GONDRO

$$au_{n+2} + b_{n+1} + cu_n = 0$$

Lemma 1: $u_n \propto x^n$.
Lemma 2: $u_n = \sum\limits_{r=1}^{m} A_r x_r^n$ (Linear Combination).
Lemma 3: $u_n = \left(\sum\limits_{r=1}^{m} A_r n^{r-1}\right) x^n$ (Repeated Roots).

$$\begin{aligned}
au_{n+2} + b_{n+1} + cu_n &= 0 \\
a\left(kx^{n+2}\right) + b\left(kx^{n+1}\right) + c\left(kx^n\right) &= 0 \\
akx^n x^2 + bkx^n x + ckx^n &= 0 \\
kx^n\left(ax^2 + bx + c\right) &= 0 \\
ax^2 + bx + c &= 0
\end{aligned}$$

Case 1: $b^2 - 4ac > 0 \implies x_1, x_2 \in \mathbb{R},\ x_1 \neq x_2$.

$$\boxed{u_n = Ax_1^n + Bx_2^n}$$

Case 2: $b^2 - 4ac = 0 \implies x_1, x_2 \in \mathbb{R},\ x_1 = x_2$.

$$\boxed{u_n = (A + Bn)x^n}$$

Case 3: $b^2 - 4ac < 0 \implies x_1, x_2 \in \mathbb{C},\ x_1 = p - iq,\ x_2 = p + iq$.

$$\begin{aligned}
u_n &= Ax_1^n + Bx_2^n \\
&= A(p - iq)^n + B(p + iq)^n \\
&= A\left(|x|e^{-i|\arg(x)|}\right)^n + B\left(|x|e^{i|\arg(x)|}\right)^n \\
&= A\left(|x|^n e^{-in|\arg(x)|}\right) + B\left(|x|^n e^{in|\arg(x)|}\right) \\
&= A|x|^n(\cos(-n|\arg(x)|) + i\sin(-n|\arg(x)|)) + B|x|^n(\cos(n|\arg(x)|) + i\sin(n|\arg(x)|)) \\
&= |x|^n(A\cos(n|\arg(x)|) - iA\sin(n|\arg(x)|) + B\cos(n|\arg(x)|) + iB\sin(n|\arg(x)|)) \\
&= |x|^n((B + A)\cos(n|\arg(x)|) + i(B - A)\sin(n|\arg(x)|)) \\
&= |x|^n(A\cos(n|\arg(x)|) + B\sin(n|\arg(x)|)) \\
&= \left(p^2 + q^2\right)^{\frac{n}{2}}\left(A\cos\left(n\left|\tan^{-1}\left(\frac{q}{p}\right)\right|\right) + B\sin\left(n\left|\tan^{-1}\left(\frac{q}{p}\right)\right|\right)\right)
\end{aligned}$$

$$\boxed{u_n = \left(p^2 + q^2\right)^{\frac{n}{2}}\left(A\cos\left(n\left|\tan^{-1}\left(\frac{q}{p}\right)\right|\right) + B\sin\left(n\left|\tan^{-1}\left(\frac{q}{p}\right)\right|\right)\right)}$$