H3 Mathematics
Problem Solving

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Outline

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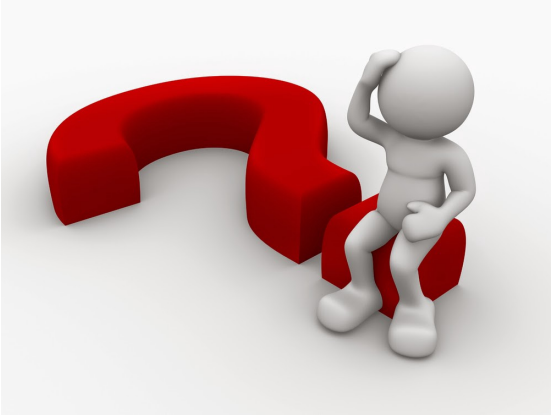
Course objectives

In this course, we learn *about* mathematical problem solving:

- A mathematician's approach to problem solving
  - exploration; and
  - not formulae only
- What to do when you are stuck
- Learning from problem solving
  - alternative solutions;
  - posing problems
- To have the language to discuss problem solving

What is a problem?



Encountering a problem

Have a look at the two questions and attempt to solve them, keeping an awareness of the meta-processes you have engaged in.

Encountering a problem

Problem (1)

A man tossed two coins. One of the coins came up 'Head'. What is the probability that the other is also 'Head'?

Coin problem

- Conducting a thought experiment may not be as concrete as conducting a real experiment.
- Real experiments are too tedious and sometimes expensive to conduct.
- We can run simulations. That's where a bit of **coding** helps.

Encountering a problem

Problem (2)

A groundhog has made an infinite number of holes one metre apart in a straight line in both directions on an infinite plane.
Every day it travels a fixed number of holes in one direction. A farmer would like to catch the groundhog by shining a torch into one of the holes at midnight when it is asleep.
What strategy can the farmer use to ensure that he catches the groundhog eventually?

Groundhog problem

Ask some clarifying questions:

- What do you mean by everyday it moves a fixed number of holes?
- Must the farmer choose a different hole at each attempt?
- Do you know which hole it was at Day 0?

Groundhog problem

Simplifying the problem:

- It is hard to keep track of the groundhog because I don't know its initial position and its velocity (forward/backward).
- Let's say we know where the initial position (label that hole as Hole No. 0) and also we know it is heading to forward.
- Reducing the number of variables.

Groundhog problem

We must devise a plan:

- Mathematize the situation: e.g., give some labels, some notations, etc.
- What if you know it was a 1-groundhog?
- No, it is not there, and we now suppose it was a 2-groundhog?
- Will we eventually catch it?

Groundhog problem

We implement the plan:

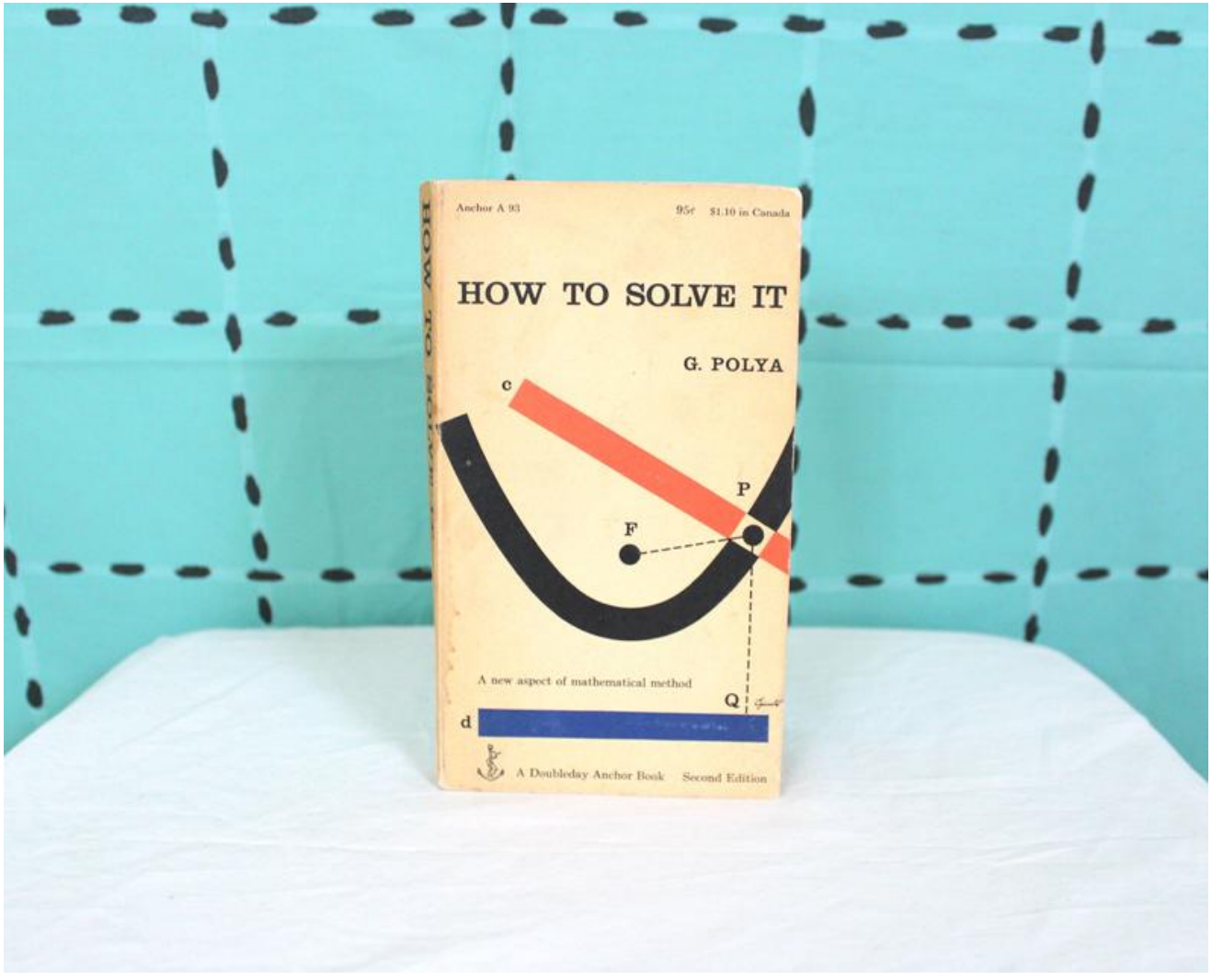
- Again we use **coding**.
- We can check whether it works.
- Then how do we solve the original problem?

What is a problem?

From our earlier experiential session, we gather that:

1. No easy access to a procedure for solving the problem.
2. There must be a desire/motivation/reason to solve the problem.
3. Problem solving begins when an effort is made to find the solution.

Polya's problem solving strategy



A model for problem solving

Advantages of having a model for problem solving:

- Know what to do when 'stuck'
- Systematic
- That is how mathematicians think

A model for problem solving

George Pólya

Figure: George Pólya (December 13, 1887 – September 7, 1985)

George Pólya was a Hungarian mathematician who tried to help people solve mathematics problems.
He wrote "How to solve it" and in it is his strategy of problem solving.

Problem solving cycle

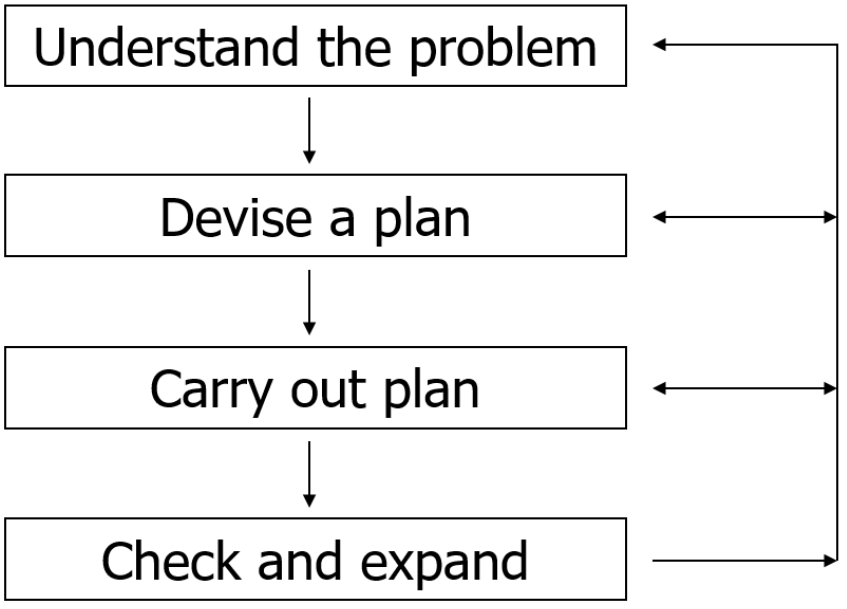


Figure: Problem solving cycle

Note: Most of the time, this is acyclic.

Some inequalities

Problem (3): H3 2020/Q1(i)

For any positive integer n and positive numbers x and y, prove that

$$((n-1)x+y)^n \geq n^n x^{n-1} y.$$

Some inequalities

Can we retrace the problem solving trajectory?

Recursively defined sequence

Problem (4): H3 2019/Q3

A sequence is defined by

$$x_1 = 1 \text{ and } x_{i+1} = \left(\frac{i+a}{i+1}\right) x_i, \ i \geq 1.$$

(i) Assume that $a \geq 0$.

(a) Prove that $x_i \geq \frac{1}{i}$, for all positive integers i.

(b) Prove that

$$\sum_{i=n+1}^{2n} x_i \geq \frac{1}{2},$$

for all positive integers n.

Recursively defined sequence

Let us go through the problem solving cycle:

- (UP) Understand the Problem
- (DP) Devise a Plan
  - We again can rely on **coding**.
- (CP) Carry out Plan
  - Pattern was recognized, and then . . .?
- (CE) Check and Extend
  - Why is the pattern periodic?

Values

- Honesty – I do not know
- Humility – I admit I do not know
- Hard work – I believe that I can make significant progress if I put in effort in an enlightened way

Princess problem

Problem (5)

There are 17 rooms in a row with a door between every two adjoining rooms. Each room has also a door opening to a corridor. A princess lives in one room each day and moves to an adjoining room the next day at 7 in the morning. A prince wants to talk to the princess and knocks on one door each day at noon. If the princess is in that room, she will open the door and they can converse. Otherwise, the prince leaves and tries again at some door at the same time the next day. Is there a strategy to guarantee that the prince will meet the princess if he has 30 days to try?

Summary of today's lesson

What did we cover today?

- **Understand the Problem**: Read through the problem carefully. Think about what the final answer would look like an equation? a diagram? a single digit answer? a set?
- **Devise a Plan**: "He who fails to plan, plans to fail." Problem solver decides on a first (and subsequent) plan(s) of attack. These plans may involve heuristics to help to discover aspects of the problem.

Summary of today's lesson

What did we cover today?

- **Carry out the Plan**: Work out the plan as decided in Stage 2. Pause at discoveries and adjust for new plans if necessary.
- **Check and Extend**: Check the solution. Look back and examine the solution and see if it can be improved or whether it can be used to solve other kinds of problems. Extend the problem.

Homework (1)

- Try all the Homework (1) problems.
- The following three questions will be discussed in the next lesson:
  - H3 2020 Question 1(ii): Some inequalities
  - H3 2019 Question 3(b): Sequence problem
  - H3 2018 Question 3: Triangle problem