ANALYSIS
TUTORIAL 3: SEQUENCES & SERIES

1 The geometric progression U has terms $u_1, u_2, u_3, \ldots$, with common ratio r, where $|r|<1$. It is given that $v_i = u_i^2$ for $i = 1, 2, 3, \ldots$.

(i) Show that $\sum_{i=1}^{n} v_i = \frac{u_1}{1+r}\sum_{i=1}^{2n} u_i$.

It is given further that $w_i = v_i - v_{i+1}$ for $i = 1, 2, 3, \ldots$.

(ii) Show that $\sum_{i=1}^{n} w_i = u_1(1-r)\sum_{i=1}^{2n} u_i$.

Let $S_U = \sum_{i=1}^{\infty} u_i$, $S_V = \sum_{i=1}^{\infty} v_i$ and $S_W = \sum_{i=1}^{\infty} w_i$. Show that

(iii) $\frac{S_u}{S_V} + \frac{1}{S_U} = \frac{2}{u_1}$,

(iv) $S_W = u_1^2$.

2 For each of the following sequences, determine if it converges or diverges.

(a) $u_n = \sin\frac{n\pi}{2}$, $n = 1, 2, 3, \ldots$.

(b) $u_n = \frac{n^3}{3^n}$, $n = 1, 2, 3, \ldots$.

(c) $u_n = \frac{n^n}{n!}$, $n = 1, 2, 3, \ldots$.

(d) $u_n = \frac{\sin n}{n}$, $n = 1, 2, 3, \ldots$.

Hint:
For (b), consider a function $\mathrm{f}(x)$ such that $\mathrm{f}(n) = u_n$ for every n, show that $\mathrm{f}'(x) < 0$ and hence deduce that f approaches certain value as $x \to \infty$.
For (c), show that u_n is strictly increasing as n increases.
For (d), recall that $-1 \le \sin n \le 1$ for every n.

3 Show that the following series is divergent.

$$5 + \sqrt{5} + \sqrt[3]{5} + \sqrt[4]{5} + \ldots.$$

4 Consider the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ (again).

Denote its k^{th} partial sum by S_k, i.e. $S_k = \sum_{n=1}^{k} \frac{1}{n}$.

(i) Show that $\sum_{n=2^{k-1}+1}^{2^k} \frac{1}{n} \geq \frac{1}{2}$ for every positive integer k.

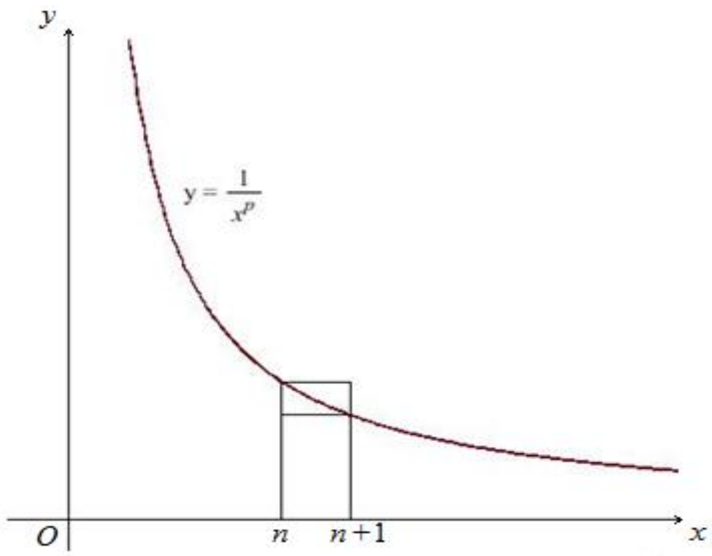
(ii) Show that $S_{2^k} \geq 1 + \frac{k}{2}$ for every positive integer k.

(iii) Hence, determine whether the series converges or diverges.

5 Prove that the following sum of series is less than $\frac{3}{2}$ using the formula for the sum of an infinite geometric series:

$$1 + \frac{1}{3} + \frac{1}{4^2} + \frac{1}{5^3} + \frac{1}{6^4} + \ldots.$$

6 The diagram below shows a sketch of the graph $y = \frac{1}{x^p}$, where $p > 0$, $p \neq 1$.



By considering the area of appropriate rectangles and the area between the graph and the x-axis, where $n \geq 1$,

Show that $$\frac{1}{(n+1)^p} < \frac{1}{1-p}(n+1)^{1-p} - \frac{1}{1-p} n^{1-p} < \frac{1}{n^p}$$

Deduce that

$$\frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \frac{1}{(n+1)^p} < \frac{1}{(1-p)}\left[(n+1)^{1-p} - 1\right] < 1 + \frac{1}{2^p} + \ldots + \frac{1}{(n-1)^p} + \frac{1}{n^p}$$

Deduce the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, when

(a) $p = \frac{1}{2}$, **(b)** $p = 2$.

7 **(i)** Sketch the graph of $y=\frac{1}{x}$ and hence explain why $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}>\int_1^{n+1}\frac{\mathrm{d}x}{x}$.

(ii) Sketch the graph of $y=\sin x$ and determine the largest constant a such that

$ax\le\sin x$ for $0\le x\le\frac{\pi}{2}$.

(iii) Part of a proof of convergence and divergence of series in a textbook is as follows:

(1) Let n be a positive integer. Then

(2) $\sum_{i=1}^{n}\sin\frac{1}{i}\ge\frac{2}{\pi}\sum_{i=1}^{n}\frac{1}{i}$

(3) $\ge\frac{2}{\pi}\ln(n+1).$

(4) $\sum_{i=1}^{n}\sin^2\frac{1}{i}\le\sum_{i=1}^{n}\frac{1}{i^2}<1+\sum_{i=2}^{n}\frac{1}{i(i-1)}<2.$

Explain the second and third lines of the proof.

Hence determine for every positive integer k, if the series $\sin^k\frac{1}{1}+\sin^k\frac{1}{2}+\sin^k\frac{1}{3}+...$ is convergent or divergent.

8 The r^{th} term u_r of an arithmetic progression U is given by

$$u_r=a+(r-1)d,\qquad \text{for } r=1,2,3,\ldots.$$

The progression is such that there exists a term of U equal to a^2. Given that a and d are positive integers, show that

(i) for each term u_r of U, there exists a term of U equal to u_r^2,

(ii) for each term u_r of U, there exists a term of U equal to u_r^3,

(iii) all the terms of the geometric progression $a,a^2,a^3,\ldots$ are terms of U.

If a and d are not required to be positive integers, show, by giving a counter-example, that the result in part **(i)** is not necessarily true.

9 A sequence u_n ($n=0,1,2,\ldots$) is defined by $u_0=X$ and $u_{n+1}=\frac{au_n+b}{cu_n+d}$ for $n\ge0$, where a,b,c and d are non-zero constants such that $ad\ne bc$. It may be assumed that $cu_n+d\ne0$ for all n. Find an expression for u_{n+2} in terms of u_n.

The sequence is said to repeat after p steps if $u_p=u_0$, whatever the value of X.

(i) Show that the sequence repeats after 2 steps if and only if $a+d=0$.
Show also that if $a^2+d^2+2bc=0$, then the sequence repeats after 4 steps.

(ii) If $u_1=Y$, find an expression for u_0 in terms of Y.
Write down an expression for u_3 in terms of Y, and hence show that if $a^2+d^2+ad+bc=0$, then the sequence repeats after 3 steps.