ANALYSIS
TUTORIAL 1: FUNCTIONS AND GRAPHS

1 f, g and h are functions with domain $\mathbb{R}$, such that $\mathrm{h}(x)=\mathrm{f}(x)+\mathrm{g}(x)$ for all $x\in\mathbb{R}$. If h is one-one, determine if this is true: either f is one-one or g is one-one.

2 The function f is defined for all $x\in\mathbb{R}$ by

$$\mathrm{f}(x)=\begin{cases}k, & |x|\le l\\ 0, & |x|>l\end{cases}$$

where k and l are positive constants.

Sketch on three separate diagrams, using the same scales for each, the graph of the function g defined by

$$\mathrm{g}(x)=\frac{1}{2}\left[\mathrm{f}(x+a)+\mathrm{f}(x-a)\right]$$

in the cases $a=\frac{1}{4}l$, $a=\frac{3}{4}l$, $a=\frac{3}{2}l$.

3 Functions f and g are defined for all $x\in\mathbb{R}$ by

$$\mathrm{f}(x)=\begin{cases}x+1, & x\ge 0\\ x^2, & x<0\end{cases}\quad\text{and}\quad \mathrm{g}(x)=\begin{cases}2x-3, & x\ge 1\\ 1-x, & x<1\end{cases}$$

Sketch the graphs of $y=\mathrm{f}(\mathrm{g}(x))$ and of $y=\mathrm{g}(\mathrm{f}(x))$, for $x\in\mathbb{R}$.

Hence or otherwise, determine the number of roots of the equation

$$\mathrm{f}(\mathrm{g}(x))=\mathrm{g}(\mathrm{f}(x))$$

4 The functions f and g are defined as follows:

$$\mathrm{f}(x)=\frac{x+|x|}{2},\ x\in\mathbb{R},\ \text{and}\ \ \mathrm{g}(x)=\begin{cases}x^2, & x\ge 0,\\ x, & x<0.\end{cases}\ .$$

Evaluate the rule and domain for the following functions:

(a) fg **(b)** gf **(c)** f^2 **(d)** g^2

5 Prove that **(a)** $\left\lfloor\sqrt{x}\right\rfloor=\left\lfloor\sqrt{\lfloor x\rfloor}\right\rfloor$ **(b)** $\left\lceil\sqrt{x}\right\rceil=\left\lceil\sqrt{\lceil x\rceil}\right\rceil$

where $\lfloor\ \bullet\ \rfloor$ and $\lceil\ \bullet\ \rceil$ refers to the floor and ceiling functions respectively.

Is $\left\lceil\sqrt{x}\right\rceil=\left\lceil\sqrt{\lfloor x\rfloor}\right\rceil$? If so, give a proof. If not, provide a counterexample.

6 A photographer for a mass event, for aesthetic reasons, will like to arrange the n participants for the event into m rows such that the first $m-r$ rows have q participants each and the remaining r rows have $q+1$ participants each, with $n = qm + r,\ 0 \le r < m$.

(a) Show that the k^{th} row has $\left\lfloor \dfrac{n+k-1}{m} \right\rfloor$ participants.

(b) Hence deduce that $\lfloor mx \rfloor = \displaystyle\sum_{r=0}^{m-1} \left\lfloor x + \frac{r}{m} \right\rfloor$ for all rational x and integer m.

(Can be extended to all real x with an additional fix)

7 Given that functions f, g are defined on $\mathbb{R}$. For any $x, y \in \mathbb{R}$,

$$\mathrm{f}(x-y) = \mathrm{f}(x)\mathrm{g}(y) - \mathrm{f}(y)\mathrm{g}(x) \text{ and } \mathrm{f}(1) \neq 0.$$

(i) Prove that f is an odd function. [2]

(ii) If $\mathrm{f}(1) = \mathrm{f}(2)$, find the value of $\mathrm{g}(1) + \mathrm{g}(-1)$. [3]

[HCI 2013/3(a)]

8 Given that $\mathrm{f} : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$, $\mathrm{f}(xy) = \mathrm{f}(x) + \mathrm{f}(y)$.

Find the values of $\mathrm{f}(1)$ and $\mathrm{f}(-1)$. Hence show that f is even.

Give an example of a function that satisfies the above properties and sketch of its graph.

9 Let the function f be defined by $\mathrm{f}(x) = \dfrac{x}{1-|x|},\quad -1 < x < 1.$

(i) Prove algebraically that f is 1–1.

(ii) Find the inverse function of f.

(iii) Is f even or odd? Justify your answer.

(iv) Give a sketch of the graph of f.

Another function g is defined by $\mathrm{g}(x) = \mathrm{e}^{2x} - 1,\ x \in S$. Find the largest possible set S for which fg is a function.

10 The function h is defined, for $x \in \mathbb{R}$, by

$$\begin{aligned} \mathrm{h}(x) &= x\cos x, \quad \text{for } 0 \leqslant x \leqslant \frac{\pi}{2}, \\ \mathrm{h}(-x) &= \mathrm{h}(x), \\ \mathrm{h}(\pi + x) &= -\mathrm{h}(x). \end{aligned}$$

Sketch the graph of h for $-2\pi \leqslant x \leqslant 2\pi$. [7]

[Math S N89]

11 Functions f and g are defined for $x \in \mathbb{R}$ by

$$\mathrm{f}(x) = ax + b,$$
$$\mathrm{g}(x) = cx + d,$$

where a, b, c and d are constants with $a \neq 0$. Given that $\mathrm{gf} = \mathrm{f}^{-1}\mathrm{g}$, show that

either	g is a constant function,	i.e. $\mathrm{g}(x)$ is constant for all $x \in \mathbb{R}$,
or	f^2 is the identity function,	i.e. $\mathrm{ff}(x) = x$ for all $x \in \mathbb{R}$,
or	g^2 is the identity function.	

[9]

[Math S N03]

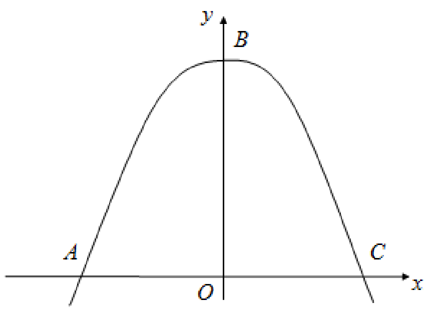
12 If f and g are convex functions, show that the function h given by $\mathrm{h}(x) = \mathrm{f}(x)\mathrm{g}(x)$ is not necessarily a convex function with a suitable example.

13 For a triangle ABC with corresponding angles a, b and c, show that

$$\sin a + \sin b + \sin c \leq \frac{3\sqrt{3}}{2}$$

and determine when equality holds (Hint: $y = \sin x$ is concave).

14



The above diagram shows a sketch of part of the curve with equation

$$y = \sin(\cos x)$$

The curve cuts the x-axis at the points A and C and the y-axis at the point B.

(a) Find the coordinates of the points A, B and C.

(b) Prove that B is a stationary point.

Given that the curve is concave for $0 \leq x \leq \frac{\pi}{2}$,

(c) Show that, for $0 \leq x \leq \frac{\pi}{2}$,

(i) $\sin(\cos x) \leq \cos x$

(ii) $\left(1 - \frac{2}{\pi}x\right)\sin 1 \leq \sin\left(\cos x\right)$

and state in each case the value or values of x for which equality is achieved.