COMBINATORICS
CHAPTER 3: DISTRIBUTION PROBLEMS (PART I)

Learning Objectives:

By the end of this chapter, students should be able to:

- Tackle 4 different types of Distribution Problems
- Apply the Bijection Principle and Principle of Inclusion and Exclusion to tackle problems
- Understand and use Stirling number of the second kind

I wish you good success, and hope to see your book soon, I am with all respect Sir

— James Sterling to Colin Maclaurin, 31 December 1728

Pre-requisites

- The Addition Principle and the Multiplication Principle (From H2 Mathematics)
- The Bijection Principle (From Chapter 1)
- The Principle of Inclusion and Exclusion (From Chapter 2)

§1 Introduction

Consider the following sandwich problems:

There are 4 types of sandwiches. How many ways are there
(a) to place an order of 6 sandwiches? (b) for 6 boys to order 1 sandwich each?
(c) to divide the sandwiches into containers?

There are now 4 egg sandwiches. How many ways are there to
(d) distribute them among 6 boys? (e) divide the sandwiches into containers?

These are examples of ***distribution problems***, which deal with the counting of ways of distributing objects into boxes. The distribution problem is a basic model for many counting problems.

In distribution problems, objects can be identical or distinct, and boxes too can be identical or distinct. Thus, there are, in general, four cases to be considered, namely

	Objects	**Boxes**
1	Identical	Distinct
2	Distinct	Distinct
3	Distinct	Identical
4	Identical	Identical

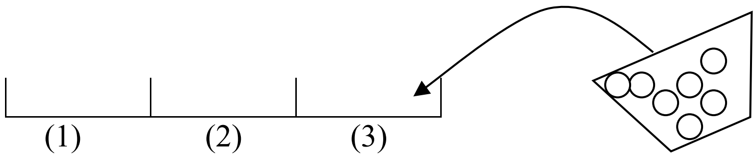
In Part I we will look at the first 2 scenarios, i.e. when the boxes are distinct, and the remaining in Part II.

§2 Identical Objects into Distinct Boxes

We first look at the case where the objects are identical (indistinguishable from each other) and the boxes to house the objects being distinct from each other. We shall see how problems of this type can be solved by applying the Bijection Principle (BP).

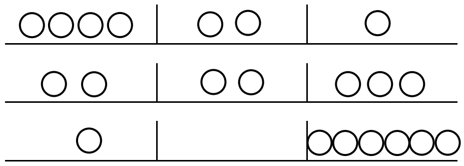
Example 1

How many different ways can we distribute seven identical balls into three distinct boxes?



Three different ways of distribution are shown in the figure below. The vertical bars denote the partition between boxes and thus denotes where each ball will be (Note that two vertical bars at the two ends are removed).

(a)
(b)
(c)



[Solution]

If we treat each vertical bar as a '1' and each ball as a '0', then each way of distribution becomes a 9-digit binary sequence with two '1's. For instance,

(a)	→	0 0 0 0 1 0 0 1 0
(b)	→	0 0 1 0 0 1 0 0 0
(c)	→	0 1 1 0 0 0 0 0 0

Clearly, this correspondence establishes a bijection between the set of ways of distributing the balls and the set of 9-digit binary sequence with two '1's. Thus by (BP), the number of ways of distributing the seven identical balls into three distinct boxes is $\binom{9}{2}$.

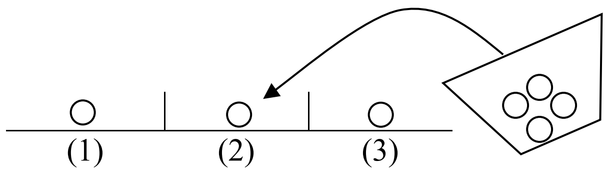
In general, we have:

> The number of ways of distributing r *identical* balls into n *distinct* boxes is given by $\binom{r+n-1}{n-1}$, which is equal to $\binom{r+n-1}{r}$.

In the distribution problem discussed above, some boxes may be empty at the end. If no box is to be empty, how many ways are there to distribute the seven identical balls into three distinct boxes?

Example 2
How many different ways can we distribute seven identical balls into three distinct boxes, with each box non-empty?

[Solution]
To meet the requirement that no box is empty, we first put a ball in each box and this can be done in one way because the balls are identical. We are then left with 4 ($= 7 - 3$) balls, but we are now free to distribute these 4 balls into any box.



By the previous result, this can be done in $\binom{4+3-1}{3-1} = \binom{6}{2}$ ways.

Thus, by (MP) the number of ways to distribute seven identical balls into three distinct boxes ***such that*** **no box is empty** is $\binom{6}{2}$.

In general, the number of ways of distributing r *identical* balls into n *distinct* boxes, where $r \geq n$, such that no box is empty can be found by the following steps:

First, we put one ball in each box. As the balls are identical, this can be done in one way. Then we distribute the remaining $r - n$ balls in the n boxes in an arbitrary way. The number of ways to perform the second step is

$$\binom{(r-n)+(n-1)}{n-1} = \binom{r-1}{n-1}$$

Thus, by (MP), we arrive at the following result.

The number of ways of distributing r *identical* objects into n *distinct* boxes, where $r \geq n$, such that no box is empty is given by $\binom{r-1}{n-1}$, which is equal to $\binom{r-1}{r-n}$.

Example 3
Consider the following linear equation:

$$x_1 + x_2 + x_3 = 7 \qquad \text{(I)}$$

If we put $x_1 = 4,\ x_2 = 1$ and $x_3 = 2,$ we see that (I) holds. Since 4, 1, 2 are nonnegative integers, we say that $(x_1, x_2, x_3) = (4, 1, 2)$ is a nonnegative integer solution to the linear equation (I). Note that $(x_1, x_2, x_3) = (1, 2, 4)$ is also a nonnegative integer solution to (I), and so are $(4, 2, 1)$ and $(1, 4, 2)$. Other nonnegative integer solutions to (I) include

$$(0, 0, 7), (0, 7, 0), (1, 6, 0), (5, 1, 1), \ldots$$

Find the number of nonnegative integer solutions to (I).

[Solution]
Let us create 3 distinct 'boxes' to represent x_1, x_2 and x_3 respectively. Then each nonnegative integer solution $(x_1,\ x_2,\ x_3) = (a,\ b,\ c)$ to (I) corresponds, in a natural way, to a way of distributing 7 identical balls into boxes so that there are *a*, *b* and *c* balls in boxes (1), (2) and (3) respectively.

$$(4,\ 1,\ 2) \longrightarrow \begin{array}{c|c|c} 0000 & 0 & 00 \\ \hline (1) & (2) & (3) \end{array}$$

$$(2,\ 5,\ 0) \longrightarrow \begin{array}{c|c|c} 00 & 00000 & \\ \hline (1) & (2) & (3) \end{array}$$

This correspondence clearly establishes a bijection between the set of nonnegative integer solutions to (I) and the set of ways of distributing **7 identical** into 3 distinct boxes.

Thus, by (BP) and the result of (I), the number of nonnegative integer solutions to (I) is $\binom{7+3-1}{3-1} = \binom{9}{2}$.

By generalizing the above argument, we can establish the following general result:

> Consider the linear equation
>
> $$x_1 + x_2 + \ \dots\ + x_n = r \qquad \text{(II)}$$
>
> where *r* is a nonnegative integer.
> Then, the number of nonnegative integer solutions to (II) is given by $\binom{r+n-1}{r}$.

Example 4
Show that the number of 3-element subsets of the set $\mathrm{N}_{10} = \{1,\ 2,\ 3,\ \dots\ ,\ 10\}$ which contain no consecutive integers is given by $\binom{8}{3}$.

[Solution]
We first establish a bijection between the set *A* of all 3-element subsets of N_{10} which contain no consecutive integers and the set *B* of 10-digit binary sequences with three '1's in which no two 1's are adjacent.

Define a mapping $\mathrm{f} : A \to B$ as follows:

Given a 3-element subset $S = \{k_1,\ k_2,\ k_3\}$ of N_{10}, where $1 \le k_1 < k_2 < k_3 \le 10$,

Let $\mathrm{f}(S) = b_1 b_2 \dots b_{10}$, where

$$b_i = \begin{cases} 1 & \text{if } i = k_1,\ k_2 \text{ or } k_3 \\ 0 & \text{otherwise} \end{cases}$$

For example,

		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
		b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}
f({1, 3, 8})	=	1	0	1	0	0	0	0	1	0	0
f({3, 6, 10})	=	0	0	1	0	0	1	0	0	0	1

It is easy to check that f is a bijection between A and B. Thus $|A| = |B|$.

But how do we count $|B|$?

Observe that a binary sequence in B can be regarded as a way of distributing 7 identical objects into 4 distinct boxes such that the 2nd and 3rd boxes are all nonempty as shown below:

		1st	2nd	3rd	4th
1010000100	$\longleftrightarrow$		0	0000	00
0010010001	$\longleftrightarrow$	00	00	000	

To get one such distribution, we first put one object each in the 2nd and 3rd boxes. The remaining 5 objects are then distributed in an arbitrary way to the 4 boxes. The first step can be done in one way while the second step can be done in $\binom{5+4-1}{4-1} = \binom{8}{3}$ ways.

Hence, $|A| = |B| = \binom{8}{3}$. (shown)

<u>Alternatively (self-reading)</u>,
Suppose that the 3-element subset $\{a, b, c\}$ of the set $\mathrm{N}_{10} = \{1, 2, 3, \ldots, 10\}$ where $a < b < c$ is such that

$$b - a \geq 2 \text{ and } c - b \geq 2 \qquad \text{(III)}.$$

In other words, $\{a, b, c\}$ does not contain any consecutive integers. *How many such 3-element subsets of* N_{10} *are there?*

Consider the mapping $\mathrm{f} : 2^{\mathrm{N}_8} \to 2^{\mathrm{N}_{10}}$ such that

$$\{p, q, r\} \to \{p, q+1, r+2\}$$

$\{p, q, r\}$ corresponds to the set C of 8-digit binary sequences with three '1's ***without restriction*** while $\{p, q+1, r+2\}$ corresponds to the set B of 10-digit binary sequences with three '1's in which no two 1's are adjacent. Such a mapping satisfies the conditions in (III).

Examples: $\{6, 7, 8\} \to \{6, 8, 10\}$ i.e. $a = 6, b = 8, c = 10$
$\{1, 2, 3\} \to \{1, 3, 5\}$ i.e. $a = 1, b = 3, c = 5$
$\{2, 4, 7\} \to \{2, 5, 9\}$ i.e. $a = 2, b = 5, c = 9$

Hence, $|B| = |C| = \binom{8}{3}$. (shown)

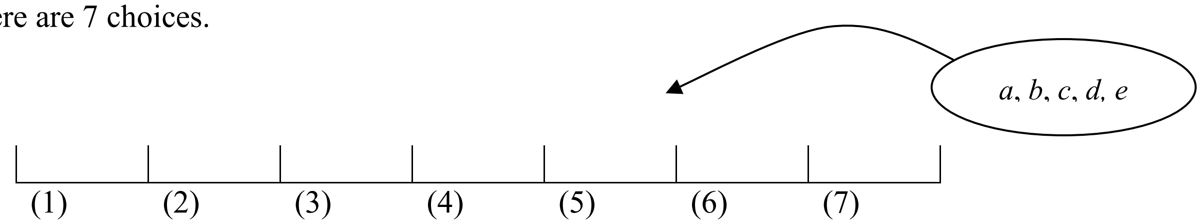
Extension: What if $a < b < c$ *such that* $b - a \geq 3$ and $c - b \geq 2$?

§3 Distinct Objects into Distinct Boxes

Suppose that 5 distinct balls are to be put into 7 distinct boxes.

Case 1: How many ways can this be done if each box can hold at most 1 ball?

Let a, b, c, d and e denote the 5 distinct balls. First, let's say we put a into one of the boxes. There are 7 choices.



Next, we consider (say) b. As each box can hold at most one ball, and one of the boxes is occupied by a, there are now 6 choices for b. Likewise, there are respectively, 5, 4 and 3 choices for c, d and e. Thus, by (MP), the number of ways of distribution is given by $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$.

Note that the above answer can be expressed as ${}^{7}P_{5}$, which is the number of ways of arranging any 5 objects from 7 distinct objects.

The fact that the above answer is ${}^{7}P_{5}$ does not surprise us as there is a 1-1 correspondence between the distribution of 5 distinct balls into 7 distinct boxes and the arrangements of 5 distinct objects from 7 distinct objects as shown below. (Find out the rule of correspondence.)

$\{a, b, c, d, e\}$ $\{1,2,3,4,5,6,7\}$

(1)	(2)	(3)	(4)	(5)	(6)	(7)		
b	c		a	e		d	$\longleftrightarrow$	41275
e	d	c	b			a	$\longleftrightarrow$	74321

In general,

> The number of ways of distributing r distinct objects into n distinct boxes such that each box can hold at most one object (and thus $r \le n$) is ${}^{n}P_{r}$ i.e. $\dfrac{n!}{(n-r)!}$.

Case 2: How many ways can this be done if each box can hold any number of balls?

Before we proceed, we would like to point out that the *ordering* of the distinct objects in each box is *not* taken into consideration.

Now, the 1st object, say a, can be put in any of the 7 boxes. As each box can hold any number of balls, there are also 7 choices for each of the remaining balls b, c, d and e. Thus, by (MP), the answer is 7^5.

In general, we have

> The number of ways of distributing r distinct objects into n distinct boxes such that each box can hold any number of objects is given by n^r.

COMBINATORICS
TUTORIAL 3 PART(I)

Section I

1 There are four types of sandwiches. A boy wishes to place an order of 3 sandwiches. How many such orders can he place?

2 Six distinct symbols are transmitted through a communication channel. A total of 18 blanks are to be inserted between the symbols with at least 2 blanks between every pair of symbols. In how many ways can the symbols and blanks be arranged?

3 Find the number of integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 50$ in each of the following cases:

(i) $x_i \geq 0$ for each $i = 1, 2, 3, 4$;

(ii) $x_1 \geq 3$, $x_2 \geq 5$ and $x_i \geq 0$ for each $i = 3, 4$;

(iii) $0 \leq x_1 \leq 8$ and $x_i \geq 0$ for each $i = 2, 3, 4$;

(iv) $x_1 + x_2 = 10$ and $x_i \geq 0$ for each $i = 1, 2, 3, 4$;

(v) x_i is positive even (respectively, odd) for each $i = 1, 2, 3, 4$.

4 An illegal gambling den has 8 rooms, each named after a different animal. The gambling lord needs to distribute 16 tables into the rooms. Find the number of ways of distributing the tables into the rooms in each of the following cases:

(i) Horse Room holds at most 3 tables.

(ii) Each of Monkey Room and Tiger Room holds at least 2 tables.

5 The number 6 can be expressed as a product of three factors in 9 ways as follows:

$$1\cdot1\cdot6,\ 1\cdot6\cdot1,\ 6\cdot1\cdot1,\ 1\cdot2\cdot3,\ 1\cdot3\cdot2,\ 2\cdot1\cdot3,\ 2\cdot3\cdot1,\ 3\cdot1\cdot2,\ 3\cdot2\cdot1.$$

In how many ways can each of the following numbers be so expressed?

(i) 2592 **(ii)** 27000

Section II

1 Find the number of ways for a teacher to distribute 6 different books to 9 students if

(i) there is no restriction;

(ii) no student gets more than one book.

2 Let A be the set of ways of distributing 5 distinct objects into 7 distinct boxes with no restriction, and let B be the set of 5-digit numbers using 1, 2, 3, 4, 5, 6, 7 as digits with repetition allowed. Establish a bijection between A and B.

3 Find the number of ways of distributing 8 distinct objects into 3 distinct boxes if each box must hold at least 2 objects.

4 Three students are discussing on the number of ways of distributing r distinct objects into n distinct boxes so that no box is empty. Who is correct?

<u>Student X:</u> First we make sure all the boxes are filled with one object each. The number of ways of doing so is $P_n^r = \dfrac{r!}{(r-n)!}$. We then distribute the remaining $r - n$ distinct objects into n distinct boxes. There are n^{r-n} ways of doing this. Thus, the number of ways = $\dfrac{r!n^{r-n}}{(r-n)!}$.

<u>Student Y:</u> We first consider the r distinct objects as if they are identical. The number of ways of distributing r identical objects into n distinct boxes so that no box is empty is given by $\binom{r-1}{n-1}$. But the objects are actually distinct, so the number of ways of permuting them is $r!$. Thus the number of ways $= \binom{r-1}{n-1} r!$.

<u>Student Z:</u> I think that it has something to do with the number of onto functions.