H3 MATH
2021 / YR 6

ANALYSIS
CHAPTER 5: INTEGRATION

Learning Objectives:

By the end of this chapter, students should be able to:

- Resolve problems involving Integration that requires more rigour

Réduites à des théories générales, les mathématiques seraient une belle forme sans contenu.

(Reduced to general theories, mathematics would be a beautiful form without content)

— Henri Léon Lebesgue (1875 – 1941)

Pre-requisites

- H2 Mathematics Integration

Setting the Context

We have seen Integration in H2 Mathematics, and we will leverage on the learning and explore the topic with more rigour.

Basic rules of integration involving polynomials, trigonometric functions, as well as interpretation of definite integrals in terms of area under graph or volume of rotation, are omitted here and students are required to read those materials given in H2 notes.

§1 Common Techniques

There are a few basic tricks involved, namely
1. Splitting the Numerator
2. Substitution or Change of Variable
3. Integration by parts and rearrangement
4. Using symmetry
5. Reducing to integrals with known solution
6. Combinations of above

1.1 Splitting the Numerator

Example 1 (Modified Math S N86(a))

Evaluate exactly $\int_0^1 \frac{x+2}{\sqrt{x+1}}\,\mathrm{d}x$.

[Solution]

$$\int_0^1 \frac{x+2}{\sqrt{x+1}}\,\mathrm{d}x$$

$$= \int_0^1 \sqrt{x+1} + \frac{1}{\sqrt{x+1}}\,\mathrm{d}x$$

$$= \left[\frac{(x+1)^{1.5}}{1.5} + \frac{(x+1)^{0.5}}{0.5} \right]_0^1$$

$$= \frac{2}{3}\left(5\sqrt{3} - 4\right)$$

§1.2 Substitution or Change of Variable

Example 2 (Modified Math S N86(a))

Without using a GC, find $\int_0^1 \frac{\sqrt{x+2}}{x+1}\,\mathrm{d}x$.

[Solution]

Remark: You may be expected to recognise basic forms of substitution as they may not be provided.

Try substitution $u = \sqrt{x+2}$ or $u^2 = x+2$, we have

$$\int_0^1 \frac{\sqrt{x+2}}{x+1}\,\mathrm{d}x = \int_{\sqrt{2}}^{\sqrt{3}} \frac{2u^2}{u^2-1}\,\mathrm{d}u$$

$$= 2\int_{\sqrt{2}}^{\sqrt{3}} \left(1 + \frac{1}{u^2-1}\right)\mathrm{d}u$$

$$= 2\left[u + \frac{1}{2}\ln\left(\frac{u-1}{u+1}\right) \right]_{\sqrt{2}}^{\sqrt{3}} = \ldots$$

Example 3

Prove the following result for $a > 0$: $\displaystyle\int_0^a \mathrm{f}(x)\,\mathrm{d}x = \int_0^a \mathrm{f}(a-x)\,\mathrm{d}x$. **(*)**

Hence evaluate

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}\,\mathrm{d}x$, where n is a real constant,

(b) $\displaystyle\int_0^a \frac{x^4}{x^4 + (x-a)^4}\,\mathrm{d}x$, where a is a positive real constant.

[Solution]

(a) Substitute $x = a - u \;\Rightarrow\; \dfrac{\mathrm{d}x}{\mathrm{d}u} = -1$.

$x = 0 \;\Rightarrow\; u = a$ and $x = a \;\Rightarrow\; u = 0$.

Using the substitution, we have

$$\int_0^a \mathrm{f}(x)\,\mathrm{d}x = \int_a^0 \mathrm{f}(a-u)\,(-1)\mathrm{d}u = \int_0^a \mathrm{f}(a-u)\,\mathrm{d}u = \int_0^a \mathrm{f}(a-x)\,\mathrm{d}x.$$

For $\mathrm{f}(x) = \dfrac{\cos^n x}{\sin^n x + \cos^n x}$, $\mathrm{f}\left(\dfrac{\pi}{2} - x\right) = \dfrac{\cos^n\left(\dfrac{\pi}{2} - x\right)}{\sin^n\left(\dfrac{\pi}{2} - x\right) + \cos^n\left(\dfrac{\pi}{2} - x\right)} = \dfrac{\sin^n x}{\cos^n x + \sin^n x}$.

Using the result, $\displaystyle\int_0^{\pi/2} \mathrm{f}(x)\,\mathrm{d}x = \int_0^{\pi/2} \mathrm{f}\left(\frac{\pi}{2} - x\right)\mathrm{d}x$

Hence $\displaystyle\int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x}\,\mathrm{d}x = \int_0^{\pi/2} \frac{\sin^n x}{\cos^n x + \sin^n x}\,\mathrm{d}x$

Let $\displaystyle I = \int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x}\,\mathrm{d}x = \int_0^{\pi/2} \frac{\sin^n x}{\cos^n x + \sin^n x}\,\mathrm{d}x$

So $\displaystyle 2I = I + I = \int_0^{\pi/2} \frac{\cos^n x + \sin^n x}{\cos^n x + \sin^n x}\,\mathrm{d}x = \int_0^{\pi/2} 1\,\mathrm{d}x = \frac{\pi}{2}$. Hence $\displaystyle\int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x}\,\mathrm{d}x = I = \frac{\pi}{4}$.

(b) Let $\mathrm{f}(x) = \dfrac{x^4}{x^4 + (x-a)^4}$. Then $\mathrm{f}(a-x) = \dfrac{(a-x)^4}{(a-x)^4 + (-x)^4} = \dfrac{(x-a)^4}{x^4 + (x-a)^4}$.

By result (*), $\displaystyle I = \int_0^a \frac{x^4}{x^4 + (x-a)^4}\,\mathrm{d}x = \int_0^a \frac{(x-a)^4}{x^4 + (x-a)^4}\,\mathrm{d}x$.

Hence $\displaystyle 2I = \int_0^a \frac{x^4}{x^4 + (x-a)^4}\,\mathrm{d}x + \int_0^a \frac{(x-a)^4}{x^4 + (x-a)^4}\,\mathrm{d}x = \int_0^a \frac{x^4 + (x-a)^4}{x^4 + (x-a)^4}\,\mathrm{d}x = \int_0^a 1\,\mathrm{d}x = a$

Therefore $I = \dfrac{a}{2}$.

§1.3 Integration by parts and rearrangement

There are some special integrals where after applying integration by parts twice in the same direction (i.e. not applying integration by parts then reversing it to get the original problem), one obtains a multiple of the original integral. Then a rearrangement will result in the required solution.

Example 4

Evaluate $\int e^x \cos x \, dx$.

[Solution]

$\int e^x \cos x \, dx$

$= e^x \cos x - \int (e^x)(-\sin x) \, dx$

(choose to differentiate the trigo function all the way)

$= e^x \cos x + \int e^x \sin x \, dx$

$= e^x \cos x + \left[e^x \sin x - \int e^x \cos x \, dx \right]$

$\therefore 2\int e^x \cos x \, dx = e^x \cos x + e^x \sin x + c$

$\int e^x \cos x \, dx = \frac{1}{2}\left(e^x \cos x + e^x \sin x\right) + C$

where c, C are arbitrary constants.

Example 5

Differentiate the following with respect to x:

(i) $\sin(\ln x)$,

(ii) $\cos(\ln x)$.

Hence evaluate $\int x^3 \sin(\ln x) \, dx$.

[Solution]

(i) $\frac{d}{dx}\left(\sin(\ln x)\right) = \frac{1}{x}\cos(\ln x)$

(ii) $\frac{d}{dx}\left(\cos(\ln x)\right) = -\frac{1}{x}\sin(\ln x)$

(Example 5 Continued)

$$\int x^3 \sin(\ln x)\ \mathrm{d}x$$
$$= \frac{x^4}{4}\sin(\ln x) - \int \frac{x^4}{4}\frac{1}{x}\cos(\ln x)\ \mathrm{d}x$$
$$= \frac{x^4}{4}\sin(\ln x) - \frac{1}{4}\int x^3 \cos(\ln x)\ \mathrm{d}x$$
$$= \frac{x^4}{4}\sin(\ln x) - \frac{1}{4}\left[\frac{x^4}{4}\cos(\ln x) - \int -\frac{x^4}{4}\frac{1}{x}\sin(\ln x)\ \mathrm{d}x\right]$$
$$= \frac{x^4}{4}\sin(\ln x) - \frac{x^4}{16}\cos(\ln x) - \frac{1}{16}\int x^3 \sin(\ln x)\ \mathrm{d}x$$

$$\therefore \frac{17}{16}\int x^3 \sin(\ln x)\ \mathrm{d}x = \frac{x^4}{4}\sin(\ln x) - \frac{x^4}{16}\cos(\ln x) + c$$
$$\int x^3 \sin(\ln x)\ \mathrm{d}x = \frac{4x^4}{17}\sin(\ln x) - \frac{x^4}{17}\cos(\ln x) + C$$
where c, C are arbitrary constants.

Alternatively,

$$\int x^3 \sin(\ln x)\ \mathrm{d}x$$
$$= \int \left(-x^4\right)\left(-\frac{1}{x}\sin(\ln x)\right)\ \mathrm{d}x$$
$$= -x^4\cos(\ln x) - \int -4x^3\cos(\ln x)\ \mathrm{d}x$$
$$= -x^4\cos(\ln x) + 4\int x^3\cos(\ln x)\ \mathrm{d}x$$
$$= -x^4\cos(\ln x) + 4\int \left(x^4\right)\left(\frac{1}{x}\cos(\ln x)\right)\ \mathrm{d}x$$
$$= -x^4\cos(\ln x) + 4\left[x^4\sin(\ln x) - \int 4x^3\sin(\ln x)\ \mathrm{d}x\right]$$
$$= 4x^4\sin(\ln x) - x^4\cos(\ln x) - 16\int x^3\sin(\ln x)\ \mathrm{d}x$$

$$\therefore 17\int x^3\sin(\ln x)\ \mathrm{d}x = 4x^4\sin(\ln x) - x^4\cos(\ln x) + c$$
$$\int x^3 \sin(\ln x)\ \mathrm{d}x = \frac{4x^4}{17}\sin(\ln x) - \frac{x^4}{17}\cos(\ln x) + C$$
where c, C are arbitrary constants.

§1.4 Integration using symmetry

There are two types of symmetry, either the limits are $\pm$ of each other (or with some symmetry about the average of the limits), and/or the function involved in the integral possess a certain level of symmetry.

Example 6

Without using the GC, evaluate $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{x\cos x - 2\sin x + 1}{\cos^2 x}\,\mathrm{d}x$

[Solution]

$$\int_{-\pi/4}^{\pi/4} \frac{x\cos x - 2\sin x + 1}{\cos^2 x}\,\mathrm{d}x$$
$$= \int_{-\pi/4}^{\pi/4} \frac{x\cos x}{\cos^2 x}\mathrm{d}x - \int_{-\pi/4}^{\pi/4} \frac{2\sin x}{\cos^2 x}\mathrm{d}x + \int_{-\pi/4}^{\pi/4} \frac{1}{\cos^2 x}\,\mathrm{d}x$$
$$= 0 - 0 + \int_{-\pi/4}^{\pi/4} \sec^2 x\,\mathrm{d}x$$
$$= \left[\tan x\right]_{-\pi/4}^{\pi/4} = 2$$

Example 7

Without using the GC, integrate a suitable Maclaurin series to obtain the value of
$$\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} \mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x.$$

[Solution]

$$\mathrm{e}^{-\frac{1}{2}x^2} = 1 + \left(-\frac{1}{2}x^2\right) + \frac{1}{2!}\left(-\frac{1}{2}x^2\right)^2 + \frac{1}{3!}\left(-\frac{1}{2}x^2\right)^3 + \frac{1}{4!}\left(-\frac{1}{2}x^2\right)^4 + \ldots$$

(standard Maclaurin series of e^x in MF 26)

Integrate with respect to x from $x = 0$ to $x = 1$ gives

$$\int_0^1 \mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x = \int_0^1 \left[1 + \left(-\frac{1}{2}x^2\right) + \frac{1}{2!}\left(-\frac{1}{2}x^2\right)^2 + \frac{1}{3!}\left(-\frac{1}{2}x^2\right)^3 + \frac{1}{4!}\left(-\frac{1}{2}x^2\right)^4 + \ldots\right]\mathrm{d}x$$
$$= \left[x - \frac{1}{2}\left(\frac{x^3}{3}\right) + \frac{1}{8}\left(\frac{x^5}{5}\right) - \frac{1}{48}\left(\frac{x^7}{7}\right) + \frac{1}{384}\left(\frac{x^9}{9}\right) + \ldots\right]_0^1 \approx 0.85562$$

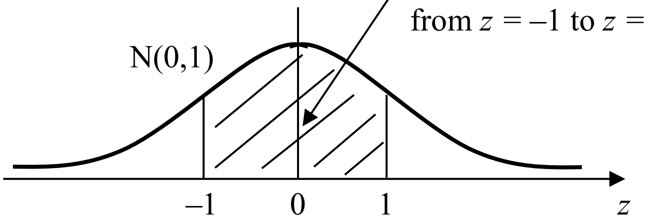
Since $\mathrm{e}^{-\frac{1}{2}x^2}$ is an **<u>even function</u>** (symmetry about the y-axis), thus

$$\int_{-1}^{1} \mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x = 2\int_0^1 \mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x \approx 2\left(0.85562\right) = 1.71124.$$

$$\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} \mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x \approx \frac{1}{\sqrt{2\pi}}\left(1.71124\right) \approx 0.683$$

Remark: The integral $\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-\frac{1}{2}x^2}\,dx$ has theoretical significance in Statistics. It is the probability that the **Standard Normal Random Variable** $Z \sim N(0,1)$ lies within 1 standard deviation from its mean 0. That is,

$\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-\frac{1}{2}x^2}\,dx = P(-1<Z<1)$ = Area under N(0,1) curve from $z = -1$ to $z = 1$



§1.5 Reducing to integrals with known solution

Example 8

Given the probability density function of the Standard Normal Distribution to be

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \qquad -\infty < x < \infty,$$

and given $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$ (see annex for this integration), show that the mean and variance of the Standard Normal Distribution is 0 and 1 respectively, with $\text{mean} = E(X) = \int_{-\infty}^{\infty} x\,f(x)\,dx$, $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$ and $\text{variance} = E(X^2) - (E(X))^2$.

[Solution]

$$\text{mean} = E(X) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-\frac{x^2}{2}}\,dx = \frac{1}{\sqrt{2\pi}}\left[-\frac{1}{2}e^{-\frac{x^2}{2}}\right]_{-\infty}^{\infty} = 0$$

$$\begin{aligned}
\text{variance} &= \mathrm{E}(X^2)-\left(\mathrm{E}(X)\right)^2 \quad = \mathrm{E}(X^2)-0 \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^2 \mathrm{e}^{-\frac{x^2}{2}}\ \mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x\left(x\mathrm{e}^{-\frac{x^2}{2}}\right) \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}\left(\left[x\mathrm{e}^{-\frac{x^2}{2}}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{x^2}{2}}\ \mathrm{d}x\right) \\
&= \frac{1}{\sqrt{2\pi}}\left(\left[x\mathrm{e}^{-\frac{x^2}{2}}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{x^2}{2}}\ \mathrm{d}x\right) \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{x^2}{2}}\ \mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \sqrt{2}\mathrm{e}^{-u^2}\ \mathrm{d}u \qquad \left(\text{with } u = \frac{x}{\sqrt{2}}\right) \\
&= 1
\end{aligned}$$

§1.6 Combinations of above

Example 9 (Question from Jonathan, 18Y6C32)

Show that for any $a \in \mathbb{R}, b \in \mathbb{Z}, \int_0^{2\pi} \sin\left(a\sin\theta + b\theta\right)\ \mathrm{d}\theta = 0.$

[Solution]

Let $\mathrm{f}\left(\theta\right) = \sin\left(a\sin\theta + b\theta\right)$

$$\begin{aligned}
\mathrm{f}\left(2\pi-\theta\right) &= \sin\left(a\sin\left(2\pi-\theta\right)+b\left(2\pi-\theta\right)\right) \\
&= \sin\left(a\sin\left(-\theta\right)+b\left(-\theta\right)\right)
\end{aligned}$$

(since sine is cyclic 2π, apply twice, once inside and once outside)

$$\begin{aligned}
&= \sin\left(-a\sin\theta - b\theta\right) \\
&= -\sin\left(a\sin\theta + b\theta\right)
\end{aligned}$$

(last two equalities due to $\sin(-t) = -\sin(t)$)

$$= -\mathrm{f}\left(\theta\right)$$

$$\begin{aligned}
\int_0^{2\pi} \mathrm{f}\left(\theta\right)\ \mathrm{d}\theta &= \int_0^{\pi} \mathrm{f}\left(\theta\right)\ \mathrm{d}\theta + \int_{\pi}^{2\pi} \mathrm{f}\left(\theta\right)\ \mathrm{d}\theta \\
&= \int_0^{\pi} \mathrm{f}\left(\theta\right)\ \mathrm{d}\theta + \int_{\pi}^{0} \left(\mathrm{f}\left(2\pi-t\right)\right)\left(-1\right)\ \mathrm{d}t
\end{aligned}$$

(substitution $t = 2\pi-\theta$ for the second portion)

$$\begin{aligned}
&= \int_0^{\pi} \mathrm{f}\left(\theta\right)\ \mathrm{d}\theta + \int_0^{\pi} \mathrm{f}\left(2\pi-t\right)\ \mathrm{d}t \\
&= \int_0^{\pi} \mathrm{f}\left(\theta\right)\ \mathrm{d}\theta - \int_0^{\pi} \mathrm{f}\left(t\right)\ \mathrm{d}t = 0
\end{aligned}$$

§2 Reduction Formula

A **Reduction Formula** is a recurrence relation involving integrals of the form I_n where n is a non-negative integer. Using the formula, I_n can be computed for any n by reduction of n.

Note: Reduction Formula is in Further Mathematics (FM) Syllabus and non-FM students are encouraged to photocopy and read your peer's notes and tutorial for further exposure.

Example 10

If $I_n = \int_0^1 x^{\frac{n}{2}}(1-x)^{\frac{1}{2}}\,\mathrm{d}x$ where n is a non-negative integer, show that

$$I_n = \left(\frac{n}{n+3}\right)I_{n-2}, \quad n \geq 2.$$

Evaluate I_0 and hence find I_6.

[Solution]

Using integration by parts with $u = x^{\frac{n}{2}}$, $\frac{\mathrm{d}v}{\mathrm{d}x} = (1-x)^{\frac{1}{2}}$ gives

$$\begin{aligned}
I_n &= -\frac{2}{3}\left[x^{\frac{n}{2}}(1-x)^{\frac{1}{2}}\right]_0^1 + \frac{n}{3}\int_0^1 x^{\frac{n-2}{2}}(1-x)^{\frac{3}{2}}\,\mathrm{d}x \\
&= \frac{n}{3}\int_0^1 x^{\frac{n-2}{2}}(1-x)(1-x)^{\frac{1}{2}}\,\mathrm{d}x \\
&= \frac{n}{3}\left[\underbrace{\int_0^1 x^{\frac{n-2}{2}}(1-x)^{\frac{1}{2}}\,\mathrm{d}x}_{I_{n-2}} - \underbrace{\int_0^1 x^{\frac{n}{2}}(1-x)^{\frac{1}{2}}\,\mathrm{d}x}_{I_n}\right] \\
&= \frac{n}{3}\left(I_{n-2} - I_n\right)
\end{aligned}$$

$$\Rightarrow I_n = \left(\frac{n}{n+3}\right)I_{n-2}, \; n \geq 2 \quad \text{-------- (*)}$$

$$I_0 = \int_0^1 (1-x)^{\frac{1}{2}}\,\mathrm{d}x = -\frac{2}{3}\left[(1-x)^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}.$$

Using the reduction formula (*):

$$I_6 = \left(\frac{6}{6+3}\right)I_{6-2} = \frac{2}{3}I_4 = \frac{2}{3}\left(\frac{4}{7}I_2\right) = \frac{2}{3}\cdot\frac{4}{7}\cdot\frac{2}{5}I_0 = \frac{2}{3}\cdot\frac{4}{7}\cdot\frac{2}{5}\cdot\frac{2}{3} = \frac{32}{315}.$$

§3 Integral with Modulus

Care is required to deal with integration involving modulus, where "area under graph" is different from "net area of graph with respect to the axes".

If $\mathrm{f}(x)$ is integrable on $[a,b]$, then $|\mathrm{f}(x)|$ is also integrable on $[a,b]$ and

$$\left|\int_a^b \mathrm{f}(x)\ \mathrm{d}x\right| \le \int_a^b |\mathrm{f}(x)|\ \mathrm{d}x.$$

[Proof]

Since $-|\mathrm{f}(x)| \le \mathrm{f}(x) \le |\mathrm{f}(x)|$,

Integrating with respect to x from $x = a$ to $x = b$,

$$-\int_a^b |\mathrm{f}(x)|\ \mathrm{d}x \le \int_a^b \mathrm{f}(x)\ \mathrm{d}x \le \int_a^b |\mathrm{f}(x)|\ \mathrm{d}x$$

$$\Rightarrow \left|\int_a^b \mathrm{f}(x)\ \mathrm{d}x\right| \le \int_a^b |\mathrm{f}(x)|\ \mathrm{d}x.$$

Remark: Equality is achieved when $\mathrm{f}(x) \ge 0$ for $x \in [a,b]$.

Note the use of the following two properties:

1. $\mathrm{f}(x) \le \mathrm{g}(x)\ \ \forall x \in [a,b] \Rightarrow \int_a^b f(x)\ dx \le \int_a^b g(x)\ dx$
2. $-n \le x \le n \Rightarrow |x| \le n$

Example 11

If n is a positive integer, prove that $\left|\int_0^1 \frac{\cos nx}{1+x^2}\ \mathrm{d}x\right| \le \frac{\pi}{4}$.

[Solution]

Firstly we note that $|\cos nx| \le 1\ \forall x \in \mathbb{R}$.

Dividing by $1+x^2$ gives $\left|\frac{\cos nx}{1+x^2}\right| \le \frac{1}{1+x^2}$.

$$\left|\int_0^1 \frac{\cos nx}{1+x^2}\ \mathrm{d}x\right| \le \int_0^1 \left|\frac{\cos nx}{1+x^2}\right|\ \mathrm{d}x \le \int_0^1 \frac{1}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \frac{\pi}{4}.$$

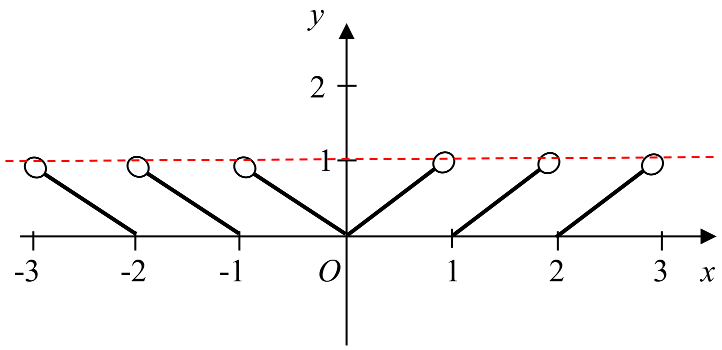
§4 Miscellaneous Functions

Example 12 (Math S N86(b))

For any real number x, the fractional part of x, denoted by $\{x\}$, is the positive difference between x and the greatest integer less than or equal to x.

Evaluate $\int_{-3}^{3} \{x\}^2 \, \mathrm{d}x$.

[Solution]
Observe the function $\{x\}$:



Hence $\int_{-3}^{3} \{x\}^2 \, \mathrm{d}x = 6\int_{0}^{1} x^2 \, \mathrm{d}x = 6\left[\frac{x^3}{3}\right]_0^1 = 2$

§5 Annex

Show that $\int_{-\infty}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x = \sqrt{\pi}$.

[Solution]

Let $I = \int_{-\infty}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x = 2\int_{0}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x$ (by symmetry since e^{-x^2} is even)

$$\begin{aligned}
I^2 &= 4\int_0^\infty \int_0^\infty \mathrm{e}^{-\left(x^2+y^2\right)} \, \mathrm{d}y \, \mathrm{d}x \\
&= 4\int_0^\infty \int_0^\infty \mathrm{e}^{-x^2\left(1+s^2\right)} x \, \mathrm{d}s \, \mathrm{d}x \quad \text{(using substitution } y = xs\text{)} \\
&= 4\int_0^\infty \int_0^\infty \mathrm{e}^{-x^2\left(1+s^2\right)} x \, \mathrm{d}x \, \mathrm{d}s \quad \text{(interchange order of integration)} \\
&= 4\int_0^\infty \left[\frac{\mathrm{e}^{-x^2\left(1+s^2\right)}}{-2(1+s^2)}\right]_{x=0}^{x=\infty} \mathrm{d}s \\
&= 2\int_0^\infty \frac{1}{1+s^2} \, \mathrm{d}s = 2\left[\tan^{-1} s\right]_0^\infty = 2\left[\frac{\pi}{2} - 0\right] = \pi
\end{aligned}$$

$\therefore \; I = \sqrt{\pi}$