CHAPTER

Introduction to Orbital Perturbations

12

CHAPTER OUTLINE

Orbital Mechanics for Engineering Students. http://dx.doi.org/10.1016/B978-0-08-097747-8.00012-8

12.1 Introduction

Keplerian orbits are the closed-form solutions of the two-body equation of relative motion (Eqn (2.22)),

$$\ddot{\mathbf{r}} = -\mu\frac{\mathbf{r}}{r^3} \tag{12.1}$$

This equation is based on the assumption that there are only two objects in space, and that their spherically symmetric gravitational fields are the only source of interaction between them. Any effect that causes the motion to deviate from a Keplerian trajectory is known as a perturbation. Common perturbations of two-body motion include a nonspherical central body, atmospheric drag, propulsive thrust, solar radiation pressure, and gravitational interactions with celestial objects like the moon and the sun. To account for perturbations, we add a term $\mathbf{p}$ to the right-hand side of Eqn (12.1) to get

$$\ddot{\mathbf{r}} = -\mu\frac{\mathbf{r}}{r^3} + \mathbf{p} \tag{12.2}$$

The vector $\mathbf{p}$ is the net perturbative acceleration from all sources other than the spherically symmetric gravitational attraction between the two bodies. The magnitude of $\mathbf{p}$ is usually small compared to the primary gravitational acceleration $a_0 = \mu/r^2$. An exception is atmospheric drag which, at an altitude of about 100 km, is large enough to deorbit a satellite. The drag effect decreases rapidly with altitude and becomes negligible ($p_{\text{drag}} < 10^{-10}a_0$) above 1000 km. The other effects depend on the altitude to various extents or, in the case of solar radiation pressure, not at all. At 1000 km altitude, their disturbing accelerations in decreasing order are (Fortescue (2011), Montenbruck (2005))

$$\begin{aligned}
&p_{\text{earth's oblateness}} \approx 10^{-2}a_0\\
&p_{\text{lunar gravity}} \approx p_{\text{solar gravity}} \approx 10^{-7}a_0\\
&p_{\text{solar radiation}} \approx 10^{-9}a_0
\end{aligned} \tag{12.3}$$

Starting with a set of initial conditions $(\mathbf{r}_0,\mathbf{v}_0)$ and the functional form of the perturbation $\mathbf{p}$, we can numerically integrate Eqn (12.2) to find the position $\mathbf{r}$ and velocity $\mathbf{v}$ at any time thereafter. The classical orbital elements at any instant are then furnished by Algorithm 4.2. Conversely, we can numerically integrate what are known as the Lagrange planetary equations to obtain the orbital elements instead of the state vector as functions of time. From the orbital elements, we obtain the state vector $(\mathbf{r},\mathbf{v})$ at any instant by using Algorithm 4.5.

We start this chapter with a look at the concept of osculating orbits and the two classical techniques for numerically integrating Eqn (12.2), namely, Cowell's method and Encke's method. These methods are then used to carry out special perturbations analyses of the effects of atmospheric drag and the earth's oblateness. Next, we discuss the method of variation of parameters, which is familiar to all students of a first course in differential equations. The method is applied to the solution of Eqn (12.2) in order to obtain the conditions that are imposed on the osculating elements of a perturbed trajectory. These conditions lead to the Lagrange planetary equations, a set of differential equations, that govern the time variation of the osculating orbital elements. A variant of the Lagrange planetary equations, the Gauss variational equations, will be derived in detail and used for our special perturbations analyses in the rest of this chapter. We will employ the Gauss variational equations to obtain the analytical expressions for the rates of change of the osculating elements. For geopotential perturbations, the method of averaging will be applied to these equations to smooth out the short period variations, leaving us with simplified formulas for only the long term secular variations. The chapter concludes with the applications of the Gauss variational equations to the special perturbations analysis of the effects of solar radiation pressure, lunar gravity, and solar gravity.

12.2 Cowell's method

Philip H. Cowell (1870–1949) was a British astronomer whose name is attached to the method of direct numerical integration of Eqn (12.2). Cowell's work at the turn of the twentieth century (e.g., predicting the time of the closest approach to the sun of Halley's Comet upon its return in 1910) relied entirely upon hand calculations to numerically integrate the equations of motion using classical methods dating from Isaac Newton's time. Today, of course, we have high-speed digital computers upon which modern, extremely accurate integration algorithms can be easily implemented. A few of these methods were presented in Section 1.8.

We have already used Cowell's method in Chapter 2 to integrate the three-body equations (derived in Appendix C) for the particular scenario depicted in Figures 2.4 and 2.5. In Section 6.10, we set $\mathbf{p} = (T/m)(\mathbf{v}/v)$ in Eqn (12.2) to simulate tangential thrust, and then we numerically integrated the ordinary differential equations to obtain the results in Example 6.15 for a high thrust situation and in Example 6.17 for a low thrust application.

Upon solving Eqn (12.2) for the state vector of the perturbed path at any time t, the orbital elements may be found by means of Algorithm 4.2. However, these orbital elements describe the *osculating orbit*, not the perturbed orbit. The osculating orbit is the two-body trajectory that would be followed after time t if at that instant the perturbing acceleration $\mathbf{p}$ were to suddenly vanish, thereby making Eqn (12.1) valid. Since the state vectors of both the perturbed orbit and the osculating orbit are identical at time t, the two orbits touch and are tangential at that point. (It is interesting to note that *osculate* has its roots in the Latin word for *kiss*.) Every point of a perturbed trajectory has its own osculating trajectory.

We illustrate the concept of osculating orbits in Figure 12.1, which shows the earth orbit of a spacecraft containing an onboard rocket engine that exerts a constant tangential thrust T starting at O. The continuous addition of energy causes the trajectory to spiral outward away from the earth. The thrust is the perturbation that forces the orbit to deviate from a Keplerian (elliptical) path. If at time t_1 the engine were to shut down, then the spacecraft would enter the elliptical osculating orbit 1 shown in the figure. Also shown are the osculating orbits 2, 3, and 4 at times t_2, t_3, and t_4. Observe that the thruster not only increases the semimajor axis but also causes the apse line to rotate counterclockwise, in the direction of the orbital motion.

12.3 Encke's method

In the method developed originally by the German astronomer Johann Franz Encke (1791–1865), the two-body motion due solely to the primary attractor is treated separately from that due to the perturbation. The two-body osculating orbit $\mathbf{r}_{osc}(t)$ is used as a reference orbit upon which the unknown deviation $\delta\mathbf{r}(t)$ due to the perturbation is superimposed to obtain the perturbed orbit $\mathbf{r}(t)$.

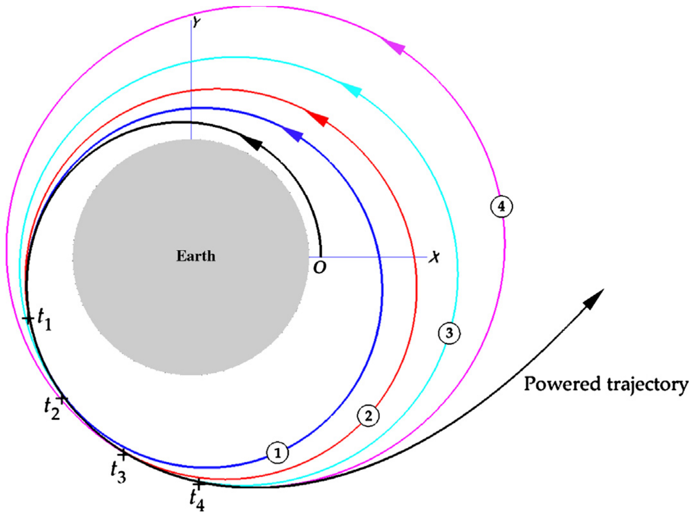

FIGURE 12.1

Osculating orbits 1–4 corresponding to times t_1 through t_4, respectively, on the powered, spiral trajectory that starts at point O. The circled labels are centered at each orbit's apogee.

Let $(\mathbf{r}_0, \mathbf{v}_0)$ be the state vector of an orbiting object at time t_0. The osculating orbit at that time is governed by Eqn (12.1),

$$\ddot{\mathbf{r}}_{\text{osc}} = -\mu \frac{\mathbf{r}_{\text{osc}}}{r^3} \tag{12.4}$$

with the initial conditions $\mathbf{r}_{\text{osc}}(t_0) = \mathbf{r}_0$ and $\mathbf{v}_{\text{osc}}(t_0) = \mathbf{v}_0$. For times $t > t_0$, the state vector $(\mathbf{r}_{\text{osc}}, \mathbf{v}_{\text{osc}})$ of the osculating, two-body trajectory may be found analytically using the Lagrange coefficients (Eqns (3.67) and (3.68)),

$$\begin{aligned} \mathbf{r}_{\text{osc}}(t) &= f(t)\mathbf{r}_0 + g(t)\mathbf{v}_0 \\ \mathbf{v}_{\text{osc}}(t) &= \dot{f}(t)\mathbf{r}_0 + \dot{g}(t)\mathbf{v}_0 \end{aligned} \tag{12.5}$$

After the initial time t_0, the perturbed trajectory $\mathbf{r}(t)$ will increasingly deviate from the osculating path $\mathbf{r}_{\text{osc}}(t)$, so that, as illustrated in Figure 12.2,

$$\mathbf{r}(t) = \mathbf{r}_{\text{osc}}(t) + \delta\mathbf{r}(t) \tag{12.6}$$

Substituting $\mathbf{r}_{\text{osc}} = \mathbf{r} - \delta\mathbf{r}$ into Eqn (12.4) and setting $\delta\mathbf{a} = \delta\ddot{\mathbf{r}}$ yields

$$\delta\mathbf{a} = \ddot{\mathbf{r}} + \mu \frac{\mathbf{r} - \delta\mathbf{r}}{r^3}$$

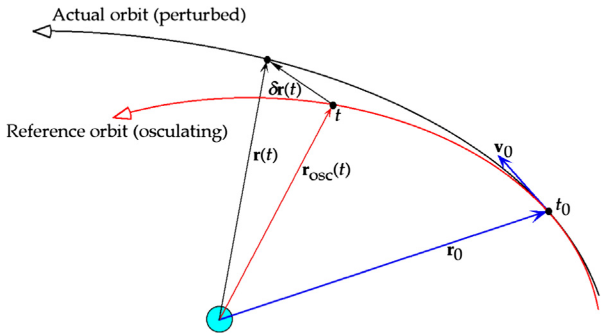

FIGURE 12.2

Perturbed and osculating orbits.

We may then substitute Eqn (12.2) into this expression to get

$$\delta\mathbf{a} = -\mu\frac{\mathbf{r}}{r^3} + \mu\frac{\mathbf{r}-\delta\mathbf{r}}{r_{osc}{}^3} + \mathbf{p} = -\frac{\mu}{r_{osc}{}^3}\delta\mathbf{r} + \mu\left(\frac{1}{r_{osc}{}^3} - \frac{1}{r^3}\right)\mathbf{r} + \mathbf{p}$$

A final rearrangement of the terms leads to

$$\delta\mathbf{a} = -\frac{\mu}{r_{osc}{}^3}\left[\delta\mathbf{r} - \left(1 - \frac{r_{osc}{}^3}{r^3}\right)\mathbf{r}\right] + \mathbf{p} \tag{12.7}$$

As is evident from Figure 12.3, r_{osc} and r may become very nearly equal, in which case accurately calculating the difference $F = 1 - (r_{osc}/r)^3$ can be problematic for a digital computer. In that case, we refer to Appendix F to rewrite Eqn (12.7) as $\delta\mathbf{a} = -\mu[\delta\mathbf{r} - F(q)\mathbf{r}]/r_{osc}{}^3 + \mathbf{p}$, where $q = \delta\mathbf{r}\cdot(2\mathbf{r} - \delta\mathbf{r})/r^2$ and $F(q)$ are given by Eqn (F.3).

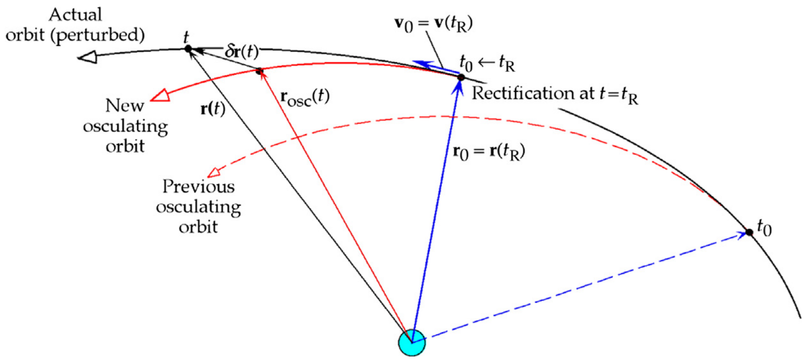

FIGURE 12.3

Resetting the reference orbit at time t_R.

Recall that $\mathbf{p}$ is the perturbing acceleration, which is a known function of time. In Encke's method, we integrate Eqn (12.7) to obtain the deviation $\delta\mathbf{r}(t)$. This is added to the osculating motion $\mathbf{r}_{osc}(t)$ to obtain the perturbed trajectory from Eqn (12.6). If at any time the ratio $\delta\mathbf{r}/r$ exceeds a preset tolerance, then the osculating orbit is redefined to be that of the perturbed orbit at time t. This process, which is called *rectification*, is illustrated in Figure 12.3.

The following is an implementation of Encke's method in which rectification occurs at the beginning of every time step Δt. Other implementations may be found in textbooks such as Bate (1971, Section 9.3), Vallado (2007, Section 8.3), and Schaub (2009, Section 12.2).

ALGORITHM 12.1

Given the functional form of the perturbing acceleration $\mathbf{p}$, and the state vector components $\mathbf{r}_0$ and $\mathbf{v}_0$ at time t, calculate $\mathbf{r}$ and $\mathbf{v}$ at time $t+\Delta t$.

1. Set $\delta\mathbf{r}=\mathbf{0}$ and $\delta\mathbf{v}=\mathbf{0}$.

2. For the time span t to $t+\Delta t$, execute Algorithm 1.3 (or an other numerical integration procedure), with $\mathbf{y}=\begin{Bmatrix}\delta\mathbf{r}\\ \delta\mathbf{v}\end{Bmatrix}$ and $\mathbf{f}=\begin{Bmatrix}\delta\mathbf{v}\\ \delta\mathbf{a}\end{Bmatrix}$, to obtain $\delta\mathbf{r}(t+\Delta t)$ and $\delta\mathbf{v}(t+\Delta t)$.

At each step i of the numerical integration from t to $t+\Delta t$:

a. $\delta\mathbf{r}_i$ and $\delta\mathbf{v}_i$ are available from the previous step.
b. Compute $\mathbf{r}_{osc_i}$ and $\mathbf{v}_{osc_i}$ from $\mathbf{r}_0$ and $\mathbf{v}_0$ using Algorithm 3.4.
c. Compute $\mathbf{r}_i=\mathbf{r}_{osc_i}+\delta\mathbf{r}_i$ from Eqn (12.6).
d. Compute $\mathbf{v}_i=\mathbf{v}_{osc_i}+\delta\mathbf{v}_i$.
e. Compute $\delta\mathbf{a}_i$ from Eqn (12.7).
f. $\delta\mathbf{v}_i$ and $\delta\mathbf{a}_i$ are used to furnish $\delta\mathbf{r}$ and $\delta\mathbf{v}$ for the next step of the numerical integration.

3. $\mathbf{r}_0\leftarrow\mathbf{r}(t+\Delta t)$ and $\mathbf{v}_0\leftarrow\mathbf{v}(t+\Delta t)$ (Rectification).

12.4 Atmospheric drag

For the earth, the commonly accepted altitude at which space "begins" is 100 km (60 miles). Although over 99.9999% of the earth's atmosphere lies below 100 km, the air density at that altitude is nevertheless sufficient to exert drag and cause aerodynamic heating on objects moving at orbital speeds. (Recall from Eqn (2.63) that the speed required for a circular orbit at 100 km altitude is 7.8 km/s.) The drag will lower the speed and the height of a spacecraft, and the heating can produce temperatures of 2000 °C or more. A spacecraft will likely burn up unless it is protected with a heat shield. Note that the altitude (*entry interface*) at which the thermally protected Space Shuttle orbiter entered the atmosphere on its return from space was considered to be 120 km.

There are a number of models that describe the variation of atmospheric properties with altitude (AIAA, 2010). One of them is USSA76, the *US Standard Atmosphere 1976* (NOAA/NASA/USAF, 1976). Figure 12.4 shows the US Standard Atmosphere density profile from the sea level to an altitude of 1000 km. This figure was obtained by selecting the density ρ_i at 28 altitudes z_i in the USSA76 table and interpolating between them with the exponential functions $\rho(z)=\rho_i e^{-(z-z_i)/H_i}$, where

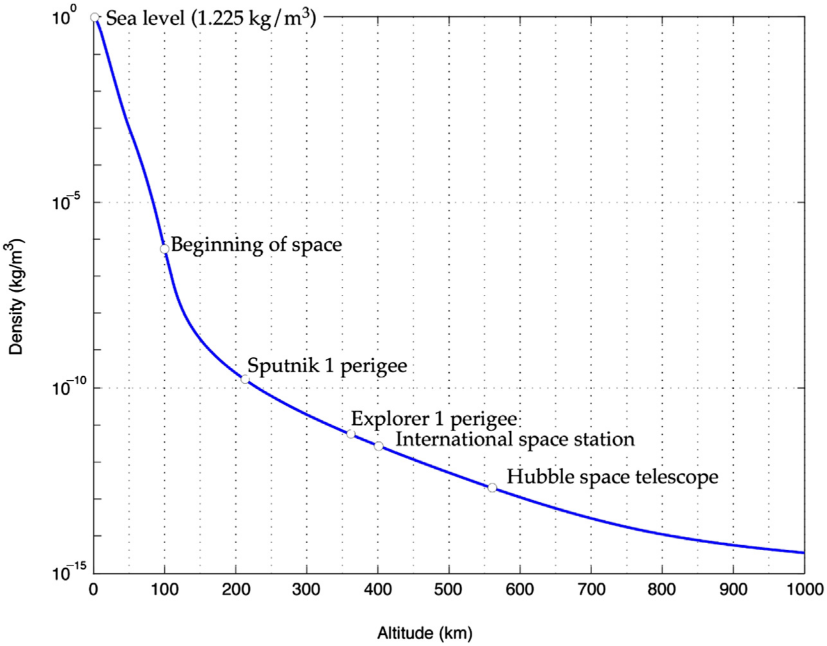

FIGURE 12.4

US Standard Atmosphere 1976: density versus altitude.

$z_i \le z < z_{i+1}$ and $H_i = -(z_{i+1} - z_i)/\ln(\rho_{i+1}/\rho_i)$. The simple procedure is implemented in the MATLAB function *atmosphere.m*, which is listed in Appendix D.41. For several equispaced altitudes (kilometers), *atmosphere* yields the following densities (kilograms per cubic meter):

```
>>
z = logspace(0,3,6);
for i = 1:6
    altitude = z(i);
    density  = atmosphere(z(i));
    fprintf('%12.3f km %12.3e kg/m^3\n', altitude, density)
    end

      1.000 km    1.068e+00 kg/m^3
      3.981 km    7.106e-01 kg/m^3
     15.849 km    1.401e-01 kg/m^3
     63.096 km    2.059e-04 kg/m^3
    251.189 km    5.909e-11 kg/m^3
   1000.000 km    3.561e-15 kg/m^3
>>
```

According to USSA76, the atmosphere is a spherically symmetric 1000 km thick gaseous shell surrounding the earth. Its properties throughout are steady state and are consistent with a period of moderate solar activity. The hypothetical variation of the properties with altitude approximately represents the year-round conditions at midlatitudes averaged over many years. The model provides realistic values of atmospheric density which, however, may not match the actual values at a given place or time.

Drag affects the life of an orbiting spacecraft. Sputnik 1, the world's first artificial satellite, had a perigee altitude of 228 km, where the air density is about five orders of magnitude greater than at its apogee altitude of 947 km. The drag force associated with repeated passage through the thicker air eventually robbed the spherical, 80-kg Sputnik of the energy needed to stay in orbit. It fell from orbit and burned up on January 4, 1958, almost 3 months to the day after it was launched by the Soviet Union. Soon thereafter, on January 31, the United States launched its first satellite, the cylindrical, 14-kg Explorer 1, into a 358 by 2550-km orbit. In this higher orbit, Explorer experienced less drag than Sputnik, and it remained in orbit for 18 years. The International Space Station's nearly circular orbit is about 400 km above the earth. At that height, drag causes orbital decay that requires frequent reboosts, usually provided by the propulsion systems of visiting supply vehicles. The altitude of the Hubble space telescope's circular orbit is 560 km, and it is also degraded by drag. Between 1993 and 2009, Space Shuttle orbiters visited Hubble five times to service it and boost its orbit. With no propulsion system of its own, the venerable Hubble is expected to deorbit around 2025, 35 years after it was launched.

If the inertial velocity of a spacecraft is $\mathbf{v}$ and that of the atmosphere at that point is $\mathbf{v}_{\text{atm}}$, then the spacecraft velocity relative to the atmosphere is

$$\mathbf{v}_{\text{rel}} = \mathbf{v} - \mathbf{v}_{\text{atm}} \tag{12.8}$$

If the atmosphere rotates with the earth, whose angular velocity is $\boldsymbol{\omega}_E$, then relative to the origin O of the geocentric equatorial frame, $\mathbf{v}_{\text{atm}} = \boldsymbol{\omega}_E \times \mathbf{r}$, where $\mathbf{r}$ is the spacecraft position vector. Thus,

$$\mathbf{v}_{\text{rel}} = \mathbf{v} - \boldsymbol{\omega}_E \times \mathbf{r} \tag{12.9}$$

Since the drag force $\mathbf{D}$ on an object acts in the direction opposite to the relative velocity vector, we can write

$$\mathbf{D} = -D\hat{\mathbf{v}}_{\text{rel}} \tag{12.10}$$

where $\hat{\mathbf{v}}_{\text{rel}} = \mathbf{v}_{\text{rel}}/v_{\text{rel}}$ is the unit vector in the direction of the relative velocity, and

$$D = \frac{1}{2}\rho v_{\text{rel}}^2 C_D A \tag{12.11}$$

ρ is the atmospheric density, A is the frontal area of the spacecraft (the area normal to the relative velocity vector), and C_D is the dimensionless drag coefficient. If the mass of the spacecraft is m, then the perturbing acceleration due to the drag force is $\mathbf{p} = \mathbf{D}/m$, so that

$$\mathbf{p} = -\frac{1}{2}\rho v_{\text{rel}}\left(\frac{C_D A}{m}\right)\mathbf{v}_{\text{rel}} \tag{12.12}$$

There apparently is no universal agreement on the name of the quantity in parentheses. We will call it the *ballistic coefficient*,

$$B = \frac{C_D A}{m} \tag{12.13}$$

The reader may encounter alternative definitions of the ballistic coefficient, such as the reciprocal, $m/(C_D A)$.

EXAMPLE 12.1

A small spherical earth satellite has a diameter of 1 m and a mass of 100 kg. At a given time t_0, its orbital parameters are

$$\begin{aligned}\text{Perigee radius:}\ & r_p = 6593\ \text{km (215-km altitude)}\\ \text{Apogee radius:}\ & r_a = 7317\ \text{km (939-km altitude)}\\ \text{Right ascension of the ascending node:}\ & \Omega = 340^\circ\\ \text{Inclination:}\ & i = 65.1^\circ\\ \text{Argument of perigee:}\ & \omega = 58^\circ\\ \text{True anomaly:}\ & \theta = 332^\circ\end{aligned}$$

Assuming a drag coefficient of $C_D = 2.2$ and using the 1976 US Standard Atmosphere rotating with the earth, employ Cowell's method to find the time for the orbit to decay to 100 km. Recall from Eqn (2.67) that the angular velocity of the earth is 72.9211×10^{-6} rad/s.

Solution

Let us first use the given data to compute some additional orbital parameters, recalling that $\mu = 398{,}600\ \text{km}^3/\text{s}^2$:

$$\begin{aligned}\text{Eccentricity:}\ & e = \frac{r_a - r_p}{r_a + r_p} = 0.052049\\ \text{Semimajor axis:}\ & a = \frac{r_a + r_p}{2} = 6955\ \text{km}\\ \text{Angular momentum:}\ & h = \sqrt{\mu a(1-e^2)} = 52{,}580.1\ \text{km}^2/\text{s}\\ \text{Period:}\ & T = \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}} = 96.207\ \text{min}\end{aligned}$$

Now we can use the six elements h, e, i, Ω, ω, and θ in Algorithm 4.5 to obtain the state vector ($\mathbf{r}_0$,$\mathbf{v}_0$) at the initial time, relative to the geocentric equatorial frame,

$$\begin{aligned}\mathbf{r}_0 &= 5873.40\hat{\mathbf{I}} - 658.522\hat{\mathbf{J}} + 3007.49\hat{\mathbf{K}}\ \text{(km)}\\ \mathbf{v}_0 &= -2.89641\hat{\mathbf{I}} + 4.09401\hat{\mathbf{J}} + 6.14446\hat{\mathbf{K}}\ \text{(km/s)}\end{aligned} \tag{a}$$

Referring to Section 1.8, we next write Eqn (12.2), which is a second-order differential equation in $\mathbf{r}$, as two first-order ordinary differential equations in $\mathbf{r}$ and $\dot{\mathbf{r}}(=\mathbf{v})$,

$$\frac{d}{dt}\begin{Bmatrix}\mathbf{r}\\ \mathbf{v}\end{Bmatrix} = \begin{Bmatrix}\mathbf{v}\\ \mathbf{a}\end{Bmatrix} = \begin{Bmatrix}\mathbf{v}\\ -\mu\dfrac{\mathbf{r}}{r^3} + \mathbf{p}\end{Bmatrix} \tag{b}$$

The drag acceleration $\mathbf{p}$ is given by Eqn (12.12) along with Eqn (12.9). With Eqn (a) as the initial conditions, we can solve for $\mathbf{r}$ and $\mathbf{v}$ on the time interval $[t_0, t_f]$ using, for example, MATLAB's built-in numerical integrator *ode45*. From the solution, we obtain the satellite's altitude, which oscillates with time between the extreme values at perigee and apogee. It is a simple matter to extract just the extrema and plot them, as shown in Figure 12.5 for $t_0 = 0$ and $t_f = 120$ days. $\boxed{\text{The orbit decays to 100 km in 108 days.}}$

Observe that the apogee starts to decrease immediately, whereas the perigee remains nearly constant until the very end. As the apogee approaches perigee, the eccentricity approaches zero. We say that atmospheric drag tends to *circularize* an elliptical orbit.

The MATLAB script *Example_12_01.m* for this problem is located in Appendix D.42.

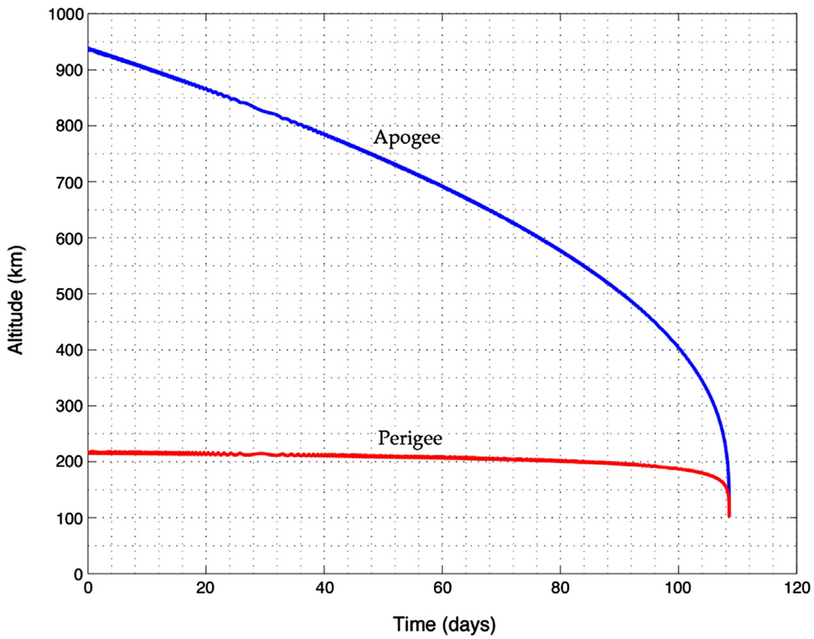

FIGURE 12.5

Perigee and apogee versus time.

12.5 Gravitational perturbations

In Appendix E, it is shown that if the central attractor (e.g., the earth) is a sphere of radius R with a spherically symmetric mass distribution, then its external gravitational potential field will be spherically symmetric, acting as though all of the mass were concentrated at the center O of the sphere. The gravitational potential energy per unit mass ($m = 1$ in Eqn (E.10)) is

$$V = -\frac{\mu}{r} \tag{12.14}$$

where $\mu = GM$, G is the universal gravitational constant, M is the sphere's mass, and r is the distance from O to a point *outside* of the sphere ($r > R$). Here r is the magnitude of the position vector $\mathbf{r}$, which, in Cartesian coordinates centered at O, is given by

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

Equipotential surfaces (those on which V is constant) are concentric spheres. The force on a unit mass placed at $\mathbf{r}$ is the acceleration of gravity, which is given by $\mathbf{a} = -\boldsymbol{\nabla}V$, where $\boldsymbol{\nabla}$ is the gradient operator. In Cartesian coordinates with the origin at O,

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right)$$

Thus, $\mathbf{a} = -\nabla(-\mu/r)$, or

$$\mathbf{a} = \mu\left(\frac{\partial(1/r)}{\partial x}\hat{\mathbf{i}} + \frac{\partial(1/r)}{\partial y}\hat{\mathbf{j}} + \frac{\partial(1/r)}{\partial z}\hat{\mathbf{k}}\right) = -\frac{\mu}{r^2}\left(\frac{\partial r}{\partial x}\hat{\mathbf{i}} + \frac{\partial r}{\partial y}\hat{\mathbf{j}} + \frac{\partial r}{\partial z}\hat{\mathbf{k}}\right)$$

But

$$r = \sqrt{x^2 + y^2 + z^2} \tag{12.15}$$

from which we can easily show that

$$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r} \qquad \frac{\partial r}{\partial z} = \frac{z}{r} \tag{12.16}$$

Therefore, we have the familiar result (cf. Eqn (2.22))

$$\mathbf{a} = -\mu\frac{\mathbf{r}}{r^3} \tag{12.17}$$

As we first observed in Section 4.7, the earth and other spinning celestial bodies are not perfect spheres. Many resemble oblate spheroids. For such a planet, the spin axis is the axis of rotational symmetry of its gravitational field. Because of the equatorial bulge caused by centrifugal effects, the gravitational field varies with the latitude as well as radius. This more complex gravitational potential is dominated by the familiar point mass contribution given by Eqn (12.14) upon which is superimposed the perturbation due to the oblateness.

It is convenient to use the spherical coordinate system shown in Figure 12.6. The origin O is at the planet's center of mass, and the z-axis of the associated Cartesian coordinate system is the axis of rotational symmetry. (The rotational symmetry means that our discussion is independent of the choice of Cartesian coordinate frame as long as each shares a common z-axis.) r is the distance of a point P

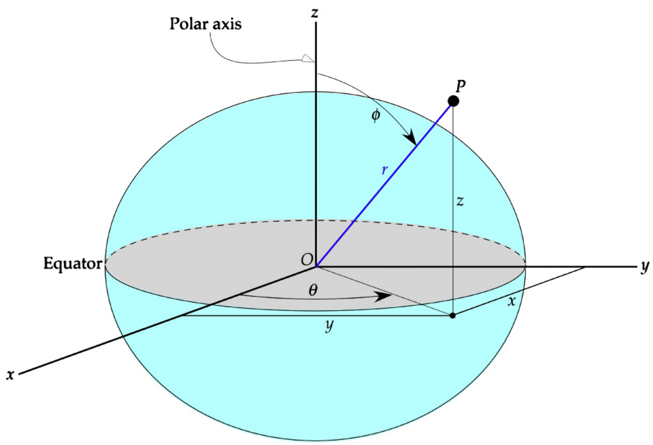

FIGURE 12.6

Spherical coordinate system.

from O, ϕ is the polar angle measured from the positive z-axis to the radial, and θ is the azimuth angle measured from the positive x-axis to the projection of the radial onto the xy plane. Observe that

$$\phi = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} \tag{12.18}$$

Since the gravitational field is rotationally symmetric, it does not depend on the azimuth angle θ. Therefore, the gravitational potential may be written as

$$V(r,\phi) = -\frac{\mu}{r} + \Phi(r,\phi) \tag{12.19}$$

Here Φ is the perturbation of the gravitational potential due to the planet's oblateness.

The rotationally symmetric perturbation $\Phi(r,\phi)$ is given by the infinite series (Battin, 1999)

$$\Phi(r,\phi) = \frac{\mu}{r}\sum_{k=2}^{\infty} J_k\left(\frac{R}{r}\right)^k P_k(\cos\phi) \tag{12.20}$$

where J_k are the *zonal harmonics* of the planet, R is its equatorial radius ($R/r < 1$), and P_k are the *Legendre polynomials* (see below). The zonal harmonics are dimensionless numbers that are not derived from mathematics but are inferred from observations of satellite motion around a planet, and they are unique to that planet. The summation starts at $k = 2$ instead of $k = 1$ because $J_1 = 0$, due to the fact that the origin of the spherical coordinate system is at the planet center of mass. For the earth, the next six zonal harmonics are (Vallado, 2007)

$$\begin{aligned} &J_2 = 0.00108263 && J_3 = -2.33936 \times 10^{-3} J_2 \\ &J_4 = -1.49601 \times 10^{-3} J_2 && J_5 = -0.20995 \times 10^{-3} J_2 \quad \text{Earth zonal harmonics} \\ &J_6 = 0.49941 \times 10^{-3} J_2 && J_7 = 0.32547 \times 10^{-3} J_2 \end{aligned} \tag{12.21}$$

This set of zonal harmonics is clearly dominated by J_2. For $k > 7$ the zonal harmonics all remain more than three orders of magnitude smaller than J_2.

The Legendre polynomials are named after the French mathematician Adrien-Marie Legendre (1752–1833). The polynomial $P_k(x)$ may be obtained from a formula derived by another French mathematician Olinde Rodrigues (1795–1851), as part of his 1816 doctoral thesis,

$$P_k(x) = \frac{1}{2^k k!}\frac{d}{dx^k}\left(x^2-1\right)^k \quad \text{Rodrigues' formula} \tag{12.22}$$

Using Rodrigues' formula, we can calculate the first few of the Legendre polynomials that appear in Eqn (12.20),

$$\begin{aligned} &P_2(x) = \tfrac{1}{2}\left(3x^2-1\right) && P_3(x) = \tfrac{1}{2}\left(5x^3-3x\right) \\ &P_4(x) = \tfrac{1}{8}\left(35x^4-30x^2+3\right) && P_5(x) = \tfrac{1}{8}\left(63x^5-70x^3+15x\right) \\ &P_6(x) = \tfrac{1}{16}\left(231x^6-315x^4+105x^2-5\right) && P_7(x) = \tfrac{1}{16}\left(429x^7-693x^5+315x^3-35x\right) \end{aligned} \tag{12.23}$$

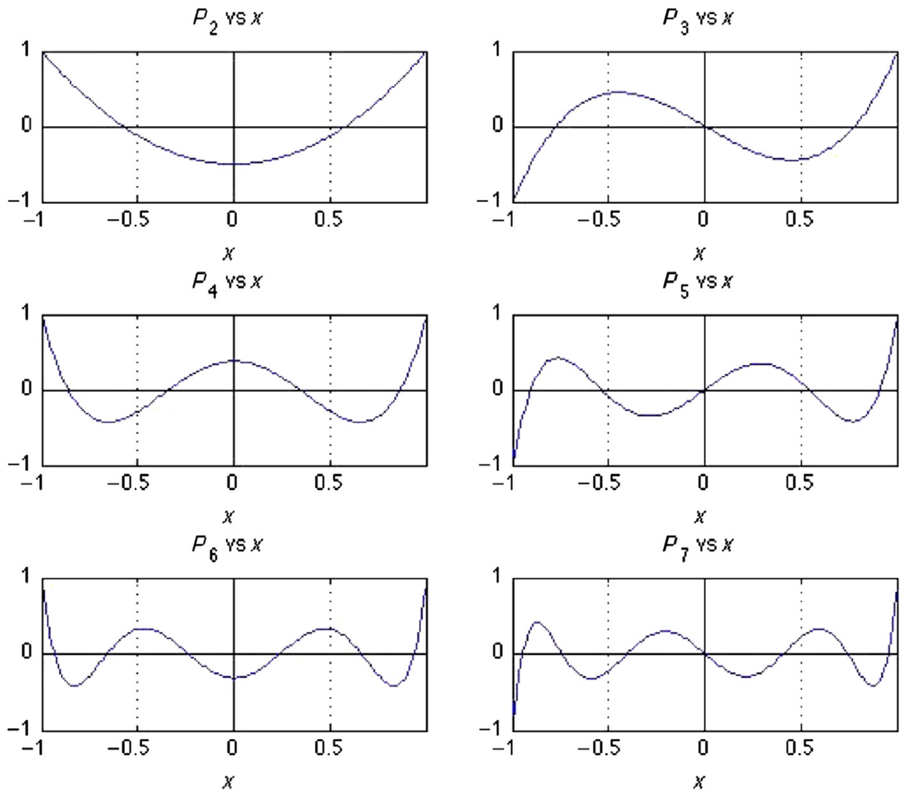

FIGURE 12.7

Legendre polynomials P_2 through P_7.

These are plotted in Figure 12.7.

Since J_2 is by far the largest zonal harmonic, we shall in the interest of simplicity focus only on its contribution to the gravitational perturbation, thereby ignoring all but the J_2 term in the series for $\Phi(r, \phi)$. In that case, Eqn (12.20) yields

$$\Phi(r, \phi) = \frac{J_2}{2}\frac{\mu}{r}\left(\frac{R}{r}\right)^2 \left(3\cos^2\phi - 1\right) \tag{12.24}$$

Observe that $\Phi = 0$ when $\cos\phi = \sqrt{1/3}$, which corresponds to about $\pm 35^\circ$ geocentric latitude. This band reflects the earth's equatorial bulge (oblateness). The perturbing acceleration is the negative of the gradient of Φ,

$$\mathbf{p} = -\nabla\Phi = -\frac{\partial\Phi}{\partial x}\hat{\mathbf{i}} - \frac{\partial\Phi}{\partial y}\hat{\mathbf{j}} - \frac{\partial\Phi}{\partial z}\hat{\mathbf{k}} \tag{12.25}$$

Noting that $\partial\Phi/\partial\theta = 0$, we have from the chain rule of calculus that

$$\frac{\partial\Phi}{\partial x} = \frac{\partial\Phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial\Phi}{\partial\phi}\frac{\partial\phi}{\partial x} \qquad \frac{\partial\Phi}{\partial y} = \frac{\partial\Phi}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial\Phi}{\partial\phi}\frac{\partial\phi}{\partial y} \qquad \frac{\partial\Phi}{\partial z} = \frac{\partial\Phi}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial\Phi}{\partial\phi}\frac{\partial\phi}{\partial z} \tag{12.26}$$

Differentiating Eqn (12.24), we obtain

$$\begin{aligned}\frac{\partial \Phi}{\partial r} &= -\frac{3}{2}J_2\frac{\mu}{r^2}\left(\frac{R}{r}\right)^2\left(3\cos^2\phi - 1\right)\\ \frac{\partial \Phi}{\partial \phi} &= -\frac{3}{2}J_2\frac{\mu}{r}\left(\frac{R}{r}\right)^2\sin\phi\cos\phi\end{aligned} \tag{12.27}$$

Use Eqn (12.18) to find the required partial derivatives of ϕ:

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= \frac{xz}{x^2+y^2+z^2}\frac{1}{\sqrt{x^2+y^2}} = \frac{xz}{r^3\sin\phi}\\ \frac{\partial \phi}{\partial y} &= \frac{yz}{x^2+y^2+z^2}\frac{1}{\sqrt{x^2+y^2}} = \frac{yz}{r^3\sin\phi}\\ \frac{\partial \phi}{\partial z} &= -\frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2} = -\frac{\sin\phi}{r}\end{aligned} \tag{12.28}$$

Substituting Eqns (12.16), (12.27), and (12.28) into Eqn (12.26) and using the fact that $\cos\phi = z/r$ leads to the following expressions for the gradient of perturbing potential Φ:

$$\begin{aligned}\frac{\partial \Phi}{\partial x} &= -\frac{3}{2}J_2\frac{\mu}{r^2}\left(\frac{R}{r}\right)^2\frac{x}{r}\left[5\left(\frac{z}{r}\right)^2 - 1\right]\\ \frac{\partial \Phi}{\partial y} &= -\frac{3}{2}J_2\frac{\mu}{r^2}\left(\frac{R}{r}\right)^2\frac{y}{r}\left[5\left(\frac{z}{r}\right)^2 - 1\right]\\ \frac{\partial \Phi}{\partial z} &= -\frac{3}{2}J_2\frac{\mu}{r^2}\left(\frac{R}{r}\right)^2\frac{z}{r}\left[5\left(\frac{z}{r}\right)^2 - 3\right]\end{aligned} \tag{12.29}$$

The perturbing gravitational acceleration **p** due to J_2 is found by substituting these equations into Eqn (12.25), yielding the vector

$$\mathbf{p} = \frac{3}{2}\frac{J_2\mu R^2}{r^4}\left[\frac{x}{r}\left(5\frac{z^2}{r^2} - 1\right)\hat{\mathbf{i}} + \frac{y}{r}\left(5\frac{z^2}{r^2} - 1\right)\hat{\mathbf{j}} + \frac{z}{r}\left(5\frac{z^2}{r^2} - 3\right)\hat{\mathbf{k}}\right] \tag{12.30}$$

The perturbing accelerations for higher zonal harmonics may be evaluated in a similar fashion. Schaub (2009, p. 553) lists the accelerations for J_3 through J_6.

Irregularities in the earth's geometry and its mass distribution cause the gravitational field to vary not only with latitude ϕ but with longitude θ as well. To mathematically account for this increased physical complexity, the series expansion of the potential function Φ in Eqn (12.19) must be generalized to include the azimuth angle θ. As a consequence, it turns out that we can identify *sectorial harmonics*, which account for the longitudinal variation over domains of the earth resembling segments of an orange. We also discover a tile-like patchwork of *tesseral harmonics*, which model how specific regions of the earth deviate locally from a perfect, homogeneous sphere. Incorporating these additional levels of detail into the gravitational model is essential for accurate long-term prediction of satellite orbits (e.g., global positioning satellite constellations). For an in-depth treatment of this subject, including the mathematical details, see Vallado (2007).

EXAMPLE 12.2

At time $t = 0$, an earth satellite has the following orbital parameters:

$$\begin{aligned}
\text{Perigee radius:}\ & r_p = 6678\ \text{km (300 km altitude)} \\
\text{Apogee radius:}\ & r_a = 9940\ \text{km (3062 km altitude)} \\
\text{Right ascension of the ascending node:}\ & \Omega = 45^\circ \\
\text{Inclination:}\ & i = 28^\circ \\
\text{Argument of perigee:}\ & \omega = 30^\circ \\
\text{True anomaly:}\ & \theta = 40^\circ
\end{aligned}$$

Use Encke's method to determine the effect of the J_2 perturbation on the variation of right ascension of the node, argument of perigee, angular momentum, eccentricity, and inclination over the next 48 hours.

Solution

Using Figure 2.18 along with Eqns (2.83), (2.84), and (2.71), we can find other orbital parameters in addition to the ones given. In particular, recalling that $\mu = 398{,}600\ \text{km}^3/\text{s}^2$ for the earth,

$$\begin{aligned}
\text{Eccentricity:}\ & e = \frac{r_a - r_p}{r_a + r_p} = 0.17136 \\
\text{Semimajor axis:}\ & a = \frac{r_a + r_p}{2} = 8059\ \text{km} \\
\text{Period:}\ & T = \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}} = 7200\ \text{s (120 min)} \\
\text{Angular momentum:}\ & h = \sqrt{\mu a(1 - e^2)} = 55{,}839\ \text{km}^2/\text{s}
\end{aligned}$$

The six elements h, e, i, Ω, ω, and θ together with Algorithm 4.5 yield the state vector ($\mathbf{r}_0$,$\mathbf{v}_0$) at the initial time, relative to the geocentric equatorial frame,

$$\begin{aligned}
\mathbf{r}_0 &= -2384.46\hat{\mathbf{I}} + 5729.01\hat{\mathbf{J}} + 3050.46\hat{\mathbf{K}}\ \text{(km)} \\
\mathbf{v}_0 &= -7.36138\hat{\mathbf{I}} - 2.98997\hat{\mathbf{J}} + 1.64354\hat{\mathbf{K}}\ \text{(km/s)}
\end{aligned} \tag{a}$$

Using this state vector as a starting point, with $t_0 = 0$, $t_f = 48 \times 3600$ s, and $\Delta t = (t_f - t_0)/1000$, we enter the Encke procedure (Algorithm 12.1) with MATLAB's *ode45* as the numerical integrator to find **r** and **v** at each of the 1001 equally spaced times. At these times, we then use Algorithm 4.2 to compute the right ascension of the node and the argument of perigee, which are plotted in Figure 12.8.

Figure 12.8 shows that the J_2 perturbation causes a drift in both Ω and ω over time. For this particular orbit, Ω decreases whereas ω increases. We see that both parameters have a straight line or *secular* variation upon which a small or *short periodic* variation is superimposed. Approximate average values of the slopes of the curves for Ω and ω are most simply found by dividing the difference between the computed values at t_f and t_0 by the time span $t_f - t_0$. In this way, we find that

$$\overline{\dot{\Omega}} = -0.172\ \text{deg/h} \tag{b}$$

$$\overline{\dot{\omega}} = 0.282\ \text{deg/h} \tag{c}$$

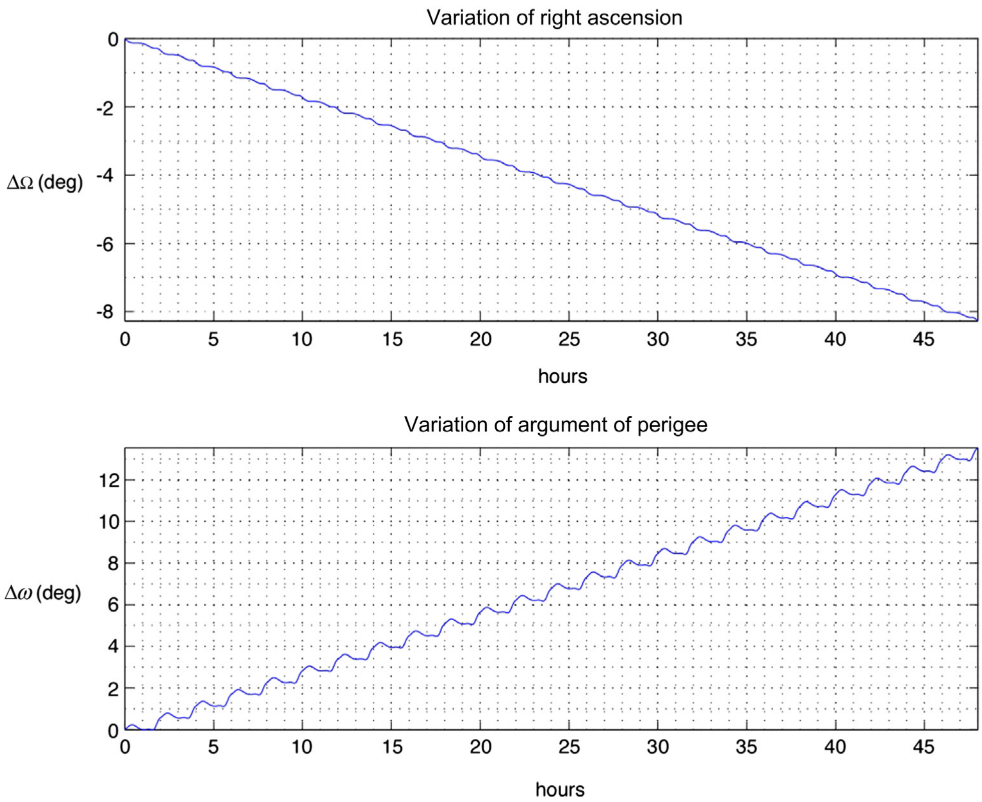

FIGURE 12.8

Histories of the right ascension of the ascending node and the argument of perigee for a time span of 48 h. The plotted points were obtained using Encke's method.

As pointed out in Section 4.7, the decrease in the node angle Ω with time is called *regression of the node*, whereas the increase of argument of perigee ω with time is called the *advance of perigee*.

The computed time histories of h, e, and i are shown in Figure 12.9. Clearly, none of these osculating elements shows a secular variation due to the earth's oblateness. They are all constant except for the short periodic perturbations evident in all three plots.

The MATLAB script *Example_12_02.m* for this problem is in Appendix D.43.

The results obtained in the above example follow from the specific initial conditions that we supplied to the Encke numerical integration procedure. We cannot generalize the conclusions of this special perturbation analysis to other orbits. In fact, whether or not the node line and the eccentricity vector advances or regresses depends on the nature of the particular orbit. We can confirm this by means of repeated numerical analyses with different initial conditions. Obtaining formulas that describe perturbation phenomena for general cases is the object of general perturbation analysis, which is beyond the scope of this book.

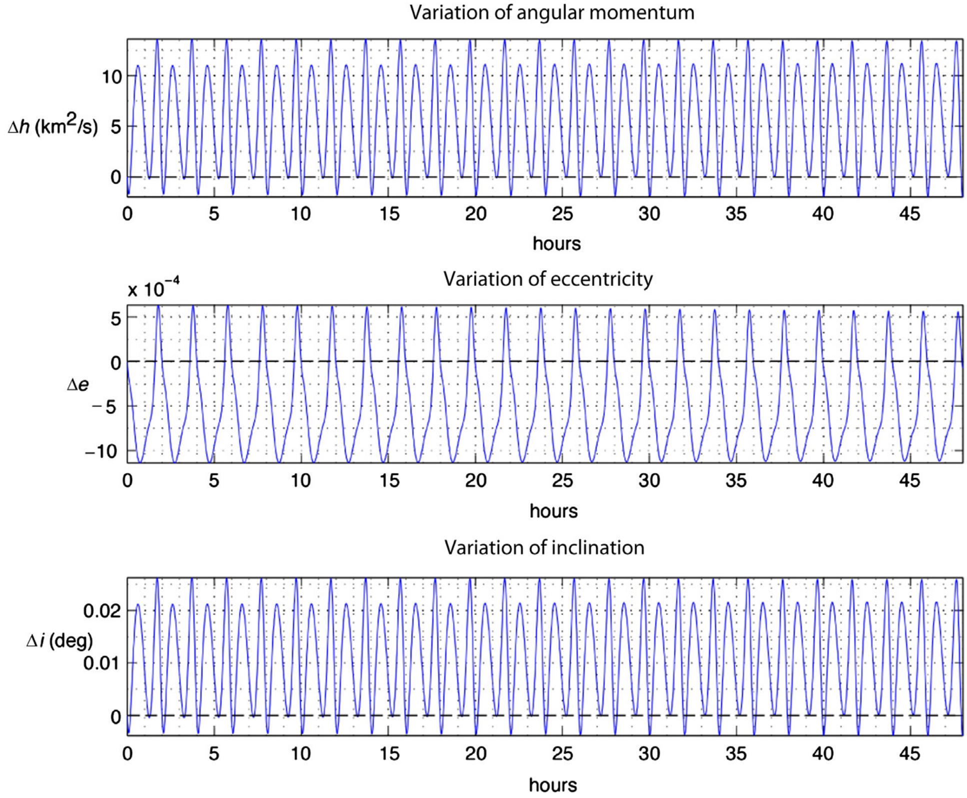

FIGURE 12.9

Histories the J_2 perturbed angular momentum, eccentricity, and inclination for a time span of 48 h.

However, a step in that direction is to derive expressions for the time variations of the osculating elements. These are the Lagrange planetary equations or their variant, the Gauss variational equations. Both are based on the variation of parameters approach to solving differential equations.

12.6 Variation of parameters

For the two-body problem, the equation of motion (Eqn (2.22))

$$\ddot{\mathbf{r}} + \mu\frac{\mathbf{r}}{r^3} = \mathbf{0} \tag{12.31}$$

yields the position vector $\mathbf{r}$ as a function of time t and six parameters or *orbital elements* $\mathbf{c}$,

$$\mathbf{r} = \mathbf{f}(\mathbf{c}, t) \tag{12.32}$$

where $\mathbf{c}$ stands for the set of six parameters (c_1, c_2, …, c_6). Keep in mind that the vector $\mathbf{f}$ has three scalar components.

EXAMPLE 12.3

At time t_0, the state vector of a spacecraft in two-body motion is $\lfloor \mathbf{r}_0 \quad \mathbf{v}_0 \rfloor$. What is $\mathbf{f}(\mathbf{c},t)$ in this case?

Solution

The constant orbital elements $\mathbf{c}$ are the six components of the state vector,

$$\mathbf{c} = \lfloor \mathbf{r}_0 \quad \mathbf{v}_0 \rfloor$$

By means of Algorithm 3.4, we find

$$\mathbf{r} = f(t)\mathbf{r}_0 + g(t)\mathbf{v}_0 \tag{a}$$

where the time-dependent Lagrange f and g functions are given by Eqn (3.69) (Examples 3.7 and 4.2). Thus,

$$\boxed{\mathbf{f}(\mathbf{c}, t) = f(t)\mathbf{r}_0 + g(t)\mathbf{v}_0}$$

As we know from Section 4.4, the orbital parameters that may alternatively be selected as the classical elements at time t_0, are the longitude of the node Ω, inclination i, argument of periapse ω, eccentricity e, angular momentum h (or semimajor axis a), and true anomaly θ (or the mean anomaly M, or the eccentric anomaly E). For $t > t_0$, only the anomalies vary with time, a fact that is reflected by the presence of the argument t in $\mathbf{f}(\mathbf{c},t)$ above. As pointed out in Section 3.2, we could use time of periapse passage t_p as the sixth orbital parameter instead of true anomaly. The advantage of doing so in the present context would be that t_p is a constant for Keplerian orbits. (Up to now we have usually set that constant equal to zero.)

The velocity $\mathbf{v}$ is the time derivative of the position vector $\mathbf{r}$, so that from Eqn (12.32),

$$\mathbf{v} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{f}(\mathbf{c}, t) = \frac{\partial}{\partial t}\mathbf{f}(\mathbf{c}, t) + \sum_{\alpha=1}^{6} \frac{\partial}{\partial c_\alpha}\mathbf{f}(\mathbf{c}, t)\frac{\mathrm{d}c_\alpha}{\mathrm{d}t} \tag{12.33}$$

Since the orbital elements are constant in two-body motion, their time derivatives are zero,

$$\frac{\mathrm{d}c_\alpha}{\mathrm{d}t} = 0 \quad \alpha = 1, \ldots, 6$$

Therefore, Eqn (12.33) becomes simply

$$\mathbf{v} = \frac{\partial}{\partial t}\mathbf{f}(\mathbf{c}, t) \quad \text{Two-body motion} \tag{12.34}$$

We find the acceleration by taking the time derivative of $\mathbf{v}$ in Eqn (12.34) while holding $\mathbf{c}$ constant,

$$\mathbf{a} = \frac{\partial^2}{\partial t^2}\mathbf{f}(\mathbf{c}, t) \quad \text{Two-body motion}$$

Substituting this and Eqn (12.32) into Eqn (12.31) yields

$$\frac{\partial^2}{\partial t^2}\mathbf{f}(\mathbf{c}, t) + \mu\frac{\mathbf{f}(\mathbf{c}, t)}{\|\mathbf{f}(\mathbf{c}, t)\|^3} = \mathbf{0} \tag{12.35}$$

A perturbing force produces a perturbing acceleration $\mathbf{p}$ that results in a perturbed motion $\mathbf{r}_\mathrm{p}$ for which the equation of motion is

$$\ddot{\mathbf{r}}_\mathrm{p} + \mu \frac{\mathbf{r}_\mathrm{p}}{r_\mathrm{p}^3} = \mathbf{p} \tag{12.36}$$

The *variation of parameters* method requires that the solution to Eqn (12.36) have the same mathematical form $\mathbf{f}$ as it does for the two-body problem, except that the six constants $\mathbf{c}$ in Eqn (12.32) are replaced by six functions of time $\mathbf{u}(t)$, so that

$$\mathbf{r}_\mathrm{p} = \mathbf{f}(\mathbf{u}(t), t) \tag{12.37}$$

where $\mathbf{u}(t)$ stands for the set of six functions $u_1(t)$,$u_2(t)$, … ,$u_6(t)$. The six parameters $\mathbf{u}(t)$ are the orbital elements of the osculating orbit that is tangent to $\mathbf{r}_\mathrm{p}$ at time t.

We find the velocity vector $\mathbf{v}_\mathrm{p}$ for the perturbed motion by differentiating Eqn (12.37) with respect to time and using the chain rule,

$$\mathbf{v}_\mathrm{p} = \frac{\mathrm{d}\mathbf{r}_\mathrm{p}}{\mathrm{d}t} = \frac{\partial \mathbf{f}(\mathbf{u}, t)}{\partial t} + \sum_{\beta=1}^{6} \frac{\partial \mathbf{f}(\mathbf{u}, t)}{\partial u_\beta} \frac{\mathrm{d}u_\beta}{\mathrm{d}t} \tag{12.38}$$

In order for $\mathbf{v}_\mathrm{p}$ to have the same mathematical form as $\mathbf{v}$ for the unperturbed case (Eqn (12.34)), we impose the following three conditions on the osculating elements $\mathbf{u}(t)$:

$$\sum_{\beta=1}^{6} \frac{\partial \mathbf{f}(\mathbf{u}, t)}{\partial u_\beta} \frac{\mathrm{d}u_\beta}{\mathrm{d}t} = \mathbf{0} \tag{12.39}$$

Eqn (12.38) therefore becomes simply

$$\mathbf{v}_\mathrm{p} = \frac{\partial \mathbf{f}(\mathbf{u}, t)}{\partial t} \tag{12.40}$$

To find $\ddot{\mathbf{r}}_\mathrm{p}$, the acceleration of the perturbed motion, we differentiate the velocity $\mathbf{v}_\mathrm{p}$ with respect to time. It follows from Eqn (12.40) that

$$\ddot{\mathbf{r}}_\mathrm{p} = \frac{\mathrm{d}\mathbf{v}_\mathrm{p}}{\mathrm{d}t} = \frac{\partial^2 \mathbf{f}(\mathbf{u}, t)}{\partial t^2} + \sum_{\beta=1}^{6} \frac{\partial^2 \mathbf{f}(\mathbf{u}, t)}{\partial u_\beta \partial t} \frac{\mathrm{d}u_\beta}{\mathrm{d}t} \tag{12.41}$$

Substituting Eqns (12.37) and (12.41) into Eqn (12.36) yields

$$\frac{\partial^2 \mathbf{f}(\mathbf{u}, t)}{\partial t^2} + \sum_{\beta=1}^{6} \frac{\partial^2 \mathbf{f}(\mathbf{u}, t)}{\partial u_\beta \partial t} \frac{\mathrm{d}u_\beta}{\mathrm{d}t} + \mu \frac{\mathbf{f}(\mathbf{u}, t)}{\|\mathbf{f}(\mathbf{u}, t)\|^3} = \mathbf{p} \tag{12.42}$$

But

$$\frac{\partial^2 \mathbf{f}(\mathbf{u}, t)}{\partial t^2} + \mu \frac{\mathbf{f}(\mathbf{u}, t)}{\|\mathbf{f}(\mathbf{u}, t)\|^3} = \mathbf{0} \tag{12.43}$$

because the six osculating orbital elements $\mathbf{u}$ evaluated at any instant of time define a Keplerian orbit for which Eqn (12.31) is valid. That means Eqn (12.42) reduces to

$$\sum_{\beta=1}^{6} \frac{\partial^2 \mathbf{f}(\mathbf{u},t)}{\partial u_\beta \partial t}\frac{\mathrm{d}u_\beta}{\mathrm{d}t} = \mathbf{p} \tag{12.44}$$

These are three conditions on the six functions $\mathbf{u}(t)$ in addition to the three conditions listed above in Eqn (12.39). If we simplify our notation by letting $\mathbf{r} = \mathbf{f}(\mathbf{u},t)$ and $\mathbf{v} = \partial\mathbf{f}(\mathbf{u},t)/\partial t$, then the six Eqns (12.39) and (12.44), respectively, become

$$\sum_{\beta=1}^{6} \frac{\partial \mathbf{r}}{\partial u_\beta}\frac{\mathrm{d}u_\beta}{\mathrm{d}t} = \mathbf{0} \tag{12.45a}$$

$$\sum_{\beta=1}^{6} \frac{\partial \mathbf{v}}{\partial u_\beta}\frac{\mathrm{d}u_\beta}{\mathrm{d}t} = \mathbf{p} \tag{12.45b}$$

These are the six *osculating conditions* imposed by the variation of parameters to ensure that the osculating orbit at each point of a perturbed trajectory is Keplerian (two-body) in nature.

In matrix form, Eqns (12.45) may be written as

$$[\mathbf{L}]\{\dot{\mathbf{u}}\} = \{\mathbf{P}\} \tag{12.46a}$$

where

$$[\mathbf{L}] = \begin{bmatrix} \partial x_i/\partial u_\alpha \\ \partial v_i/\partial u_\alpha \end{bmatrix} \quad \{\dot{\mathbf{u}}\} = \{\dot{u}_\alpha\} \quad \{\mathbf{P}\} = \begin{Bmatrix} 0 \\ p_i \end{Bmatrix} \quad \begin{matrix} i = 1,2,3 \\ \alpha = 1,2,\ldots,6 \end{matrix} \tag{12.46b}$$

$[\mathbf{L}]$ is the 6 by 6 *Lagrangian matrix*. x_1, x_2, and x_3 are the *xyz* components of the position vector $\mathbf{r}$ in a Cartesian inertial reference. Likewise, v_1, v_2, and v_3 are the inertial velocity components, whereas p_1, p_2, and p_3 are the *xyz* components of the perturbating acceleration. The solution of Eqn (12.46) yields the time variations of the osculating elements

$$\{\dot{\mathbf{u}}\} = [\mathbf{L}]^{-1}\{\mathbf{P}\} \tag{12.47}$$

An alternate straightforward manipulation of Eqns (12.45) reveals that the Lagrange matrix $[\mathbf{L}]$ and the acceleration vector $\{\mathbf{P}\}$ may be written as

$$[\mathbf{L}] = \left[\sum_{i=1}^{3}\left(\frac{\partial x_i}{\partial u_\alpha}\frac{\partial v_i}{\partial u_\beta} - \frac{\partial v_i}{\partial u_\alpha}\frac{\partial x_i}{\partial u_\beta}\right)\right] \quad \{\mathbf{P}\} = \left\{\sum_{i=1}^{3} p_i \frac{\partial x_i}{\partial u_\alpha}\right\} \quad \alpha,\beta = 1,2,\ldots,6 \tag{12.48}$$

These are the forms attributable to Lagrange, and in that case Eqn (12.48) are known as the *Lagrange planetary equations*. If the perturbing forces, and hence the perturbing accelerations, are conservative, like gravity, then **p** is the spatial gradient of a scalar potential function R, that is,

$$p_i = \frac{\partial R}{\partial x_i} \quad i = 1, 2, 3$$

It follows from Eqn (12.48) that $\{\mathbf{P}\}$ is the gradient of a function of the orbital elements,

$$\{\mathbf{P}\} = \left\{ \sum_{i=1}^{3} \frac{\partial R}{\partial x_i} \frac{\partial x_i}{\partial u_\alpha} \right\} = \left\{ \frac{\partial R}{\partial u_\alpha} \right\} \quad \alpha = 1, 2, \ldots, 6 \tag{12.49}$$

The Lagrange planetary equations for the variation of the six classical orbital elements a, e, t_p, Ω, i, and ω, as derived by Battin (1999) and others, may be written as follows:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{2a^2}{\mu} \frac{\partial R}{\partial t_\mathrm{p}} \tag{12.50a}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = -\frac{\sqrt{1-e^2}}{\sqrt{\mu a} e} \frac{\partial R}{\partial \omega} - \frac{a\left(1-e^2\right)}{\mu e} \frac{\partial R}{\partial t_\mathrm{p}} \tag{12.50b}$$

$$\frac{\mathrm{d}t_\mathrm{p}}{\mathrm{d}t} = \frac{2a^2}{\mu} \frac{\partial R}{\partial a} + \frac{a\left(1-e^2\right)}{\mu e} \frac{\partial R}{\partial e} \tag{12.50c}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{1}{\sqrt{\mu a(1-e^2)} \sin i} \frac{\partial R}{\partial i} \tag{12.50d}$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{1}{\sqrt{\mu a(1-e^2)}} \left(\frac{1}{\tan i} \frac{\partial R}{\partial \omega} - \frac{1}{\sin i} \frac{\partial R}{\partial \Omega} \right) \tag{12.50e}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{1}{\sqrt{\mu a(1-e^2)} \tan i} \frac{\partial R}{\partial i} + \frac{\sqrt{1-e^2}}{\sqrt{\mu a} e} \frac{\partial R}{\partial e} \tag{12.50f}$$

The Lagrange planetary equations were obtained by writing the state vector components **r** and **v** in terms of the orbital parameters **u**, then taking partial derivatives to form the Lagrangian matrix [**L**], and finally inverting [**L**] to find the time variations of **u**, as in Eqn (12.50). The more direct Gauss approach is to obtain the orbital elements **u** from the state vector, as in Algorithm 4.2, and then differentiate those expressions with respect to time to get the equations of variation. Gauss's form of the Lagrange planetary equations relaxes the requirement for perturbations to be conservative and avoids the lengthy though systematic procedure devised by Lagrange for computing the Lagrange matrix [**L**] in Eqn (12.48). We will pursue the Gauss approach in the next section.

12.7 Gauss variational equations

Let u be an osculating element. Its time derivative is

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial \mathbf{r}} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} + \frac{\partial u}{\partial \mathbf{v}} \cdot \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \tag{12.51}$$

The acceleration d**v**/dt consists of the two-body part plus that due to the perturbation,

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\mu\frac{\mathbf{r}}{r^3} + \mathbf{p} \tag{12.52}$$

Therefore, Eqn (12.51) becomes

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial \mathbf{r}}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} + \frac{\partial u}{\partial \mathbf{v}}\left(-\mu\frac{\mathbf{r}}{r^3}\right) + \frac{\partial u}{\partial \mathbf{v}}\cdot\mathbf{p}$$

or

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \left.\frac{\mathrm{d}u}{\mathrm{d}t}\right)_{\text{two-body}} + \frac{\partial u}{\partial \mathbf{v}}\cdot\mathbf{p} \tag{12.53}$$

Except for the true, mean, and eccentric anomalies, the Keplerian elements are constant, so that $\mathrm{d}u/\mathrm{d}t)_{\text{two-body}} = 0$, whereas

$$\left.\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)_{\text{two-body}} = \frac{h}{r^2} \tag{12.54}$$

$$\left.\frac{\mathrm{d}M}{\mathrm{d}t}\right)_{\text{two-body}} = n \tag{12.55}$$

$$\left.\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\text{two-body}} = \frac{na}{r} \tag{12.56}$$

We usually do orbital mechanics and define the orbital elements relative to a Cartesian inertial frame with origin at the center of the primary attractor. For earth-centered missions, we have employed the geocentric equatorial frame extensively throughout this book. Although not necessary, it may be convenient to imagine it as our inertial frame in what follows. The orthogonal unit vectors of the inertial frame are $\hat{\mathbf{I}}$, $\hat{\mathbf{J}}$, and $\hat{\mathbf{K}}$. As illustrated in Figure 4.7, they form a right-hand triad, so that $\hat{\mathbf{K}} = \hat{\mathbf{I}} \times \hat{\mathbf{J}}$.

Another Cartesian inertial reference that we have employed for motion around a central attractor is the perifocal frame illustrated in Figure 2.29. The orthogonal unit vectors along its $\bar{x}$-, $\bar{y}$-, and $\bar{z}$-axes are $\hat{\mathbf{p}}$, $\hat{\mathbf{q}}$, and $\hat{\mathbf{w}}$, respectively. The unit vector $\hat{\mathbf{p}}$ (not to be confused with the perturbing acceleration **p**) lies in the direction of the eccentricity vector of the orbiting body, $\hat{\mathbf{w}}$ is normal to the orbital plane, and $\hat{\mathbf{q}}$ completes the right-handed triad: $\hat{\mathbf{w}} = \hat{\mathbf{p}} \times \hat{\mathbf{q}}$. Equation (4.48), repeated here, gives the direction cosine matrix for the transformation from XYZ to pqw:

$$[\mathbf{Q}]_{X\bar{x}} = \begin{bmatrix} -\sin\Omega\cos i\sin\omega + \cos\Omega\cos\omega & \cos\Omega\cos i\sin\omega + \sin\Omega\cos\omega & \sin i\sin\omega \\ -\sin\Omega\cos i\cos\omega - \cos\Omega\sin\omega & \cos\Omega\cos i\cos\omega - \sin\Omega\sin\omega & \sin i\cos\omega \\ \sin\Omega\sin i & -\cos\Omega\sin i & \cos i \end{bmatrix} \tag{12.57}$$

Ω, ω, and i are, respectively, the right ascension, argument of periapsis, and inclination.

We have also made use of noninertial LVLH frames in connection with the analysis of relative motion (see Figure 7.1). It will be convenient to use such a reference for the perturbation analysis in this section. The orthogonal unit vectors of this frame are $\hat{\mathbf{r}}$, $\hat{\mathbf{s}}$, and $\hat{\mathbf{w}}$. $\hat{\mathbf{r}}$ is directed radially outward from the central attractor to the orbiting body and defines the direction of the local vertical. As in the perifocal frame, $\hat{\mathbf{w}}$ is the unit vector normal to the osculating orbital plane of the orbiting body. $\hat{\mathbf{w}}$ lies in the direction of the angular momentum vector $\mathbf{h}$, so that $\hat{\mathbf{w}} = \mathbf{h}/h$. The transverse unit vector $\hat{\mathbf{s}}$ (which we have previously denoted $\hat{\mathbf{u}}_\perp$) is normal to both $\hat{\mathbf{r}}$ and $\hat{\mathbf{w}}$, and it therefore points in the direction of the orbiting body's local horizon: $\hat{\mathbf{s}} = \hat{\mathbf{w}} \times \hat{\mathbf{r}}$. The *rsw* and *pqr* frames are illustrated in Figure 12.10.

The transformation from *pqw* to *rsw* is simply a rotation about the normal $\hat{\mathbf{w}}$ through the true anomaly θ. According to Eqn (4.34), the direction cosine matrix for this rotation is

$$[\mathbf{R}_3(\theta)] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, the transformation from *XYZ* to *rsw* is represented by $[\mathbf{Q}]_{Xr} = [\mathbf{R}_3(\theta)][\mathbf{Q}]_{X\bar{x}}$. Carrying out the matrix multiplication and using the trig identities $\sin(\omega+\theta) = \sin\omega\cos\theta + \cos\omega\sin\theta$ and $\cos(\omega+\theta) = \cos\omega\cos\theta - \sin\omega\sin\theta$ leads to

$$[\mathbf{Q}]_{Xr} = \begin{bmatrix} -\sin\Omega\cos i\sin u + \cos\Omega\cos u & \cos\Omega\cos i\sin u + \sin\Omega\cos u & \sin i\sin u \\ -\sin\Omega\cos i\cos u - \cos\Omega\sin u & \cos\Omega\cos i\cos u - \sin\Omega\sin u & \sin i\cos u \\ \sin\Omega\sin i & -\cos\Omega\sin i & \cos i \end{bmatrix} \tag{12.58}$$

where $u = \omega + \theta$. u is known as *the argument of latitude*. The direction cosine matrix $[\mathbf{Q}]_{Xr}$ could of course be obtained from Eqn (12.57) by simply replacing the argument of periapsis ω with the argument of latitude u.

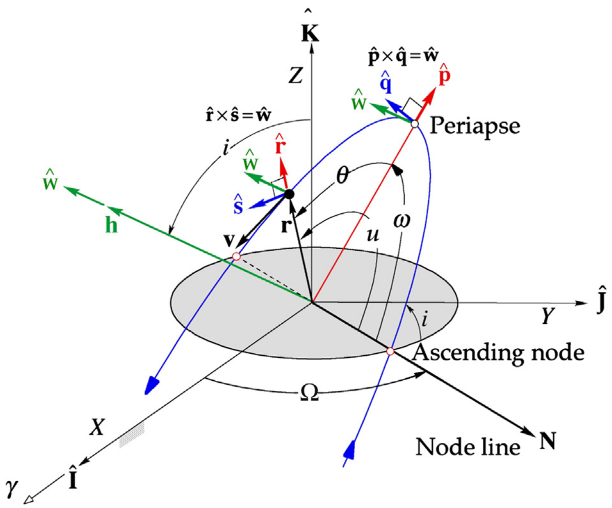

FIGURE 12.10

The perifocal *pqw* frame and the local vertical–local horizon *rsw* frame. u is the argument of latitude.

In terms of its components in the inertial *XYZ* frame, the perturbing acceleration **p** is expressed analytically as follows:

$$\mathbf{p} = p_X\hat{\mathbf{I}} + p_Y\hat{\mathbf{J}} + p_Z\hat{\mathbf{K}} \tag{12.59}$$

whereas in the noninertial *rsw* frame

$$\mathbf{p} = p_r\hat{\mathbf{r}} + p_s\hat{\mathbf{s}} + p_w\hat{\mathbf{w}} \tag{12.60}$$

The transformation between these two sets of components is

$$\begin{Bmatrix} p_r \\ p_s \\ p_w \end{Bmatrix} = [\mathbf{Q}]_{Xr}\begin{Bmatrix} p_X \\ p_Y \\ p_Z \end{Bmatrix} \tag{12.61}$$

and

$$\begin{Bmatrix} p_X \\ p_Y \\ p_Z \end{Bmatrix} = [\mathbf{Q}]_{Xr}^T\begin{Bmatrix} p_r \\ p_s \\ p_w \end{Bmatrix} \tag{12.62}$$

$[\mathbf{Q}]_{Xr}^T$ is the transpose of the direction cosine matrix in Eqn (12.58).

Each row of $[\mathbf{Q}]_{Xr}$ comprises the direction cosines of the unit vectors $\hat{\mathbf{r}}$, $\hat{\mathbf{s}}$, and $\hat{\mathbf{w}}$, respectively, relative to the *XYZ* axes. Therefore, from Eqn (12.58), it is apparent that

$$\hat{\mathbf{r}} = (-\sin\Omega\cos i\sin u + \cos\Omega\cos u)\hat{\mathbf{I}} + (\cos\Omega\cos i\sin u + \sin\Omega\cos u)\hat{\mathbf{J}} + \sin i\sin u\hat{\mathbf{K}} \tag{12.63a}$$

$$\hat{\mathbf{s}} = (-\sin\Omega\cos i\cos u - \cos\Omega\sin u)\hat{\mathbf{I}} + (\cos\Omega\cos i\cos u - \sin\Omega\sin u)\hat{\mathbf{J}} + \sin i\cos u\hat{\mathbf{K}} \tag{12.63b}$$

$$\hat{\mathbf{w}} = \sin\Omega\sin i\hat{\mathbf{I}} - \cos\Omega\sin i\hat{\mathbf{J}} + \cos i\hat{\mathbf{K}} \tag{12.63c}$$

These will prove useful in the following derivation of the time derivatives of the osculating orbital elements h, e, θ, Ω, i, and ω. The formulas that we obtain for these derivatives will be our version of the Gauss planetary equations.

We will employ familiar orbital mechanics formulas from Chapters 1, 2, and 4 to find the time derivatives that we need. The procedure involves the use of basic differential calculus, some vector operations, and a lot of algebra. Those who prefer not to read through the derivations can skip to the summary listing of the Gauss planetary equations in Eqn (12.84).

Variation of the specific angular momentum *h*

The time derivative of the angular momentum $\mathbf{h} = \mathbf{r} \times \mathbf{v}$ due to the perturbing acceleration **p** is

$$\frac{d\mathbf{h}}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{v} + \mathbf{r} \times \frac{d\mathbf{v}}{dt} = \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \left(-\mu\frac{\mathbf{r}}{r^3} + \mathbf{p}\right)$$

But $\mathbf{v} \times \mathbf{v} = \mathbf{r} \times \mathbf{r} = \mathbf{0}$, so this becomes

$$\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t} = \mathbf{r} \times \mathbf{p} \tag{12.64}$$

Since the magnitude of the angular momentum is $h = \sqrt{\mathbf{h}\cdot\mathbf{h}}$, its time derivative is

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\mathbf{h}\cdot\mathbf{h}} = \frac{1}{2}\frac{1}{\sqrt{\mathbf{h}\cdot\mathbf{h}}}\left(2\mathbf{h}\cdot\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t}\right) = \frac{\mathbf{h}}{h}\cdot\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t} = \hat{\mathbf{w}}\cdot\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t}$$

Substituting Eqn (12.64) yields

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \hat{\mathbf{w}}\cdot(\mathbf{r} \times \mathbf{p}) \tag{12.65}$$

Using the vector identity in Eqn (1.21) (interchange of the dot and the cross), we can modify this to read

$$\frac{\mathrm{d}h}{\mathrm{d}t} = (\hat{\mathbf{w}} \times \mathbf{r})\cdot\mathbf{p}$$

Since $\mathbf{r} = r\hat{\mathbf{r}}$ and $\hat{\mathbf{w}} \times \hat{\mathbf{r}} = \hat{\mathbf{s}}$, it follows that

$$\frac{\mathrm{d}h}{\mathrm{d}t} = rp_s \tag{12.66}$$

where $p_s = \mathbf{p}\cdot\hat{\mathbf{s}}$. Clearly, the variation of the angular momentum depends only on perturbation components that lie in the transverse (local horizon) direction.

Variation of the eccentricity *e*

The eccentricity may be found from Eqn (4.11),

$$e = \sqrt{1 + \frac{h^2}{\mu^2}\left(v^2 - \frac{2\mu}{r}\right)} \tag{12.67}$$

To find its time derivative, we will use Eqn (12.53),

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{\partial e}{\partial \mathbf{v}}\cdot\mathbf{p} \tag{12.68}$$

since $\mathrm{d}e/\mathrm{d}t)_{\text{two-body}} = 0$. Differentiating Eqn (12.67) with respect to $\mathbf{v}$ and using $\partial v^2/\partial\mathbf{v} = 2\mathbf{v}$ together with $\partial h^2/\partial\mathbf{v} = 2\mathbf{h} \times \mathbf{r}$ yields

$$\frac{\partial e}{\partial \mathbf{v}} = \frac{1}{2e}\frac{\partial}{\partial \mathbf{v}}\left[\frac{h^2}{\mu^2}\left(v^2 - \frac{2\mu}{r}\right)\right] = \frac{1}{\mu^2 e}\left[h^2\mathbf{v} + \left(v^2 - \frac{2\mu}{r}\right)(\mathbf{h} \times \mathbf{r})\right]$$

Substituting $\mathbf{v} = v_r\hat{\mathbf{r}} + v_s\hat{\mathbf{s}}$ and $\mathbf{h} \times \mathbf{r} = h\hat{\mathbf{w}} \times r\hat{\mathbf{r}} = hr\hat{\mathbf{s}}$ we get

$$\frac{\partial e}{\partial \mathbf{v}} = \frac{1}{\mu^2 e}\left\{h^2 v_r\hat{\mathbf{r}} + \left[hr\left(v^2 - \frac{2\mu}{r}\right) + h^2 v_s\right]\hat{\mathbf{s}}\right\}$$

Keeping in mind that v_s is the same as $v_\perp$, we can use Eqn (2.31) ($v_s = h/r$) along with Eqn (2.49) ($v_r = \mu e \sin\theta/h$) to write this as

$$\frac{\partial e}{\partial \mathbf{v}} = \frac{h}{\mu}\sin\theta\hat{\mathbf{r}} + \frac{h}{\mu^2 e}\left[r\left(v^2 - \frac{2\mu}{r}\right) + \frac{h^2}{r}\right]\hat{\mathbf{s}} \tag{12.69}$$

According to Problem 2.10, $v = (\mu/h)\sqrt{1 + 2e\cos\theta + e^2}$. Using the orbit formula (Eqn (2.45)) in the form $e\cos\theta = (h^2/\mu r) - 1$, we can write this as $v = \sqrt{2\mu/r - \mu^2(1-e^2)/h^2}$ and substitute it into Eqn (12.69) to get

$$\frac{\partial e}{\partial \mathbf{v}} = \frac{h}{\mu}\sin\theta\hat{\mathbf{r}} + \frac{1}{\mu h}\left[\left(h^2 + \mu r\right)\cos\theta + \mu e r\right]\hat{\mathbf{s}}$$

Finally, it follows from Eqn (12.68) that

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{h}{\mu}\sin\theta p_r + \frac{1}{\mu h}\left[\left(h^2 + \mu r\right)\cos\theta + \mu e r\right]p_s \tag{12.70}$$

where $p_r = \mathbf{p}\cdot\hat{\mathbf{r}}$ and $p_s = \mathbf{p}\cdot\hat{\mathbf{s}}$. Clearly, the eccentricity is affected only by perturbations that lie in the orbital plane.

Variation of the true anomaly θ

According to Eqns (12.53) and (12.54),

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{h}{r^2} + \frac{\partial\theta}{\partial\mathbf{v}}\cdot\mathbf{p} \tag{12.71}$$

To find $\partial\theta/\partial\mathbf{v}$, we start with the orbit formula (Eqn (2.45)), writing it as

$$\mu e r = \frac{h^2 - \mu r}{\cos\theta} \tag{12.72}$$

Another basic equation containing the true anomaly is the radial speed formula (Eqn (2.49)), from which we obtain

$$\mu e r = \frac{h\mathbf{r}\cdot\mathbf{v}}{\sin\theta} \tag{12.73}$$

Equating these two expressions for $\mu e r$ yields

$$\left(h^2 - \mu r\right)\sin\theta = h(\mathbf{r}\cdot\mathbf{v})\cos\theta$$

Applying the partial derivative with respect to **v** and rearranging the terms leads to

$$\left[\left(h^2-\mu r\right)\cos\theta + h(\mathbf{r}\cdot\mathbf{v})\sin\theta\right]\frac{\partial\theta}{\partial\mathbf{v}} = h\frac{\partial(\mathbf{r}\cdot\mathbf{v})}{\partial\mathbf{v}}\cos\theta + \left[(\mathbf{r}\cdot\mathbf{v})\cos\theta - 2h\sin\theta\right]\frac{\partial h}{\partial\mathbf{v}}$$

The use of Eqns (12.72) and (12.73) simplifies the square brackets on each side, so that

$$\mu e r\frac{\partial\theta}{\partial\mathbf{v}} = h\frac{\partial(\mathbf{r}\cdot\mathbf{v})}{\partial\mathbf{v}}\cos\theta - \left(h^2+\mu r\right)\frac{\sin\theta}{h}\frac{\partial h}{\partial\mathbf{v}}$$

Making use of $\partial(\mathbf{r}\cdot\mathbf{v})/\partial\mathbf{v} = \mathbf{r}$ and $\partial h/\partial\mathbf{v} = (\mathbf{h}\times\mathbf{r})/h$ yields

$$\frac{\partial\theta}{\partial\mathbf{v}} = \frac{h}{\mu e}\cos\theta\hat{\mathbf{r}} - \frac{1}{e}\left(\frac{h^2}{\mu}+r\right)\frac{\sin\theta}{h}\hat{\mathbf{s}} \tag{12.74}$$

where it is to be recalled that $\hat{\mathbf{s}} = \hat{\mathbf{w}}\times\hat{\mathbf{r}}$. Substituting this expression for $\partial\theta/\partial\mathbf{v}$ into Eqn (12.71) finally yields the time variation of true anomaly,

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{h}{r^2} + \frac{1}{eh}\left[\frac{h^2}{\mu}\cos\theta p_r - \left(\frac{h^2}{\mu}+r\right)\sin\theta p_s\right] \tag{12.75}$$

Like the eccentricity, true anomaly is unaffected by perturbations that are normal to the orbital plane.

Variation of right ascension Ω

As we know, and as can be seen from Figure 12.10, the right ascension of the ascending node is the angle between the inertial X-axis and the node line vector **N**. Therefore, it may be found by using the vector dot product,

$$\cos\Omega = \frac{\mathbf{N}}{\sqrt{\mathbf{N}\cdot\mathbf{N}}}\cdot\hat{\mathbf{I}}$$

We employed this formula in Algorithm 4.2. Alternatively, it should be evident from Figure 12.11 that we can use the equation $\tan\Omega = -h_X/h_Y$. Since $h_X = \mathbf{h}\cdot\hat{\mathbf{I}}$ and $h_Y = \mathbf{h}\cdot\hat{\mathbf{J}}$, this can be written as

$$\tan\Omega = -\frac{\mathbf{h}\cdot\hat{\mathbf{I}}}{\mathbf{h}\cdot\hat{\mathbf{J}}} \tag{12.76}$$

The time derivative of Eqn (12.76) is

$$\frac{1}{\cos^2\Omega}\frac{\mathrm{d}\Omega}{\mathrm{d}t} = -\frac{\left(\mathbf{h}\cdot\hat{\mathbf{J}}\right)\left(\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t}\cdot\hat{\mathbf{I}}\right) - \left(\mathbf{h}\cdot\hat{\mathbf{I}}\right)\left(\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t}\cdot\hat{\mathbf{J}}\right)}{\left(\mathbf{h}\cdot\hat{\mathbf{J}}\right)^2} = -\frac{h_Y\left(\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t}\cdot\hat{\mathbf{I}}\right) - h_X\left(\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t}\cdot\hat{\mathbf{J}}\right)}{{h_Y}^2}$$

or, more simply,

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \cos^2\Omega\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t}\cdot\frac{h_X\hat{\mathbf{J}} - h_Y\hat{\mathbf{I}}}{{h_Y}^2} \tag{12.77}$$

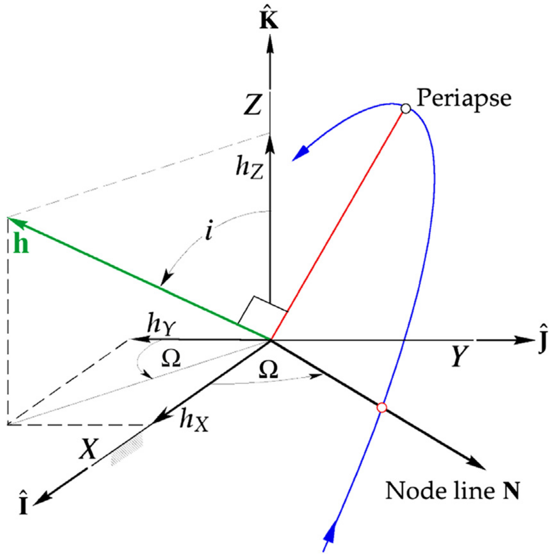

FIGURE 12.11

Relation between the components of angular momentum and the node angle Ω.

Since $\mathbf{h} = h\hat{\mathbf{w}}$ and $\hat{\mathbf{w}}$ is given by Eqn (12.63c), we have

$$\mathbf{h} = h\sin\Omega\sin i\hat{\mathbf{I}} - h\cos\Omega\sin i\hat{\mathbf{J}} + h\cos i\hat{\mathbf{K}}$$

This shows that $h_X = h\sin\Omega\sin i$ and $h_Y = -h\cos\Omega\sin i$, so that Eqn (12.77) becomes

$$\frac{d\Omega}{dt} = \frac{d\mathbf{h}}{dt}\cdot\frac{h\sin i\left(\cos\Omega\hat{\mathbf{I}} + \sin\Omega\hat{\mathbf{J}}\right)}{(h\sin i)^2} = \frac{1}{h\sin i}\hat{\mathbf{N}}\cdot\frac{d\mathbf{h}}{dt}$$

in which $\hat{\mathbf{N}}$ is the unit vector along the node line. Recalling that $d\mathbf{h}/dt = \mathbf{r}\times\mathbf{p}$ yields

$$\frac{d\Omega}{dt} = \frac{1}{h\sin i}\hat{\mathbf{N}}\cdot(\mathbf{r}\times\mathbf{p}) = \frac{1}{h\sin i}\left(\hat{\mathbf{N}}\times\mathbf{r}\right)\cdot\mathbf{p} \tag{12.78}$$

where we interchanged the dot and the cross by means of the identity in Eqn (1.21). The angle between the node line $\hat{\mathbf{N}}$ and the radial $\mathbf{r}$ is the argument of latitude u, and since $\hat{\mathbf{w}}$ is normal to the plane of $\hat{\mathbf{N}}$ and $\mathbf{r}$, it follows from the definition of the cross product operation that $\hat{\mathbf{N}}\times\mathbf{r} = r\sin u\hat{\mathbf{w}}$. Therefore, Eqn (12.78) may be written as

$$\frac{d\Omega}{dt} = \frac{r\sin u}{h\sin i}\hat{\mathbf{w}}\cdot\mathbf{p} \tag{12.79}$$

or, since $\hat{\mathbf{w}}\cdot\mathbf{p} = p_w$,

$$\frac{d\Omega}{dt} = \frac{r\sin u}{h\sin i}p_w \tag{12.80}$$

Clearly, the variation of right ascension Ω is influenced only by perturbations that are normal to the orbital plane.

Variation of the inclination *i*

The orbital inclination, which is the angle i between the Z-axis and the normal to the orbital plane (Figure 12.10), may be found from Eqn (4.7),

$$\cos i = \frac{\mathbf{h}\cdot\hat{\mathbf{K}}}{h}$$

Differentiating this expression with respect to time yields

$$-\frac{\mathrm{d}i}{\mathrm{d}t}\sin i = \frac{1}{h}\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t}\cdot\hat{\mathbf{K}} - \frac{1}{h^2}\frac{\mathrm{d}h}{\mathrm{d}t}\left(\mathbf{h}\cdot\hat{\mathbf{K}}\right)$$

Using Eqns (12.64) and (12.65), along with $\mathbf{h}\cdot\hat{\mathbf{K}} = h\cos i$, we find that

$$\frac{\mathrm{d}i}{\mathrm{d}t}\sin i = \frac{r}{h}\left(\hat{\mathbf{w}}\cos i - \hat{\mathbf{K}}\right)\cdot(\hat{\mathbf{r}}\times\mathbf{p})$$

Replacing $\hat{\mathbf{w}}$ by the expression in Eqn (12.63c) yields

$$\begin{aligned}\frac{\mathrm{d}i}{\mathrm{d}t} &= \frac{r}{h}\left(\sin\Omega\cos i\hat{\mathbf{I}} - \cos\Omega\cos i\hat{\mathbf{J}} - \sin i\hat{\mathbf{K}}\right)\cdot(\hat{\mathbf{r}}\times\mathbf{p})\\ &= \frac{r}{h}\left[\left(\sin\Omega\cos i\hat{\mathbf{I}} - \cos\Omega\cos i\hat{\mathbf{J}} - \sin i\hat{\mathbf{K}}\right)\times\hat{\mathbf{r}}\right]\cdot\mathbf{p}\end{aligned}$$

where we once again used Eqn (1.21) to interchange the dot and the cross. Replacing the unit vector $\hat{\mathbf{r}}$ by Eqn (12.63a) and using the familiar determinant formula for the cross product, we get

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{r}{h}\begin{vmatrix}\hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}}\\ \sin\Omega\cos i & -\cos\Omega\cos i & -\sin i\\ -\sin\Omega\cos i\sin u + \cos\Omega\cos u & \cos\Omega\cos i\sin u + \sin\Omega\cos u & \sin i\sin u\end{vmatrix}\cdot\mathbf{p}$$

Expanding the determinant and recognizing $\hat{\mathbf{w}}$ from Eqn (12.63c), we find

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{r}{h}\cos u\left(\sin\Omega\sin i\hat{\mathbf{I}} - \cos\Omega\sin i\hat{\mathbf{J}} + \cos i\hat{\mathbf{K}}\right)\cdot\mathbf{p} = \frac{r}{h}\cos u(\hat{\mathbf{w}}\cdot\mathbf{p})$$

Since $\hat{\mathbf{w}}\cdot\mathbf{p} = p_w$,

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{r}{h}\cos u\, p_w \tag{12.81}$$

Like Ω, the orbital inclination is affected only by perturbation components that are normal to the orbital plane.

Variation of argument of periapsis ω

The arguments of periapsis and latitude are related by $\omega = u - \theta$. Let us first seek an expression for du/dt and then obtain the variation $d\omega/dt$ from the fact that $d\omega/dt = du/dt - d\theta/dt$, where we found $d\theta/dt$ above in Eqn (12.74).

Since the argument of latitude is the angle u between the node line vector $\mathbf{N}$ and the position vector $\mathbf{r}$, it is true that $\cos u = \hat{\mathbf{r}}\cdot\hat{\mathbf{N}}$. Differentiating this expression with respect to the velocity vector $\mathbf{v}$ and noting that $\hat{\mathbf{N}} = \cos\Omega\hat{\mathbf{I}} + \sin\Omega\hat{\mathbf{J}}$, we get

$$\frac{\partial u}{\partial \mathbf{v}} = -\frac{1}{\sin u}\hat{\mathbf{r}}\cdot\frac{\partial \hat{\mathbf{N}}}{\partial \mathbf{v}} = \frac{1}{\sin u}\left(\sin\Omega\hat{\mathbf{r}}\cdot\hat{\mathbf{I}} - \cos\Omega\hat{\mathbf{r}}\cdot\hat{\mathbf{J}}\right)\frac{\partial \Omega}{\partial \mathbf{v}}$$

Using the expression for the radial unit vector $\hat{\mathbf{r}}$ in Eqn (12.63a), we conclude that

$$\sin\Omega\hat{\mathbf{r}}\cdot\hat{\mathbf{I}} - \cos\Omega\hat{\mathbf{r}}\cdot\hat{\mathbf{J}} = -\cos i \sin u$$

which simplifies our result,

$$\frac{\partial u}{\partial \mathbf{v}} = -\cos i\frac{\partial \Omega}{\partial \mathbf{v}} \tag{12.82}$$

From Eqn (12.53), we know that $d\Omega/dt = (\partial\Omega/\partial\mathbf{v})\cdot\mathbf{p}$, whereas according to Eqn (12.79), $d\Omega/dt = (r\sin u/h\sin i)\hat{\mathbf{w}}\cdot\mathbf{p}$. It follows that

$$\frac{\partial \Omega}{\partial \mathbf{v}} = \frac{r\sin u}{h\sin i}\hat{\mathbf{w}}$$

Therefore, Eqn (12.82) yields

$$\frac{\partial u}{\partial \mathbf{v}} = -\frac{r\sin u}{h\tan i}\hat{\mathbf{w}}$$

Finally, since $du/dt = (\partial u/\partial\mathbf{v})\cdot\mathbf{p}$, we conclude that

$$\frac{du}{dt} = -\frac{r\sin u}{h\tan i}p_w$$

Substituting this result into $d\omega/dt = du/dt - d\theta/dt$ and making use of Eqn (12.74) (the variation of θ due solely to perturbations) yields

$$\frac{d\omega}{dt} = -\frac{1}{eh}\left[\frac{h^2}{\mu}\cos\theta p_r - \left(r + \frac{h^2}{\mu}\right)\sin\theta p_s\right] - \frac{r\sin(\omega+\theta)}{h\tan i}p_w \tag{12.83}$$

We see that $d\omega/dt$ depends on all three components of the perturbing acceleration.

For convenience, let us summarize the Gauss form of the planetary equations that govern the variations of the orbital elements angular momentum h, eccentricity e, true anomaly θ, node angle Ω, inclination i, and argument of periapsis ω.

$$\frac{dh}{dt} = rp_s \tag{12.84a}$$

$$\frac{de}{dt}=\frac{h}{\mu}\sin\theta p_r+\frac{1}{\mu h}\left[\left(h^2+\mu r\right)\cos\theta+\mu er\right]p_s \tag{12.84b}$$

$$\frac{d\theta}{dt}=\frac{h}{r^2}+\frac{1}{eh}\left[\frac{h^2}{\mu}\cos\theta p_r-\left(r+\frac{h^2}{\mu}\right)\sin\theta p_s\right] \tag{12.84c}$$

$$\frac{d\Omega}{dt}=\frac{r}{h\sin i}\sin(\omega+\theta)p_w \tag{12.84d}$$

$$\frac{di}{dt}=\frac{r}{h}\cos(\omega+\theta)p_w \tag{12.84e}$$

$$\frac{d\omega}{dt}=-\frac{1}{eh}\left[\frac{h^2}{\mu}\cos\theta p_r-\left(r+\frac{h^2}{\mu}\right)\sin\theta p_s\right]-\frac{r\sin(\omega+\theta)}{h\tan i}p_w \tag{12.84f}$$

where $r=h^2/[\mu(1+e\cos\theta)]$ (Eqn (2.45)).

Given the six orbital elements h_0, e_0, θ_0, Ω_0, i_0, and ω_0 at time t_0 and the functional form of the perturbing acceleration **p**, we can numerically integrate the six planetary equations to obtain the osculating orbital elements and therefore the state vector at subsequent times. Formulas for the variations of alternative orbital elements can be derived, but it is not necessary here because at any instant all other osculating orbital parameters are found in terms of the six listed above. For example,

$$\text{Semimajor axis (Eqn 2.72): } a=\frac{h^2}{\mu}\frac{1}{1-e^2}$$

$$\text{Eccentric anomaly (Eqn 3.13): } E=2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right)$$

$$\text{Mean anomaly (Eqn 3.6): } M=2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right)-\frac{e\sqrt{1-e^2}\sin\theta}{1+e\cos}$$

EXAMPLE 12.4

Obtain the form of the Gauss planetary equations for a perturbing force that is tangent to the orbit.

Solution

If the perturbing force is tangent to the orbit, then the perturbing acceleration **p** lies in the direction of the velocity vector **v**,

$$\mathbf{p}=p_v\frac{\mathbf{v}}{v}$$

The radial and transverse components of the perturbing acceleration are found by projecting **p** onto the radial and transverse directions

$$p_r=\left(p_v\frac{\mathbf{v}}{v}\right)\cdot\hat{\mathbf{r}}=\frac{p_v}{v}(\mathbf{v}\cdot\hat{\mathbf{r}})=\frac{p_v}{v}v_r$$

$$p_s=\left(p_v\frac{\mathbf{v}}{v}\right)\cdot\hat{\mathbf{s}}=\frac{p_v}{v}(\mathbf{v}\cdot\hat{\mathbf{s}})=\frac{p_v}{v}v_s$$

The component p_w normal to the orbital plane is zero. From Eqns (2.31) and (2.49), we have

$$v_s = \frac{h}{r} \quad v_r = \frac{\mu}{h} e \sin\theta$$

Therefore, for a tangential perturbation, the *rsw* components of acceleration are

$$\begin{aligned} p_r &= \frac{\mu}{vh} e p_v \sin\theta \\ p_s &= \frac{h}{vr} p_v \\ p_w &= 0 \end{aligned}$$

Substituting these expressions into Eqn (12.84) leads to the Gauss planetary equations for tangential perturbations,

$$\boxed{\begin{array}{lll} \dfrac{dh}{dt} = \dfrac{h}{v} p_v & \dfrac{de}{dt} = \dfrac{2}{v}(e + \cos\theta) p_v & \dfrac{d\theta}{dt} = \dfrac{h}{r^2} - \dfrac{2}{ev} p_v \sin\theta \\ \dfrac{d\Omega}{dt} = 0 & \dfrac{di}{dt} = 0 & \dfrac{d\omega}{dt} = \dfrac{2}{ev} p_v \sin\theta \end{array}} \quad \text{Tangential perturbation} \qquad (12.85)$$

Clearly, the right ascension and the inclination are unaffected.

If the perturbing acceleration is due to a tangential thrust T, then $p_v = T/m$, where m is the instantaneous mass of the spacecraft. If the thrust is in the direction of the velocity $\mathbf{v}$, then p_v is positive and tends to speed up the spacecraft, producing an outwardly spiraling trajectory like that shown in Figure 12.1. According to Eqn (12.85), a positive tangential acceleration causes the eccentricity to increase and the perigee to advance (rotating the apse line counterclockwise). Both of these effects are evident in Figure 12.1. If the thrust is opposite to the direction of $\mathbf{v}$, then the effects are opposite: the spacecraft slows down, spiraling inward.

Atmospheric drag acts opposite to the direction of motion. If we neglect the rotation of the atmosphere, then according to Eqn (12.12), the tangential acceleration is

$$p_v = -\frac{1}{2}\rho v^2 B \tag{12.86}$$

in which v is the inertial speed and B is the ballistic coefficient. Substituting this expression into Eqn (12.85) yields the Gauss planetary equations for atmospheric drag,

$$\begin{array}{lll} \dfrac{dh}{dt} = -\dfrac{B}{2} h\rho v & \dfrac{de}{dt} = -B(e + \cos\theta)\rho v & \dfrac{d\theta}{dt} = \dfrac{h}{r^2} + B\dfrac{\rho v}{e}\sin\theta \\ \dfrac{d\Omega}{dt} = 0 & \dfrac{di}{dt} = 0 & \dfrac{d\omega}{dt} = -B\dfrac{\rho v}{e}\sin\theta \end{array} \tag{12.87}$$

EXAMPLE 12.5

Find the components p_r, p_s, and p_w of the gravitational perturbation of Eqn (12.30) in the *rsw* frame of Figure 12.10.

Solution

The transformation from *XYZ* to *rsw* is given by Eqn (12.61). Using the direction cosine matrix in Eqn (12.58) we therefore have

$$\begin{Bmatrix} p_r \\ p_s \\ p_w \end{Bmatrix} = \frac{3}{2}\frac{J_2 \mu R^2}{r^4}\begin{bmatrix} -\sin\Omega\cos i\sin u + \cos\Omega\cos u & \cos\Omega\cos i\sin u + \sin\Omega\cos u & \sin i\sin u \\ -\sin\Omega\cos i\cos u - \cos\Omega\sin u & \cos\Omega\cos i\cos u - \sin\Omega\sin u & \sin i\cos u \\ \sin\Omega\sin i & -\cos\Omega\sin i & \cos i \end{bmatrix} \cdot \begin{Bmatrix} \dfrac{X}{r}\left(5\dfrac{Z^2}{r^2} - 1\right) \\ \dfrac{Y}{r}\left(5\dfrac{Z^2}{r^2} - 1\right) \\ \dfrac{Z}{r}\left(5\dfrac{Z^2}{r^2} - 3\right) \end{Bmatrix} \qquad \text{(a)}$$

Before carrying out the multiplication, note that in the spherical coordinate system of Figure 12.12,

$$\begin{aligned} Z &= r\sin\delta \\ X &= r\cos\delta\cos\alpha \\ Y &= r\cos\delta\sin\alpha \end{aligned} \qquad \text{(b)}$$

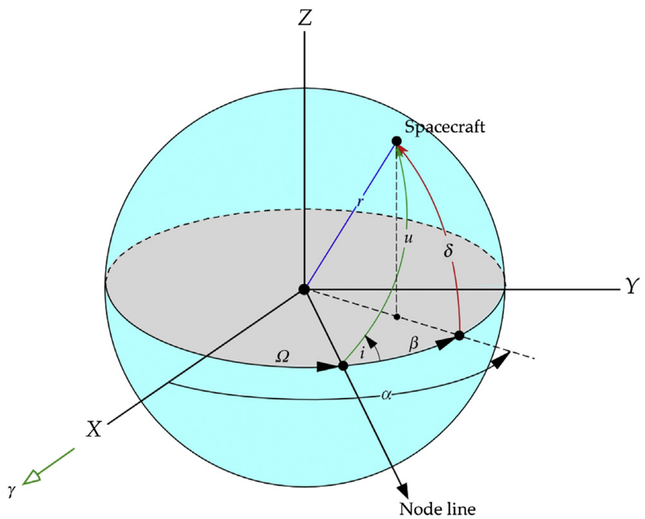

FIGURE 12.12

Spherical coordinates and the geocentric equatorial frame.

Here α is the azimuth angle measured in the XY plane positive from the X-axis. (We use α instead of the traditional θ to avoid confusion with true anomaly.) The declination δ is the complement of the usual polar angle ϕ, which is measured positive from the polar (Z) axis toward the equator. As shown in Figure 12.6, δ is measured positive northward from the equator. Using δ instead of ϕ makes it easier to take advantage of spherical trigonometry formulas. The angle β in Figure 12.12 is the difference between the azimuth angle α and the right ascension of the ascending node,

$$\beta = \alpha - \Omega \tag{c}$$

On the unit sphere, the angles i, u, β, and δ appear as shown in Figure 12.12. Spherical trigonometry (Beyer, 1999) yields the following relations among these four angles:

$$\sin\delta = \sin i \sin u \tag{d}$$
$$\cos u = \cos\delta\cos\beta \tag{e}$$
$$\sin\beta = \tan\delta\cot i \tag{f}$$
$$\cos u = \cos\delta\cos\beta \tag{g}$$

From Eqns (d) and (f), we find

$$\sin\beta = (\sin\delta/\cos\delta)(\cos i/\sin i) = (\sin i\sin u/\cos\delta)(\cos i/\sin i)$$

or

$$\sin\beta = \frac{\sin u\cos i}{\cos\delta} \tag{h}$$

whereas Eqn (g) may be written as

$$\cos\beta = \frac{\cos u}{\cos\delta} \tag{i}$$

Using Eqns (b), Eqn (a) becomes

$$\begin{Bmatrix} p_r \\ p_s \\ p_w \end{Bmatrix} = \frac{3}{2}\frac{J_2\mu R^2}{r^4}\begin{bmatrix} -\sin\Omega\cos i\sin u + \cos\Omega\cos u & \cos\Omega\cos i\sin u + \sin\Omega\cos u & \sin i\sin u \\ -\sin\Omega\cos i\cos u - \cos\Omega\sin u & \cos\Omega\cos i\cos u - \sin\Omega\sin u & \sin i\cos u \\ \sin\Omega\sin i & -\cos\Omega\sin i & \cos i \end{bmatrix} \cdot \begin{Bmatrix} \cos\delta\cos\alpha\left(5\sin^2\delta - 1\right) \\ \cos\delta\sin\alpha\left(5\sin^2\delta - 1\right) \\ \sin\delta\left(5\sin^2\delta - 3\right) \end{Bmatrix} \tag{j}$$

Carrying out the multiplication on the right for p_r yields

$$p_r = \frac{3}{2}\frac{J_2\mu R^2}{r^4}\left[\cos\delta(\cos u\cos\beta + \cos i\sin u\sin\beta)\left(1 - 5\sin^2\delta\right) + \sin^2\delta\left(3 - 5\sin^2\delta\right)\right]$$

Substituting Eqns (h) and (i) leads to

$$p_r = \frac{3}{2}\frac{J_2\mu R^2}{r^4}\left[\left(\cos^2 u + \cos^2 i\sin^2 u\right)\left(1 - 5\sin^2\delta\right) + \sin^2\delta\left(3 - 5\sin^2\delta\right)\right]$$

After substituting Eqn (d) and $\cos^2 i = 1 - \sin^2 i$ and simplifying we get

$$p_r = -\frac{3}{2}\frac{J_2\mu R^2}{r^4}\left(1 - 3\sin^2 i \sin^2 u\right)$$

In a similar fashion we find p_s and p_w, so that in summary

$$\boxed{\begin{aligned} p_r &= -\frac{3}{2}\frac{J_2\mu R^2}{r^4}\left[1 - 3\sin^2 i\cdot\sin^2(\omega+\theta)\right] \\ p_s &= -\frac{3}{2}\frac{J_2\mu R^2}{r^4}\sin^2 i\cdot\sin 2(\omega+\theta) \\ p_s &= -\frac{3}{2}\frac{J_2\mu R^2}{r^4}\sin 2i\cdot\sin(\omega+\theta) \end{aligned}} \tag{12.88}$$

We may now substitute p_r, p_s, and p_w from Eqn (12.88) into Gauss's planetary equations (Eqn (12.84)) to obtain the variation of the osculating elements due to the J_2 gravitational perturbation (keeping in mind that $u = \omega + \theta$). After some straightforward algebraic manipulations, we obtain:

$$\frac{dh}{dt} = -\frac{3}{2}\frac{J_2\mu R^2}{r^3}\sin^2 i \sin 2u \tag{12.89a}$$

$$\frac{de}{dt} = \frac{3}{2}\frac{J_2\mu R^2}{hr^3}\left\{\frac{h^2}{\mu r}\sin\theta\left(3\sin^2 i\sin^2 u - 1\right) - \sin 2u\sin^2 i[(2 + e\cos\theta)\cos\theta + e]\right\} \tag{12.89b}$$

$$\frac{d\theta}{dt} = \frac{h}{r^2} + \frac{3}{2}\frac{J_2\mu R^2}{ehr^3}\left[\frac{h^2}{\mu r}\cos\theta\left(3\sin^2 i\sin^2 u - 1\right) + (2 + e\cos\theta)\sin 2u\sin^2 i\sin\theta\right] \tag{12.89c}$$

$$\frac{d\Omega}{dt} = -3\frac{J_2\mu R^2}{hr^3}\sin^2 u\cos i \tag{12.89d}$$

$$\frac{di}{dt} = -\frac{3}{4}\frac{J_2\mu R^2}{hr^3}\sin 2u\sin 2i \tag{12.89e}$$

$$\frac{d\omega}{dt} = \frac{3}{2}\frac{J_2\mu R^2}{ehr^3}\left[\frac{h^2}{\mu r}\cos\theta\left(1 - 3\sin^2 i\sin^2 u\right) - (2 + e\cos\theta)\sin 2u\sin^2 i\sin\theta + 2e\cos^2 i\sin^2 u\right] \tag{12.89f}$$

EXAMPLE 12.6

At time $t = 0$, an earth satellite has the following orbital parameters:

Perigee radius: $r_p = 6678$ km (300 km altitude)

Apogee radius: $r_a = 9440$ km (3062 km altitude)

Right ascension of the ascending node: $\Omega = 45°$ (a)

Inclination: $i = 28°$ (b)

Argument of perigee: $\omega = 30°$ (c)

True anomaly: $\theta = 40°$ (d)

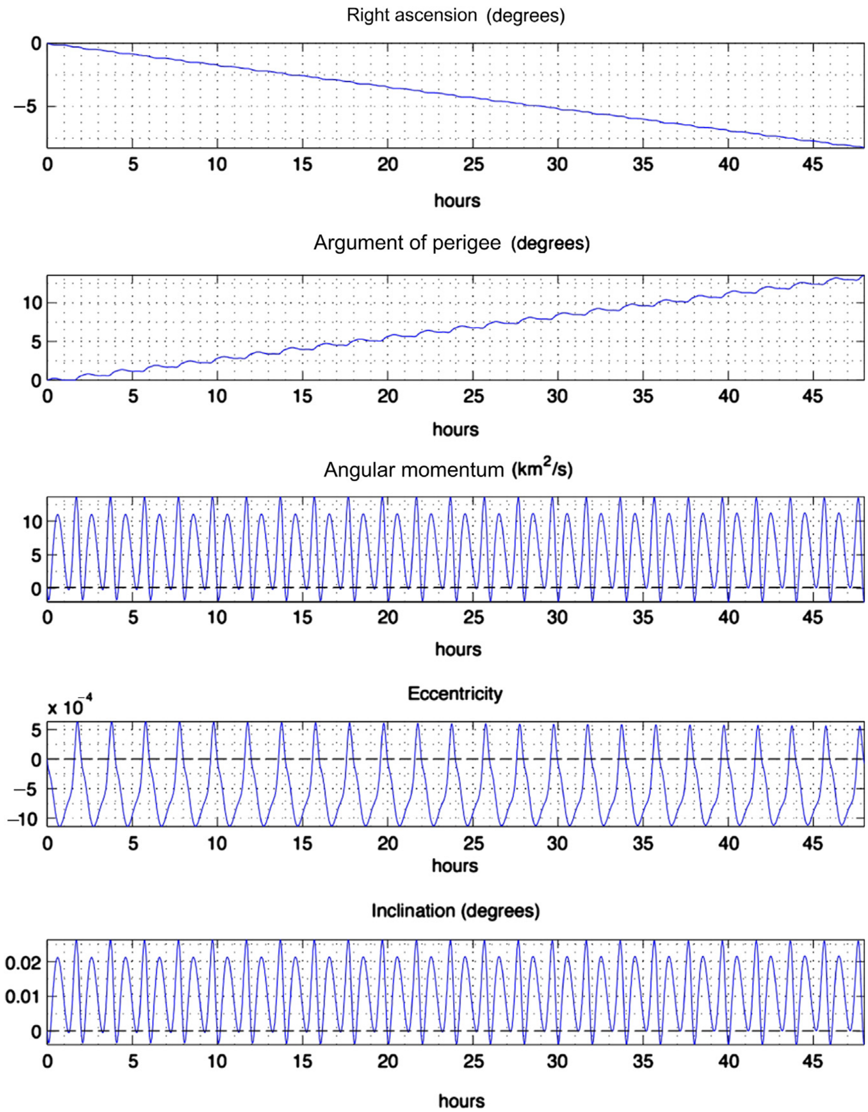

FIGURE 12.13

Variation of the osculating elements due to J_2 perturbation.

Use the Gauss variational Eqns (12.89) to determine the effect of the J_2 perturbation on the variation of orbital elements h, e, Ω, i, and ω over the next 48 hours.

Solution

From the given information, we find the eccentricity and the angular momentum from Eqns (2.84) and (2.50) with $\mu = 398{,}600\ \text{km}^3/\text{s}^2$,

$$e = \frac{r_a - r_p}{r_a + r_p} = 0.17136 \tag{e}$$

$$h = \sqrt{\mu(1+e)r_p} = 55{,}839\ \text{km}^2/\text{s} \tag{f}$$

We write the differential equations in Eqns (12.89) as $\text{d}\mathbf{y}/\text{d}t = \mathbf{f}(\mathbf{y})$, where

$$\mathbf{y} = \lfloor h \quad e \quad \theta \quad \Omega \quad i \quad \omega \rfloor^T$$
$$\mathbf{f} = \lfloor \dot{h} \quad \dot{e} \quad \dot{\theta} \quad \dot{\Omega} \quad \dot{i} \quad \dot{\omega} \rfloor^T$$

and the rates $\dot{h}$, $\dot{e}$, etc., are given by Eqns (12.89), with $r = h^2/[\mu(1+e\cos\theta)]$ and $u = \omega + \theta$. With Eqns (a) through (f) comprising the initial conditions vector $\mathbf{y}_0$, we can solve for $\mathbf{y}$ on the time interval $[t_0, t_f]$ using MATLAB's numerical integrator *ode45* (or one of those described in Section 1.8). For $t_0 = 0$ and $t_f = 48$ h, the solutions are plotted in Figure 12.13. The MATLAB script *Example_12_06.m* is in Appendix D.45.

The data for this problem are the same as for Example 12.2, wherein the results are identical to what we found here. In Example 12.2, the perturbed state vector $(\mathbf{r},\mathbf{v})$ was found over the two-day time interval, at each point of which the osculating elements were then derived from Algorithm 4.2. Here, on the other hand, we directly obtained the osculating elements, from which at any time on the solution interval we can calculate the state vector by means of Algorithm 4.5.

12.8 Method of averaging

One advantage of the Gauss planetary equations for the J_2 perturbation is that they show explicitly the dependence of the element variations on the elements themselves and the position of the spacecraft in the gravitational field. We see in Eqns (12.89) that the short period "ripples" superimposed on the long-term secular trends are due to the presence of the trigonometric terms in the true anomaly θ. In order to separate the secular terms from the short period terms the *method of averaging* is used. Let f be an osculating element and $\dot{f}$ its variation as given in Eqn (12.89). Since $\dot{f} = (\text{d}f/\text{d}\theta)\dot{\theta}$, the average of $\dot{f}$ over one orbit is

$$\bar{\dot{f}} = \overline{\frac{\text{d}f}{\text{d}\theta}}\, n \tag{12.90}$$

where n is the mean motion (Eqn (3.9))

$$n = \bar{\dot{\theta}} = \frac{1}{T}\int_0^T \frac{\text{d}\theta}{\text{d}t}\text{d}t = \frac{1}{T}\int_0^{2\pi} \text{d}\theta = \frac{2\pi}{T}$$

Substituting the formula for the period (Eqn (2.83)), we can write n as

$$n = \sqrt{\frac{\mu}{a^3}} \tag{12.91}$$

For the average value of $\mathrm{d}f/\mathrm{d}\theta$ we have

$$\overline{\frac{\mathrm{d}f}{\mathrm{d}\theta}} = \frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}f}{\mathrm{d}\theta}\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}f}{\mathrm{d}t}\frac{1}{\dot{\theta}}\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}f}{\mathrm{d}t}\frac{r^2}{h}\mathrm{d}\theta$$

where we made use of Eqn (2.47). Substituting this into Eqn (12.90) yields a formula for the time-averaged variation,

$$\bar{\dot{f}} = \frac{n}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}f}{\mathrm{d}t}\frac{r^2}{h}\mathrm{d}\theta \tag{12.92}$$

In doing the integral, the only variable is θ; all of the other orbital elements are held fixed.

Let us use Eqn (12.92) to compute the orbital averages of each of the rates in Eqns (12.89). In doing so, we will make frequent use of the orbit formula $r = h^2/[\mu(1 + e\cos\theta)]$.

Angular momentum

$$\begin{aligned}\bar{\dot{h}} &= \frac{n}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}h}{\mathrm{d}t}\frac{r^2}{h}\mathrm{d}\theta = \frac{n}{2\pi}\int_0^{2\pi}\left(-\frac{3}{2}\frac{J_2\mu R^2}{r^3}\sin^2 i\sin 2u\right)\frac{r^2}{h}\mathrm{d}\theta \\ &= \frac{n}{2\pi}\left(-\frac{3}{2}\frac{J_2\mu R^2}{h}\sin^2 i\right)\int_0^{2\pi}\left(\frac{1}{r}\sin 2u\right)\mathrm{d}\theta \\ &= \frac{n}{2\pi}\left(-\frac{3}{2}\frac{J_2\mu^2 R^2}{h^3}\sin^2 i\right)\int_0^{2\pi}(1 + e\cos\theta)\sin 2(\omega+\theta)\mathrm{d}\theta\end{aligned}$$

Evaluate the integral as follows, remembering to hold the orbital element ω constant.

$$\begin{aligned}\int_0^{2\pi}(1 + e\cos\theta)\sin 2(\omega+\theta)\mathrm{d}\theta &= \int_0^{2\pi}(1 + e\cos\theta)(\sin 2\omega\cos 2\theta + \cos 2\omega\sin 2\theta)\mathrm{d}\theta \\ &= \sin 2\omega\int_0^{2\pi}\cos 2\theta\mathrm{d}\theta + \cos 2\omega\int_0^{2\pi}\sin 2\theta\mathrm{d}\theta \\ &\quad + e\sin 2\omega\int_0^{2\pi}\cos\theta\cos 2\theta\mathrm{d}\theta + e\cos 2\omega\int_0^{2\pi}\cos\theta\sin 2\theta\mathrm{d}\theta \\ &= \sin 2\omega\overbrace{\left(\frac{\sin 2\theta}{2}\right)_0^{2\pi}}^{=0} - \cos 2\omega\overbrace{\left(\frac{\cos 2\theta}{2}\right)_0^{2\pi}}^{=0} \\ &\quad + e\sin 2\omega\overbrace{\left(\frac{\sin\theta}{2} + \frac{\sin 3\theta}{6}\right)_0^{2\pi}}^{=0} - e\cos 2\omega\overbrace{\left(\frac{\cos\theta}{2} + \frac{\cos 3\theta}{6}\right)_0^{2\pi}}^{=0} = 0\end{aligned}$$

Therefore, $\bar{\dot{h}} = 0$.

Eccentricity

$$\bar{\dot{e}} = \frac{n}{2\pi}\int_0^{2\pi} \frac{de}{dt}\frac{r^2}{h}d\theta$$

$$= \frac{n}{2\pi}\int_0^{2\pi} \frac{3}{2}\frac{J_2\mu R^2}{hr^3}\left\{\frac{h^2}{\mu r}\sin\theta\left[3\sin^2 i\sin^2 u - 1\right] - [(2+e\cos\theta)\cos\theta + e]\sin 2u\sin^2 i\right\}\frac{r^2}{h}d\theta$$

$$= \frac{n}{2\pi}\left(\frac{3}{2}\frac{J_2\mu R^2}{h^2}\right)\int_0^{2\pi} \frac{1}{r}\left\{\frac{h^2}{\mu r}\sin\theta\left[3\sin^2 i\sin^2 u - 1\right] - [(2+e\cos\theta)\cos\theta + e]\sin 2u\sin^2 i\right\}d\theta$$

$$= \frac{n}{2\pi}\left(\frac{3}{2}\frac{J_2\mu^2 R^2}{h^4}\right)\int_0^{2\pi} (1+e\cos\theta)\big\{(1+e\cos\theta)\sin\theta\left[3\sin^2 i\sin^2 u - 1\right]$$

$$- [(2+e\cos\theta)\cos\theta + e]\sin 2u\sin^2 i\big\}d\theta$$

But

$$\int_0^{2\pi} (1+e\cos\theta)\big\{(1+e\cos\theta)\sin\theta\left[3\sin^2 i\sin^2 u - 1\right] - [(2+e\cos\theta)\cos\theta + e]\sin 2u\sin^2 i\big\}d\theta = 0$$

A convenient way to evaluate this lengthy integral is to use MATLAB's symbolic math feature, as illustrated in the following Command Window session, in which `w` and `q` represent ω and θ, respectively:

```
>>
syms w q e i positive
f = (1+e*cos(q))*((1+e*cos(q))*sin(q)*(3*sin(i)^2*sin(w+q)^2-1) ...
                    -((2+e*cos(q))*cos(q) + e)*sin(2*(w+q))*sin(i)^2);
integral = int(f, q, 0, 2*pi)
integral =
0
>>
```

It follows that $\bar{\dot{e}} = 0$.

True anomaly

$$\bar{\dot{\theta}} = \frac{n}{2\pi}\int_0^{2\pi} \frac{d\theta}{dt}\frac{r^2}{h}d\theta$$

$$= \frac{n}{2\pi}\int_0^{2\pi}\left\{\frac{h}{r^2} + \frac{3}{2}\frac{J_2\mu R^2}{ehr^3}\left[\frac{h^2}{\mu r}\cos\theta\left(3\sin^2 i\sin^2 u - 1\right) + (2 + e\cos\theta)\sin 2u\sin^2 i\sin\theta\right]\right\}\frac{r^2}{h}d\theta$$

$$= \frac{n}{2\pi}\int_0^{2\pi}\left\{1 + \frac{3}{2}\frac{J_2\mu R^2}{eh^2 r}\left[\frac{h^2}{\mu r}\cos\theta\left(3\sin^2 i\sin^2 u - 1\right) + (2 + e\cos\theta)\sin 2u\sin^2 i\sin\theta\right]\right\}d\theta$$

$$= n + \frac{n}{2\pi}\frac{3}{2}\frac{J_2\mu^2 R^2}{eh^4}\int_0^{2\pi}\left\{(1 + e\cos\theta)\left[(1 + e\cos\theta)\cos\theta\left(3\sin^2 i\sin^2 u - 1\right)\right.\right.$$
$$\left.\left. + (2 + e\cos\theta)\sin 2u\sin^2 i\sin\theta\right]\right\}d\theta$$

The integral evaluates to $(1-3\cos^2 i)\pi e$, as we see in this MATLAB Command Window session:

```
>>
syms w q e i positive
f = (1+e*cos(q))*((1+e*cos(q))*cos(q)*(3*sin(i)^2*sin(w+q)^2-1) ...
     +(2+e*cos(q))*sin(2*(w+q))*sin(i)^2*sin(q));
f = collect(expand(f));
integral = int(f, q, 0, 2*pi)
integral =
pi*e - 3*pi*e*cos(i)^2
>>
```

Thus,

$$\bar{\dot{\theta}} = n\left[1 + \frac{3}{4}\frac{J_2\mu^2 R^2}{h^4}\left(1 - 3\cos^2 i\right)\right]$$

Substituting Eqn (12.91), $h^2 = \mu a(1 - e^2)$, and $\cos^2 i = 1 - \sin^2 i$, we can write this as

$$\bar{\dot{\theta}} = n + \frac{3}{4}\frac{J_2 R^2\sqrt{\mu}}{a^{7/2}(1 - e^2)^2}\left(3\sin^2 i - 2\right)$$

Right ascension of the ascending node

$$
\begin{aligned}
\overline{\dot{\Omega}} &= \frac{n}{2\pi}\int_0^{2\pi}\frac{d\Omega}{dt}\frac{r^2}{h}d\theta \\
&= \frac{n}{2\pi}\int_0^{2\pi}\left(-3\frac{J_2\mu R^2}{hr^3}\sin^2 u\cos i\right)\frac{r^2}{h}d\theta \\
&= -\frac{3n}{2\pi}\frac{J_2\mu R^2}{h^2}\cos i\int_0^{2\pi}\frac{1}{r}\sin^2 u d\theta \\
&= -\frac{3n}{2\pi}\frac{J_2\mu^2 R^2}{h^4}\cos i\int_0^{2\pi}(1+e\cos\theta)\sin^2(\omega+\theta)d\theta
\end{aligned}
$$

But

$$
\int_0^{2\pi}(1+e\cos\theta)\sin^2(\omega+\theta)d\theta = \pi
$$

as is evident from the MATLAB session:

```
>>
syms w q e positive
integral = int('(sin(w + q))^2*(1 + e*cos(q))', q, 0, 2*pi)
integral =
pi
>>
```

Hence,

$$
\overline{\dot{\Omega}} = -\frac{3n}{2}\frac{J_2\mu^2 R^2}{h^4}\cos i
$$

Substituting Eqn (12.91) along with $h^4 = \mu^2 a^2(1-e^2)^2$ from Eqn (2.71), we obtain

$$
\overline{\dot{\Omega}} = -\left[\frac{3}{2}\frac{J_2\sqrt{\mu}R^2}{a^{7/2}(1-e^2)^2}\right]\cos i
$$

Inclination

$$
\begin{aligned}
\overline{\dot{i}} &= \frac{n}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}i}{\mathrm{d}t}\frac{r^2}{h}\mathrm{d}\theta \\
&= \frac{n}{2\pi}\int_0^{2\pi}\left(-\frac{3}{4}\frac{J_2\mu R^2}{hr^3}\sin 2u\sin 2i\right)\frac{r^2}{h}\mathrm{d}\theta \\
&= -\frac{n}{2\pi}\frac{3}{4}\frac{J_2\mu R^2}{h^2}\sin 2i\int_0^{2\pi}\frac{1}{r}\sin 2u\mathrm{d}\theta \\
&= -\frac{n}{2\pi}\frac{3}{4}\frac{J_2\mu R^2}{h^4}\sin 2i\int_0^{2\pi}(1+e\cos\theta)\sin 2(\omega+\theta)\mathrm{d}\theta
\end{aligned}
$$

Using MATLAB, we see that the integral vanishes:

```
>>
syms w q e positive
integral = int('(1 + e*cos(q))*sin(2*(w + q))', q, 0, 2*pi)
integral = 0
0
>>
```

Therefore, $\overline{\dot{i}} = 0$.

Argument of perigee

$$
\begin{aligned}
\overline{\dot{\omega}} &= \frac{n}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\omega}{\mathrm{d}t}\frac{r^2}{h}\mathrm{d}\theta \\
&= \frac{n}{2\pi}\int_0^{2\pi}\frac{3}{2}\frac{J_2\mu R^2}{ehr^3}\left[\frac{h^2}{\mu r}\cos\theta\left(1-3\sin^2 i\sin^2 u\right)-(2+e\cos\theta)\sin 2u\sin^2 i\sin\theta+2e\cos^2 i\sin^2 u\right]\frac{r^2}{h}\mathrm{d}\theta \\
&= \frac{n}{2\pi}\frac{3}{2}\frac{J_2\mu R^2}{eh^2}\int_0^{2\pi}\frac{1}{r}\left[\frac{h^2}{\mu r}\cos\theta\left(1-3\sin^2 i\sin^2 u\right)-(2+e\cos\theta)\sin 2u\sin^2 i\sin\theta+2e\cos^2 i\sin^2 u\right]\mathrm{d}\theta \\
&= \frac{n}{2\pi}\frac{3}{2}\frac{J_2\mu^2 R^2}{eh^4}\int_0^{2\pi}(1+e\cos\theta)\big[\cos\theta(1+e\cos\theta)\left(1-3\sin^2 i\sin^2 u\right) \\
&\qquad -(2+e\cos\theta)\sin 2u\sin^2 i\sin\theta+2e\cos^2 i\sin^2 u\big]\mathrm{d}\theta
\end{aligned}
$$

With the aid of MATLAB, we find that integral evaluates to $\pi e(5\cos^2 i - 1)$.

```
>>
syms w q e i positive
f = (1+e*cos(q))*...
     ((1+e*cos(q))*cos(q)*(1-3*sin(i)^2*sin(w+q)^2)...
      -(2+e*cos(q))*sin(2*(w+q))*sin(i)^2*sin(q)...
      + 2*e*cos(i)^2*sin(w+q)^2);
f = collect(expand(f));
integral = int(f, q, 0, 2*pi)
integral = 5*pi*e*cos(i)^2 - pi*e
>>
```

Therefore,

$$\overline{\dot{\omega}} = \frac{3}{2}\frac{J_2\mu^2R^2}{h^4}\frac{n}{2}\left(5\cos^2 i - 1\right)$$

Substituting Eqn (12.91), along with $h^4 = \mu^2a^2(1-e^2)^2$ from Eqn (2.71) and the trig identity $\cos^2 i = 1 - \sin^2 i$, we obtain

$$\overline{\dot{\omega}} = -\left[\frac{3}{2}\frac{J_2\sqrt{\mu}R^2}{a^{7/2}(1-e^2)^2}\right]\left(\frac{5}{2}\sin^2 i - 2\right)$$

Let us summarize our calculations of the average rates of variation of the orbital elements due to the J_2 perturbation.

$$\overline{\dot{h}} = \overline{\dot{e}} = \overline{\dot{i}} = 0 \tag{12.93a}$$

$$\overline{\dot{\Omega}} = -\left[\frac{3}{2}\frac{J_2\sqrt{\mu}R^2}{a^{7/2}(1-e^2)^2}\right]\cos i \tag{12.93b}$$

$$\overline{\dot{\omega}} = -\left[\frac{3}{2}\frac{J_2\sqrt{\mu}R^2}{a^{7/2}(1-e^2)^2}\right]\left(\frac{5}{2}\sin^2 i - 2\right) \tag{12.93c}$$

$$\overline{\dot{\theta}} = n - \left[\frac{3}{2}\frac{J_2\sqrt{\mu}R^2}{a^{7/2}(1-e^2)^2}\right]\left(1 - \frac{3}{2}\sin^2 i\right) \tag{12.93d}$$

Formulas such as these for the average rates of variation of the orbital elements are useful for the design of frozen orbits. Frozen orbits are those whose size, shape, and/or orientation remain, on average, constant over long periods of time. Careful selection of the orbital parameters can minimize or eliminate the drift caused by perturbations. For example, the apse line will be frozen in space ($\overline{\dot{\omega}} = 0$) if the orbital inclination i is such that $\sin i = \sqrt{4/5}$. Similarly, the J_2 gravitational

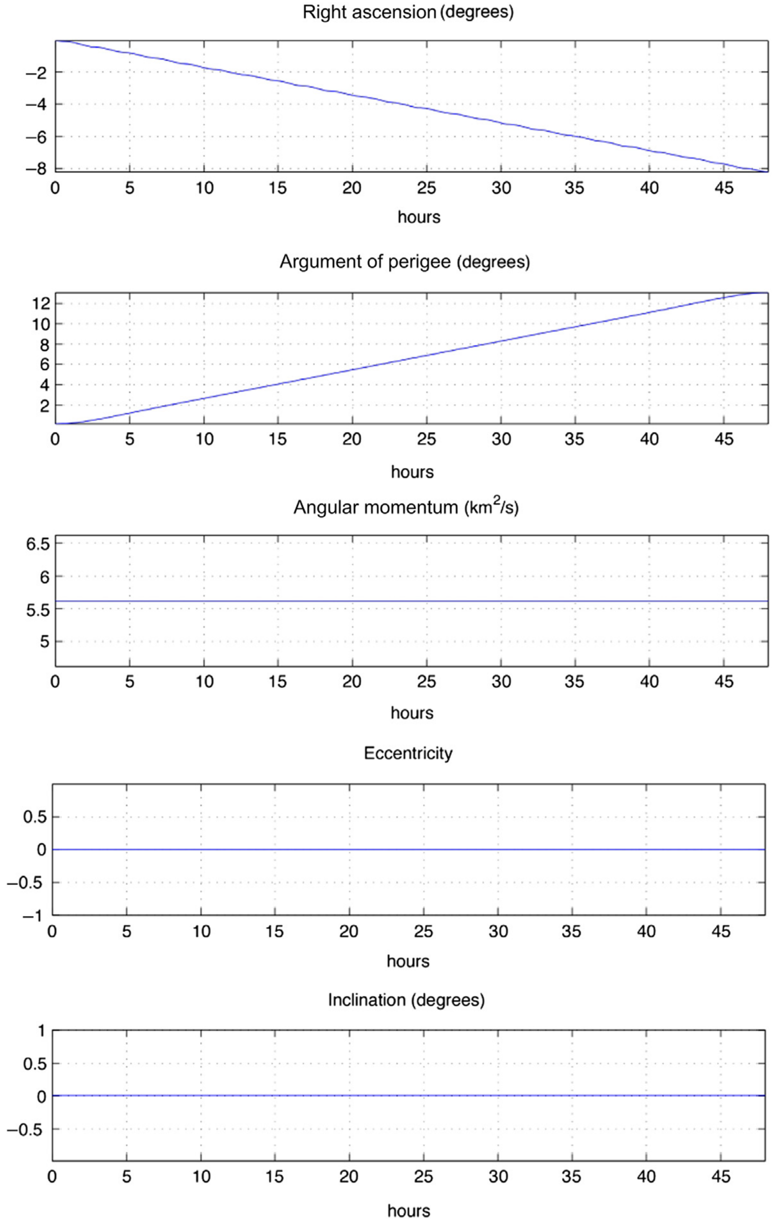

FIGURE 12.14

Numerically smoothed plots of the data in Figure 12.13, showing only the long-term trends.

perturbation on the mean motion $\overline{\dot{\theta}}$ vanishes if $\sin i = \sqrt{2/3}$. The precession $\overline{\dot{\Omega}}$ of an orbital plane is prevented if $\cos i = 0$. Practical applications of Molniya, sun-synchronous, and polar orbits are discussed in Section 4.7.

Equation (12.93) agrees with the plotted results of Examples 12.2 and 12.6. We can average out the high-frequency components ("ripples") of the curves in Figures 12.8, 12.9, and 12.13 with a numerical smoothing technique such as that presented by Garcia (2010), yielding the curves in Figure 12.14.

12.9 Solar radiation pressure

According to quantum physics, solar radiation comprises photons, which are massless elementary particles traveling at the speed of light ($c = 2.998 \times 10^8$ m/s). Even though a photon's mass is zero, its energy and momentum are not. The energy (in Joules) of a photon is hf, where f is the frequency of its electromagnetic wave (in Hertz) and h is the *Planck constant* ($h = 6.626 \times 10^{-34}$ J·s). The momentum of a photon is hf/c, its energy is divided by the speed of light.

The visible surface of the sun is the *photosphere*, which acts like a black body emitting radiation that spans most of the electromagnetic spectrum, from low-energy radiowaves on up the visible spectrum and beyond to high energy ultraviolet light and X-rays. According to the Stefan–Boltzmann law, the intensity of the radiated power is σT^4, where T is the absolute temperature of the black body and σ is the Stefan–Boltzmann constant,

$$\sigma = 5.670 \times 10^{-8}\ \mathrm{W/m^2K^4}$$

The effective temperature of the photosphere is 5777 K so that at its surface the radiated power intensity is

$$S_0 = 5.670 \times 10^{-8}(5777)^4 = 63.15 \times 10^6\ \mathrm{W/m^2}$$

Electromagnetic radiation follows the inverse square law. That is, if R_0 is the radius of the photosphere, then the radiation intensity S at a distance R from the sun's center is

$$S = S_0\left(\frac{R_0}{R}\right)^2$$

The radius of the photosphere is 696,000 km and the mean earth-to-sun distance is 149.6×10^6 km (1 Astronomical Unit). It follows that at the earth's orbit the radiation intensity S, known as the *solar constant*, is

$$S = 63.15 \times 10^6\left(\frac{696{,}000}{149.6 \times 10^6}\right)^2 = 1367\ \mathrm{W/m^2} \tag{12.94}$$

This is the energy flux (the energy per unit time per unit area) transported by photons across a surface normal to the radiation direction. As mentioned above, we must divide S by the speed of light to find the momentum flux, which is the *solar radiation pressure* P_{SR},

$$P_{SR} = \frac{S}{c} = \frac{1367\ (\mathrm{N \cdot m/s})/\mathrm{m}^2}{2.998 \times 10^8\ \mathrm{m/s}} = 4.56 \times 10^{-6}\ \mathrm{N/m}^2 (4.56\ \mu\mathrm{Pa}) \tag{12.95}$$

Compare this to sea level atmospheric pressure (101 kPa), which exceeds P_{SR} by more than ten orders of magnitude.

In the interest of simplicity, let us adopt the *cannonball model* for solar radiation, which assumes that the satellite is a sphere of radius R. Then the perturbing force $\mathbf{F}$ on the satellite due to the radiation pressure S/c is

$$\mathbf{F} = -\nu \frac{S}{c} C_R A_s \hat{\mathbf{u}} \tag{12.96}$$

where $\hat{\mathbf{u}}$ is the unit vector pointing from the satellite toward the sun. The negative sign shows that the solar radiation force is directed away from the sun. A_s is the absorbing area of the satellite, which is πR^2 for the cannonball model. ν is the *shadow function*, which has the value 0 if the satellite is in the earth's shadow; otherwise, $\nu = 1$. C_R is the radiation pressure coefficient, which lies between 1 and 2. C_R equals 1 if the surface is a black body, absorbing all of the momentum of the incident photon stream and giving rise to the pressure in Eqn (12.95). When C_R equals 2, all the incident radiation is reflected so that the incoming photon momentum is reversed in direction, doubling the force on the satellite.

Because the sun is so far from the earth, the angle between the earth-to-sun line and the satellite-to-sun line is less than 0.02 degree, even for GEO satellites. Therefore, it will be far simpler and sufficiently accurate for our purposes to let $\hat{\mathbf{u}}$ in Eqn (12.96) be the unit vector pointing toward the sun from the earth instead of from the satellite. Then $\hat{\mathbf{u}}$ tracks only the relative motion of the sun around the earth and does not include the motion of the satellite around the earth.

If m is the mass of the satellite, then the perturbing acceleration $\mathbf{p}$ due to solar radiation is $\mathbf{F}/m$, or

$$\mathbf{p} = -p_{SR} \hat{\mathbf{u}} \tag{12.97}$$

where the magnitude of the perturbation is

$$p_{SR} = \nu \frac{S}{c} \frac{C_R A_s}{m} \tag{12.98}$$

The magnitude of the solar radiation pressure perturbation clearly depends on the satellite's area-to-mass ratio A_s/m. Very large spacecraft with a very low mass (like solar sails) are the most affected by solar radiation pressure. Extreme examples of such spacecraft were the Echo 1, Echo 2, and Pageos passive communication balloon satellites launched by the United States in the 1960s. They were very thin walled, highly reflective spheres about 100 ft (30 m) in diameter, and they had area-to-mass ratios on the order of 10 m^2/kg.

The influence of solar radiation pressure is more pronounced at higher orbital altitudes where atmospheric drag is comparatively negligible. To get an idea of where the tradeoff between the two perturbations occurs, set the magnitude of the drag perturbation equal to that of the solar radiation perturbation, $p_D = p_{SR}$, or

$$\frac{1}{2}\rho v^2\left(\frac{C_D A}{m}\right) = \frac{S}{c}\frac{C_R A_s}{m}$$

Solving for the atmospheric density and assuming that the orbit is circular ($v^2 = \mu/r$), we get

$$\rho = 2\frac{A_s}{A}\frac{C_R}{C_D}\frac{S/c}{\mu}r$$

If $A_s/A = 1$, $C_R = 1$, $C_D = 2$, and $r = 6378 + z$, where z is the altitude in kilometers, then this becomes

$$\rho = 2\cdot 1\cdot\frac{1}{2}\cdot\frac{\left(4.56\times 10^{-6}\ \mathrm{kg/m\cdot s^2}\right)}{\left(398.6\times 10^{12}\ \mathrm{m^3/s^2}\right)}[(6378+z)(\mathrm{km})](1000\ \mathrm{m/km})$$

or

$$\rho\left(\mathrm{kg/m^3}\right) = 1.144\times 10^{-17}(6378+z)\,(\mathrm{km})$$

When $z = 625$ km, this formula gives $\rho = 8.01 \times 10^{-14}$ kg/m^3, whereas according to the US Standard Atmosphere, $\rho = 7.998 \times 10^{-14}$ kg/m^3 at that altitude. So 625 km is a rough estimate of the altitude of the transition from the dominance of the perturbative effect of atmospheric drag to that of solar radiation pressure. This estimate is about 20% lower than the traditionally accepted value of 800 km (Vallado, 2007).

Recall from Section 4.2 that the angle between earth's equatorial plane and the ecliptic plane is the obliquity of the ecliptic ε. The obliquity varies slowly with time and currently is 23.44°. Therefore, the plane of the sun's apparent orbit around the earth is inclined 23.44° to the earth's equator. In the geocentric ecliptic ($X'Y'Z'$) frame, the Z'-axis is normal to the ecliptic and the X'-axis lies along the vernal equinox direction. In this frame, the unit vector $\hat{\mathbf{u}}$ along the earth-to-sun line is provided by the solar ecliptic longitude λ,

$$\hat{\mathbf{u}} = \cos\lambda\hat{\mathbf{I}}' + \sin\lambda\hat{\mathbf{J}}' \tag{12.99}$$

λ is the angle between the vernal equinox line and the earth-sun line.

The geocentric equatorial frame (XYZ) and the geocentric ecliptic frame ($X'Y'Z'$) share the vernal equinox line as their common X-axis. Transformation from one frame to the other is therefore simply a rotation through the obliquity ε around the positive X-axis. The transformation from $X'Y'Z'$ to XYZ is represented by the direction cosine matrix found in Eqn (4.32) with $\phi = -\varepsilon$. It follows that the components in XYZ of the unit vector $\hat{\mathbf{u}}$ in Eqn (12.99) are

$$\{\hat{\mathbf{u}}\}_{XYZ} = [\mathbf{R}_1(-\varepsilon)]\{\hat{\mathbf{u}}\}_{X'Y'Z'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{bmatrix}\begin{Bmatrix} \cos\lambda \\ \sin\lambda \\ 0 \end{Bmatrix} = \begin{Bmatrix} \cos\lambda \\ \cos\varepsilon\sin\lambda \\ \sin\varepsilon\sin\lambda \end{Bmatrix} \tag{12.100}$$

Substituting this vector expression back into Eqn (12.97) yields the components of the solar radiation perturbation in the geocentric equatorial frame,

$$\{\mathbf{p}\}_{XYZ} = -p_{SR}\begin{Bmatrix} \cos\lambda \\ \cos\varepsilon\sin\lambda \\ \sin\varepsilon\sin\lambda \end{Bmatrix} \tag{12.101}$$

In order to use the Gauss planetary equations (Eqn (12.84)) to determine the effects of solar radiation pressure on the variation of the orbital elements, we must find the components of the perturbation **p** in the *rsw* frame of Figure 12.10. To do so, we use Eqn (12.61),

$$\begin{Bmatrix} p_r \\ p_s \\ p_w \end{Bmatrix} = [\mathbf{Q}]_{Xr}\begin{Bmatrix} -p_{SR}\cos\lambda \\ -p_{SR}\sin\lambda\cos\varepsilon \\ -p_{SR}\sin\lambda\sin\varepsilon \end{Bmatrix} \tag{12.102}$$

The direction cosine matrix $[\mathbf{Q}]_{Xr}$ of the transformation from *XYZ* to *rsw* is given by Eqn (12.58). Thus,

$$\begin{Bmatrix} p_r \\ p_s \\ p_w \end{Bmatrix} = -p_{SR}\begin{bmatrix} -\sin\Omega\cos i\sin u + \cos\Omega\cos u & \cos\Omega\cos i\sin u + \sin\Omega\cos u & \sin i\sin u \\ -\sin\Omega\cos i\cos u - \cos\Omega\sin u & \cos\Omega\cos i\cos u - \sin\Omega\sin u & \sin i\cos u \\ \sin\Omega\sin i & -\cos\Omega\sin i & \cos i \end{bmatrix} \cdot \begin{Bmatrix} \cos\lambda \\ \sin\lambda\cos\varepsilon \\ \sin\lambda\sin\varepsilon \end{Bmatrix} \tag{12.103}$$

Carrying out the matrix multiplications leads to

$$p_r = -p_{SR}u_r \quad p_s = -p_{SR}u_s \quad p_w = -p_{SR}u_w \tag{12.104}$$

where u_r, u_s, and u_w are the components of the unit vector $\hat{\mathbf{u}}$ in the *rsw* frame, namely,

$$\begin{aligned} u_r = {} & \sin\lambda\cos\varepsilon\cos\Omega\cos i\sin u + \sin\lambda\cos\varepsilon\sin\Omega\cos u \\ & -\cos\lambda\sin\Omega\cos i\sin u + \cos\lambda\cos\Omega\cos u + \sin\lambda\sin\varepsilon\sin i\sin u \end{aligned} \tag{12.105a}$$

$$\begin{aligned} u_s = {} & \sin\lambda\cos\varepsilon\cos\Omega\cos i\cos u - \sin\lambda\cos\varepsilon\sin\Omega\sin u \\ & -\cos\lambda\sin\Omega\cos i\cos u - \cos\lambda\cos\Omega\sin u + \sin\lambda\sin\varepsilon\sin i\cos u \end{aligned} \tag{12.105b}$$

$$u_w = -\sin\lambda\cos\varepsilon\cos\Omega\sin i + \cos\lambda\sin\Omega\sin i + \sin\lambda\sin\varepsilon\cos i \tag{12.105c}$$

Substituting Eqn (12.104) into Eqn (12.84) yields the Gauss planetary equations for solar radiation pressure, where it is to be recalled that $p_{SR} = \nu(S/c)C_R(A_s/m)$:

$$\frac{dh}{dt} = -p_{SR} r u_s \tag{12.106a}$$

$$\frac{de}{dt} = -p_{SR}\left\{\frac{h}{\mu}\sin\theta u_r + \frac{1}{\mu h}\left[\left(h^2 + \mu r\right)\cos\theta + \mu e r\right]u_s\right\} \tag{12.106b}$$

$$\frac{d\theta}{dt} = \frac{h}{r^2} - \frac{p_{SR}}{eh}\left[\frac{h^2}{\mu}\cos\theta u_r - \left(r + \frac{h^2}{\mu}\right)\sin\theta u_s\right] \tag{12.106c}$$

$$\frac{d\Omega}{dt} = -p_{SR}\frac{r}{h\sin i}\sin(\omega+\theta)u_w \tag{12.106d}$$

$$\frac{di}{dt} = -p_{SR}\frac{r}{h}\cos(\omega+\theta)u_w \tag{12.106e}$$

$$\frac{d\omega}{dt} = -p_{SR}\left\{-\frac{1}{eh}\left[\frac{h^2}{\mu}\cos\theta u_r - \left(r + \frac{h^2}{\mu}\right)\sin\theta u_s\right] - \frac{r\sin(\omega+\theta)}{h\tan i}u_w\right\} \tag{12.106f}$$

To numerically integrate Eqn (12.106) requires that we know the time variation of the obliquity ε and solar ecliptic longitude λ, both of which appear throughout the expressions for u_r, u_s, and u_w. We also need the time history of the earth-to-sun distance r_S in order to compute the geocentric equatorial position vector of the sun, namely, $\mathbf{r}_S = r_S\hat{\mathbf{u}}$. $\mathbf{r}_S$ together with the geocentric position vector of the satellite allow us to determine when the satellite is in the earth's shadow ($\nu = 0$ in Eqn (12.98)).

According to *The Astronomical Almanac* (National Almanac Office, 2013), the apparent solar ecliptic longitude (in degrees) is given by the formula

$$\lambda = L + 1.915°\sin M + 0.0200°\sin 2M \,(0° \le \lambda \le 360°) \tag{12.107}$$

L and M are, respectively, the mean longitude and mean anomaly of the sun, both in degrees:

$$L = 280.459° + 0.98564736°n \,(0° \le L \le 360°) \tag{12.108}$$

$$M = 357.529° + 0.98560023°n \,(0° \le M \le 360°) \tag{12.109}$$

n is the number of days since J2000,

$$n = \text{JD} - 2{,}451{,}545.0 \tag{12.110}$$

The concepts of Julian day number JD and the epoch J2000 are explained in Section 5.4. The above formulas for L, M, and λ may deliver angles outside of the range 0° to 360°. In those cases, the angle should be reduced by appropriate multiples of 360° so as to place it in that range. (If the **angle** is a number, then the MATLAB function `mod(angle,360)` yields a number in the range 0° to 360°.)

In terms of n, the obliquity is

$$\varepsilon = 23.439° - 3.56\left(10^{-7}\right)n \tag{12.111}$$

Finally, the *Almanac* gives the distance r_S from the earth to the sun in terms of the mean anomaly,

$$r_S = (1.00014 - 0.01671\cos M - 0.000140\cos 2M)\ \text{AU} \tag{12.112}$$

where AU is the *astronomical unit* (1 AU = 149,597,870.691 km).

The following algorithm delivers ε, λ, and $\mathbf{r}_S$.

ALGORITHM 12.2

Given the year, month, day, and universal time, calculate the obliquity of the ecliptic ε, ecliptic longitude of the sun λ, and the geocentric position vector of the sun $\mathbf{r}_S$.

1. Compute the Julian day number JD using Eqns (5.47) and (5.48).
2. Calculate n, the number of days since J2000 from Eqn (12.110).
3. Calculate the mean anomaly M using Eqn (12.109).
4. Calculate the mean solar longitude L by means of Eqn (12.108).
5. Calculate the longitude λ using Eqn (12.107).
6. Calculate the obliquity ε from Eqn (12.111).
7. Calculate the unit vector $\hat{\mathbf{u}}$ from the earth to the sun (Eqn (12.100)):
$$\hat{\mathbf{u}} = \cos\lambda\hat{\mathbf{I}} + \sin\lambda\cos\varepsilon\hat{\mathbf{J}} + \sin\lambda\sin\varepsilon\hat{\mathbf{K}}$$
8. Calculate the distance r_S of the sun from the earth using Eqn (12.112).
9. Calculate the sun's geocentric position vector $\mathbf{r}_S = r_S\hat{\mathbf{u}}$.

EXAMPLE 12.7

Use Algorithm 12.2 to find the apparent ecliptic longitude of the sun at 0800 UT on July 25, 2013.

Solution

Step 1:

According to Eqn (5.48), with $y = 2013$, $m = 7$, and $d = 25$, the Julian day number at 0 h UT is

$$J_0 = 2{,}456{,}498.5$$

Therefore, from Eqn (5.47), the Julian day number at 0800 UT is

$$\text{JD} = 2{,}456{,}498.5 + \frac{8}{24} = 2{,}456{,}498.8333\ \text{days}$$

Step 2:

$$n = 2{,}456{,}498.8333 - 2{,}451{,}545.0 = 4953.8333$$

Step 3:

$$M = 357.529^\circ + 0.98560023^\circ(4953.8333) = 5240.03^\circ(200.028^\circ)$$

Step 4:

$$L = 280.459^\circ + 0.98564736^\circ(4953.8333) = 5163.19^\circ(123.192^\circ)$$

Step 5:

$$\lambda = 123.192^\circ + 1.915^\circ \sin(200.028^\circ) + 0.020^\circ \sin(2\cdot 200.028^\circ) = \boxed{122.549^\circ}$$

Step 6:

$$\varepsilon = 23.439^\circ - 3.56^\circ\left(10^{-7}\right)(4953.8333) = \boxed{23.4372^\circ}$$

Step 7:

$$\begin{aligned}\hat{\mathbf{u}} &= \cos 122.549^\circ\hat{\mathbf{I}} + (\sin 122.549^\circ)(\cos 23.4372^\circ)\hat{\mathbf{J}} + (\sin 122.549^\circ)(\sin 23.4372^\circ)\hat{\mathbf{K}}\\ &= -0.538017\hat{\mathbf{I}} + 0.773390\hat{\mathbf{J}} + 0.335269\hat{\mathbf{K}}\end{aligned}$$

Step 8:

$$\begin{aligned}r_S &= [1.00014 - 0.01671\cos 200.028^\circ - 0.000140\cos(2\cdot 200.028^\circ)](149{,}597{,}870.691)\\ &= 151{,}951{,}387\ \text{km}\end{aligned}$$

Step 9:

$$\begin{aligned}\mathbf{r}_S &= (151{,}951{,}387)\left(-0.5380167\hat{\mathbf{I}} + 0.7733903\hat{\mathbf{J}} + 0.3352693\hat{\mathbf{K}}\right)\\ &= \boxed{-81{,}752{,}385\hat{\mathbf{I}} + 117{,}517{,}729\hat{\mathbf{J}} + 50{,}944{,}632\hat{\mathbf{K}}\ (\text{km})}\end{aligned}$$

In order to determine when a satellite is in the earth's shadow (so that the solar radiation pressure perturbation is "off"), we can use the following elementary procedure (Vallado, 2007). First, consider two spacecraft A and B orbiting a central body of radius R. The two position vectors $\mathbf{r}_A$ and $\mathbf{r}_B$ define a plane, which is the plane of Figure 12.15. That plane contains the circular profile C of the central body. The angle θ between the two position vectors may be found from the dot product operation,

$$\theta = \cos^{-1}\left(\frac{\mathbf{r}_A\cdot\mathbf{r}_B}{r_A r_B}\right) \tag{12.113}$$

In Figure 12.15(a), T_1 and T_2 are points of tangency to C of lines drawn from A and B, respectively. The radii OT_1 and OT_2 along with the tangent lines AT_1 and BT_2 and the position vectors $\mathbf{r}_A$ and $\mathbf{r}_B$ comprise the two right triangles OAT_1 and OBT_2. The angles at the vertex O of these two triangles are obtained from

$$\theta_1 = \cos^{-1}\frac{R}{r_A} \qquad \theta_2 = \cos^{-1}\frac{R}{r_B} \tag{12.114}$$

If, as in Figure 12.15(a), the line AB intersects the central body, which means there is no line of sight, then $\theta_1 + \theta_2 < \theta$. If the AB is tangent to C (Figure 12.15(b)) or lies outside it, then $\theta_1 + \theta_2 \geq \theta$ and there is line of sight.

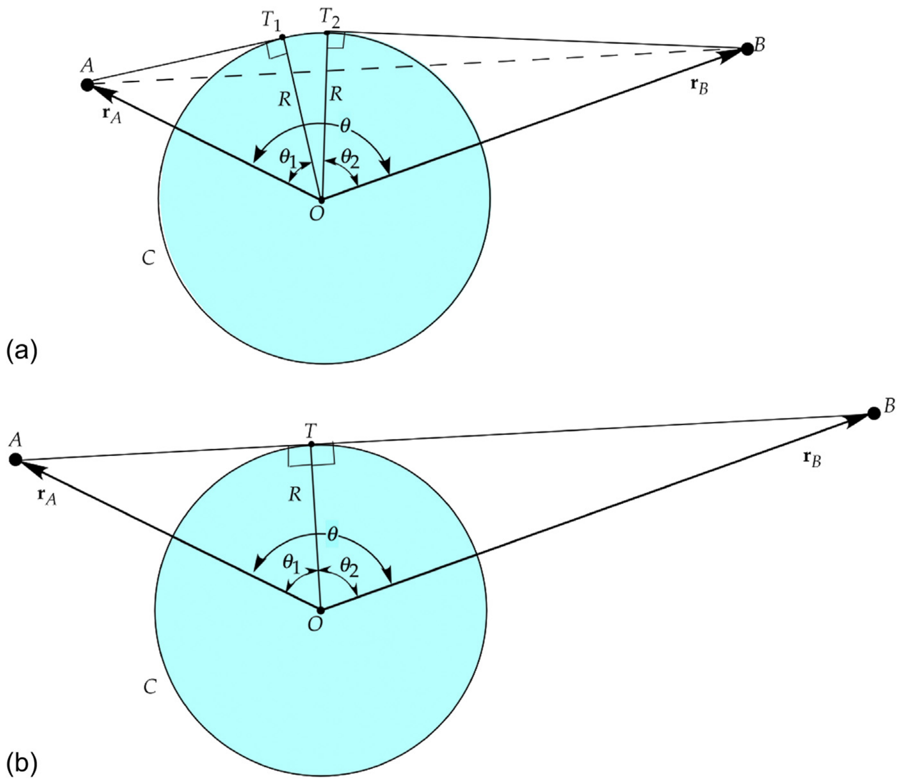

FIGURE 12.15

(a) *AB* intersects the central body ($\theta_1 + \theta_2 < \theta$). (b) *AB* is tangent to the central body ($\theta_1 + \theta_2 = \theta$).

ALGORITHM 12.3

Given the position vector $\mathbf{r} = X\hat{\mathbf{I}} + Y\hat{\mathbf{J}} + Z\hat{\mathbf{K}}$ of a satellite and the apparent position vector $\mathbf{r}_s = X_s\hat{\mathbf{I}} + Y_s\hat{\mathbf{J}} + Z_s\hat{\mathbf{K}}$ of the sun, both in the geocentric equatorial frame, determine the value of ν (0 or 1) of the shadow function. $R = 6378$ km (the radius of the earth).

1. $r = \|\mathbf{r}\| \quad r_s = \|\mathbf{r}_s\|$
2. $\theta = \cos^{-1}\left(\dfrac{\mathbf{r}_s \cdot \mathbf{r}}{r_s r}\right)$
3. $\theta_1 = \cos^{-1}\dfrac{R}{r} \quad \theta_2 = \cos^{-1}\dfrac{R}{r_s}$
4. If $\theta_1 + \theta_2 \le \theta$ then $\nu = 0$. Otherwise, $\nu = 1$.

EXAMPLE 12.8

At a given instant, the geocentric position vector of an earth satellite is

$$\mathbf{r} = 2817.899\hat{\mathbf{I}} - 14{,}110.473\hat{\mathbf{J}} - 7502.672\hat{\mathbf{K}} \text{ (km)}$$

and the geocentric position vector of the sun is

$$\mathbf{r}_S = -11{,}747{,}041\hat{\mathbf{I}} + 139{,}486{,}985\hat{\mathbf{J}} + 60{,}472{,}278\hat{\mathbf{K}} \text{ (km)}$$

Determine whether or not the satellite is in earth's shadow.

Solution

Step 1:

$$r = \left\|2817.899\hat{\mathbf{I}} - 14{,}110.473\hat{\mathbf{J}} - 7502.672\hat{\mathbf{K}}\right\| = 16{,}227.634 \text{ km}$$

$$r_S = \left\| - 11{,}747{,}041\hat{\mathbf{I}} + 139{,}486{,}985\hat{\mathbf{J}} + 60{,}472{,}278\hat{\mathbf{K}}\right\| = 152{,}035{,}836 \text{ km}$$

Step 2:

$$\theta = \cos^{-1}\frac{\left(2817.899\hat{\mathbf{I}} - 14{,}110.473\hat{\mathbf{J}} - 7502.672\hat{\mathbf{K}}\right)\cdot\left(-11{,}747{,}041\hat{\mathbf{I}} + 139{,}486{,}985\hat{\mathbf{J}} + 60{,}472{,}278\hat{\mathbf{K}}\right)}{(16{,}227.634)\cdot(152{,}035{,}836)}$$

$$= \cos^{-1}\left(\frac{-2{,}425{,}241{,}207{,}229}{2{,}467{,}181{,}827{,}406}\right)$$

$$= 169.420°$$

Step 3:

$$\theta_1 = \cos^{-1}\frac{6378}{16{,}227.634} = 66.857°$$

$$\theta_2 = \cos^{-1}\frac{6378}{152{,}035{,}836} = 89.998°$$

Step 4:

$$\theta_1 + \theta_2 = 156.85° < \theta. \boxed{\text{Therefore, the spacecraft is in earth's shadow.}}$$

EXAMPLE 12.9

A spherical earth satellite has an absorbing area-to-mass ratio (A_S/m) of 2 m^2/kg. At time t_0 (Julian date $JD_0 = 2{,}438{,}400.5$) its orbital parameters are

$$\text{Angular momentum: } h_0 = 63{,}383.4 \text{ km}^2/\text{s} \tag{a}$$

$$\text{Eccentricity: } e_0 = 0.025422 \tag{b}$$

$$\text{Right ascension of the node: } \Omega_0 = 45.3812° \tag{c}$$

$$\text{Inclination: } i_0 = 88.3924° \tag{d}$$

$$\text{Argument of perigee: } \omega_0 = 227.493° \tag{e}$$

$$\text{True anomaly: } \theta_0 = 343.427° \tag{f}$$

Use numerical integration to plot the variation of these orbital parameters over the next three years ($t_f = 1095$ days) due just to solar radiation pressure. Assume that the radiation pressure coefficient is 2.

Solution

We write the system of differential Eqn (12.84) as $d\mathbf{y}/dt = \mathbf{f}(\mathbf{y}, t)$, where the components of $\mathbf{y}$ are the orbital elements

$$\mathbf{y} = \lfloor h \quad e \quad \theta \quad \Omega \quad i \quad \omega \rfloor \tag{g}$$

and

$$\mathbf{f} = \lfloor \dot{h} \quad \dot{e} \quad \dot{\theta} \quad \dot{\Omega} \quad \dot{i} \quad \dot{\omega} \rfloor \tag{h}$$

The osculating element rates in $\mathbf{f}$ are given by Eqn (12.106) with $p_{SR} = \nu(S/c)C_R(A_s/m)$.

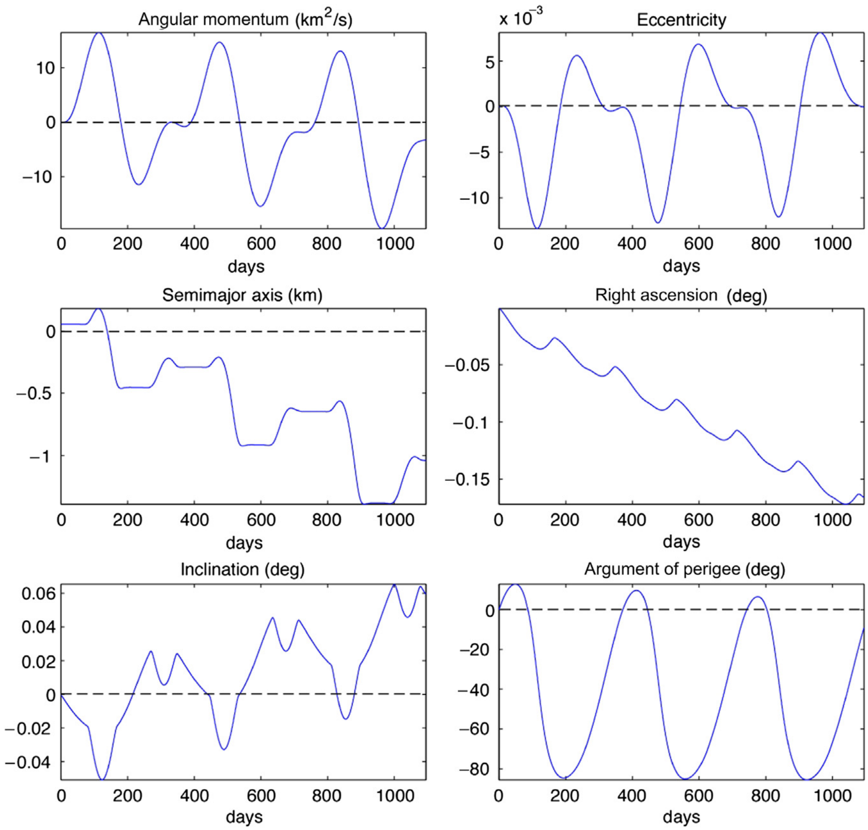

FIGURE 12.16

Solar radiation perturbations of h, e, a, Ω, i, and ω during a 3-year interval following Julian date 2,438,400.5 (January 1, 1964).

The initial conditions vector $\mathbf{y}_0$ comprises Eqns (a)–(f). We can solve the system $d\mathbf{y}/dt = \mathbf{f}(\mathbf{y}, t)$ for $\mathbf{y}$ on the interval $[t_0, t_f]$ by using a numerical integrator such as MATLAB's *ode45*. At each time step, the integrator relies upon a subroutine to compute the rates $\mathbf{f}$ from the current values of $\mathbf{y}$ and the time t, as follows:

Update the Julian day number: $\text{JD} \leftarrow \text{JD}_0 + t(\text{days})$.

Use $\mathbf{y}$ to compute the position vector $\mathbf{r}$ of the satellite by means of Algorithm 4.5.

Compute the magnitude of $\mathbf{r}$ ($r = \|\mathbf{r}\|$).

Using JD, compute the sun's apparent ecliptic longitude λ, the obliquity of the ecliptic ε, and the geocentric equatorial position vector $\mathbf{r}_S$ of the sun by means of Algorithm 12.2.

Compute the components u_r, u_s, and u_w of the earth-to-sun unit vector from Eqn (12.105).

Calculate the value of the shadow function ν from $\mathbf{r}$ and $\mathbf{r}_S$ using Algorithm 12.3.

Compute the solar radiation perturbation $p_{SR} = \nu(S/c)C_R(A_s/m)$.

Calculate the components of $\mathbf{f}$ in Eqn (h).

Plots of the orbital parameter variations $h(t) - h_0$, $e(t) - e_0$, etc., are shown in Figure 12.16. The MATLAB M-file *Example_12_09.m* is listed in Appendix D.47.

12.10 Lunar gravity

Figure C.1 of Appendix C shows a three-body system comprising masses m_1, m_2, and m_3. The position vectors of the three masses relative to the origin of an inertial XYZ frame are $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$, respectively. Eqn (C.2) give the absolute accelerations $\mathbf{a}_1 = d^2\mathbf{R}_1/dt^2$, $\mathbf{a}_2 = d^2\mathbf{R}_2/dt^2$, and $\mathbf{a}_3 = d^2\mathbf{R}_3/dt^2$ of the three masses due to the mutual gravitational attraction among them. The acceleration $\mathbf{a}_{2/1}$ of body 2 relative to body 1 is $\mathbf{a}_2 - \mathbf{a}_1$. Therefore, from Eqns (C.2a) and (C.2b), we obtain

$$\mathbf{a}_{2/1} = \overbrace{\left(Gm_1\frac{\mathbf{R}_1 - \mathbf{R}_2}{\|\mathbf{R}_1 - \mathbf{R}_2\|^3} + Gm_3\frac{\mathbf{R}_3 - \mathbf{R}_2}{\|\mathbf{R}_3 - \mathbf{R}_2\|^3}\right)}^{\mathbf{a}_2} - \overbrace{\left(Gm_2\frac{\mathbf{R}_2 - \mathbf{R}_1}{\|\mathbf{R}_2 - \mathbf{R}_1\|^3} + Gm_3\frac{\mathbf{R}_3 - \mathbf{R}_1}{\|\mathbf{R}_3 - \mathbf{R}_1\|^3}\right)}^{\mathbf{a}_1}$$

Rearranging terms yields

$$\mathbf{a}_{2/1} = -\mu\frac{\mathbf{R}_2 - \mathbf{R}_1}{\|\mathbf{R}_2 - \mathbf{R}_1\|^3} + \mu_3\left(\frac{\mathbf{R}_3 - \mathbf{R}_2}{\|\mathbf{R}_3 - \mathbf{R}_2\|^3} - \frac{\mathbf{R}_3 - \mathbf{R}_1}{\|\mathbf{R}_3 - \mathbf{R}_1\|^3}\right) \tag{12.115}$$

where $\mu = G(m_1 + m_2)$ and $\mu_3 = Gm_3$.

Suppose body 1 is the earth, body 2 is an artificial earth satellite (s), and body 3 is the moon (m). Let us simplify the notation so that, as pictured in Figure 12.17,

$\mathbf{r} = \mathbf{R}_2 - \mathbf{R}_1$	Position of the spacecraft relative to the earth
$\ddot{\mathbf{r}} = \mathbf{a}_{2/1}$	Acceleration of the spacecraft relative to the earth
$\mathbf{r}_m = \mathbf{R}_3 - \mathbf{R}_1$	Position of the moon relative to the earth
$\mathbf{r}_{m/s} = \mathbf{R}_3 - \mathbf{R}_2$	Position of the moon relative to the spacecraft

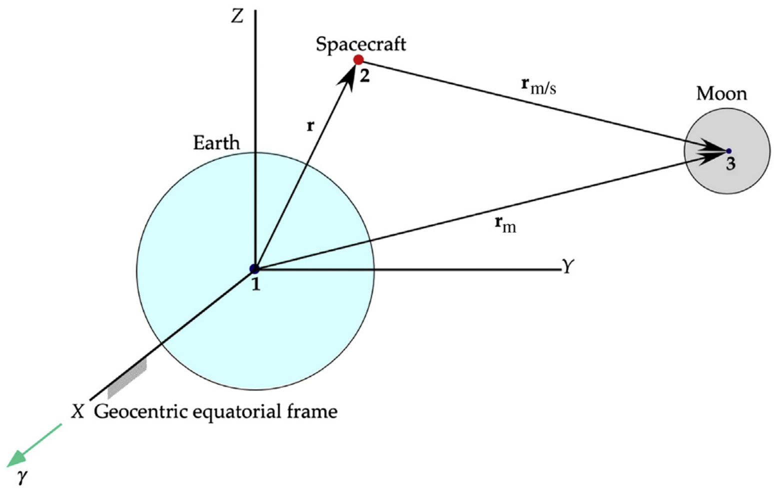

FIGURE 12.17

Perturbation of a spacecraft's earth orbit by a third body (the moon).

Then, Eqn (12.115) becomes

$$\ddot{\mathbf{r}} = -\mu\frac{\mathbf{r}}{r^3} + \mu_m\left(\frac{\mathbf{r}_{m/s}}{{r_{m/s}}^3} - \frac{\mathbf{r}_m}{{r_m}^3}\right) \tag{12.116}$$

where $\mu = \mu_{earth} = 398{,}600\ \text{km}^3/\text{s}^2$ and $\mu_m = \mu_{moon} = 4903\ \text{km}^3/\text{s}^2$. The last term is the perturbing acceleration due to lunar gravity,

$$\mathbf{p} = \mu_m\left(\frac{\mathbf{r}_{m/s}}{{r_{m/s}}^3} - \frac{\mathbf{r}_m}{{r_m}^3}\right) \tag{12.117}$$

If $\mathbf{p} = 0$, then Eqn (12.116) reduces to Eqn (2.22), the fundamental equation of Keplerian motion.

The unit vector $\hat{\mathbf{u}}$ from the center of the earth to that of the moon is given in the geocentric ecliptic frame by an expression similar to Eqn (12.99),

$$\hat{\mathbf{u}} = \cos\delta\cos\lambda\hat{\mathbf{I}}' + \cos\delta\sin\lambda\hat{\mathbf{J}}' + \sin\delta\hat{\mathbf{K}}' \tag{12.118}$$

in which λ is the lunar ecliptic longitude and δ is the lunar ecliptic latitude. If $\delta = 0$, then this expression reduces to that for the sun, which does not leave the ecliptic plane. The components of $\hat{\mathbf{u}}$ in the geocentric equatorial system are found as in Eqn (12.100),

$$\{\hat{\mathbf{u}}\}_{XYZ} = [\mathbf{R}_1(-\varepsilon)]\{\hat{\mathbf{u}}\}_{X'Y'Z'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{bmatrix} \begin{Bmatrix} \cos\delta\cos\lambda \\ \cos\delta\sin\lambda \\ \sin\delta \end{Bmatrix}$$
$$= \begin{Bmatrix} \cos\delta\cos\lambda \\ \cos\varepsilon\cos\delta\sin\lambda - \sin\varepsilon\sin\delta \\ \sin\varepsilon\cos\delta\sin\lambda + \cos\varepsilon\sin\delta \end{Bmatrix} \tag{12.119}$$

The geocentric equatorial position of the moon is $\mathbf{r}_\mathrm{m} = r_\mathrm{m}\hat{\mathbf{u}}$, so that

$$\mathbf{r}_\mathrm{m} = r_\mathrm{m}\cos\delta\cos\lambda\hat{\mathbf{I}} + r_\mathrm{m}(\cos\varepsilon\cos\delta\sin\lambda - \sin\varepsilon\sin\delta)\hat{\mathbf{J}} + r_\mathrm{m}(\sin\varepsilon\cos\delta\sin\lambda + \cos\varepsilon\sin\delta) \tag{12.120}$$

The distance to the moon may be obtained from the formula

$$r_\mathrm{m} = \frac{R_\mathrm{E}}{\sin HP} \tag{12.121}$$

where R_E is the earth's equatorial radius (6378 km) and HP is the horizontal parallax, defined in Figure 12.18.

The formulas presented in *The Astronomical Almanac* (National Almanac Office, 2013) for the time variation of lunar ecliptic longitude λ, lunar ecliptic latitude δ, and lunar horizontal parallax HP are

$$\lambda = b_0 + c_0T_0 + \sum_{i=1}^{6} a_i\sin(b_i + c_iT_0) \quad (0^\circ \le \lambda < 360^\circ) \tag{12.122}$$

$$\delta = \sum_{i=1}^{4} d_i\sin(e_i + f_iT_0) \quad (0^\circ \le \lambda < 360^\circ) \tag{12.123}$$

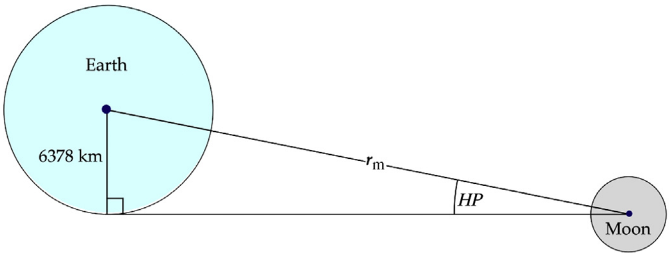

FIGURE 12.18

Horizontal parallax *HP*.

$$\mathrm{HP} = g_0 + \sum_{i=1}^{4} g_i \cos(h_i + k_i T_0) \quad (0^\circ \le \lambda < 180^\circ) \tag{12.124}$$

where T_0 is the number of Julian centuries since J2000 for the current Julian day JD,

$$T_0 = \frac{\mathrm{JD} - 2{,}451{,}545.0}{36{,}525} \tag{12.125}$$

The coefficients in these formulas are listed in Table 12.1.

Recall from the previous section that for the apparent motion of the sun around the earth the *Almanac* uses n, the number of days since J2000, for the time variable instead of the number of centuries T_0 that is employed for the moon's motion. It is obvious from Eqns (12.110) and (12.125) that the relation between n and T_0 is simply

$$n = 36{,}525 T_0 \tag{12.126}$$

In terms of T_0, the formula for obliquity of the ecliptic (Eqn (12.111)) is

$$\varepsilon = 23.439^\circ - 0.0130042 T_0 \tag{12.127}$$

ALGORITHM 12.4

Given the year, month, day, and universal time, calculate the geocentric position vector of the moon.

1. Compute the Julian day number JD using Eqns (5.47) and (5.48).
2. Calculate T_0, the number of Julian centuries since J2000, using Eqn (12.125).
3. Calculate the obliquity ε using Eqn (12.127).
4. Calculate the lunar ecliptic longitude λ by means of Eqn (12.122).
5. Calculate the lunar ecliptic latitude δ using Eqn (12.123).
6. Calculate the lunar horizontal parallax HP by means of Eqn (12.124).
7. Calculate the distance r_m from the earth to the moon using Eqn (12.121).
8. Compute the geocentric equatorial position $\mathbf{r}_m$ of the moon from Eqn (12.120).

EXAMPLE 12.10

Use Algorithm 12.4 to find the geocentric equatorial position vector of the moon at 0800 UT on July 25, 2013. This is the same epoch as used to compute the sun's apparent position in Example 12.7.

Solution

Step 1:
According to Step 1 of Example 12.7, the Julian day number is

$$\mathrm{JD} = 2{,}456{,}498.8333 \text{ days}$$

Step 2:

$$T_0 = \frac{2{,}456{,}498.8333 - 2{,}451{,}545.0}{36{,}525} = 0.135629 \text{ Cy}$$

Table 12.1 Coefficients for Computing Lunar Position

	Longitude, λ			Latitude, δ			Horizontal Parallax, *HP*		
i	a_i	b_i	c_i	d_i	e_i	f_i	g_i	h_i	k_i
0	–	218.32	481,267.881	–	–	–	0.9508	–	–
1	6.29	135.0	477,198.87	5.13	93.3	483,202.03	0.0518	135.0	477,198.87
2	−1.27	259.3	−413,335.36	0.28	220.2	960,400.89	0.0095	259.3	−413,335.38
3	0.66	235.7	890,534.22	−0.28	318.3	6003.15	0.0078	253.7	890,534.22
4	0.21	269.9	954,397.74	−0.17	217.6	−407,332.21	0.0028	269.9	954,397.70
5	−0.19	357.5	35,999.05	–	–	–	–	–	–
6	−0.11	106.5	966,404.03	–	–	–	–	–	–

Step 3:

$$\varepsilon = 23.439^\circ - 0.0130042(0.135629) = 23.4375^\circ$$

Step 4:

$$\lambda = b_0 + c_0(0.135629) + \sum_{i=1}^{6} a_i \sin[b_i + c_i(0.135629)] = 338.155^\circ$$

Step 5:

$$\delta = \sum_{i=1}^{4} d_i \sin[e_i + f_i(0.135629)] = 4.55400^\circ$$

Step 6:

$$HP = g_0 + \sum_{i=1}^{4} g_i \cos[h_i + k_i(0.135629)] = 0.991730^\circ$$

Step 7:

$$r_m = \frac{6378}{\sin(0.991730^\circ)} = 368{,}498 \text{ km}$$

Step 8:

$$\begin{aligned}\mathbf{r}_m = (368{,}498)\Big\{&(\cos 4.55400^\circ)(\cos 338.155^\circ)\hat{\mathbf{I}}\\ &+ [(\cos 23.4375^\circ)(\cos 4.55400^\circ)(\sin 338.155^\circ) - (\sin 23.4375^\circ)(\sin 4.55400^\circ)]\hat{\mathbf{J}}\\ &+ [(\sin 23.4375^\circ)(\cos 4.55400^\circ)(\sin 338.155^\circ) + (\cos 23.4375^\circ)(\sin 4.55400^\circ)]\hat{\mathbf{K}}\Big\}\end{aligned}$$

$$\boxed{\mathbf{r}_m = 340{,}958\hat{\mathbf{I}} - 137{,}043\hat{\mathbf{J}} - 27{,}521.3\hat{\mathbf{K}} \text{ (km)}}$$

EXAMPLE 12.11

The orbital parameters of three earth satellites at time t_0 (Julian date $JD_0 = 2{,}454{,}283.0$) are as Ω follows. For each orbit, find the variation of Ω, ω, and i over the following 60 days due to lunar gravity.

Low earth orbit (LEO)	**Highly elliptic earth orbit (HEO)**	**Geostationary earth orbit (GEO)**
$h_0 = 51{,}591.1 \text{ km}^2/\text{s}$	$h_0 = 69{,}084.1 \text{ km}^2/s$	$h_0 = 129{,}640 \text{ km}^2/s$ (a)
$e_0 = 0.01$	$e_0 = 0.741$	$e_0 = 0.0001$ (b)
$\Omega_0 = 0^\circ$	$\Omega_0 = 0^\circ$	$\Omega_0 = 0^\circ$ (c)
$i_0 = 28.5^\circ$	$i_0 = 63.4^\circ$	$i_0 = 1^\circ$ (d)
$\omega_0 = 0^\circ$	$\omega_0 = 270^\circ$	$\omega_0 = 0^\circ$ (e)
$\theta_0 = 0^\circ$	$\theta_0 = 0^\circ$	$\theta_0 = 0^\circ$ (f)
$a_0 = 6678.126$ km	$a_0 = 26{,}553.4$ km	$a_0 = 42{,}164$ km (g)
$T_0 = 1.50866$ h	$T_0 = 11.9616$ h	$T_0 = 23.9343$ h (h)

Solution

For each orbit in turn we write the system of differential Eqn (12.84) as $d\mathbf{y}/dt = \mathbf{f}(\mathbf{y}, t)$, where the six components of $\mathbf{y}$ are the orbital elements

$$\mathbf{y} = \lfloor h \quad e \quad \theta \quad \Omega \quad i \quad \omega \rfloor^T \tag{i}$$

and

$$\mathbf{f} = \lfloor \dot{h} \quad \dot{e} \quad \dot{\theta} \quad \dot{\Omega} \quad \dot{i} \quad \dot{\omega} \rfloor^T \tag{j}$$

where the six time rates are given by the Gauss planetary equations (Eqn (12.84)).

The initial conditions vector $\mathbf{y}_0$ for each of the three orbits consists of Eqns (a) through (f). The system $dy/dt = \mathbf{f}(\mathbf{y}, t)$ is solved for $\mathbf{y}$ on the interval $[t_0, t_f]$ using a numerical integrator such as MATLAB's *ode45*. At each time step, the integrator relies upon a subroutine to compute the rates $\mathbf{f}$ from the current values of $\mathbf{y}$ and the time t, as follows:

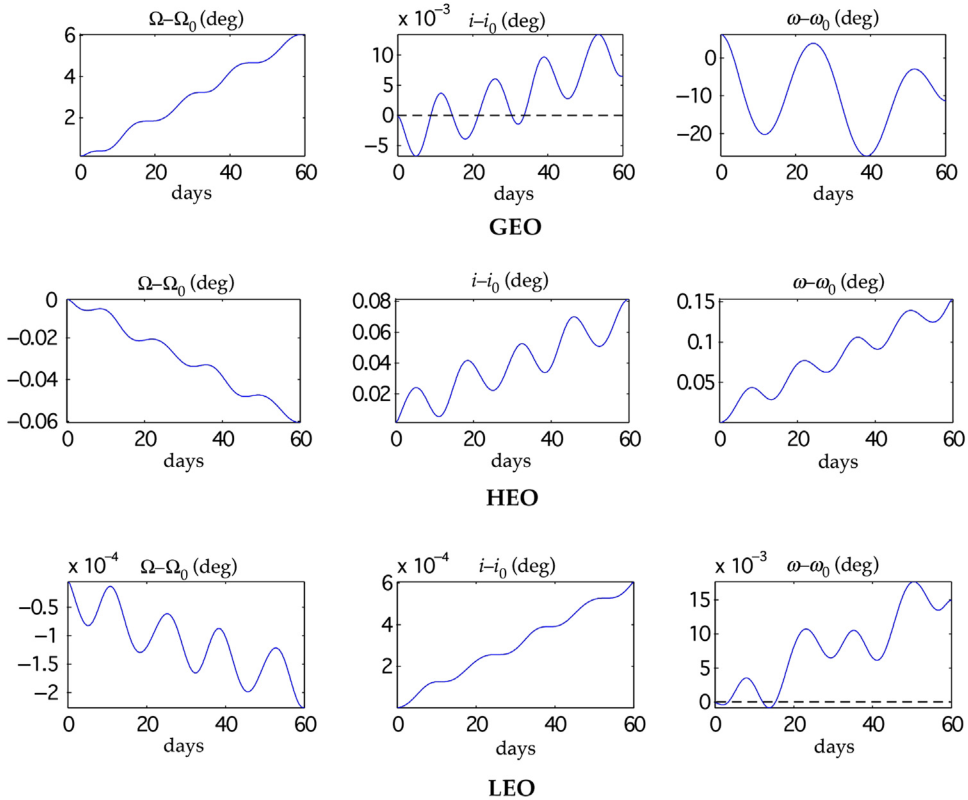

FIGURE 12.19

Lunar gravity perturbations of Ω, i, and ω of the three orbits during a 60-day interval following Julian date 2,454,283.0 (July 1, 2007).

Update the Julian day number: JD ← $JD_0 + t$(days).
Using JD, compute the geocentric equatorial position vector $\mathbf{r}_m$ of the moon by means of Algorithm 12.4.
Use **y** to compute the state vector (**r**,**v**) of the satellite by means of Algorithm 4.5.
Compute the position of the moon relative to the satellite: $\mathbf{r}_{m/s} = \mathbf{r}_m - \mathbf{r}$.

Compute the perturbing acceleration of the moon: $\mathbf{p} = \mu_m\left(\dfrac{\mathbf{r}_{m/s}}{r^3_{m/s}} - \dfrac{\mathbf{r}_m}{r^3_m}\right)$

Compute the unit vectors of the *rsw* frame (Figure 12.10):

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{\|\mathbf{r}\|} \quad \hat{\mathbf{w}} = \frac{\mathbf{r}\times\mathbf{v}}{\|\mathbf{r}\times\mathbf{v}\|} \quad \hat{\mathbf{s}} = \frac{\hat{\mathbf{w}}\times\hat{\mathbf{r}}}{\|\hat{\mathbf{w}}\times\hat{\mathbf{r}}\|}$$

Compute the components of the perturbing acceleration along the *rsw* axes:

$$p_r = \mathbf{p}\cdot\hat{\mathbf{r}} \quad p_s = \mathbf{p}\cdot\hat{\mathbf{s}} \quad p_w = \mathbf{p}\cdot\hat{\mathbf{w}}$$

Use Eqn (12.84) to calculate the components of **f** in Eqn (j).

Plots of the orbital parameter variations $\delta\Omega = \Omega(t) - \Omega_0$, $\delta i = i(t) - i_0$, and $\delta\omega = \omega(t) - \omega_0$ are shown in Figures 12.19 and 12.20. The MATLAB M-file *Example_12_11.m* is listed in Appendix D.49.

12.11 Solar gravity

The special perturbations approach to assessing the sun's influence on the orbits of earth satellites proceeds as it did for the moon in the previous section. The sun replaces the moon as the third body. The perturbing acceleration of the sun may be inferred from that of the moon in Eqn (12.117),

$$\mathbf{p} = \mu_\odot\left(\frac{\mathbf{r}_{\odot/s}}{r^3_{\odot/s}} - \frac{\mathbf{r}_\odot}{r^3_\odot}\right) \tag{12.128}$$

where it is convenient here to use the subscript $\odot$, the astronomical symbol for the *sun*, in the place of S in order to avoid confusion with subscript *s*, which we have been using for the *satellite*. According to Table A.2, the gravitational parameter $\mu_\odot$ of the sun is 132.712(10^9) km^3/s^2. The sun's geocentric position vector $\mathbf{r}_\odot$ in its apparent motion around the earth is found by using Algorithm 12.2, as we did in our study of solar radiation pressure effects.

Using the fact that $\mathbf{r}_\odot = \mathbf{r} + \mathbf{r}_{\odot/s}$, we may rewrite Eqn (12.128) as

$$\mathbf{p} = \frac{\mu_\odot}{r_{\odot/s}{}^3}\left[\mathbf{r}_\odot\left(1 - \frac{r_{\odot/s}{}^3}{r_\odot{}^3}\right) - \mathbf{r}\right] \tag{12.129}$$

Because the sun is so far from the earth, the ratio $r_{\odot/s}{}^3/r_\odot{}^3$ is very nearly 1. Therefore, evaluating Eqn (12.129) involves subtracting two nearly equal numbers, which should be avoided in a digital computer. We do so by referring to Appendix F to write Eqn (12.129) as

$$\mathbf{p} = \frac{\mu_\odot}{r_{\odot/s}{}^3}\left[F(q)\mathbf{r}_\odot - \mathbf{r}\right] \tag{12.130}$$

where, according to Eqn (F.4),

$$q = \frac{\mathbf{r}\cdot(2\mathbf{r}_\odot - \mathbf{r})}{r_\odot{}^2} \tag{12.131}$$

and $F(q)$ is given by Eqn (F.3)

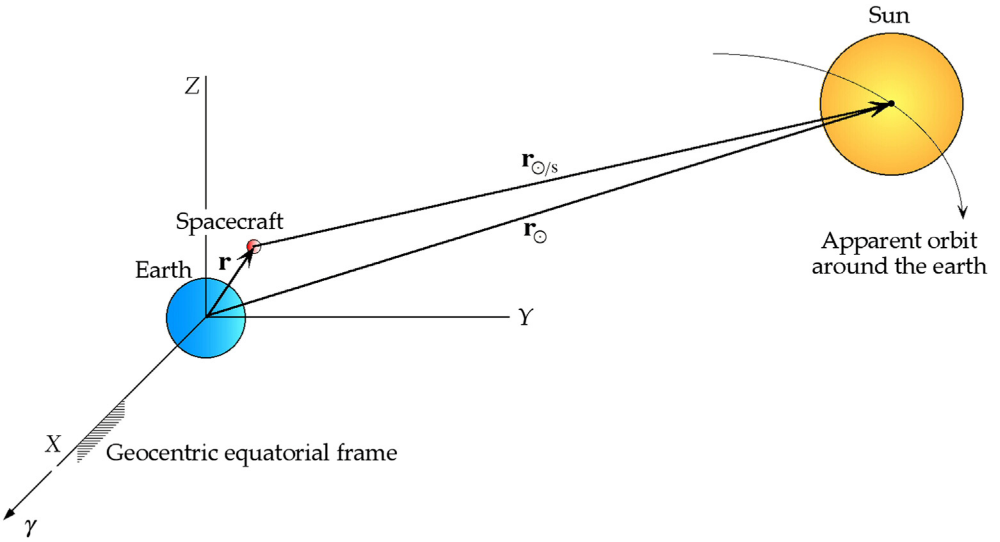

FIGURE 12.20

Perturbation of a spacecraft's earth orbit by solar gravity.

EXAMPLE 12.12

For the three orbits of Example 12.11, find the variation of the node angle Ω, argument of perigee ω, and inclination i due to solar gravity for a period of 720 days following the given initial conditions.

Solution

We will use MATLAB's function o*de45* to numerically integrate the system d**y**/dt = **f**(**y**,t) on the interval [t_0,t_f], where t_0 is Julian Day 2,454,283.0 and t_f is $t_0 + 720$ days. **y** is the six component vector of orbital elements,

$$\mathbf{y} = \lfloor h \quad e \quad \theta \quad \Omega \quad i \quad \omega \rfloor^T \tag{a}$$

The vector **f**,

$$\mathbf{f} = \lfloor \dot{h} \quad \dot{e} \quad \dot{\theta} \quad \dot{\Omega} \quad \dot{i} \quad \dot{\omega} \rfloor^T \tag{b}$$

contains the time derivatives of the orbital elements as given by the Gauss planetary equations (Eqn (12.84)). The initial conditions vector $\mathbf{y}_0$ for each of the three orbits consists of Eqns (a) through (f) of Example 12.11.

At each time step, a numerical integrator like *ode45* relies upon a subroutine to compute the rates **f** of the osculating elements from their current **y** and the time t, as follows:

Update the Julian day number: $\text{JD} \leftarrow \text{JD}_0 + t(\text{days})$.
Using JD, compute the geocentric equatorial position vector $\mathbf{r}_\odot$ of the sun by means of Algorithm 12.2.
Use **y** to compute the state vector (**r**,**v**) of the satellite by means of Algorithm 4.5.
Compute the position vector of the sun relative to the satellite: $\mathbf{r}_{\odot/s} = \mathbf{r}_\odot - \mathbf{r}$
Compute q from Eqn (12.131): $q = \mathbf{r}\cdot(2\mathbf{r}_\odot - \mathbf{r})/r_\odot^2$
Compute $F(q)$ from Appendix F.3: $F(q) = q(q^2 - 3q + 3)/\left[1 + (1 - q)^{\frac{3}{2}}\right]$

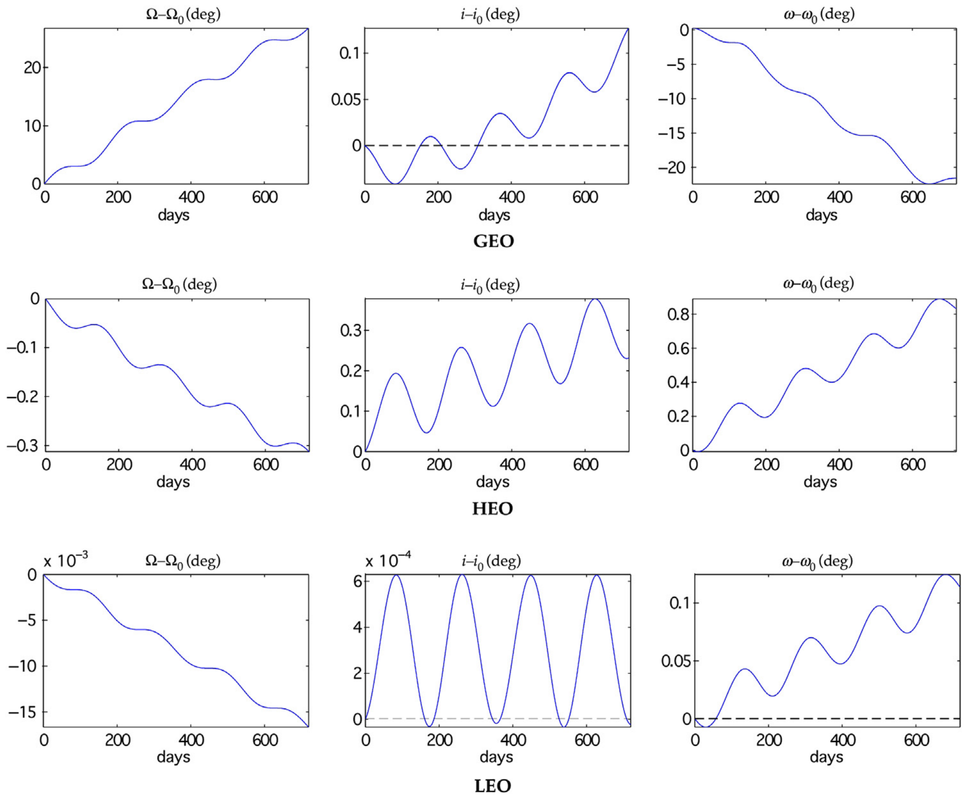

FIGURE 12.21

Solar gravity perturbations of Ω, i, and ω of the three orbits in Example 12.11 during a 720-day interval following Julian date 2,454,283.0 (July 1, 2007).

Compute the perturbing acceleration of the sun: $\mathbf{p} = \mu_\odot[F(q)\mathbf{r}_\odot - \mathbf{r}]/r^3_{\odot/s}$
Compute the unit vectors of the *rsw* frame (Figure 12.10):

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{\|\mathbf{r}\|} \quad \hat{\mathbf{w}} = \frac{\mathbf{r}\times\mathbf{v}}{\|\mathbf{r}\times\mathbf{v}\|} \quad \hat{\mathbf{s}} = \frac{\hat{\mathbf{w}}\times\hat{\mathbf{r}}}{\|\hat{\mathbf{w}}\times\hat{\mathbf{r}}\|}$$

Compute the components of the perturbing acceleration along the *rsw* axes:

$$p_r = \mathbf{p}\cdot\hat{\mathbf{r}} \quad p_s = \mathbf{p}\cdot\hat{\mathbf{s}} \quad p_w = \mathbf{p}\cdot\hat{\mathbf{w}}$$

Use Eqn (12.84) to calculate the components of **f**.

Plots of the orbital parameter variations $\delta\Omega = \Omega(t) - \Omega_0$, $\delta i = i(t) - i_0$, and $\delta\omega = \omega(t) - \omega_0$ are shown in Figure 12.21. The MATLAB script file *Example_12_12.m* is listed in Appendix D.50.

PROBLEMS

Section 12.2

12.1 In Figure 12.1, the radius at perigee O is 7000 km and the speed is $v_0 = \sqrt{\mu/r_0}$. The initial mass of the spacecraft is $m_0 = 2000$ kg and the thrust of the propulsion system is 0.5 kN. Using Cowell's method and *ode45*, find the perigee and the eccentricity of the osculating orbits at the following times after t_0: (a) 1 h, (b) 1.2 h, (c) 1.4 h, (d) 1.6 h.
{Ans.: (a) 0.1856, 7903 km; (b) 0.2046, 8450 km; (c) 0.2272, 9123 km; (d) 0.2825, 9895 km}

Section 12.3

12.2 Repeat Problem 12.1 using *ode45* and Encke's method.

Section 12.4

12.3 Solve Example 12.1 using Encke's method.

Section 12.5

12.4 Find the zeros of each of the Legendre polynomials in Figure 12.7.
12.5 Use Rodrigues' formula to calculate Legendre polynomials $P_8(x)$ and $P_9(x)$.
12.6 Plot and find the zeros of the Legendre polynomials found in Problem 12.5.
12.7 Verify that the third zonal harmonic of the perturbing gravitational potential is

$$\Phi = \frac{J_3}{2}\frac{\mu}{r}\left(\frac{R}{r}\right)^3\left(5\cos^3\phi - 3\cos\phi\right)$$

12.8 Show that the perturbing acceleration $\mathbf{p} = -\nabla\Phi$ due to the J_3 zonal harmonic is

$$\mathbf{p} = \frac{1}{2}\frac{J_3\mu R^3}{r^5}\left\{5\frac{x}{r}\left[7\left(\frac{z}{r}\right)^3 - 3\frac{z}{r}\right]\hat{\mathbf{i}} + 5\frac{y}{r}\left[7\left(\frac{z}{r}\right)^3 - 3\frac{z}{r}\right]\hat{\mathbf{j}} + 3\left[\frac{35}{3}\left(\frac{z}{r}\right)^4 - 10\left(\frac{z}{r}\right)^2 + 1\right]\hat{\mathbf{k}}\right\}$$

where $z/r = \cos\phi$.

12.9 For the orbit of Example 12.2, use Cowell's method to determine the J_3 effect on the orbital parameters Ω, ω, h, e, and i for 48 h after the initial epoch.

Section 12.6

12.10 Use the method of variation of parameters to solve Eqn (1.113).

12.11 Use the method of variation of parameters to solve the differential equation $\ddot{x} + 2\dot{x} + 2x = t\sin t$.
{Ans.: $x = a_1 e^{-t}\sin t + a_2 e^{-t}\cos t + (-2t/5 + 14/25)\cos t + (t/5 - 2/25)\sin t$, where a_1 and a_2 are constants of integration that are determined from the initial conditions}

12.12 Show that the variation of parameters solution of the differential equation $\ddot{x} + a_1(t)\dot{x} + a_2(t)x = f(t)$ is $x = u_1(t)x_1(t) + u_2(t)x_2(t)$, where

$$u_1 = \int \frac{x_2 f(t)}{x_2\dot{x}_1 - \dot{x}_2 x_1}\mathrm{d}t + C_1 \quad u_2 = \int \frac{x_1 f(t)}{x_1\dot{x}_2 - \dot{x}_1 x_2}\mathrm{d}t + C_2$$

in which x_1 and x_2 are solutions of the reduced homogeneous equation $\ddot{x} + a_1(t)\dot{x} + a_2(t)x = 0$ and C_1 and C_2 are constants.

12.13 Using the definition of the dot product of two vectors ($\mathbf{b}\cdot\mathbf{c} = \sum_{i=1}^{3} b_i c_i$) it is easy to see from Eqn (12.48) that the 36 components of the Lagrange matrix, called *Lagrange brackets*, may be written

$$L_{\alpha\beta} = \frac{\partial\mathbf{r}}{\partial u_\alpha}\cdot\frac{\partial\mathbf{v}}{\partial u_\beta} - \frac{\partial\mathbf{v}}{\partial u_\alpha}\cdot\frac{\partial\mathbf{r}}{\partial u_\beta} \quad \alpha, \beta = 1, \ldots, 6$$

Show that $L_{\alpha\beta} = -L_{\beta\alpha}$ and that $L_{\alpha\beta} = 0$ when $\alpha = \beta$, so that there are only 15 independent Lagrange brackets.

12.14 Using the facts that $\partial\mathbf{r}/\partial t = \mathbf{v}$ and $\partial\mathbf{v}/\partial t = \mathbf{a}$, where $\mathbf{v}$ is the velocity and $\mathbf{a}$ is the acceleration, show that the time derivative of $L_{\alpha\beta}$ is

$$\frac{\partial L_{\alpha\beta}}{\partial t} = \frac{\partial\mathbf{a}}{\partial u_\beta}\cdot\frac{\partial\mathbf{r}}{\partial u_\alpha} - \frac{\partial\mathbf{a}}{\partial u_\alpha}\cdot\frac{\partial\mathbf{r}}{\partial u_\beta}$$

12.15 If $\mathbf{a}$ is the two-body relative acceleration vector, $\mathbf{a} = -\mu\mathbf{r}/r^3$, show that

$$\frac{\partial\mathbf{a}}{\partial u_\alpha} = -\frac{\mu}{r^3}\left(\frac{\partial\mathbf{r}}{\partial u_\alpha} - 3\frac{\mathbf{r}}{r}\frac{\partial r}{\partial u_\alpha}\right)$$

where u_α ($\alpha = 1,\ldots,6$) are a set of osculating orbital elements.

12.16 Use the formula for $\partial\mathbf{a}/\partial u_\alpha$ in Problem 12.15 plus the fact that $\mathbf{r}\cdot(\partial\mathbf{r}/\partial u_\beta) = r(\partial r/\partial u_\beta)$ to verify that

$$\frac{\partial\mathbf{a}}{\partial u_\alpha}\cdot\frac{\partial\mathbf{r}}{\partial u_\beta} = -\frac{\mu}{r^3}\left(\frac{\partial\mathbf{r}}{\partial u_\alpha}\cdot\frac{\partial\mathbf{r}}{\partial u_\beta} - 3\frac{\partial r}{\partial u_\alpha}\frac{\partial r}{\partial u_\beta}\right)$$

12.17 Combine the results of Problems 12.14 and 12.16 to show that

$$\frac{\partial L_{\alpha\beta}}{\partial t} = 0 \quad \alpha, \beta = 1, \ldots, 6$$

That is, on a given osculating orbit (i.e., for a given set of orbital elements u_α), the Lagrangian brackets are constant. This means that [**L**] may be computed at a point where the orbit formulas have their simplest algebraic form, which usually is at periapsis (as it was for obtaining the specific energy formula in Section 2.6).

12.18 For the orbital elements

$$u_1 = h \quad u_2 = e \quad u_3 = \theta \quad u_4 = \Omega \quad u_5 = i \quad u_6 = \omega$$

it can be shown that the Lagrange matrix is

$$[\mathbf{L}] = \begin{bmatrix} 0 & 0 & -\dfrac{1-e}{1+e} & -\cos i & 0 & -1 \\ 0 & 0 & -\dfrac{eh}{(1+e)^2} & 0 & 0 & 0 \\ \dfrac{1-e}{1+e} & \dfrac{eh}{(1+e)^2} & 0 & 0 & 0 & 0 \\ \cos i & 0 & 0 & 0 & -h\sin i & 0 \\ 0 & 0 & 0 & h\sin i & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solve Eqn (12.47), where

$$\{\mathbf{P}\} = \lfloor \partial R/\partial h \quad \partial R/\partial e \quad \partial R/\partial \theta \quad \partial R/\partial \Omega \quad \partial R/\partial \omega \quad \partial R/\partial i \rfloor^T$$

for the element rates $\dot{h}$, $\dot{e}$, $\dot{\theta}$, $\dot{\Omega}$, $\dot{\omega}$, and $\dot{i}$.

12.19 For the orbital elements

$$u_1 = \Omega \quad u_2 = i \quad u_3 = \omega \quad u_4 = a \quad u_5 = e \quad u_6 = t_p$$

where t_p is the time of periapse passage, it can be shown that the Lagrange matrix is

$$[\mathbf{L}] = \begin{bmatrix} 0 & -nab\sin i & 0 & \dfrac{1}{2}nb\cos i & -\dfrac{na^3e}{b}\cos i & 0 \\ nab\sin i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1}{2}nb & -\dfrac{na^3e}{b} & 0 \\ -\dfrac{1}{2}nb\cos i & 0 & -\dfrac{1}{2}nb & 0 & 0 & \dfrac{1}{2}n^2a \\ \dfrac{na^3e}{b}\cos i & 0 & \dfrac{na^3e}{b} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{1}{2}n^2a & 0 & 0 \end{bmatrix}$$

in which a and b are the semimajor and semiminor axes and n is the mean motion. Solve Eqn (12.47) with

$$\{\mathbf{P}\} = \lfloor \partial R/\partial \Omega \quad \partial R/\partial i \quad \partial R/\partial \omega \quad \partial R/\partial a \quad \partial R/\partial e \quad \partial R/\partial t_{\mathrm{p}} \rfloor^T$$

to obtain the Lagrange planetary equations listed in Eqn (12.50).

Section 12.7

12.20 Show that $\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{0}$ implies that $\dfrac{\mathrm{d}r}{\mathrm{d}t} = 0$.

12.21 Verify that for unperturbed two-body motion, $\mathrm{d}h/\mathrm{d}t = h/r^2$.

12.22 Verify that for unperturbed two-body motion, $\mathrm{d}M/\mathrm{d}t = n$.

12.23 Verify that for unperturbed two-body motion, $\mathrm{d}E/\mathrm{d}t = na/r$.

12.24 Show that $\dfrac{\mathrm{d}a}{\mathrm{d}t} = \dfrac{2a^2}{h}\left(e \sin\theta p_r + \dfrac{h^2}{\mu r} p_s\right)$.

12.25 Show that there is no J_2 perturbation of the semimajor axis ($\mathrm{d}a/\mathrm{d}t = 0$).

12.26 Show that $\partial v^2/\partial \mathbf{v} = 2\mathbf{v}$.

12.27 Show that $\partial(\mathbf{r}\cdot\mathbf{v})/\partial\mathbf{v} = \mathbf{r}$.

12.28 Show that $\partial h/\partial \mathbf{v} = (\mathbf{h} \times \mathbf{r})/h$.

12.29 Show that $\partial h^2/\partial \mathbf{v} = 2\mathbf{h} \times \mathbf{r}$.

12.30 Find the Gauss variational equation for $\mathrm{d}r_{\mathrm{p}}/\mathrm{d}t$.

12.31 Find the Gauss variational equation for $\mathrm{d}r_{\mathrm{a}}/\mathrm{d}t$.

12.32 Find the Gauss variational equations for a radial acceleration perturbation, $\mathbf{p} = p_r\hat{\mathbf{r}}$.

12.33 Numerically integrate the Gauss planetary equations for a given perturbation $\mathbf{p}$ and set of initial conditions.

12.34 Find the p_r, p_s, and p_w components in the rsw frame of the J_3 gravitational perturbation $\mathbf{p}$ found in Problem 12.8.

12.35 Find the expression for $\mathrm{d}h/\mathrm{d}t$ due to J_3.

12.36 Find the expression for $\mathrm{d}e/\mathrm{d}t$ due to J_3.

12.37 Find the expression for $\mathrm{d}\Omega/\mathrm{d}t$ due to J_3.

12.38 Find the expression for $\mathrm{d}i/\mathrm{d}t$ due to J_3.

12.39 Find the expression for $\mathrm{d}\omega/\mathrm{d}t$ due to J_3.

Section 12.8

12.40–12.44 Find the orbital averages of the J_3 rates found in Problems 12.34–12.38.

Section 12.9

12.45 An earth satellite has the following orbital parameters on Julian date 2,456,793:

$$r_p = 10{,}000 \text{ km}$$

$$r_a = 15{,}000 \text{ km}$$

$$\theta = 40^\circ$$

$$\Omega = 300^\circ$$

$$\omega = 110^\circ$$

$$i = 55^\circ$$

Assuming an absorbing area-to-mass ratio of 2 m^2/kg and a radiation pressure coefficient of 2, use Cowell's method and MATLAB's *ode45* (or similar) to plot the variation of these orbital parameters over the next three years due just to solar radiation pressure.

Section 12.10

12.46 Solve Problem 12.44, neglecting solar radiation pressure and including only the perturbing effect of lunar gravity.

12.47 Solve Problem 12.44, retaining the effect of solar radiation pressure and adding that of lunar gravity as well.

12.48 For the orbits of Example 12.11, plot the variations a, e, and θ over the same time interval.

12.49 For the orbits of Example 12.11, plot the variations of r_p and r_a over the same time interval.

12.50 For the data of Example 12.11, use Cowell's method to integrate Eqn (12.2) with lunar gravity as the perturbation and then use Algorithm 4.2 to obtain the time histories of the orbital parameters. Compare the results with Figure 12.19.

Section 12.11

12.51 For the data of Example 12.11, use Encke's method to integrate Eqn (12.2) with lunar gravity as the perturbation and then use Algorithm 4.2 to obtain the time histories of the orbital parameters. Compare the results with Figure 12.19.

12.52 Solve Problem 12.44, neglecting solar radiation pressure and including only the perturbing effect of solar gravity.

12.53 Solve Problem 12.44, retaining the effect of solar radiation pressure and adding that of solar gravity as well.

12.54 Solve Problem 12.44, retaining the effect of solar radiation pressure and adding those of lunar and solar gravity as well.

12.55 Plot the variation of a, e, and è in Example 12.12.

12.56 Plot the variations of r_p and r_a in Example 12.12.

12.57 For the data of Example 12.12, use Cowell's method to integrate Eqn (12.2) with solar gravity as the perturbation and then use Algorithm 4.2 to obtain the time histories of the orbital parameters. Compare the results with Figure 12.21.

12.58 For the data of Example 12.12, use Encke's method to integrate Eqn (12.2) with solar gravity as the perturbation and then use Algorithm 4.2 to obtain the time histories of the orbital parameters. Compare the results with Figure 12.21.