Suppose $(x_0, y_0) = (0,0)$ and directrix is $\ell : x = -d$. The locus of point $P(x,y)$ has eccentricity e. Clearly, $F(de, 0)$ is the focus of such locus. From the definition of eccentricity,

$$\begin{aligned}
e &= \frac{PF}{P\ell} \\
e &= \frac{\sqrt{(x-de)^2 + y^2}}{x - (-d)} \\
e^2 &= \left(\frac{\sqrt{(x-de)^2 + y^2}}{x+d}\right)^2 \\
e^2 &= \frac{(x-de)^2 + y^2}{(x+d)^2} \\
e^2 &= \frac{x^2 - 2dex + d^2e^2 + y^2}{x^2 + 2dx + d^2} \\
e^2x^2 + 2de^2x + d^2e^2 &= x^2 - 2dex + d^2e^2 + y^2 \\
e^2x^2 + 2de^2x &= x^2 - 2dex + y^2 \\
\left(1 - e^2\right)x^2 - 2de(1+e)x + y^2 &= 0 \\
(1-e)(1+e)x^2 - 2de(1+e)x + y^2 &= 0 \\
(1+e)x((1-e)x - 2de) + y^2 &= 0
\end{aligned}$$

For $e = 1$,

$$\begin{aligned}
(1+1)x((1-1)x - 2d(1)) + y^2 &= 0 \\
2x(-2d) + y^2 &= 0 \\
y^2 &= 4dx
\end{aligned}$$

For $e \neq 1$, the centre is in the middle of the two x-intercepts. Clearly, one of the x-intercepts is $(0,0)$. To find the other one, set $y = 0 \neq x$.

$$\begin{aligned}
(1+e)x((1-e)x - 2de) + 0^2 &= 0 \\
(1+e)x((1-e)x - 2de) &= 0 \\
(1-e)x - 2de &= 0 \\
(1-e)x &= 2de \\
x &= \frac{2de}{1-e}
\end{aligned}$$

Thus, the centre is at $\left(\frac{de}{1-e}, 0\right)$. Now, replacing x with $\left(x + \frac{de}{1-e}\right)$ to make origin the centre.

$$
\begin{aligned}
(1+e)\left(x+\frac{de}{1-e}\right)\left((1-e)\left(x+\frac{de}{1-e}\right)-2de\right)+y^2&=0\\
\left((1+e)x+de\frac{1+e}{1-e}\right)((1-e)x+de-2de)+y^2&=0\\
\frac{1+e}{1-e}((1-e)x+de)((1-e)x-de)+y^2&=0\\
\frac{1+e}{1-e}\left((1-e)^2x^2-d^2e^2\right)+y^2&=0\\
(1-e)(1+e)x-d^2e^2\frac{1+e}{1-e}+y^2&=0\\
(1-e)(1+e)x+y^2&=d^2e^2\frac{1+e}{1-e}\\
\frac{(1-e)(1+e)x}{d^2e^2\frac{1+e}{1-e}}+\frac{y^2}{d^2e^2\frac{1+e}{1-e}}&=1\\
\frac{x^2}{\frac{d^2e^2}{(1-e)^2}}+\frac{y^2}{\frac{d^2e^2}{(1-e)^2}(1-e)(1+e)}&=1\\
\frac{x^2}{\left(\frac{de}{1-e}\right)^2}+\frac{y^2}{\left(\frac{de}{1-e}\right)^2(1-e^2)}&=1
\end{aligned}
$$

Notice that the semi-major axis, $a=\frac{de}{1-e}$, as it is the distance from leftmost vertex to the centre of the locus of points.

For $0<e<1$,

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1 \implies b^2=a^2\left(1-e^2\right)$$

OR

For $e>1$,

$$\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1 \implies b^2=a^2\left(e^2-1\right)$$

Focus is at:

$$x=de-a=a(1-e)-a=a-ae-a=-ae$$

Similarly, directrix is at:

$$x=-d-a=-a\frac{(1-e)}{e}-a=-\frac{a}{e}+a-a=-\frac{a}{e}$$