# ANALYSIS
# CHAPTER 2: INEQUALITIES

**Learning Objectives:**

By the end of this chapter, students should be able to:

- Prove properties and results, and solve non-routine problems involving inequalities in general
- Apply the Triangle Inequality, Cauchy-Schwarz Inequality and AM-GM Inequality to tackle problems

*There is no greater inequality than the equal treatment of unequals.*

— *Thomas Jefferson*

## Pre-requisites

- All H2 Mathematics content in general and previously taught H3 content as they will be used from time to time

## Setting the Context

We have seen equations and equalities in H2 Mathematics, and the next natural step is to consider inequalities. In fact, this line of thinking is not foreign to even mathematicians from ancient times and many useful results have been obtained with the use of various inequalities discovered. In this chapter, we will leverage on the learning from various topics previously taught in H2 and H3 Mathematics and introduce inequalities and see how they are used in higher mathematics.

## §1 Introduction

In equations and equalities, we have seen the usage of the symbol "=" as well as some usage of inequalities in the form of "<" (lesser than), ">" (greater than), "≤" (lesser than or equals to) and "≥" (greater than or equals to). We shall re-introduce the basic properties of inequalities and note that all of these properties also hold if all of the non-strict inequalities (≤ and ≥) are replaced by their corresponding strict inequalities (< and >) and (in the case of applying a function) monotonic functions are limited to *strictly* monotonic functions.

## §1.1 Basic Definition

The basic ordering of numbers has the following properties:

For two real numbers $a$ and $b$:

- $a > b$ if $a$ is greater than $b$, that is, $a-b$ is positive
- $a < b$ if $a$ is smaller than $b$, that is, $a-b$ is negative
- $a \geq b$ if $a$ is greater than or equal to $b$, that is, $a-b$ is either positive or 0
- $a \leq b$ if $a$ is smaller than or equal to $b$, that is, $a-b$ is either negative or 0

Note that $a > b \Leftrightarrow b < a$ and $a \geq b \Leftrightarrow b \leq a$.

## §1.2 Transitivity

Next, we have the transitive property of inequality which states that

For any real numbers $a$, $b$, $c$:

- if $a \geq b$ and $b \geq c$, then $a \geq c$
- if $a \leq b$ and $b \leq c$, then $a \leq c$

If either of the premises is a strict inequality, then the conclusion is a strict inequality:

- if $a \geq b$ and $b > c$, then $a > c$
- if $a > b$ and $b \geq c$, then $a > c$

Further modification of the premises from $a \geq b$ to $a = b$ or from $b \geq c$ to $b = c$ retains the strict inequality:

- if $a = b$ and $b > c$, then $a > c$
- if $a > b$ and $b = c$, then $a > c$

## §1.3 Order

Addition of positive values to the side with a larger value or negative values to the side with the smaller value retains the order:

- if $a > b$, then $a + c > b$, where $c \geq 0$
- if $a \geq b$, then $a + c \geq b$, where $c \geq 0$
- if $a \geq b$, then $a + c > b$, where $c > 0$
- if $a > b$, then $a > b + c$, where $c \leq 0$
- if $a \geq b$, then $a \geq b + c$, where $c \leq 0$
- if $a \geq b$, then $a > b + c$, where $c < 0$

A common constant $c$ may be added to or subtracted from both sides of an inequality without changing the order of the direction of the inequality:

For any real numbers $a$, $b$, $c$,

- if $a \geq b$, then $a + c \geq b + c$ and $a - c \geq b - c$
- if $a > b$, then $a + c > b + c$ and $a - c \geq b - c$
- if $a \leq b$, then $a + c \leq b + c$ and $a - c \leq b - c$
- if $a < b$, then $a + c < b + c$ and $a - c < b - c$

Multiplication or division of positive numbers throughout does not change the order of the inequality:

- If $a \geq$ b and $c > 0$, then $ac \geq bc$ and $\frac{a}{c} \geq \frac{b}{c}$
- If $a \leq$ b and $c > 0$, then $ac \leq bc$ and $\frac{a}{c} \leq \frac{b}{c}$

But multiplication or division of negative numbers throughout changes the order of the inequality:

- If $a \geq b$ and $c < 0$, then $ac \leq bc$ and $\frac{a}{c} \leq \frac{b}{c}$
- If $a \leq b$ and $c < 0$, then $ac \geq bc$ and $\frac{a}{c} \geq \frac{b}{c}$

Lastly, the trickiest property involves the reciprocal:

For any non-zero real numbers $a$ and $b$ that are both positive or both negative, the order is reversed when the reciprocal is involved:

- if $a \geq b$, then $\frac{1}{a} \leq \frac{1}{b}$
- if $a > b$, then $\frac{1}{a} < \frac{1}{b}$
- if $a \leq b$, then $\frac{1}{a} \geq \frac{1}{b}$
- if $a < b$, then $\frac{1}{a} > \frac{1}{b}$

Furthermore, the position of the zero depends on the sign of $a$ and $b$:

- if $a \geq b > 0$, then $0 < \frac{1}{a} \leq \frac{1}{b}$
- if $0 > a \geq b$, then $\frac{1}{a} \leq \frac{1}{b} < 0$

However, if one of $a$ and $b$ is positive and the other is negative, then the order stays:

- if $a > 0 > b$, then $\frac{1}{a} > \frac{1}{b}$

### §1.4 Applying a monotone function to the inequality

Any monotone increasing function may be applied to both sides of an inequality (provided they are in the domain of that function) and it will still hold. Applying a monotone decreasing function to both sides of an inequality reverses the order of the inequality:

Given real values $a$ and $b$ and a monotone increasing function f,

- if $a \geq b$, then $\mathrm{f}(a) \geq \mathrm{f}(b)$

with equality holding if and only if $a = b$.

Given real values $a$ and $b$ and a monotone decreasing function f,

- if $a \geq b$, then $\mathrm{f}(a) \leq \mathrm{f}(b)$

with equality holding if and only if $a = b$.

## §2 Solving H2 inequalities

There are two types of solution methods, namely graphical and analytical, in the H2 notes and left for the student to recap.

## §3 H3 Inequalities

There are a few inequalities involving in the functions involved in Chapter 1, namely the floor function and the convex/concave functions.

For the floor function, the inequality is straightforward as the function is increasing:

- if $a \geq b$, then $\lfloor a \rfloor \geq \lfloor b \rfloor$

This can be considered an example of the usage of section 1.3.

**Example 1**

**(a)** Show that $\lfloor 2x \rfloor - 2\lfloor x \rfloor$ is either 0 or 1 for any real number $x$.

**(b)** Also show $\lfloor 2x \rfloor + \lfloor 2y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x+y \rfloor$.

[Solution]

**(a)** Let $\alpha = x - \lfloor x \rfloor$, then we have

$$\begin{aligned}
\lfloor 2x \rfloor - 2\lfloor x \rfloor &= \lfloor 2(\lfloor x \rfloor + \alpha) \rfloor - 2\lfloor x \rfloor \\
&= \lfloor 2\lfloor x \rfloor + 2\alpha \rfloor - 2\lfloor x \rfloor \\
&= \begin{cases} 1 & \text{if } 0.5 \leq \alpha < 1, \\ 0 & \text{if } 0 \leq \alpha < 0.5. \end{cases}
\end{aligned}$$

**(b)** Let $\alpha = x - \lfloor x \rfloor$ and $\beta = y - \lfloor y \rfloor$, then we have

$$\lfloor 2x \rfloor + \lfloor 2y \rfloor = 2\lfloor x \rfloor + \lfloor 2\alpha \rfloor + 2\lfloor y \rfloor + \lfloor 2\beta \rfloor$$

$$\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor = 2\lfloor x \rfloor + 2\lfloor y \rfloor + \lfloor \alpha + \beta \rfloor$$

Hence we only need to show that $\lfloor 2\alpha \rfloor + \lfloor 2\beta \rfloor \geq \lfloor \alpha + \beta \rfloor$ for $0 \leq \alpha < 1$ and $0 \leq \beta < 1$.
There are two cases to show,
Case 1: $0 \leq \alpha < 0.5$ and $0 \leq \beta < 0.5$, then both LHS and RHS are zero and hence the inequality holds.

Case 2: $\alpha \geq 0.5$ or $\beta \geq 0.5$, in this case the LHS is minimally 1 and since the RHS is at most 1, the inequality holds.

With this we shown the inequality required.

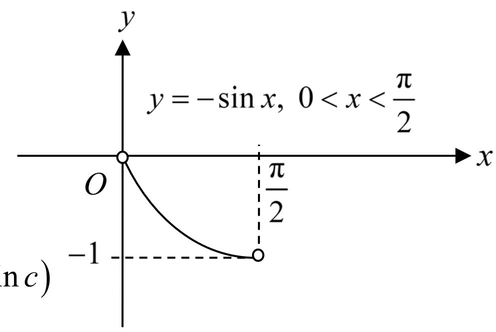
**Example 2**

Given that $0 < a < b < c < \dfrac{\pi}{2}$, by considering the graph of $y = -\sin x$, show that

$$\sin b \geq \min(\sin a, \sin c),$$

where min ($A$, $B$) gives the smaller value of $A$ and $B$.

**[Solution]**
Curve is concave up (convex) by considering the sketch of $y = -\sin x$ for $0 < x < \dfrac{\pi}{2}$,




Since $0 < a < b < c < \dfrac{\pi}{2}$,
therefore $b = ta + (1-t)c$ with $0 \leq t \leq 1$,

$$\begin{aligned}
-\sin b &\leq t(-\sin a) + (1-t)(-\sin c) \\
\sin b &\geq t\sin a + (1-t)(\sin c) \\
&\geq t\min(\sin a, \sin c) + (1-t)\min(\sin a, \sin c) \\
&= \min(\sin a, \sin c)
\end{aligned}$$

(shown)

## §3.1 Triangle Inequality

| If $a$ and $b$ are any real numbers, then $\lvert a+b\rvert \le \lvert a\rvert + \lvert b\rvert$. |
| --- |

Observe that if the numbers $a$ and $b$ are either both positive or both negative at the same time, then both sides of the Triangle Inequality are equal.

**[Proof]**
By properties of absolute value functions we can obtain $-\lvert a\rvert \le a \le \lvert a\rvert$ and $-\lvert b\rvert \le b \le \lvert b\rvert$.

Adding both inequalities we have $-\left(\lvert a\rvert + \lvert b\rvert\right) \le a+b \le \lvert a\rvert + \lvert b\rvert$.

Using the properties of absolute value functions again we obtain the Triangle Inequality.

**Example 3**
If $\lvert x-4\rvert < 0.1$ and $\lvert y-7\rvert < 0.2$, use the Triangle Inequality to estimate $\lvert x+y-11\rvert$.

**[Solution]**

$$\begin{aligned} \lvert x+y-11\rvert &= \lvert (x-4)+(y-7)\rvert \\ &\le \lvert x-4\rvert + \lvert y-7\rvert \\ &< 0.1+0.2 = 0.3 \end{aligned}$$

Thus $\lvert x+y-11\rvert < 0.3$.

**Example 4** (2018 H3 Mathematics A-Level Paper Q3)
A triangle has sides of length $a$, $b$, $c$ units. In each of the following cases, prove that there is a triangle having sides of the given lengths.

**(i)** $\dfrac{a}{1+a}$, $\dfrac{b}{1+b}$ and $\dfrac{c}{1+c}$ units.

**(ii)** $\sqrt{a}$, $\sqrt{b}$ and $\sqrt{c}$ units.

**(iii)** $\sqrt{a(b+c-a)}$, $\sqrt{b(c+a-b)}$ and $\sqrt{c(a+b-c)}$ units.

**[Solution] (For Example 4)**

**(i)** Without loss of generality, we may assume that $a \le b \le c$. By monotonicity of the function $y = \frac{x}{1+x}$, $x > 0$, we have $\frac{a}{1+a} \le \frac{b}{1+b} \le \frac{c}{1+c}$. It remains to be shown that $\frac{a}{1+a} + \frac{b}{1+b} > \frac{c}{1+c}$, as the remaining other triangle inequalities are trivially true.

$$\begin{aligned}
\frac{a}{1+a} + \frac{b}{1+b} &\ge \frac{a}{1+b} + \frac{b}{1+b} \quad \text{(bigger denominator } 1+b\text{)} \\
&= \frac{a+b}{1+b} \\
&> \frac{c}{1+b} \quad \text{(bigger numerator } c,\ \text{triangle inequality)} \\
&\ge \frac{c}{1+c} \quad \text{(bigger denominator } 1+c\text{)} \qquad \text{(shown)}
\end{aligned}$$

**(ii)** Without loss of generality, we may assume that $a \le b \le c$. By monotonicity of the function $y = \sqrt{x}$, $x > 0$, we have $\sqrt{a} \le \sqrt{b} \le \sqrt{c}$. It remains to be shown that $\sqrt{a} + \sqrt{b} > \sqrt{c}$, as the remaining other triangle inequalities are trivially true.

$$\begin{aligned}
\left(\sqrt{a} + \sqrt{b}\right)^2 &= a + b + 2\sqrt{ab} \\
&> c + 2\sqrt{ab} \\
&> c > 0
\end{aligned}$$

$$\sqrt{a} + \sqrt{b} > \sqrt{c} \quad \text{(shown)}$$

**(iii)** We first show that a triangle with sides $a(b+c-a)$, $b(c+a-b)$, $c(a+b-c)$ exist then a direct application of the result from (ii) will suffice.

WLOG similarly, we only need to show that $a(b+c-a) + b(c+a-b) > c(a+b-c)$ due to the symmetricity of the terms involved.

$$\begin{aligned}
& a(b+c-a) + b(c+a-b) > c(a+b-c) \\
\Leftrightarrow \quad & ab + ac - a^2 + bc + ab - b^2 > ac + bc - c^2 \\
\Leftrightarrow \quad & c^2 - a^2 + 2ab - b^2 > 0 \\
\Leftrightarrow \quad & c^2 - (a-b)^2 > 0 \\
\Leftrightarrow \quad & (c+a-b)(c-a+b) > 0 \qquad \text{which is true by triangle inequality}
\end{aligned}$$

Hence, using $(a+c-b)(a-c+b) > 0$ and $(b+a-c)(b-a+c) > 0$, we obtain the three necessary and sufficient conditions for a triangle with sides $a(b+c-a)$, $b(c+a-b)$, $c(a+b-c)$ to exist, and as mentioned earlier, using the result from **(ii)**, we will have the existence of a triangle with the required three sides.

## §3.2 AM – GM Inequality

**Arithmetic Mean (AM)**

The Arithmetic Mean of $x_1, x_2, \ldots, x_n$ is defined as $\frac{1}{n}\sum_{i=1}^{n} x_i$

**Geometric Mean (GM)**

The Geometric Mean of $x_1, x_2, \ldots, x_n$ is defined as $\sqrt[n]{x_1 x_2 \cdots x_n}$

For example,

The arithmetic mean of 2, 4, 5, 9, 10 (5 numbers in total) is $\dfrac{2+4+5+9+10}{5} = 6$.

The geometric mean of 2, 4, 5, 9, 10 is $\sqrt[5]{(2)(4)(5)(9)(10)} = 5.14 \text{ (3 s.f.)}$.

Notice that the arithmetic mean of the 5 positive numbers is greater than its geometric mean.

### §3.2.1 AM – GM Inequality for *n* = 2 (see §3.2.2)

$$\frac{a+b}{2} \geq \sqrt{ab} \quad \forall a,b \in \mathbb{R}_0^+.$$

**[Proof]**

Using $(a-b)^2 \geq 0 \quad \forall a,b \in \mathbb{R}_0^+$, $(a-b)^2 = a^2 - 2ab + b^2$, $(a+b)^2 = a^2 + 2ab + b^2$

$$\begin{aligned}
& (a-b)^2 \geq 0 \\
\Rightarrow \quad & a^2 - 2ab + b^2 \geq 0 \\
\Rightarrow \quad & a^2 + 2ab + b^2 \geq 4ab \\
\Rightarrow \quad & (a+b)^2 \geq 4ab \\
\Rightarrow \quad & a + b \geq 2\sqrt{ab} \qquad (\text{Since } a \geq 0,\ b \geq 0)
\end{aligned}$$

Note that equality holds when $a = b$.

**Example 5**
If $a$, $b$, $c$ are positive numbers, show that $(a+b)(b+c)(c+a) \geq 8abc$.

**[Solution]**

Apply AM – GM Inequality three times, we have

$$\frac{(a+b)}{2} \geq \sqrt{ab}$$

$$\frac{(b+c)}{2} \geq \sqrt{bc}$$

$$\frac{(c+a)}{2} \geq \sqrt{ca}$$

Hence, $\left(\frac{(a+b)}{2}\right)\left(\frac{(b+c)}{2}\right)\left(\frac{(c+a)}{2}\right) \geq \sqrt{(ab)(bc)(ca)}$

Since $a$, $b$, $c$ are positive numbers,

$$(a+b)(b+c)(c+a) \geq 8abc$$

### §3.2.2 AM – GM Inequality for general *n*

$$\frac{1}{n}\sum_{i=1}^{n} x_i \geq \sqrt[n]{x_1 x_2 \cdots x_n} \text{ for } x_1, x_2, \ldots, x_n \in \mathbb{R}_0^+$$

Note: Equality holds if and only if $x_1 = x_2 = \ldots = x_n$.

The proof using induction is left as an exercise in the tutorial.

**Example 6**
Show that $A$, the maximum area of a triangle $XYZ$ with a fixed perimeter $P$, happens when the triangle is equilateral. You may use the Heron's formula which states that $A$, the area of a triangle is $A = \sqrt{\frac{P}{2}\left(\frac{P}{2}-x\right)\left(\frac{P}{2}-y\right)\left(\frac{P}{2}-z\right)}$ where $x$, $y$, $z$ are the lengths of the sides for the triangle.

**[Solution]**
Using the AM-GM inequality,

$$\left(\frac{P}{2}-x\right)\left(\frac{P}{2}-y\right)\left(\frac{P}{2}-z\right) \leq \left(\frac{\frac{P}{2}-x+\frac{P}{2}-y+\frac{P}{2}-z}{3}\right)^3 = \left(\frac{P}{6}\right)^3$$

Hence the maximum area is $A = \sqrt{\frac{P}{2}\left(\frac{P}{6}\right)^3} = \frac{P^2}{12\sqrt{3}}$ and occurs when $\frac{P}{2}-x = \frac{P}{2}-y = \frac{P}{2}-z$, i.e. $x = y = z$ which means that the triangle is equilateral.

## §3.3 Cauchy-Schwarz Inequality

There are 2 versions for the Cauchy-Schwarz Inequality, one is for summation and one is for integration.

### §3.3.1 Cauchy-Schwarz Inequality for Sums

For real values $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$,

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)$$

with equality holding if and only if $a_k = tb_k$ for some real value *t* and for all $k = 1,\ 2,\ ...\ ,\ n$.

**[Proof]**
(Side Note: While it can be seen directly from using dot product in vectors in H2, we did not establish the dot product definition in terms of why it works.)

Let $\mathbf{v} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ for real values $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$.

Note that the inequality holds trivially for $\mathbf{v} = \mathbf{0}$

Let $\lambda \in \mathbb{R}$, since $|\mathbf{u} - \lambda \mathbf{v}|^2 \geq 0$,

$$(\mathbf{u} - \lambda \mathbf{v}) \bullet (\mathbf{u} - \lambda \mathbf{v}) \geq 0$$

$$\mathbf{u} \bullet \mathbf{u} - \lambda \mathbf{v} \bullet \mathbf{u} - \lambda \mathbf{u} \bullet \mathbf{v} + \lambda^2 \mathbf{v} \bullet \mathbf{v} \geq 0$$

$$(\mathbf{v} \bullet \mathbf{v})\lambda^2 - (2\mathbf{u} \bullet \mathbf{v})\lambda + (\mathbf{u} \bullet \mathbf{u}) \geq 0$$

This is a quadratic function in $\lambda \in \mathbb{R}$, hence the discriminant must be less than or equal to zero.

Therefore

$$(2\mathbf{u} \bullet \mathbf{v})^2 - 4(\mathbf{v} \bullet \mathbf{v})(\mathbf{u} \bullet \mathbf{u}) \leq 0$$

$$(\mathbf{u} \bullet \mathbf{v})^2 \leq |\mathbf{u}|^2 |\mathbf{v}|^2$$

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) \qquad \text{(proven)}$$

Note that for the equality to hold, the discriminant is zero. This means that quadratic function $\mathrm{f}(\lambda) = (\mathbf{v} \bullet \mathbf{v})\lambda^2 - (2\mathbf{u} \bullet \mathbf{v})\lambda + (\mathbf{u} \bullet \mathbf{u})$ has 1 repeated root. The existence of the root means that there exists a real value *t* such that $(\mathbf{v} \bullet \mathbf{v})t^2 - (2\mathbf{u} \bullet \mathbf{v})t + (\mathbf{u} \bullet \mathbf{u}) = 0$. Since the LHS is equivalent to $|\mathbf{u} - t\mathbf{v}|^2$, we have $|\mathbf{u} - t\mathbf{v}|^2 = 0$ i.e. $a_k = tb_k$ for some real value *t* and for all $k = 1,\ 2,\ ...\ ,\ n$.

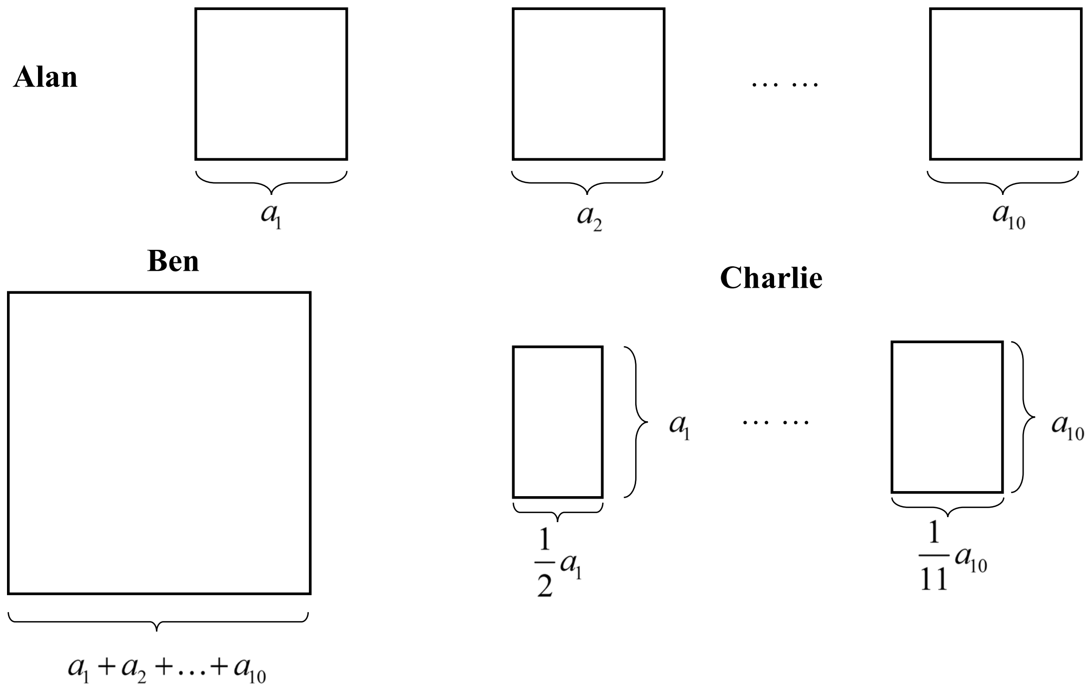
**Example 7 (2018 DHS H3 Prelim)**

**(i)** Let $n$ be a positive integer. Show that $\frac{x_1^{\,2}}{y_1}+\frac{x_2^{\,2}}{y_2}+\ldots+\frac{x_n^{\,2}}{y_n} \geq \frac{\left(x_1+x_2+\ldots+x_n\right)^2}{y_1+y_2+\ldots+y_n}$ for all positive real numbers $y_1, y_2, \ldots, y_n$. [4]

**(ii)** Alan has ten square pieces of paper with lengths of each side being $a_1, a_2, \ldots, a_{10}$ respectively.

Ben cuts out a square piece of paper with length of each side being the combined lengths of Alan's ten pieces of paper.

Charlie then cuts Alan's first piece vertically into two equal strips and takes a strip. He then cuts Alan's second piece vertically into three equal strips and takes a strip. This continues until he cuts Alan's 10th piece vertically into 11 equal strips and takes a strip. (See diagrams below. Diagrams are not drawn to scale).




The area of Ben's square paper is 65 times that of the total area of the 10 pieces of paper that Charlie has taken. By using the result in **(i)** and $a_1 = 1$, find $a_{10}$. [4]

**[Solution] (For Example 7)**

**(i)** By Cauchy Schwarz Inequality $\left(\sum_{i=1}^{n} c_i^2\right)\left(\sum_{i=1}^{n} d_i^2\right) \ge \left(\sum_{i=1}^{n} c_i d_i\right)^2$

Using $c_i = \dfrac{x_i}{\sqrt{y_i}},\ d_i = \sqrt{y_i}$,

$$\left(\frac{x_1^2}{y_1}+\frac{x_2^2}{y_2}+\ldots+\frac{x_n^2}{y_n}\right)\left(y_1+y_2+\ldots+y_n\right) \ge \left(x_1+x_2+\ldots+x_n\right)^2 \qquad (*)$$

Since $y_i$ are all positive so $y_1+y_2+\ldots+y_n > 0$,

$$\frac{x_1^2}{y_1}+\frac{x_2^2}{y_2}+\ldots+\frac{x_n^2}{y_n} \ge \frac{\left(x_1+x_2+\ldots+x_n\right)^2}{y_1+y_2+\ldots+y_n} \quad \text{(shown)}$$

**(ii)** Using **(i)**,

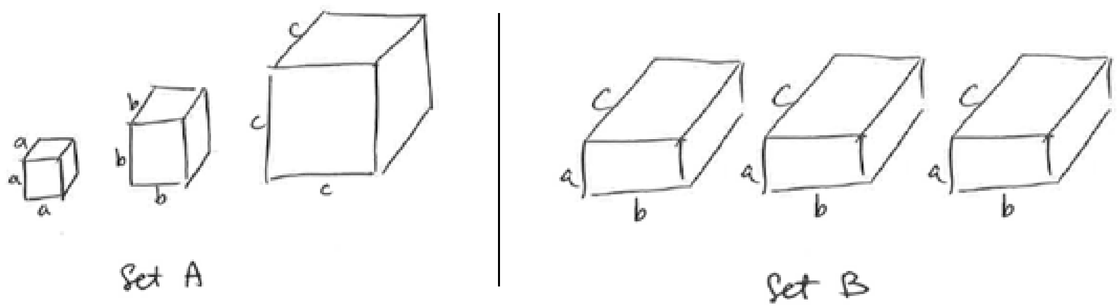
$$\frac{a_1^2}{2}+\frac{a_2^2}{3}+\ldots+\frac{a_{10}^2}{11} \ge \frac{\left(a_1+a_2+\ldots+a_{10}\right)^2}{2+3+\ldots+11}$$

Since $\dfrac{a_1^2}{2}+\dfrac{a_2^2}{3}+\ldots+\dfrac{a_{10}^2}{11} = \dfrac{\left(a_1+a_2+\ldots+a_{10}\right)^2}{65}$ equality holds,

$\dfrac{a_i}{\sqrt{i+1}} = k\sqrt{i+1} \therefore a_i = k(i+1)$ for all $1 \le i \le 10, i \in \mathbb{Z}$.

Since $a_1 = 1,\ k = \dfrac{1}{2}\ \ a_{10} = \dfrac{1}{2}(10+1) = \dfrac{11}{2}$

**Example 8**
Two sets of 3 boxes are given as follows:




Show that the total surface area of the boxes in set A is not less than that of the boxes in set B.

**[Solution]**

Total Surface Area of boxes in set A

$= 6\left(a^2 + b^2 + c^2\right)$

$= 6\left(a^2 + b^2 + c^2\right)^{1/2}\left(b^2 + c^2 + a^2\right)^{1/2}$

$\geq 6\left(ab + bc + ca\right)$ $\qquad$ (by Cauchy-Schwarz Inequality)

$= 3\left(2\left(ab + bc + ca\right)\right)$

$=$ Total Surface Area of boxes in set B $\qquad$ (Shown)

### §3.3.2 Cauchy-Schwarz Inequality (Extensions)

There are a few simple extensions and variations:

1) $\left(\left|\sum_{k=1}^{n} a_k b_k\right|\right)^2 \leq \left(\sum_{k=1}^{n} \left|a_k\right|^2\right)\left(\sum_{k=1}^{n} \left|b_k\right|^2\right)$

(Applied to complex numbers with triangle inequality)

2) $\left(\int_a^b \mathrm{f}(t)\mathrm{g}(t)\ \mathrm{d}t\right)^2 \leq \left(\int_a^b \left(\mathrm{f}(t)\right)^2\ \mathrm{d}t\right)\left(\int_a^b \left(\mathrm{g}(t)\right)^2\ \mathrm{d}t\right)$ (Applied to integrals)

Side Note: Cauchy-Schwarz Inequality for summation is a special case of Holder's Inequality (Out of syllabus)

**[Proof]**
(1) is left as an exercise for students. The hint given is sufficiently explicit.

(2): For all $x \in \mathbb{R}$,

$$\left(x\ \mathrm{f}(t) + \mathrm{g}(t)\right)^2 \geq 0$$

$$\int_a^b \left(x\ \mathrm{f}(t) + \mathrm{g}(t)\right)^2\ \mathrm{d}t \geq 0 \qquad \text{(Area under graph of positive function is } \geq 0\text{)}$$

$$\int_a^b x^2 \left(\mathrm{f}(t)\right)^2\ \mathrm{d}t + \int_a^b 2x\ \mathrm{f}(t)\mathrm{g}(t)\ \mathrm{d}t + \int_a^b \left(\mathrm{g}(t)\right)^2\ \mathrm{d}t \geq 0 \qquad \text{(Expand)}$$

$$x^2 \int_a^b \left(\mathrm{f}(t)\right)^2\ \mathrm{d}t + 2x \int_a^b \mathrm{f}(t)\mathrm{g}(t)\ \mathrm{d}t + \int_a^b \left(\mathrm{g}(t)\right)^2\ \mathrm{d}t \geq 0 \qquad (x \text{ is independent of } t)$$

The rest is to apply the discriminant analysis on quadratic real valued functions, similar to the proof in section 3.3.1.

**Example 9**
Prove the Cauchy-Schwarz Inequality for Expectations in Probability, namely

$$\left(E(XY)\right)^2 \leq E\left(X^2\right)E\left(Y^2\right)$$

**[Solution]**
Unless $Y = -tX$ for some constant *t*, in which equality holds for the given inequality, it follows that for all $t \in \mathbb{R}$,

$$0 < E\left[\left(tX + Y\right)^2\right] = t^2 E\left(X^2\right) + 2tE(XY) + E\left(Y^2\right)$$

Hence, the roots of the quadratic equation in *t*

$$t^2 E\left(X^2\right) + 2tE(XY) + E\left(Y^2\right) = 0$$

must be imaginary, i.e. the discriminant must be less than zero:

$$\left(2E(XY)\right)^2 - 4E\left(X^2\right)E\left(Y^2\right) < 0$$

$$4\left(E(XY)\right)^2 < 4E\left(X^2\right)E\left(Y^2\right)$$

$$\left(E(XY)\right)^2 < E\left(X^2\right)E\left(Y^2\right)$$

Combining cases, we obtain $\left(E(XY)\right)^2 \leq E\left(X^2\right)E\left(Y^2\right)$ (shown)

## §4 References

https://en.wikipedia.org/wiki/Inequality_(mathematics)

http://artofproblemsolving.com/wiki/index.php/Inequality