1.

Let $\mathbf{M} = \begin{bmatrix} 1 & b & a \\ a & 1 & b \\ b & a & 1 \end{bmatrix}$.

$$\begin{aligned} \det(\mathbf{M}) &= 1(1-ab) + b\left(b^2 - a\right) + a\left(a^2 - b\right) \\ &= 1 - ab + b^3 - ab + a^3 - ab \\ &= a^3 - 3ab + b^3 + 1 \end{aligned}$$

Since $\left\{ \begin{bmatrix} 1 \\ a \\ b \end{bmatrix}, \begin{bmatrix} a \\ 1 \\ b \end{bmatrix}, \begin{bmatrix} b \\ a \\ 1 \end{bmatrix} \right\}$ is not a basis for $\mathbb{R}^3$, $\operatorname{rank}(\mathbf{M}) < 3$.

Thus, $\det(\mathbf{M}) = 0$.

$$a^3 - 3ab + b^3 + 1 = 0 \quad \square$$

2. (i)

$$\mathbf{A0} = \mathbf{0} \implies \mathbf{0} \in W$$

Let $\mathbf{u}, \mathbf{v} \in W$. ($\implies \mathbf{Au} = \mathbf{u} \wedge \mathbf{Av} = \mathbf{v}$)

$$\mathbf{A}(a\mathbf{u} + b\mathbf{v}) = \mathbf{A}(a\mathbf{u}) + \mathbf{A}(b\mathbf{v}) = a\mathbf{Au} + b\mathbf{Av} = a\mathbf{u} + b\mathbf{v} \implies a\mathbf{u} + b\mathbf{v} \in W$$

Since $W \subset \mathbb{R}^n$, $\mathbf{0} \in W$ and $\mathbf{u}, \mathbf{v} \in W \implies a\mathbf{u} + b\mathbf{v} \in W$, W is a subspace of $\mathbb{R}^n$. $\square$

2. (ii)

$$\mathbf{Au} = \mathbf{u} \iff \mathbf{Au} - \mathbf{Iu} = \mathbf{0} \iff (\mathbf{A} - \mathbf{I})\mathbf{u} = \mathbf{0}$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0 \implies u_3 = 0$$

Let $u_1 = \lambda$ and $u_2 = \mu$.

$$\mathbf{u} = \begin{bmatrix} \lambda \\ \mu \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

A basis of W, is

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}}$$

3. (i)

Let $\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 \end{bmatrix}$.

$$\det(\mathbf{X}) = 1(1(1(1))) = 1 \neq 0 \implies \operatorname{rank}(\mathbf{X}) = 4$$

Since $\mathbf{X}$ is full rank, the columns of $\mathbf{X}$, $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ and $\mathbf{x}_4$ form a basis of $\mathbb{R}^4$. $\square$

3. (ii)

$$\text{rref}(\mathbf{A}) = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \implies \text{rank}(A) = \boxed{2}$$

$$\text{nullity}(\mathbf{A}) = 4 - 2 = \boxed{2}$$

3. (iii)

$$\mathbf{A}\mathbf{x}_1 = \begin{bmatrix} 1 \\ -3 \\ 4 \\ 6 \end{bmatrix}$$

$$\mathbf{A}\mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 9 \\ 4 \end{bmatrix}$$

Since $\text{rank}(\mathbf{A}) = 2$, all $\mathbf{A}\mathbf{x}_i$ are coplanar. Therefore, all linear combinations of these vector, ie $\sum \lambda \mathbf{A}\mathbf{x}$, exist on the same two dimensional space. Moreover, since $\mathbf{A}\mathbf{x}_1$ cannot be expressed as a scalar multiple of $\mathbf{A}\mathbf{x}_2$, the dimension of V cannot be 0 or 1. Thus, $\dim(V) = 2$. □

A basis of V, is

$$\boxed{\left\{ \begin{bmatrix} 1 \\ -3 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 9 \\ 4 \end{bmatrix} \right\}}$$

3. (iv)

$$\begin{bmatrix} p \\ q \\ 23 \\ 6 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ -3 \\ 4 \\ 6 \end{bmatrix} + \mu \begin{bmatrix} 2 \\ 1 \\ 9 \\ 4 \end{bmatrix}$$

$$\begin{cases} 4\lambda + 9\mu = 23 \\ 6\lambda + 4\mu = 6 \end{cases} \iff \begin{cases} 12\lambda + 27\mu = 69 \\ 12\lambda + 8\mu = 12 \end{cases} \implies 19\mu = 57 \iff \mu = 3$$

$$6\lambda + 4(3) = 6 \iff 6\lambda = -6 \iff \lambda = -1$$

$$p = (-1)1 + (3)2 = \boxed{5}$$

$$q = (-1)(-3) + (3)1 = \boxed{6}$$