I}} + 4555.1\hat{\mathbf{J}} + 4577.0\hat{\mathbf{K}}$ (km) and $\mathbf{v} = -4.7689\hat{\mathbf{I}} - 5.6113\hat{\mathbf{J}} + 3.0535\hat{\mathbf{K}}$ (km/s). Find $\mathbf{r}$ and $\mathbf{v}$ precisely 72 h later, taking into consideration the node line regression and the advance of perigee.
{Ans.: $\mathbf{r} = 4596\hat{\mathbf{I}} + 5759\hat{\mathbf{J}} - 1266\hat{\mathbf{K}}$ (km), $\mathbf{v} = -3.601\hat{\mathbf{I}} + 3.179\hat{\mathbf{J}} + 5.617\hat{\mathbf{K}}$ (km/s)}

Section 4.8

4.27 A spacecraft is in a circular orbit of 180 km altitude and inclination 30°. What is the spacing, in kilometers, between successive ground tracks at the equator, including the effect of earth's oblateness?
{Ans.: 2511 km}