ANALYSIS
CHAPTER 4: DIFFERENTIATION

Learning Objectives:

By the end of this chapter, students should be able to:

- Understand Differentiation with more rigour
- Apply theorems to solve problems related to Differentiation

He was a shining example of a man of the highest social distinction whose love of learning drove him to devote much of his short life to scientific writing.

— Julian Coolidge (1873 – 1954) assessment of L'Hôpital (1661 – 1704)

Pre-requisites

- H2 Mathematics Differentiation

Setting the Context

We have seen Differentiation in H2 Mathematics, and we will leverage on the learning and explore the topic with more rigour.

§1 Recap

See H2 Notes for Differentiation. Restating the First Principles here, where we have

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{f}(x)\right] = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}}$$

A function f is called differentiable at a point $x = x_0$ if it has a derivative at this point, i.e. $\mathrm{f}'\left(x_0\right)$ exist. Note that if f is differentiable at a point $x = x_0$ then it must be continuous there. However the converse is not necessarily true.

Do also recap section 3.3 of the H2 notes for the graphical interpretation of the derivative.

§1.1 Right and Left Hand Derivatives

The right hand derivative of f(x) at $x = x_0$ is defined as

$$\mathrm{f}'_+(x_0) = \lim_{h \to 0^+} \frac{\mathrm{f}(x_0 + h) - \mathrm{f}(x_0)}{h}$$

Similarly, the left hand derivative of f(x) at $x = x_0$ is defined as

$$\mathrm{f}'_-(x_0) = \lim_{h \to 0^-} \frac{\mathrm{f}(x_0 + h) - \mathrm{f}(x_0)}{h}$$

A function f has a derivative at $x = x_0$ if and only if $\mathrm{f}'_+(x_0) = \mathrm{f}'_-(x_0)$.

§1.2 Rules for Differentiation (Section 4 H2 Notes)

Let $k \in \mathbb{R}$, and suppose u and v be functions of x, i.e. $u = u(x)$, $v = v(x)$. Then

(i) If $y = ku$, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = k\dfrac{\mathrm{d}u}{\mathrm{d}x}$.

(ii) Sum/Difference Rule

If $y = u \pm v$, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}u}{\mathrm{d}x} \pm \dfrac{\mathrm{d}v}{\mathrm{d}x}$.

(iii) Product Rule

If $y = uv$, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = v\dfrac{\mathrm{d}u}{\mathrm{d}x} + u\dfrac{\mathrm{d}v}{\mathrm{d}x}$.

(iv) Quotient Rule

If $u = \dfrac{u}{v}$, then $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{u}{v}\right] = \dfrac{v\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$.

(v) Chain Rule

If $y = \mathrm{g}(u)$, and $u = u(x)$, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}u} \times \dfrac{\mathrm{d}u}{\mathrm{d}x}$.

(vi) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\left(\dfrac{\mathrm{d}x}{\mathrm{d}y}\right)}$ (**Note:** This is NOT true for higher derivatives, e.g. $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} \neq \dfrac{1}{\left(\dfrac{\mathrm{d}^2x}{\mathrm{d}y^2}\right)}$.)

An exercise for the student is to try to derive the above rules using First Principles and Limit Rules.

§1.3 Higher Order Derivatives

If $\mathrm{f}'(x)$ is further differentiable in the interval given, its derivative is denoted by $\mathrm{f}''(x)$, and in general the n^{th} derivative is denoted by $\mathrm{f}^{(n)}(x)$.

By repeated usage of the First Principle on f and f', we can show that

$$\mathrm{f}''(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - 2\mathrm{f}(x) + \mathrm{f}(x-h)}{h^2}$$

§2 Mean Value Theorems

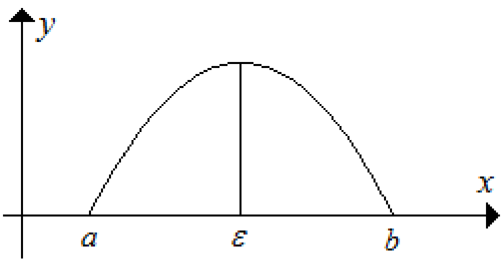
There are three theorems good to know in H3. Questions of this nature will be guided although pre-exposure is always an asset.

1. Rolle's Theorem
2. The Theorem of the Mean
3. Cauchy's Generalized Theorem of the Mean

§2.1 Rolle's Theorem

If f(x) is continuous in [a, b] and differentiable in (a, b) and if f(a) = f(b) = 0, then there exist a point ε in (a, b) such that $\mathrm{f}'(\varepsilon) = 0$.

In layman terms, basically if you have to draw a smooth curve in which the starting point and the ending point is level (i.e. the y value is the same), there must at least be a min/max point, or the function is a constant zero.



Proof is left as an exercise for the student to try.

§2.2 The Theorem of the Mean

If f(x) is continuous in [a, b] and differentiable in (a, b), then there exist a point ε in (a, b) such that $\mathrm{f}'(\varepsilon) = \dfrac{\mathrm{f}(b) - \mathrm{f}(a)}{b - a}, \quad a < \varepsilon < b.$

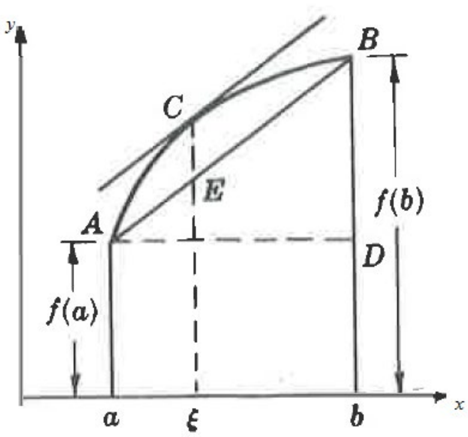
Rolle's Theorem is a special case of this where f(a) = f(b) = 0.

[Proof]

Apply Rolle's Theorem on the function $\mathrm{F}(x) = \mathrm{f}(x) - \mathrm{f}(a) - (x - a)\left(\dfrac{\mathrm{f}(b) - \mathrm{f}(a)}{b - a}\right)$.

Details are left to the student as an exercise.

The geometric interpretation is as follows:



Observe that the gradient of the graph at C is the same as the gradient of the line AB.

§2.3 Cauchy's Generalized Theorem of the Mean

If f(x) and g(x) are continuous in [a, b] and differentiable in (a, b), then there exist a point ε in (a, b) such that $\dfrac{\mathrm{f}'(\varepsilon)}{\mathrm{g}'(\varepsilon)} = \dfrac{\mathrm{f}(b) - \mathrm{f}(a)}{\mathrm{g}(b) - \mathrm{g}(a)}$, $a < \varepsilon < b$, where we assume $\mathrm{g}(a) \neq \mathrm{g}(b)$ and $\mathrm{f}'(x),\ \mathrm{g}'(x)$ are not simultaneously zero.

Note that the case $\mathrm{g}(x) = x$ reduces to the Theorem of the Mean in section 2.2.

[Proof]

Apply Rolle's Theorem on the function $\mathrm{G}(x) = \mathrm{f}(x) - \mathrm{f}(a) - \left(\dfrac{\mathrm{f}(b) - \mathrm{f}(a)}{\mathrm{g}(b) - \mathrm{g}(a)}\right)(\mathrm{g}(x) - \mathrm{g}(a))$.

Details are left to the student as an exercise.

§3 L'Hopital's Rule

Given functions f and g which are differentiable on an open interval I except possibly at a point c contained in I, and either

1. $\lim_{x\to c} \mathrm{f}(x) = \lim_{x\to c} \mathrm{g}(x) = 0$,
2. $\lim_{x\to c} \mathrm{f}(x) = \lim_{x\to c} \mathrm{g}(x) = \pm\infty$,

with $\mathrm{g}'(x) \neq 0$ for all $x \notin I/\{c\}$, and $\lim_{x\to c} \frac{\mathrm{f}'(x)}{\mathrm{g}'(x)}$ exists, then

$$\lim_{x\to c} \frac{\mathrm{f}(x)}{\mathrm{g}(x)} = \lim_{x\to c} \frac{\mathrm{f}'(x)}{\mathrm{g}'(x)}.$$

The use of the rule is to be able to compute limits of more functions otherwise difficult to compute. The proof is to apply the Cauchy's generalized theorem of the mean, but is omitted to focus on the application of the rule.

Example 1
Evaluate

(a) $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)$, **(b)** $\lim_{x\to 0}\left(\frac{\ln(\tan 2x)}{\ln(\tan 3x)}\right)$, **(c)** $\lim_{x\to 0}\left((\cos x)^{1/x^2}\right)$.

[Solution]

(a) $\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = \lim_{x\to 0}\left(\frac{\cos x}{1}\right) = 1$

(b)
$$\begin{aligned}
\lim_{x\to 0}\left(\frac{\ln(\tan 2x)}{\ln(\tan 3x)}\right) &= \lim_{x\to 0}\left(\frac{(2\sec^2 2x)/(\tan 2x)}{(3\sec^2 3x)/(\tan 3x)}\right)\\
&= \lim_{x\to 0}\left(\frac{(2\sec^2 2x)(\tan 3x)}{(3\sec^2 3x)(\tan 2x)}\right)\\
&= \lim_{x\to 0}\left(\frac{2\sec^2 2x}{3\sec^2 3x}\right)\lim_{x\to 0}\left(\frac{\tan 3x}{\tan 2x}\right)\\
&= \frac{2}{3}\lim_{x\to 0}\left(\frac{\sec^2 3x}{\sec^2 2x}\right) = \frac{2}{3}
\end{aligned}$$

(c) Let $\mathrm{F}(x) = (\cos x)^{1/x^2}$, and we have $\ln(\mathrm{F}(x)) = \frac{\ln(\cos x)}{x^2}$, and

$$\lim_{x\to 0}(\ln(\mathrm{F}(x))) = \lim_{x\to 0}\left(\frac{-\sin x}{2x\cos x}\right) = \lim_{x\to 0}\left(\frac{-\cos x}{2x\sin x + 2\cos x}\right) = -\frac{1}{2}$$

(L'Hopital applied twice)

Since ln is a continuous function, $\ln\left(\lim_{x\to 0}(\mathrm{F}(x))\right) = \lim_{x\to 0}(\ln(\mathrm{F}(x)))$ and $\lim_{x\to 0}\left((\cos x)^{1/x^2}\right) = \mathrm{e}^{-0.5}$.