Theorem. $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

Proof. Let $\mathbf{A}$ and $\mathbf{B}$ be $N \times N$ matrices.

Case 1. $\mathbf{A}$ is singular.

$$\det(\mathbf{A}) = 0 \implies \operatorname{rank}(\mathbf{A}) < N$$

$$\operatorname{rank}(\mathbf{AB}) \leqslant \operatorname{rank}(\mathbf{A}) < N$$

Thus, $\mathbf{AB}$ is singular.

$$\det(\mathbf{AB}) = 0 = 0\det(\mathbf{B}) = \det(\mathbf{A})\det(\mathbf{B})$$

Case 2. $\mathbf{A}$ is elementary.

Type 1. Let $\mathbf{A} = \mathbf{T}_{ij}$, where $\mathbf{T}_{ij}$ is the matrix produced by exchanging row i and j of $\mathbf{I}_N$. It is clear that the following is true.

$$\det(\mathbf{AB}) = -\det(\mathbf{B})$$

$$\det(\mathbf{A}) = -1$$

Thus,

$$\det(\mathbf{AB}) = -\det(\mathbf{B}) = (-1)\det(\mathbf{B}) = \det(\mathbf{A})\det(\mathbf{B})$$

Type 2. Let $\mathbf{A} = \mathbf{D}_i(m)$, where $\mathbf{D}_i(m)$ is the matrix produced from $\mathbf{I}_N$ by multiplying row i by m. It is clear that the following is true.

$$\det(\mathbf{AB}) = m\det(\mathbf{B})$$

$$\det(\mathbf{A}) = m$$

Thus,

$$\det(\mathbf{AB}) = m\det(\mathbf{B}) = \det(\mathbf{A})\det(\mathbf{B})$$

Type 3. Let $\mathbf{A} = \mathbf{L}_{ij}(m)$, where $\mathbf{L}_{ij}(m)$ is the matrix produced from $\mathbf{I}_N$ by adding m times row i to row j. It is clear that the following is true.

$$\det(\mathbf{AB}) = \det(\mathbf{B})$$

$$\det(\mathbf{A}) = 1$$

Thus,

$$\det(\mathbf{AB}) = \det(\mathbf{B}) = 1\det(\mathbf{B}) = \det(\mathbf{A})\det(\mathbf{B})$$

Case 3. $\mathbf{A}$ is general. Notice that $\mathbf{A}$ can be rewritten as a product of k elementary matrices, $\mathbf{E}$, such that

$$\mathbf{A} = \prod_{r=1}^{k} \mathbf{E}_r$$

$$\begin{aligned}
\det(\mathbf{AB}) &= \det\left(\left(\prod_{r=1}^{k} \mathbf{E}_r\right)\mathbf{B}\right) \\
&= \det\left(\mathbf{E}_k\left(\prod_{r=1}^{k-1} \mathbf{E}_r\right)\mathbf{B}\right) \\
&= \det(\mathbf{E}_k)\det\left(\left(\prod_{r=1}^{k-1} \mathbf{E}_r\right)\mathbf{B}\right) \\
&= \cdots \\
&= \left(\prod_{r=1}^{k} \det(\mathbf{E}_r)\right)\det(\mathbf{B}) \\
&= \left(\prod_{r=3}^{k} \det(\mathbf{E}_k)\right)\det(\mathbf{E}_2)\det(\mathbf{E}_1)\det(\mathbf{B}) \\
&= \left(\prod_{r=3}^{k} \det(\mathbf{E}_k)\right)\det(\mathbf{E}_2\mathbf{E}_1)\det(\mathbf{B}) \\
&= \cdots \\
&= \det\left(\prod_{r=1}^{k} \mathbf{E}_k\right)\det(\mathbf{B}) \\
&= \det(\mathbf{A})\det(\mathbf{B})
\end{aligned}$$

Therefore,

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$$

□