MOE H3 Math Numbers and Proofs

Problem Set 1

1. Determine whether each of the following statements is true or false? Give a direct proof if it is true, and give a counter-example if it is false.

   (a) The set of prime numbers is closed under addition.

   (b) The set of positive rational numbers is closed under division.

2. Let a, b and c be nonzero integers. Use the definition of divisibility and write down a direct proof for each of the following statements. (Indicate every step clearly.)

   (a) If a divides b, then ac divides bc.

   (b) If a divides b and b divides a, then $a = \pm b$.

3. Show that 3 divides $n(n+1)(2n+1)$ for any integer n.

4. Prove that for all integers a, if the remainder is NOT 2 when a is divided by 4, then $4 \mid a^3 + 23a$.

5. Prove the following bi-conditional statement:

   For all integers a and b, $3 \mid ab$ if and only if $3 \mid a$ or $3 \mid b$.

6. Let a, b, n be integers with $n > 1$. Suppose $a \equiv b \mod n$. Prove the following :

   (a) $ka \equiv kb \mod kn$ for any positive integer k.

   (b) If m is a common divisor of a, b and n, and $1 < m < n$, then

   $$\frac{a}{m} \equiv \frac{b}{m} \mod \frac{n}{m}.$$

7. Let $a, b, c \in \mathbb{Z}$, with a not zero. Show that:

   (a) $\gcd(a, b) = \gcd(a, a + b)$;

   (b) if $a \mid bc$, then $a \mid \gcd(a, b) \times c$.

   (c) if $\gcd(a, b) = 1$ and $\gcd(a, c) = 1$, then $\gcd(a, bc) = 1$.

   (d) if $\gcd(a, b) = 1$ and $a \mid c$ and $b \mid c$, then $ab \mid c$.