MOE H3 Math   Numbers and Proofs

Problem Set 5 (Miscellaneous)

1. Let a, b and $n > 1$ be integers. Prove that if $m > 1$ is a divisor of n and $a \equiv b \mod n$, then $a \equiv b \mod m$.

2. Prove that, if $3 \nmid a$, then $3 \mid a^2 + 5$.

3. Prove the following statement:
   If m and n are any two integers with the same parity, then $4 \mid m^2 - n^2$.

4. Prove that, for any real number x, either $\sqrt{2} - x$ or $\sqrt{2} + x$ is irrational.

5. Show that:
   For any integer n, if 3 divides $1 + n + n^2$, then $n \equiv 1 \mod 3$.

6. Prove that there is no smallest positive rational number.

7. Use mathematical induction to prove that $n^3 < n!$ for all integers $n \geq 6$.

8. The Fibonacci sequence is defined by

$$f_1 = 1, \quad f_2 = 1, \qquad f_{n+2} = f_{n+1} + f_n \quad \text{for all } n \geq 1.$$

   Prove that $2f_n + 3f_{n+1} = f_{n+4}$ for all $n \in \mathbb{Z}^+$.

9. A sequence is defined by

$$a_1 = 2, \quad a_2 = 4, \qquad a_{n+2} = 5a_{n+1} - 6a_n \quad \text{for all } n \geq 1.$$

   Prove that $a_n = 2^n$ for all $n \in \mathbb{Z}^+$.

10. Let $f_1, f_2, \ldots, f_n, \ldots$ denote the Fibonacci sequence defined by

$$f_1 = 1, \quad f_2 = 1, \quad \text{and} \quad f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 3.$$

    Show that, for all $n \in \mathbb{Z}^+$, $f_2 + f_4 + \cdots + f_{2n} = f_{2n+1} - 1$.

11. Show that if m and $m + 2$ are both primes with $m > 3$, then $m + 1$ is divisible by 6.

12. If $\gcd(a, b) = 1$, show that the possible values for $\gcd(a + b, a - b)$ can be 1 and 2.

13. Show that, for any integers a and b, not both zero, $\gcd(a, b) = ax_0 + by_0$ for some integers x_0 and y_0.

14. Let a, b be integers such that $\gcd(a, b) = 1$. Prove that $\gcd(ab, a + b) = 1$.

15. Show that, if p is an odd prime, then $\gcd(p^2+1,(p+1)^2) = 2$.

16. Let n be a positive integer. Show that, if there are exactly k positive integers that divide n where k is odd, then there exists a positive integer m such that $m^2 = n$.

17. Let $p_1, p_2, \ldots, p_{2n+1}$ be the <u>first</u> $2n+1$ prime numbers (in increasing order). Let

$$M = p_1p_3p_5 \cdots p_{2n-1} + p_2p_4p_6 \cdots p_{2n}.$$

Prove that, if $M < (p_{2n+1})^2$, then M is a prime.

18. Let p be a prime number. Prove that $(p-1)!+1$ is divisible by p.

Hints

1. Direct proof
2. Consider two cases.
3. Consider two cases.
4. Proof by contradiction.
5. Proof by contrapositive.
6. Proof by contradiction.
7. You need to show and use the inequality $(k+1)^2 \leq k^3$ for $k \geq 3$.
8. Use a variation of mathematical induction.
9. Use a variation of mathematical induction.
10. Use mathematical induction.
11. Consider the possible remainders of m and $m+2$.
12. Show that $\gcd(a+b, a-b)$ divides $2a$ and $2b$.
13. Use well ordering principle and consider the set $S = \{x \in \mathbb{Z}^+ \mid x = as + bt \text{ for some integers } s \text{ and } t\}$.
14. Proof by contradiction.
15. Prove that the gcd is divisible by 2, and 2 is divisible by the gcd.
16. Consider all the k divisors of n and try to pair them up in a specific way.
17. Prove by contradiction, and use the fact that every integer greater than 1 has a prime divisor.
18. Show $(p-1)! \equiv -1 \mod p$. Pair up the elements in $\{1, 2, \ldots, p-1\}$ with its inverse modulo p.