**MOE H3 Math    Numbers and Proofs**

**Problem Set 3**

1. Prove that for every pair of irrational numbers $p$ and $q$ such that $p < q$, there is an irrational $x$ such that $p < x < q$.

2. Show that there is one and only one integer $t$ such that $t, t+2, t+4$ are all prime numbers.

3. Show that there exists integers $x$ and $y$ that satisfy

$$(2n+1)x + (9n+4)y = 1$$

for every integer $n$.

4. Given $n$ real numbers $a_1, a_2, \ldots, a_n$. Show that there exists an $a_i$ $(1 \leq i \leq n)$ such that $a_i$ is greater than or equal to the mean (average) value of the $n$ numbers.

5. Determine whether the two statements below are true or false. Justify your answers.

   (a) There is an irrational number $a$ such that for all irrational number $b$, $ab$ is rational.

   (b) For every irrational number $a$, there is an irrational number $b$ such that $ab$ is rational.

6. Prove that there are infinitely many prime numbers that are congruent to 3 modulo 4.

7. Prove the following two statements.

   (a) There exists a rational number $a$ and an irrational number $b$ such that $a^b$ is rational.

   (b) There exists a rational number $a$ and an irrational number $b$ such that $a^b$ is irrational.

8. Prove that, for any positive integer $n$, there is a perfect square $m^2$ ($m$ is an integer) such that $n \leq m^2 \leq 2n$.

9. For every integer $n \geq 1$, show that we can find an odd integer $h$ and a non-negative integer $t$ such that $n = 2^t h$. Are $h$ and $t$ unique for each $n$? Justify your answer.

**Hints:**

1. Consider the average of $p$ and $q$. Proof by construction.
2. Existence part is by construction; uniqueness part: show that one of $t, t+2, t+4$ must be divisible by 3.
3. Use Constructive proof. Your same example for $x$ and $y$ should work for all integer $n$.
4. Prove by contradiction by assuming all the $n$ numbers are less than the average value.
5. If the statement is false, prove its negation. Proof by construction.
6. The idea is similar to the proof for infinitely many primes. Use the fact that integers of the form $4k+1$ are closed under multiplication.
7. One part is easy (don't try too hard!); the other part use the same idea as example 9 in the lecture notes.
8. Prove by contradiction by assuming there exists a positive integer $n$ such that for all perfect squares $m^2$, we either have $m^2 < n$ or $m^2 > 2n$. With this assumption, you may choose the largest perfect square $m_0^2 < n$ such that $(m_0 + 1)^2 > 2n$.
9. Use method of infinite descent similar to example 6 in the lecture notes. For the uniqueness part, write $n = 2^{t_1} h_1$ and $n = 2^{t_2} h_2$. Then show that $t_1 = t_2$ and $h_1 = h_2$.