Application of Diagonalisation: Markov Chain

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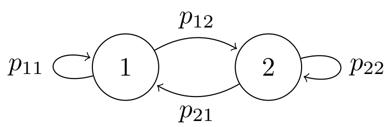
Suppose there are two states, 1 and 2. The probabilities of change of state can be summarised by the diagram below:



This can be represented by the matrix $\mathbf{P}$.

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

Let p_{12} and p_{21} be a and b respectively, where $a, b \in (0, 1)$. For ease of calculations, let $\mathbf{A}$ be the transpose of $\mathbf{P}$, such that:

$$\mathbf{A} = \mathbf{P}^{\mathrm{T}} = \begin{bmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix}$$

Let λ be the eigenvalues of $\mathbf{A}$. $\forall \mathbf{v} \in \mathbb{R}^2 \backslash \{\mathbf{0}\}$,

$$\begin{aligned}
\lambda \mathbf{v} &= \mathbf{A}\mathbf{v} \\
\lambda \mathbf{v} - \mathbf{A}\mathbf{v} &= \mathbf{0} \\
(\lambda \mathbf{I} - \mathbf{A})\mathbf{v} &= \mathbf{0} \\
\det(\lambda \mathbf{I} - \mathbf{A}) &= 0 \\
\begin{vmatrix} \lambda + a - 1 & -b \\ -a & \lambda + b - 1 \end{vmatrix} &= 0 \\
(\lambda + a - 1)(\lambda + b - 1) - (-b)(-a) &= 0 \\
\lambda^2 + b\lambda - \lambda + a\lambda + ab - a - \lambda - b + 1 - ab &= 0 \\
\lambda^2 + (a + b - 2)\lambda + 1 - a - b &= 0
\end{aligned}$$

Let the discriminant of the quadratic in λ be Δ.

$$\begin{aligned}
\Delta &= (a + b - 2)^2 - 4(1)(1 - a - b) \\
&= a^2 + ab - 2a + ab + b^2 - 2b - 2a - 2b + 4 - 4 + 4a + 4b \\
&= a^2 + 2ab + b^2 \\
&= (a + b)^2
\end{aligned}$$

$$\lambda = \frac{-(a+b-2) \pm \sqrt{\Delta}}{2(1)} = \frac{2-a-b \pm (a+b)}{2}$$
$$\lambda \in \{1, 1-a-b\}$$

For $\lambda = 1$,

$$\begin{aligned}(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} &= \mathbf{0} \\ \begin{bmatrix} a & -b \\ -a & b \end{bmatrix}\mathbf{x} &= \mathbf{0} \\ \begin{bmatrix} a & -b \\ 0 & 0 \end{bmatrix}\mathbf{x} &= \mathbf{0} \\ \begin{bmatrix} a & -b \\ 0 & 0 \end{bmatrix}\mathbf{x} &= \mathbf{0} \\ ax_1 - bx_2 &= 0 \\ ax_1 &= bx_2 \\ x_1 &= \frac{b}{a}x_2 \end{aligned}$$

Let $x_2 = k$. $x_1 = \frac{b}{a}k$.

$$\mathbf{x} = \begin{bmatrix} \frac{b}{a}k \\ k \end{bmatrix} = bk\begin{bmatrix} b \\ a \end{bmatrix}$$

$$\text{null}(\mathbf{I} - \mathbf{A}) = \text{span}\left\{\begin{bmatrix} b \\ a \end{bmatrix}\right\}$$

For $\lambda = 1 - a - b$,

$$\begin{aligned}(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} &= \mathbf{0} \\ \begin{bmatrix} -b & -b \\ -a & -a \end{bmatrix}\mathbf{x} &= \mathbf{0} \\ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\mathbf{x} &= \mathbf{0} \\ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{x} &= \mathbf{0} \\ x_1 + x_2 &= 0 \\ x_1 &= -x_2 \end{aligned}$$

Let $x_2 = k$. $x_1 = -k$.

$$\mathbf{x} = \begin{bmatrix} -k \\ k \end{bmatrix} = -k\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\text{null}((1-a-b)\mathbf{I} - \mathbf{A}) = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

Let $\mathbf{A} = \mathbf{QDQ}^{-1}$.

$$\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 1-a-b \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} b & 1 \\ a & -1 \end{bmatrix}$$

Check:

$$\begin{aligned}
\mathbf{AQ} &= \mathbf{A}\begin{bmatrix} b & 1 \\ a & -1 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{A}\begin{bmatrix} b \\ a \end{bmatrix} & \mathbf{A}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} 1\begin{bmatrix} b \\ a \end{bmatrix} & (1-a-b)\begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} b & 1 \\ a & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1-a-b \end{bmatrix} \\
&= \mathbf{QD}
\end{aligned}$$

$$\mathbf{AQ} = \mathbf{QD} \iff \mathbf{A} = \mathbf{QDQ}^{-1}$$

$$\mathbf{Q}^{-1} = \frac{\operatorname{adj}(\mathbf{Q})}{\det(\mathbf{Q})} = \frac{1}{-b-a}\begin{bmatrix} -1 & -1 \\ -a & b \end{bmatrix} = \frac{1}{a+b}\begin{bmatrix} 1 & 1 \\ a & -b \end{bmatrix}$$

To find to equilibrium distribution of states, $\lim\limits_{n\to\infty} \mathbf{A}^n$ is wanted.

$$\lim_{n\to\infty} \mathbf{D}^n = \lim_{n\to\infty} \begin{bmatrix} 1 & 0 \\ 0 & 1-a-b \end{bmatrix}^n = \lim_{n\to\infty} \begin{bmatrix} 1^n & 0 \\ 0 & (1-a-b)^n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned}
\lim_{n\to\infty} \mathbf{A}^n &= \lim_{n\to\infty} \left(\mathbf{QDQ}^{-1}\right)^n \\
&= \lim_{n\to\infty} \mathbf{QD}^n\mathbf{Q}^{-1} \\
&= \mathbf{Q}\left(\lim_{n\to\infty} \mathbf{D}^n\right)\mathbf{Q}^{-1} \\
&= \begin{bmatrix} b & 1 \\ a & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{a+b}\begin{bmatrix} 1 & 1 \\ a & -b \end{bmatrix} \\
&= \frac{1}{a+b}\begin{bmatrix} b & 1 \\ a & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ a & -b \end{bmatrix} \\
&= \frac{1}{a+b}\begin{bmatrix} b & 1 \\ a & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \\
&= \frac{1}{a+b}\begin{bmatrix} b & b \\ a & a \end{bmatrix} \\
&= \begin{bmatrix} \frac{b}{a+b} & \frac{b}{a+b} \\ \frac{a}{a+b} & \frac{a}{a+b} \end{bmatrix}
\end{aligned}$$

Let $\mathbf{s}_0$ be the initial distribution of states. It is obvious that.

$$\begin{aligned}
s_1 + s_2 &= 1 \\
s_1 &= 1 - s_2
\end{aligned}$$

Let $s_2 = m$, $s_1 = 1 - m$.

The equilibrium distribution can be calculated as such:

$$\lim_{n\to\infty} \mathbf{s}_n = \lim_{n\to\infty} \mathbf{A}^n\mathbf{s}_0 = \left(\lim_{n\to\infty} \mathbf{A}^n\right)\mathbf{s}_0 = \begin{bmatrix} \frac{b}{a+b} & \frac{b}{a+b} \\ \frac{a}{a+b} & \frac{a}{a+b} \end{bmatrix}\begin{bmatrix} 1-m \\ m \end{bmatrix} = \begin{bmatrix} \frac{b}{a+b} \\ \frac{a}{a+b} \end{bmatrix} = \boxed{\begin{bmatrix} \frac{p_{21}}{p_{12}+p_{21}} \\ \frac{p_{12}}{p_{12}+p_{21}} \end{bmatrix}}$$