$$
\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}t} &= (\text{per-capita birth rate})P - (\text{per-capita death rate})P \\
&= \beta P - (\alpha + \gamma P)P \\
&= P(\beta - \alpha - \gamma P) \\
&= P\Big(k - k\frac{\gamma}{k}P\Big) \\
&= kP\Big(1 - \frac{\gamma}{k}P\Big) \\
&= kP\left(1 - \frac{P}{N}\right),\ N = \frac{k}{\gamma}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}t} &= kP\left(1 - \frac{P}{N}\right) \\
\frac{\mathrm{d}P}{\mathrm{d}t} &= k\frac{P(N-P)}{N} \\
\frac{N}{P(N-P)}\,\mathrm{d}P &= k\,\mathrm{d}t \\
\int\left(\frac{1}{P} + \frac{1}{N-P}\right)\mathrm{d}P &= \int k\,\mathrm{d}t \\
\ln|P| - \ln|N-P| &= kt + c \\
\ln\left|\frac{P}{N-P}\right| &= kt + c \\
\frac{P}{N-P} &= Ae^{kt} \implies A = \frac{P_0}{N-P_0} \\
P &= ANe^{kt} - APe^{kt} \\
P + APe^{kt} &= ANe^{kt} \\
P\big(1 + Ae^{kt}\big) &= ANe^{kt} \\
P &= \frac{ANe^{kt}}{1 + Ae^{kt}} \\
&= \frac{N}{\frac{1}{A}e^{-kt} + 1} \\
&= \frac{N}{1 + \frac{N-P_0}{P_0}e^{-kt}} \\
&= \frac{N}{1 + be^{-kt}},\ b = \frac{N}{P_0} - 1
\end{aligned}
$$

$$
\therefore \boxed{P(t) = \frac{N}{1 + be^{-kt}}}
$$