Differentiation

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1 Formal Definition

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

2 Example

$$\begin{aligned}
\ln(x)' &= \lim_{h\to 0} \frac{\ln(x+h) - \ln(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \ln\left(\frac{x+h}{x}\right) \\
&= \lim_{h\to 0} \ln\left(1 + \frac{h}{x}\right)^{1/h}, \ t = \frac{x}{h} \\
&= \lim_{t\to\infty} \ln\left(1 + \frac{1}{t}\right)^{t/x} \\
&= \lim_{t\to\infty} \ln\left(1 + \frac{1}{t}\right)^{t/x} \\
&= \ln\left(\lim_{t\to\infty}\left(1 + \frac{1}{t}\right)^{t}\right)^{\frac{1}{x}} \\
&= \ln\left(\lim_{t\to\infty}\left(1 + \frac{1}{t}\right)^{t}\right)^{\frac{1}{x}} \\
&= \ln(e)^{\frac{1}{x}} \\
&= \frac{1}{x} \quad \square
\end{aligned}$$

3 Product Rule

$$(f(x)g(x))' = f(x)g'(x) + g(x)f'(x)$$

4 Chain Rule

$$g(f(x))' = g'(f(x))f'(x)$$

5 Quotient Rule

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

5.1 Derivation from Product Rule and Chain Rule

$$\begin{aligned}
\left(\frac{f(x)}{g(x)}\right)' &= \left(f(x)\frac{1}{g(x)}\right)' \\
&= f(x)\left(\frac{1}{g(x)}\right)' + \frac{1}{g(x)}f'(x) \\
&= f(x)\left(-\frac{1}{g(x)^2}g'(x)\right) + \frac{f'(x)}{g(x)} \\
&= -\frac{f(x)g'(x)}{g(x)^2} + \frac{g(x)f'(x)}{g(x)^2} \\
&= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \quad \square
\end{aligned}$$

6 Table of Derivatives

$f(x)$	$f'(x)$
$\sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1}(x)$	$\frac{1}{1+x^2}$
$\text{cosec}^{-1}(x)$	$-\text{cosec}(x)\cot(x)$
$\sec^{-1}(x)$	$\sec(x)\tan(x)$