ANALYSIS
TUTORIAL 4: DIFFERENTIATION

1 Given that $y = x\sin x$, find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and $\dfrac{\mathrm{d}^4y}{\mathrm{d}x^4}$, simplifying your results as far as possible, and show that

$$\frac{\mathrm{d}^6y}{\mathrm{d}x^6} = -x\sin x + 6\cos x$$

Use induction to establish an expression for $\dfrac{\mathrm{d}^{2n}y}{\mathrm{d}x^{2n}}$, where n is a positive integer.

2 The polynomial $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, of degree n, is denoted by $\mathrm{P}_n(x)$.

Show that $-2x\mathrm{P}_n(x) + \dfrac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{P}_n(x)\right]$ is a polynomial of degree $(n+1)$.

Let $\mathrm{f}_n(x) = \mathrm{e}^{x^2}\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\mathrm{e}^{-x^2}\right)$ (A)

(i) Show that $\mathrm{f}_1(x)$ and $\mathrm{f}_2(x)$ are polynomials of degree 1 and 2 respectively.

(ii) Differentiate (A) with respect to x to show that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{f}_n(x)\right] = \mathrm{f}_{n+1}(x) + 2x\,\mathrm{f}_n(x)$.

(iii) Hence prove by induction that, for every positive integer n, $\mathrm{f}_n(x)$ is a polynomial of degree n.

3 Let $\mathrm{f}(x) = \dfrac{x^{\mathrm{e}}}{\mathrm{e}^x}$ $(x > 0)$. Find the maximum value of $\mathrm{f}(x)$ and hence prove that $\mathrm{e}^{\pi} > \pi^{\mathrm{e}}$.

4 Provide a proof for the Product Rule and Chain Rule by First Principles.

5 Show that $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\dfrac{\mathrm{d}^2x}{\mathrm{d}y^2}\Bigg/\left(\dfrac{\mathrm{d}x}{\mathrm{d}y}\right)^3$, stating precise conditions under which it holds.

6 Let

$$\mathrm{f}(x)=\begin{cases} x\sin\left(\dfrac{1}{x}\right) & \text{if } x\neq 0,\\ 0 & x=0.\end{cases}$$

Show that f is continuous at $x=0$ but f does not have a derivative at $x=0$.
(This shows that a function maybe continuous at a point but need not have a derivative at the point.)

7 Use Mathematical Induction to prove the Leibniz's rule for the n^{th} derivative of a product:

$$(\mathrm{fg})^{(n)}(x)=\sum_{k=0}^{n}\binom{n}{k}\left((\mathrm{f})^{(n-k)}(x)\right)\left((\mathrm{g})^{(k)}(x)\right),$$

where $(\mathrm{f})^{(n)}(x)=\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\mathrm{f}(x)\right)$ and $(\mathrm{fg})^{(n)}(x)=\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\mathrm{f}(x)\mathrm{g}(x)\right)$.

8 By applying the Rolle's Theorem on the function $\mathrm{f}(x)=\mathrm{e}^{-x}-\sin x$, show that there is at least one real root of $\mathrm{e}^x\cos x=-1$ between any two real roots of $\mathrm{e}^x\sin x=1$.

9 By using the Theorem of the Mean, show that

$$\left(\frac{\pi}{6}+\frac{\sqrt{3}}{15}\right)<\sin^{-1}(0.6)<\left(\frac{\pi}{6}+\frac{1}{8}\right).$$

10 **(i)** Given that $0<a<b$, show that $\dfrac{b-a}{1+b^2}<\tan^{-1}(b)-\tan^{-1}(a)<\dfrac{b-a}{1+a^2}$.

(ii) Hence show that $\dfrac{\pi}{4}+\dfrac{3}{25}<\tan^{-1}\left(\dfrac{4}{3}\right)<\dfrac{\pi}{4}+\dfrac{1}{6}$.

11 Let f be a differentiable function on $(0,\infty)$ and suppose that $\lim\limits_{x\to\infty}\left(\mathrm{f}(x)+\mathrm{f}'(x)\right)=L$.

By considering $\mathrm{f}(x)=\dfrac{\mathrm{e}^x\mathrm{f}(x)}{\mathrm{e}^x}$, show that $\lim\limits_{x\to\infty}\mathrm{f}(x)=L$ and $\lim\limits_{x\to\infty}\mathrm{f}'(x)=0$.

12 Show that $\dfrac{\ln(1+x)}{\sin^{-1}(x)}<1$ if $0<x<1$.