9. (i)

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 & 3 \\ 2 & 1 & 5 & -7 \\ 4 & -3 & 7 & {}^{-}1 \\ 3 & 14 & 15 & -43 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & \frac{11}{5} & -\frac{11}{5} \\ 0 & 1 & \frac{3}{5} & -\frac{13}{5} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{K} = \ker(\mathbf{A}) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : \begin{bmatrix} 1 & 0 & \frac{11}{5} & -\frac{11}{5} \\ 0 & 1 & \frac{3}{5} & -\frac{13}{5} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\implies \begin{cases} x_4 = \mu \\ x_3 = \lambda \\ x_2 = -\frac{3}{5}\lambda + \frac{13}{5}\mu \\ x_1 = -\frac{11}{5}\lambda + \frac{11}{5}\mu \end{cases}, \ \lambda, \mu \in \mathbb{R}$$

$$\mathbf{K} = \left\{ \lambda \begin{bmatrix} -11 \\ -3 \\ 5 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 11 \\ 13 \\ 0 \\ 5 \end{bmatrix}, \ \lambda, \mu \in \mathbb{R} \right\}$$

When $\lambda = \mu = 0$, $\lambda \begin{bmatrix} -11 \\ -3 \\ 5 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 11 \\ 13 \\ 0 \\ 5 \end{bmatrix} = \mathbf{0} \in \mathbf{K}$.

Let $\mathbf{k}_{i,j} = \lambda_i \begin{bmatrix} -11 \\ -3 \\ 5 \\ 0 \end{bmatrix} + \mu_j \begin{bmatrix} 11 \\ 13 \\ 0 \\ 5 \end{bmatrix} \in \mathbf{K}$.

$$\begin{aligned} \alpha\mathbf{k}_{1,1} + \beta\mathbf{k}_{2,2} &= \alpha \left(\lambda_1 \begin{bmatrix} -11 \\ -3 \\ 5 \\ 0 \end{bmatrix} + \mu_1 \begin{bmatrix} 11 \\ 13 \\ 0 \\ 5 \end{bmatrix} \right) + \beta \left(\lambda_2 \begin{bmatrix} -11 \\ -3 \\ 5 \\ 0 \end{bmatrix} + \mu_2 \begin{bmatrix} 11 \\ 13 \\ 0 \\ 5 \end{bmatrix} \right) \\ &= \alpha\lambda_1 \begin{bmatrix} -11 \\ -3 \\ 5 \\ 0 \end{bmatrix} + \alpha\mu_1 \begin{bmatrix} 11 \\ 13 \\ 0 \\ 5 \end{bmatrix} + \beta\lambda_2 \begin{bmatrix} -11 \\ -3 \\ 5 \\ 0 \end{bmatrix} + \beta\mu_2 \begin{bmatrix} 11 \\ 13 \\ 0 \\ 5 \end{bmatrix} \\ &= (\alpha\lambda_1 + \beta\lambda_2) \begin{bmatrix} -11 \\ -3 \\ 5 \\ 0 \end{bmatrix} + (\alpha\mu_1 + \beta\mu_2) \begin{bmatrix} 11 \\ 13 \\ 0 \\ 5 \end{bmatrix} \in \mathbf{K} \end{aligned}$$

Therefore, $\mathbf{K}$ is a vector space. $\square$

$$\dim(\mathbf{K}) = \boxed{2}$$

(ii)

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & 3 \\ 2 & 1 & 5 & -7 & 1 \\ 4 & -3 & 7 & -1 & 7 \\ 3 & 14 & 15 & -43 & -11 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & 0 & \frac{11}{5} & -\frac{11}{5} & 1 \\ 0 & 1 & \frac{3}{5} & -\frac{13}{5} & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : \begin{bmatrix} 1 & 0 & \frac{11}{5} & -\frac{11}{5} \\ 0 & 1 & \frac{3}{5} & -\frac{13}{5} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\implies \begin{cases} x_4 = \mu \\ x_3 = \lambda \\ x_2 = -1 - \frac{3}{5}\lambda + \frac{13}{5}\mu \\ x_1 = 1 - \frac{11}{5}\lambda + \frac{11}{5}\mu \end{cases}, \lambda, \mu \in \mathbb{R}$$

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -11 \\ -3 \\ 5 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 11 \\ 13 \\ 0 \\ 5 \end{bmatrix}, \lambda, \mu \in \mathbb{R} \right\}$$

Suppose $(\exists s_p \in S)(s_p = \mathbf{0})$.

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \lambda_p \begin{bmatrix} -11 \\ -3 \\ 5 \\ 0 \end{bmatrix} + \mu_p \begin{bmatrix} 11 \\ 13 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{cases} 0 + 5\lambda_p + 0\mu_p = 0 \\ 0 + 0\lambda_p + 5\mu_p = 0 \\ 1 - 11\lambda_p + 11\mu_p = 0 \end{cases} \implies \begin{cases} \lambda_p = 0 \\ \mu_p = 0 \\ 1 = 0 \end{cases} \implies \Leftarrow$$

$$\implies \mathbf{0} \notin S$$

Therefore, S is not a vector space.