ANALYSIS
CHAPTER 3: SEQUENCES & SERIES

Learning Objectives:

By the end of this chapter, students should be able to:

- understand that an infinite sequence and series may converge or diverge, and find:
  - the limit of the sequence,
  - the sum to infinity when the series is convergent.
- Prove/explain properties of sequences and series involving:
  - convergence and divergence,
  - recurrence formula.

§1 Recap

A ***sequence*** is a set of numbers $u_1, u_2, u_3, \ldots, u_n, \ldots$ (in short, we may denote them using a capital letter like U, or $\{u_n\}_{n\in\mathbb{N}}$, or simply $\{u_n\}$) arranged in some definite order.

A sequence can also be called a ***progression***. A sequence is **finite** if it terminates; otherwise it is an **infinite sequence**.

A ***series*** is formed when the terms of a sequence are added,

e.g. $u_1 + u_2 + \ldots + u_n = \sum_{i=1}^{n} u_i$ for a finite series, and $\sum_{i=1}^{\infty} u_i$ for an infinite series.

An **arithmetic progression** (**AP**) is a sequence in which each term differs from the preceding term by a constant called the **common difference**.

In general, the first term of an AP is denoted by $\boldsymbol{a}$ and the common difference by $\boldsymbol{d}$.

A **geometric progression** (**GP**) is a sequence in which each term other than the first is obtained from the preceding one by multiplying by a non-zero constant, called the **common ratio**, $\boldsymbol{r}$.

Example 1

The sum of the first n terms $u_1, u_2, u_3, \ldots, u_n$ of a sequence U is denoted by S_n.

(i) It is given that $S_n - kS_{n-1} = c$, for all values of n greater than or equal to 1, where k and c are constants and S_0 is defined to be zero. Write down the corresponding equation with n replaced by $n-1$, and hence show that U is a geometric progression with common ratio k.

(ii) It is given instead that $\dfrac{S_n}{n} - \dfrac{S_{n-1}}{n-1} = \dfrac{d}{2}$, for all values of n greater than 1, where d is a constant. Show that U is an arithmetic progression with common difference d.

[Solution]

(i)

$$S_n - kS_{n-1} = c$$
$$S_{n-1} - kS_{n-2} = c$$
$$S_n - kS_{n-1} = S_{n-1} - kS_{n-2}$$
$$S_n - S_{n-1} = kS_{n-1} - kS_{n-2} = k\left(S_{n-1} - S_{n-2}\right)$$
$$u_n = ku_{n-1}$$
$$\frac{u_n}{u_{n-1}} = k$$

Thus U is a geometric progression with common ratio k.

(ii)

$$\frac{S_n}{n} - \frac{S_{n-1}}{n-1} = \frac{d}{2}$$
$$(n-1)S_n - nS_{n-1} = \frac{d}{2}(n)(n-1)$$
$$nu_n - S_n = \frac{d}{2}(n)(n-1)$$
$$u_n = \frac{S_n}{n} + \frac{d}{2}(n-1)$$
$$u_n - u_{n-1} = \frac{S_n}{n} + \frac{d}{2}(n-1) - \frac{S_{n-1}}{n-1} + \frac{d}{2}(n-2)$$
$$= \frac{S_n}{n} - \frac{S_{n-1}}{n-1} + \frac{d}{2}$$
$$= \frac{d}{2} + \frac{d}{2} = d$$

Thus, U is an arithmetic progression with common difference d.

Example 2

The function g is defined by the series

$$\mathrm{g}(x) = a_0 + \binom{n}{1} a_1 x + \binom{n}{2} a_2 x^2 + \ldots + a_n x^n, \text{ for } x \in \mathbb{R},$$

where a_i $(i = 0,1,2,\ldots,n)$ are constants with $a_n \neq 0$, and n is an integer such that $n \geq 2$.

(i) If $a_0,\ a_1,\ \ldots,\ a_n$ are in geometric progression with a non-zero common ratio *r*, show that the equation $\mathrm{g}(x) = 0$ has exactly one real root, and express it in terms of *r*.

(ii) Write down a simplified expression for $\mathrm{g}(x)$ in the case when $a_i = 1$ $(i = 0,\ 1,\ \ldots,\ n)$, and verify that, in the case $a_i = i$ $(i = 0,\ 1,\ \ldots,\ n)$,

$$\mathrm{g}(x) = nx(1+x)^{n-1}.$$

(iii) If $a_0,\ a_1,\ \ldots,\ a_n$ are in arithmetic progression with non-zero common difference d, deduce from part **(ii)** that the equation $\mathrm{g}(x) = 0$ has exactly two real roots, one of which is independent of the constants $a_0,\ a_1,\ \ldots,\ a_n$, and the other of which can be expressed in terms of a_0 and a_n only.

(iv) Show that the function g has an inverse in only one of the two cases **(i)** and **(iii)**, and then only for certain values of *n*, which should be specified. Give a formula for $\mathrm{g}^{-1}(x)$ in the relevant case.

[Solution]

(i) Given $a_i = a_0 r^i$,

$$\mathrm{g}(x) = a_0 + \binom{n}{1} a_0 r x + \binom{n}{2} a_0 r^2 x^2 + \ldots + a_0 r^n x^n.$$

Setting $\mathrm{g}(x) = 0$:

$$0 = a_0 + \binom{n}{1} a_0 r x + \binom{n}{2} a_0 r^2 x^2 + \ldots + a_0 r^n x^n$$

$$0 = 1 + \binom{n}{1} r x + \binom{n}{2} r^2 x^2 + \ldots + r^n x^n$$

$$0 = (1+rx)^n$$

i.e. exactly one real root at $x = -\dfrac{1}{r}$.

(ii) Case where $a_i = 1$ ($i = 0,\ 1,\ \ldots,\ n$):

$$\mathrm{g}(x) = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + x^n = (1+x)^n$$

Case where $a_i = i$ ($i = 0,\ 1,\ \ldots,\ n$):

$$\begin{aligned}\mathrm{g}(x) &= 0 + \binom{n}{1}x + 2\binom{n}{2}x^2 + 3\binom{n}{3}x^3 + \ldots + nx^n \\ &= nx + 2\cdot\frac{n}{2}\binom{n-1}{1}x^2 + 3\cdot\frac{n}{3}\binom{n-1}{2}x^3 + \ldots + nx^n \\ &= nx\left[1 + \binom{n-1}{1}x + \binom{n-1}{2}x^2 + \ldots + x^{n-1}\right] \\ &= nx(1+x)^{n-1} \quad \text{(verified)}\end{aligned}$$

Alternatively,

$$\begin{aligned}(1+x)^n &= 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + x^n \\ n(1+x)^{n-1} &= \frac{\mathrm{d}}{\mathrm{d}x}\left((1+x)^n\right) = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \ldots + nx^{n-1} \\ nx(1+x)^{n-1} &= \binom{n}{1}x + 2\binom{n}{2}x^2 + 3\binom{n}{3}x^3 + \ldots + nx^n = \mathrm{g}(x) \qquad \text{(verified)}\end{aligned}$$

(iii) Given $a_i = a_0 + (i)d$,

$$\begin{aligned}\mathrm{g}(x) &= a_0 + \binom{n}{1}(a_0 + d)x + \binom{n}{2}(a_0 + 2d)x^2 + \ldots + (a_0 + nd)x^n \\ &= \left[a_0 + \binom{n}{1}a_0 x + \ldots + a_0 x^n\right] + \left[\binom{n}{1}dx + \binom{n}{2}2dx^2 + \ldots + ndx^n\right] \\ &= a_0(1+x)^n + dnx(1+x)^{n-1} \\ &= (1+x)^{n-1}\left[a_0(1+x) + dnx\right] \\ &= (1+x)^{n-1}\left[a_0 + (a_0 + dn)x\right]\end{aligned}$$

Thus $\mathrm{g}(x) = 0 \iff x = -1$, or $x = -\dfrac{a_0}{a_0 + dn} = -\dfrac{a_0}{a_n}$.

(iv) It is shown that in case (iii), there are two real roots,
i.e. the line $y = 0$ cuts the graph at 2 points,
i.e. the function is not one-one and hence the inverse does not exist.

In case (i), $\mathrm{g}(x) = (1+rx)^n$ satisfies the *horizontal line test* on $\mathbb{R}$,
if and only if $n = 2k+1, k \in \mathbb{Z}^+$ (odd numbers ≥ 3)

Hence the inverse exists *if and only if n* is odd.

Let $y = (1+rx)^n$. Then $x = \dfrac{1}{r}\left(y^{\frac{1}{n}} - 1\right)$. Thus $\mathrm{g}^{-1}(x) = \dfrac{1}{r}\left(x^{\frac{1}{n}} - 1\right)$.

§1.1 Limiting Behavior of Sequences & Series

Let U be an infinite sequence $u_1, u_2, \ldots, u_n, \ldots$.

U is said to be **convergent** if u_n approaches some limit ℓ. This limit ℓ is called the **limit of the sequence**.

A sequence U converges to ℓ means:

a) As n tends to ∞, u_n tends to ℓ, u_{n+1} tends to ℓ; or
b) As $n \to \infty$, $u_n \to \ell$, $u_{n+1} \to \ell$; or
c) $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} u_{n+1} = \ell$.

A sequence that fails to converge to a **finite** number ℓ is **divergent** (or sequence diverges).

Example 3
Determine if the following sequence converges or diverges. Try deducing from the GC first.

(a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$.

(b) $u_n = \dfrac{4n^2 - 3n + 2}{7n^2 + 6n - 5}$, $n = 1, 2, 3, \ldots$.

(c) $u_n = \dfrac{\ln n}{n}$, $n = 1, 2, 3, \ldots$.

(d) $\left\{ \dfrac{n!}{2^n} \right\}_{n \in \mathbb{N}}$.

[Solution]

(a) $u_n = \dfrac{n}{n+1} = 1 - \dfrac{1}{n+1} \to 1$ as $n \to \infty$.

(b) Method 1:

$$u_n = \frac{4n^2 - 3n + 2}{7n^2 + 6n - 5} = \frac{\left(4 - \frac{3}{n} + \frac{2}{n^2}\right)}{\left(7 + \frac{6}{n} - \frac{5}{n^2}\right)} \to \frac{4}{7} \text{ as } n \to \infty.$$

Method 2:

Consider $\mathrm{f}(x) = \dfrac{4x^2 - 3x + 2}{7x^2 + 6x - 5}$.

From Graphing Techniques 1, we can show (using long division) that $\mathrm{f}(x)$ has the horizontal asymptote $y = \dfrac{4}{7}$.

Thus the $u_n \to \dfrac{4}{7}$ as $n \to \infty$.

(c) $0 < \dfrac{\ln n}{n} = \dfrac{y}{\mathrm{e}^y} \le \dfrac{y}{y^2} = \dfrac{1}{y}$ for $n = \mathrm{e}^y \ge \mathrm{e}$

Taking limits throughout and apply Squeeze Theorem,

$$0 \le \lim_{n\to\infty} \frac{\ln n}{n} \le \lim_{y\to\infty} \frac{1}{y} = 0$$

$$\therefore \lim_{n\to\infty} \frac{\ln n}{n} = 0$$

(d) $u_n = \dfrac{n!}{2^n} = \dfrac{1\cdot 2\cdot 3\cdots n}{2\cdot 2\cdot 2\cdots 2} = \dfrac{1}{2}\dfrac{3}{2}\dfrac{4}{2}\cdots\dfrac{n}{2} > \dfrac{n}{2}$ for $n > 4$,

Let $v_n = \dfrac{n}{2}$,

clearly $v_n \to \infty$ as $n \to \infty$,

since $u_n > v_n$, $u_n \to \infty$ as $n \to \infty$

i.e. the sequence diverges.

From the infinite sequence U (i.e. $u_1, u_2, \ldots, u_n, \ldots$), we can form the infinite sequence of *partial sums* $\{S_n\}$ as follows:

$$\begin{aligned}
S_1 &= u_1 \\
S_2 &= u_1 + u_2 \\
S_3 &= u_1 + u_2 + u_3 \\
&\vdots \\
S_n &= \sum_{i=1}^{n} u_i \\
&\vdots
\end{aligned}$$

The series $\sum_{i=1}^{\infty} u_i = u_1 + u_2 + u_3 + \ldots$ is said to be **convergent** if the sum to infinity S_∞ exists and is **unique** and **finite**. In other words, $\sum_{i=1}^{\infty} u_i$ is convergent if the sequence $S_1, S_2, \ldots, S_n, \ldots$ is convergent, i.e. S_n approaches some limit ℓ.

§1.2 Some guidelines to determine the convergence of series

1) Divergence Theorem

If $u_n \not\to 0$ as $n \to \infty$, then $\sum_{i=1}^{\infty} u_i$ diverges (proof left to student, should be easy by considering $u_n \to k \neq 0$ and then consider the partial sum).

Do note that the inverse is not true, since $\sum_{i=1}^{\infty} \frac{1}{i}$ diverges but $\frac{1}{i} \to 0$ as $i \to \infty$.

2) Sequence with bounded partial sums

Suppose $u_i \geq 0$ for all i. Then, $\sum_{i=1}^{\infty} u_i$ converges if and only if its sequence of partial sums is bounded.

[Proof]

Suppose $u_i \geq 0$ for all i. If $S_n = \sum_{i=1}^{n} u_i < M$ for some positive integer M, then $S_\infty = \lim_{n\to\infty} S_n \leq M$.

Is it still true without the condition of $u_i \geq 0$? Hint: consider $\sum (-1)^n$.

3) Comparison Tests

Suppose $u_i \geq 0$ and $v_i \geq 0$ for all i, and that there is a positive integer m for which $u_k \leq v_k$ for all integers $k \geq m$. Then

- If $\sum_{i=1}^{\infty} v_i$ converges, so does $\sum_{i=1}^{\infty} u_i$.
- If $\sum_{i=1}^{\infty} u_i$ diverges, so does $\sum_{i=1}^{\infty} v_i$.

4) Absolute Convergence

If $\sum_{i=1}^{\infty} |u_i|$ converges, then $\sum_{i=1}^{\infty} u_i$ converges.

[Proof]

$0 \le u_i + |u_i| \le 2|u_i|$ for all *i*.

Using the previous guideline, since $\sum_{i=1}^{\infty} 2|u_i| = 2\sum_{i=1}^{\infty} |u_i|$ converges,

so does $\sum_{i=1}^{\infty} (u_i + |u_i|)$.

We see that $\sum_{i=1}^{\infty} u_i = \sum_{i=1}^{\infty} (u_i + |u_i|) - \sum_{i=1}^{\infty} |u_i|$, which is finite, thus convergent.

Example 4

Determine the convergence of the following series: $\frac{1}{3} + \frac{2}{5} + \frac{3}{7} + \frac{4}{9} + \dots$.

[Solution]

Denote the n^{th} term of the series by u_n.

$u_n = \frac{n}{2n+1} = \frac{1}{2} - \frac{1}{2(2n-1)} \to \frac{1}{2}$ as $n \to \infty$.

Since $u_n \not\to 0$ as $n \to \infty$, the series diverges.

Alternatively,

$u_n = \frac{n}{2n+1} \ge \frac{n}{2n+n} = \frac{1}{3}$.

Since $u_n \not\to 0$ as $n \to \infty$, the series diverges.

Example 5

Use the S_∞ formula of a geometric series to prove that the series

$$\frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} + \dots$$

is bounded by a constant k. Hence, deduce whether the series converges or diverges.

[Solution]

Denote the n^{th} term of the series by u_n.

Observe that

$u_n = \frac{1}{n2^n} \le \frac{1}{2^n}$ for all *n*

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n2^n} \le \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\text{geometric series with } a = \frac{1}{2} \text{ and } r = \frac{1}{2} \right)$$

$$= 1$$

Hence, the series converges.

Example 6

Determine the convergence of the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$.

[Solution]

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n} &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\ldots\\ &= 1+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\frac{1}{9}+\ldots\\ &> 1+\left(\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}+\ldots+\frac{1}{16}\right)+\ldots\\ &= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots\end{aligned}$$

Since the infinite sum of $\frac{1}{2}$ diverges, so does the harmonic series.

Note: The fact that $\frac{1}{n}\to 0$ as $n\to\infty$ doesn't give us any information about the convergence of $\sum_{n=1}^{\infty}\frac{1}{n}$. However, for a series $\sum_{n=1}^{\infty}u_n$, if we know that $u_n \not\to 0$, we know for sure that the series $\sum_{n=1}^{\infty}u_n$ diverges.

Example 7

Determine the convergence of the series $\sum_{n=2}^{\infty}\frac{1}{\ln n}$.

[Solution]

$v_n=\frac{1}{\ln n}>\frac{1}{n}=u_n$ for every $n\geq 2$. (note that both $\frac{1}{\ln n}$ and $\frac{1}{n}$ are positive for all $n\geq 2$)

Since $\sum_{n=2}^{\infty}\frac{1}{n}$ diverges (harmonic series without the first term),

the given series $\sum_{n=2}^{\infty}\frac{1}{\ln n}$ diverges.

§2 Recurrence Relation

A **recurrence relation** is one which shows how the $(n+1)^{\text{th}}$ term, u_{n+1}, is related to the previous term, such as the n^{th} term, u_n.

Start with a given term (usually the first term u_1), and use it to calculate the next term u_2. The second term u_2 is then used to find the third term u_3 and so on.

To define a sequence using a recurrence relation, **we need to have the initial conditions**, i.e. the values of the first term u_1 (or the first few terms u_1, u_2,...) of the sequence.

Example 8 (Self Read)

The sequence $u_1,\ u_2,\ u_3,\ \ldots$ is defined by

$$u_1 = 1 \quad \text{and} \quad u_r = 2u_{r-1} + 1 \ \text{ for } r = 2,\ 3,\ \ldots.$$

Show by induction that $u_n = 2^n - 1$.

The sequence $v_1,\ v_2,\ v_3,\ \ldots$ is defined by

$$v_1 = u_1,\ v_2 = u_2 \quad \text{and} \quad v_r = 2v_{r-2} + 3 \ \text{ for } r = 3,\ 4,\ \ldots.$$

Show by induction that

$$v_n = \begin{cases} 3(2^{\frac{1}{2}n} - 1) & \text{if } n \text{ is even,} \\ 2^{\frac{1}{2}(n+3)} - 3 & \text{if } n \text{ is odd.} \end{cases}$$

[Solution]

Let $\mathrm{P}(n)$ be the proposition $u_n = 2^n - 1$.

When $n = 1$, $\text{RHS} = 2^1 - 1 = 1 = u_1 = \text{LHS}$, $\therefore \mathrm{P}(1)$ is true.

Assume $\mathrm{P}(k)$ is true for some $k \in \mathbb{Z}^+$,

i.e. $u_k = 2^k - 1$.

Consider $\mathrm{P}(k+1)$:

$u_{k+1} = 2u_k + 1 = 2(2^k - 1) + 1 = 2^{k+1} - 1$.

$\therefore \mathrm{P}(k)$ is true $\Rightarrow \mathrm{P}(k+1)$ is true.

Since $\mathrm{P}(1)$ is true and $\mathrm{P}(k)$ is true $\Rightarrow$ $\mathrm{P}(k+1)$ is true, hence by mathematical induction, $\mathrm{P}(n)$ is true for every $n \in \mathbb{Z}^+$.

Let $\mathrm{Q}(n)$ be the proposition $v_n = \begin{cases} 3(2^{\frac{1}{2}n} - 1) & \text{if } n \text{ is even,} \\ 2^{\frac{1}{2}(n+3)} - 3 & \text{if } n \text{ is odd.} \end{cases}$

When $n = 1$, $v_1 = 2^{\frac{1}{2}(1+3)} - 3 = 1 = u_1$.

When $n = 2$, $v_2 = 3(2^{\frac{1}{2}(2)} - 1) = 3 = 2^2 - 1 = u_2$.

$\therefore \mathrm{Q}(1), \mathrm{Q}(2)$ are true.

Assume $\mathrm{Q}(k)$ is true for some $k \geq 3$,

i.e. $v_k = \begin{cases} 3(2^{\frac{1}{2}k} - 1) & \text{if } k \text{ is even,} \\ 2^{\frac{1}{2}(k+3)} - 3 & \text{if } k \text{ is odd.} \end{cases}$

Consider $\mathrm{Q}(k+2)$:

$$v_{k+2} = 2v_k + 3$$
$$= \begin{cases} 2[3(2^{\frac{1}{2}k} - 1)] + 3 & \text{if } k \text{ is even,} \\ 2[2^{\frac{1}{2}(k+3)} - 3] + 3 & \text{if } k \text{ is odd} \end{cases}$$
$$= \begin{cases} 3 \cdot 2^{\frac{1}{2}k+1} - 6 + 3 & \text{if } k \text{ is even,} \\ 2^{\frac{1}{2}(k+3)+1} - 6 + 3 & \text{if } k \text{ is odd} \end{cases}$$
$$= \begin{cases} 3(2^{\frac{1}{2}(k+2)} - 1) & \text{if } k \text{ is even,} \\ 2^{\frac{1}{2}(k+2)+3} - 3 & \text{if } k \text{ is odd.} \end{cases}$$

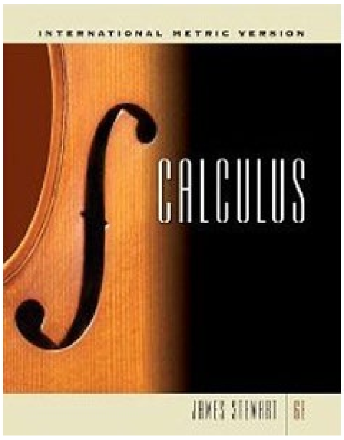
$\therefore \mathrm{Q}(k)$ is true $\Rightarrow \mathrm{Q}(k+2)$ is true.

Since $\mathrm{Q}(1)$ & $\mathrm{Q}(2)$ is true and $\mathrm{Q}(k)$ is true $\Rightarrow \mathrm{Q}(k+2)$ is true, hence by mathematical induction,

$$v_n = \begin{cases} 3(2^{\frac{1}{2}n} - 1) & \text{if } n \text{ is even,} \\ 2^{\frac{1}{2}(n+3)} - 3 & \text{if } n \text{ is odd} \end{cases} \qquad \text{for all } n \in \mathbb{Z}^+.$$

§3 References

- *Calculus* by James Stewart (chapter 12)



- http://www.stewartcalculus.com/media/10_inside_chapters.php?subaction=showfull&id=1090822719&archive=&start_from=&ucat=2&show_cat=2