The Catenary

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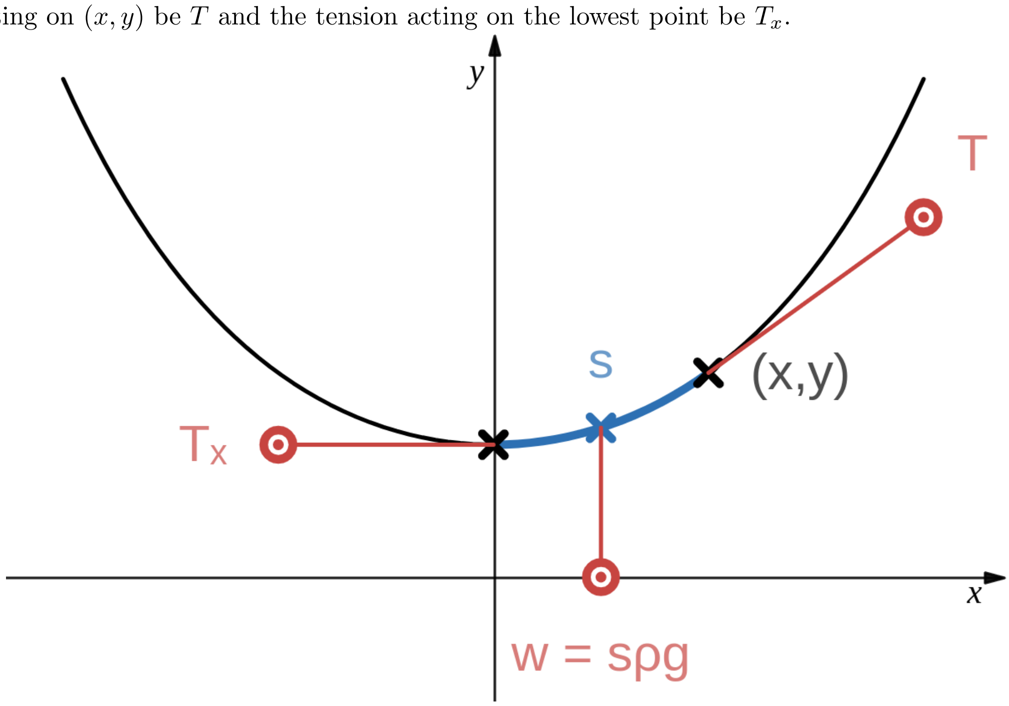
1 Assumptions and Details

- the catenary has a finite length S
- gravity is uniform and downwards with magnitude g
- the catenary has a uniform linear density ρ such that it has total mass $m = S\rho$
- the catenary is suspended at its two end points that are distance $0 < d < S$ apart horizontally and have the same vertical level

2 Derivation

To determine the equation of the catenary $y(x)$, let the point (x, y) be a point on the catenary that is distance s along the cable from the lowest point of the catenary. Also, let the tangent of the catenary at (x, y) have an angle of θ with respect to the horizontal. Further let the tension acting on (x, y) be T and the tension acting on the lowest point be T_x.



By equilibrium, it can be deduced that,

$$\sum F_x = 0 \implies T\cos\theta - T_x = 0 \implies T\cos\theta = T_x \tag{1}$$

$$\sum F_y = 0 \implies T\sin\theta - w = 0 \implies T\sin\theta = s\rho g \tag{2}$$

By dividing the two equations,

$$\tan\theta = \frac{s\rho g}{T_x} \tag{3}$$

Let $\varphi = \frac{\rho g}{T_x}$. Since at (x, y), $\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \varphi s \tag{4}$$

By the Pythagoras' theorem,

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 \tag{5}$$

By dividing $\mathrm{d}x^2$ and taking the square root on both sides of the equation,

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \tag{6}$$

By differentiating the first derivative,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \varphi\frac{\mathrm{d}s}{\mathrm{d}x} \tag{7}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \varphi\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \tag{8}$$

Let $m = \frac{\mathrm{d}y}{\mathrm{d}x}$,

$$\frac{\mathrm{d}m}{\mathrm{d}x} = \varphi\sqrt{1 + m^2} \tag{9}$$

$$\frac{1}{\sqrt{1 + m^2}}\,\mathrm{d}m = \varphi\,\mathrm{d}x \tag{10}$$

By integrating both sides,

$$\sinh^{-1} m = \varphi x + C \tag{11}$$

By intuition, $m(0) = 0$, thus,

$$\sinh^{-1} 0 = \varphi \cdot 0 + C \implies C = 0 \tag{12}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sinh\varphi x \tag{13}$$

By integration,

$$y = \frac{1}{\varphi}\cosh\varphi x + C \tag{14}$$

Let $a = \frac{1}{\varphi}$ and $b = C$.

$$\therefore \boxed{y(x) = b + a\cosh\frac{x}{a}}$$