Solution for The Pure Mathematics Portion of GCE 'A' Levels Further Mathematics Paper 2

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November 30, 2020

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(i)

Let $\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 & -1 & 5 \\ 1 & 3 & -2 & 1 & -1 \\ 1 & 0 & 1 & 3 & 0 \\ 0 & 1 & -1 & 1 & 2 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$.

$$\begin{aligned}
\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 & -1 & 5 \\ 1 & 3 & -2 & 1 & -1 \\ 1 & 0 & 1 & 3 & 0 \\ 0 & 1 & -1 & 1 & 2 \end{bmatrix} &\to \begin{bmatrix} 1 & -1 & 2 & -1 & 5 \\ 0 & 4 & -4 & 2 & -6 \\ 0 & 1 & -1 & 4 & -5 \\ 0 & 1 & -1 & 1 & 2 \end{bmatrix} \\
\to \begin{bmatrix} 1 & -1 & 2 & -1 & 5 \\ 0 & 1 & -1 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 & -1 & 4 & -5 \\ 0 & 1 & -1 & 1 & 2 \end{bmatrix} &\to \begin{bmatrix} 1 & 0 & 1 & -\frac{1}{2} & \frac{7}{2} \\ 0 & 1 & -1 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 0 & \frac{7}{2} & -\frac{7}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{7}{2} \end{bmatrix} \\
\to \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & \frac{7}{2} & -\frac{7}{2} \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} &\to \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
\to \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} &= \operatorname{rref}(\mathbf{A})
\end{aligned}$$

$$\mathbf{Ax} = \mathbf{0} \implies \operatorname{rref}(\mathbf{A})\mathbf{x} = \mathbf{0} \implies \begin{cases} x_1 + x_3 = 0 \\ x_2 - x_3 = 0 \\ x_4 = 0 \\ x_5 = 0 \end{cases} \implies \begin{cases} x_1 = -x_3 \\ x_2 = x_3 \\ x_4 = 0 \\ x_5 = 0 \end{cases}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \\ 0 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \; x_3 \in \mathbb{R}$$

$$S = \ker(\mathbf{A}) \implies B_S = B_{\ker(\mathbf{A})} = \boxed{\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}}$$

(ii)

$$\dim(S) = \dim(\ker \mathbf{A}) = \boxed{1}$$

2

(i)

$$\begin{aligned} z^7 &= 1 \\ z^7 &\equiv e^{i(2\pi m)}, \; m \in \{0, \pm 1, \pm 2, \pm 3\} \\ z &\in \left\{ 1, e^{i\left(\pm\frac{2\pi}{7}\right)}, e^{i\left(\pm\frac{4\pi}{7}\right)}, e^{i\left(\pm\frac{6\pi}{7}\right)} \right\} \end{aligned}$$

$$\begin{aligned} z^7 - 1 &= (z-1) \prod_{n=1}^{3} \left(z - e^{i\left(\frac{2n\pi}{7}\right)} \right) \left(z - e^{i\left(-\frac{2n\pi}{7}\right)} \right) \\ \frac{z^7 - 1}{z - 1} &= \prod_{n=1}^{3} \left(z^2 - \left(2 \cos \frac{2n\pi}{7} \right) z + e^{i(0)} \right) \\ \sum_{n=0}^{6} z^n &= \boxed{\left(z^2 - \left(2 \cos \frac{2\pi}{7} \right) z + 1 \right) \left(z^2 - \left(2 \cos \frac{4\pi}{7} \right) z + 1 \right) \left(z^2 - \left(2 \cos \frac{6\pi}{7} \right) z + 1 \right)} \end{aligned}$$

(ii)

$$\begin{aligned} &\cos\left(\frac{1}{7}\pi\right) - \cos\left(\frac{2}{7}\pi\right) + \cos\left(\frac{3}{7}\pi\right) \\ &= \cos\left(\frac{1}{7}\pi\right) - 2\cos^2\left(\frac{1}{7}\pi\right) + 1 + 4\cos^3\left(\frac{1}{7}\pi\right) - 3\cos\left(\frac{1}{7}\pi\right) \\ &= 1 - () \end{aligned}$$

(iii)

$$\begin{aligned} &\cos\left(\frac{1}{7}\pi\right) \cos\left(\frac{2}{7}\pi\right) \cos\left(\frac{3}{7}\pi\right) \\ &= \frac{2 \sin\left(\frac{1}{7}\pi\right)}{2 \sin\left(\frac{1}{7}\pi\right)} \cos\left(\frac{1}{7}\pi\right) \cos\left(\frac{2}{7}\pi\right) \cos\left(\frac{3}{7}\pi\right) \end{aligned}$$

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(i)

(a)

$$T(\mathbf{v}) = \mathbf{A}\mathbf{v} = \begin{bmatrix} -1 & 7 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$