H3 Mathematics
Problem Solving

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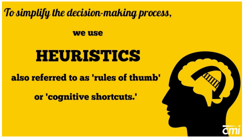

Outline

Tower power problem

Figure: Leaning tower of Pisa

Tower power problem

Problem (1): Homework (2) Question 3

Let $a^{(n)}$ denote $a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}$, a tower of n of the a's.
Find the smallest integer m in terms of n such that

$$9^{(n)} < 3^{(m)}.$$

UP

First thing first, it is easy to be engaged in misconceptions; in particular, it is not clear what the tower notation is.
What are the values of:

$$9, \ 9^9, \ \ 9^{9^9}, \ldots ?$$

UP

If you write a recurrence relation, it is perhaps easier to understand what is going on.
Let $a_n = 9^{(n)}$, where n is a positive integer.

UP

Since $a_n = 9^{(n)}$, it follows that

$$a_{n+1} = 9^{(n+1)} = 9^{a_n}.$$

This way, one can see that the terms of the sequence grow very quickly in size.

UP

We call up another sequence b_n that computes the nth tower of 3. The question then becomes:

Problem (3) rephrased

Given a positive integer m, find in terms of m, the least integer value n for which

$$a_m < b_n.$$

UP

Because both sequence grow large very quickly, and in particular, the base 9 on LHS seems a bit overwhelming for the base 3 on RHS to 'cope with', we expect the value of n to be very large in comparison to m.
But the astronomical values of the respective towers make computations impossible.

DP

It is good to try out what happens when the tower power is small. Let us see the least height of the 3-tower that is needed to beat the first level 9-tower.

DP

- 3 cannot beat 9.
- $3^{(2)} = 3^3 = 27$ already beats 9.

So, for $m = 1$, the least value of n is 2.

DP

- Clearly, 3 cannot beat $9^{(2)} = 9^9$.
- Also, $3^{(2)} = 27$ cannot overcome $9^{(2)} = 9^9$.

Your intuition tells you that you need to go a lot higher for a 3-tower to beat the second tower of 9.

Warning!

Do not be fooled by your intuition!

DP

We shall now argue that $3^{(3)} = 3^{3^3}$ is enough to beat 9^9.

Proof.

$$9^9 = (3^{2\times 9}) = 3^{18} < 3^{27} = 3^{3^3}.$$

$\square$

The important inequality here is

$$2 \times 9^{(1)} < 3^3.$$

CP

We thus suspect that $m = n + 1$ based on the above (rather limited) evidence.

- Does this argument work for higher towers?
- Let us see whether the property holds for higher towers simply because it holds for lower towers.

In short, we appeal to *mathematical induction* for proving this claim.

CP

We now argue that

$$9^{(3)} = 9^{9^9} = 3^{2\times 9^9} = 3^{2\times 9^{(2)}} < 3^{3^{(3)}} = 3^{(4)}.$$

Expanding out the details in the red part,

$$\begin{aligned} 2\times 9^{(2)} &< 1 + 2\times\left(3^{2\times 9^{(1)}}\right) \\ &< 3\left(3^{2\times 9^{(1)}}\right) \\ &= 3^{1+2\times 9^{(1)}} \\ &< 3^{3^{(2)}} \\ &= 3^{(3)}. \end{aligned}$$

CP

We now propose to prove by induction that

$$P(n): \; 1 + 2 \times 9^{(n)} < 3^{(n+1)}$$

holds for all $n \in \mathbb{N}$.
In the midst of proving, we should need to use the obvious lemma that if $k \geq 1$, then $3k \geq 2k + 1$.

CP

- Base case: $n = 1$.
- Is it true that $1 + 2 \times 9^{(1)} < 3^{(2)}$?
- This is true since LHS is 19, while RHS is 27.
- So, $P(1)$ holds.

CP

- Inductive step
- Assume that $P(k)$ holds, i.e.,

$$1 + 2 \times 9^{(k)} < 3^{(k+1)}.$$

- We want to prove that $P(k+1)$ holds, i.e.,

$$1 + 2 \times 9^{(k+1)} < 3^{(k+2)}.$$

CP

The proof is given below for $P(k) \implies P(k+1)$:

Proof.

$$\begin{aligned} 1 + 2 \times 9^{(k+1)} &\leq 3 \times 9^{(k+1)} \\ &= 3 \times (3^{2 \times 9^{(k)}}) \\ &= 3^{1+2\times 9^{(k)}} \\ &< 3^{3^{(k+1)}} \\ &= 3^{(k+2)}. \end{aligned}$$

$\square$

CP

Now we shall make use of the proven statement to settle the initial conjecture: For any positive integer, it holds that

$$9^{(n)} < 3^{(n+1)}.$$

CP

Proof.

$$\begin{aligned} 9^{(n)} &= 3^{2\times 9^{(n-1)}} \\ &< 3^{1+2\times 9^{(n-1)}} \\ &< 3^{3^{(n)}} \\ &= 3^{(n+1)}. \end{aligned}$$

□

A counting problem

Problem (2): H3 2019 Question 4

An n-digit number uses no digits other than 1, 2 and 3. It does not have any 2s adjacent to each other, and it does not have any 3s adjacent to each other. Let there be T_n such numbers, with X_n of these having first digit 1 and Y_n having digit 2.

(i) Prove that, for any $n \geq 2$,

(a) $Y_n = X_{n-1} + Y_{n-1}$,
(b) $X_n = X_{n-1} + 2Y_{n-1}$,
(c) $X_{n+1} = 2X_n + X_{n-1}$.

(a)

We proceed to prove that

$$Y_n = X_{n-1} + Y_{n-1}.$$

UP

Let us call those n digit numbers that use only digits '1', '2' and '3', with no adjacent 2's and no adjacent 3's *nice numbers*.
We have three notations defined in the question:

- T_n the number of n-digit nice numbers;
- X_n the number of n-digit nice numbers starting with 1;
- Y_n the number of n-digit nice numbers starting with 2.

UP

Why so unfair? Why not define one more:

- T_n the number of n-digit nice numbers;
- X_n the number of n-digit nice numbers starting with 1;
- Y_n the number of n-digit nice numbers starting with 2;
- Z_n the number of n-digit nice numbers starting with 3.

UP

Hang on. If you think carefully, the role played by 2 and 3 is the same. By symmetry, switching between 2 and 3, we easily deduce that

$$Y_n = Z_n.$$

So the symbol Z_n is not needed.

DP

Let us find a 'recursive' definition for Y_n the number of n-digit nice numbers that starts with '2'. Now such a number looks like:

$$2 \quad \underbrace{* \cdots}_{(n-1)\text{ digits}}$$

We call the appending number of $n-1$ digits the tail of the original number, and remark that the tail

1. is an $n-1$ digit number;
2. begins with '$*$' which cannot be '2'; and
3. the tail itself must be nice.

CP

This means that

$$Y_n = T_{n-1} - Y_{n-1},$$

i.e., the total number of $n-1$ digit nice numbers take away the number of $n-1$ digit nice numbers which start with 2.

CP

Clearly,

$$T_{n-1} = X_{n-1} + Y_{n-1} + Z_{n-1} = X_{n-1} + 2Y_{n-1}$$

because $Z_{n-1} = Y_{n-1}$ actually.

CP

Substituting into the first equation, we have

$$\begin{aligned} Y_n &= T_{n-1} - Y_{n-1} \\ &= (X_{n-1} + 2Y_{n-1}) - Y_{n-1} \\ &= X_{n-1} + Y_{n-1}. \end{aligned}$$

This completes (a).

(b)

We now prove that

$$X_n = X_{n-1} + 2Y_{n-1}.$$

UP

Again this part is demanding a recursive set-up.
Based on our earlier success, we can mimic our previous method of solution.

DP

Unlike the previous part, this time we are counting those n-digit nice numbers starting with 1.
Notice that there is no requirement that 1 may not appear consecutively.
So the second digit (i.e., the first digit of the tail) can be any one of the numbers $1,\ 2$ or 3.

CP

Thus,

$$X_n = X_{n-1} + Y_{n-1} + Z_{n-1} = X_{n-1} + 2Y_{n-1}$$

because $Z_{n-1} = Y_{n-1}$.
So we have managed to prove (b).

(c)

We now proceed to prove

$$X_{n+1} = 2X_n + X_{n-1}.$$

This equation is unlike the previous two because there are no more Y's.

Mutual recursion

We realize that X_n and Y_n have been defined mutually in a recursive manner:

$$\begin{align} Y_n &= X_{n-1} + Y_{n-1} \tag{1} \\ X_n &= X_{n-1} + 2Y_{n-1} \tag{2} \end{align}$$

Now consider $2 \times (1) - (2)$ so as to obtain

$$2Y_n - X_n = 2X_{n-1} + 2Y_{n-1} - X_{n-1} - 2Y_{n-1}$$

which reduces to

$$X_n + X_{n-1} = 2Y_n.$$

CP

Beginning with the right hand side of (c), we have

$$2X_n + X_{n-1} = X_n + (X_n + X_{n-1}) = X_n + 2Y_n.$$

But by (b) applied at the index $n+1$, we have

$$X_n + 2Y_n = X_{n+1}.$$

So all in all we proved that

$$X_{n+1} = 2X_n + X_{n-1}.$$

A counting problem

Homework (2): Problem (4)

An n-digit number uses no digits other than 1, 2 and 3. It does not have any 2s adjacent to each other, and it does not have any 3s adjacent to each other. Let there be T_n such numbers, with X_n of these having first digit 1 and Y_n having digit 2.

(ii) Use mathematical induction to prove that, for $n \geq 1$,

$$X_n \equiv n^2 - n + 1 \pmod 4.$$

(ii)

The proof proceeds by induction on n.

- Base cases: $n = 1$ and $n = 2$.
- For $n = 1$, $X_1 = 1$, and

$$1^2 - 1 + 1 \equiv 1 \pmod 4$$

and hence $X_1 \equiv 1^2 - 1 + 1 \pmod 4$.
- For $n = 2$, $X_2 = 3$, and

$$3^2 - 3 + 1 = 7 \equiv 3 \pmod 4$$

and hence $X_3 \equiv 3^2 - 3 + 1 \pmod 4$.

DP

I plan to make use of strong induction, i.e., we may assume that for all $k < n+1$,

$$X_k \equiv k^2 - k + 1 \pmod 4.$$

To prove that $X_{n+1} \equiv (n+1)^2 - (n+1) + 1 \pmod 4$.

DP

OK. Let's make the end point more explicit, i.e., to prove that

$$X_{n+1} \equiv n^2 + n + 1 \pmod{4}.$$

CP

By the induction hypotheses for $n-1$ and n, we have

$$X_{n-1} \equiv (n-1)^2 - (n-1) + 1 \pmod 4$$

and

$$X_n \equiv n^2 - n + 1 \pmod 4.$$

CP

Since $X_{n+1} = 2X_n + X_{n-1}$, it follows that

$$\begin{aligned} X_{n+1} &\equiv 2(n^2 - n + 1) + (n-1)^2 - (n-1) + 1 \\ &= 3n^2 - 5n + 5 \pmod 4. \end{aligned}$$

Oops, it is still a far cry from

$$X_{n+1} \equiv n^2 + n + 1 \pmod 4.$$

DP

Although $3n^2 - 5n + 5 \neq n^2 + n + 1$, it could be (for some reasons) that

$$3n^2 - 5n + 5 \equiv n^2 + n + 1 \pmod 4.$$

DP

We probably have to work backwards by formulating an equivalent statement:

$$(3n^2 - 5n + 5) - (n^2 + n + 1) = 2n^2 - 6n + 4 \equiv 0 \pmod 4.$$

This means we just need to prove that

$$2n^2 - 6n = 2n(n-3) \equiv 0 \pmod 4.$$

CP

We know that

$$2n(n-3) \equiv 0 \pmod{4}$$

if and only if

$$n(n-3) \equiv 0 \pmod{2}.$$

This, however, is obvious by parity consideration for n.

CP

Once we have found out the reason why this is correct, we will have to write out the proof by reversing all the above reasoning involved.
Perhaps it will read like this...

CP

For any positive integer n, note that n and $n-3$ are of opposite parity, i.e., n is even if and only if $n-3$ is odd.
Hence
$$n(n-3) \equiv 0 \pmod{2}.$$

CP

It follows that

$$2n(n-3) \equiv 0 \pmod{4},$$

and so

$$2n^2 - 6n + 4 \equiv 0 \pmod{4}.$$

CP

Thus,

$$\begin{aligned} X_{n+1} &\equiv 2(n^2 - n + 1) + (n-1)^2 - (n-1) + 1 \\ &\equiv 3n^2 - 5n + 5 \pmod 4 \\ &\equiv 3n^2 - 5n + 5 - (2n^2 - 6n + 4) \pmod 4 \\ &\equiv n^2 + n + 1 \pmod 4 \\ &\equiv (n+1)^2 - (n+1) + 1 \pmod 4, \end{aligned}$$

as desired.

Pause and think

Apart from the proof by induction, what is the key heuristic that I used in solving this problem?

A counting problem

Problem (2): H3 2019 Question 4

An n-digit number uses no digits other than 1, 2 and 3. It does not have any 2s adjacent to each other, and it does not have any 3s adjacent to each other. Let there be T_n such numbers, with X_n of these having first digit 1 and Y_n having digit 2.

(iii) Find and simplify an expression for $T_n \pmod 4$.

UP

This part asks for the residue of T_n modulo 4.
But have we even found the residue of Y_n modulo 4 (or related expression)?

Pause and think

Here what heuristic am I making use of?

CP

Recall that

$$2Y_n = X_n + X_{n-1},$$

and so

$$2Y_n \equiv (n^2 - n + 1) + [(n-1)^2 - (n-1) + 1] = 2n(n-2) \pmod 4.$$

CP

Since $T_n = X_n + Y_n + Z_n = X_n + 2Y_n$, we have

$$\begin{aligned} T_n &\equiv (n^2 - n + 1) + 2n(n-2) \pmod 4 \\ &\equiv 3n^2 - 5n + 1 \pmod 4. \end{aligned}$$

CE

I have two question now:

1. How do I know if my answer is correct?
2. How can this problem be extended?

CE

To the first question, I use some coding via EXCEL.

CE

To the second question, there are many possibilities:

1. Can I generalize this approach if I can have more elements in the alphabet, e.g., $\Sigma = \{1, 2, 3, \ldots, 9\}$?
2. Are there closed formulae for X_n and Y_n in terms of n? Can these formulae also give the required residues modulo 4?
3. Why modulo 4? Can I change the number to something else? Is there a special meaning to 4?

Heuristics

- Heuristics is a Greek word that means ‘serving to discover’. Think Eureka!
- A heuristic is not a magic method but it is something problem solvers do to help them make some headway in attacking the problem.
- Quite often, the use of heuristics will result in the problem solver discovering something new or important about the problem.

Using heuristics

- Use heuristics to Understand the Problem
- Use heuristics as a Plan

List of heuristics

- Restate the problem in another way
- Think of a related problem
- Work backwards
- Aim for sub-goals
- Divide into cases
- Use suitable numbers (instead of algebra)
- Consider a simpler problem
  - smaller numbers
  - special case – tighten conditions
  - fewer variables

List of heuristics

- Consider a more general case – loosen conditions
- Act it out
- Guess-and-check
- Make a systematic list
- Make a table
- Look for patterns
- Use equations/algebra
- Draw a diagram
- Use a suitable representation
- Use suitable notation

MA	Mathematically Appropriate Content
Depiction of firearms and implied shooting	

Phoney Russian roulette problem

Click on the barrel! 

Phoney Russian roulette problem

Problem (3)

Two bullets are placed in two consecutive chambers of a 6-chamber revolver. The cylinder is then spun. Two persons play a safe version of Russian Roulette. The first points the gun at his mobile phone and pulls the trigger. The shot is blank. Suppose you are the second person and it is now your turn to point the gun at your mobile phone and pull the trigger.
Should you pull the trigger or spin the cylinder another time before pulling the trigger?

DP

There are only **two** decisions, and so we do a case analysis to check out on the probability of surviving.

CP: Dividing into cases

- Case 1: Spin and pull
- Note that there are two bullets there, and so you refresh situation, and the probability of surviving is

$$\frac{4}{6} = \frac{2}{3} \approx 67\%.$$

CP: Dividing into cases

- Case 2: Don't spin, and pull the trigger
- The two bullets are next to each other.
- The revolver only spins in one direction.
- Hence there are four empty chambers.
- We now focus arguing about the adjacent chambers.

CP: Dividing into cases

- Now, we label the loaded chambers 5 and 6.
- Of the 4 empty chambers $(1, 2, 3, 4)$, only one is adjacent to a loaded chamber. We can call that the bad chamber.
- First pull of the trigger, you survived.
- This means you were at one of the four empty chambers $(1, 2, 3, 4)$, and hence you are left with only three empty chambers.
- The chambers have all shifted one space.
- This means that the probability that you survive on the second pull of the trigger is

$$\frac{3}{4} = 75\%.$$

CE

Pause and think

Suppose that the second person survives, and passes the gun back to the first person. Now, spin and pull, or pull?

Discussion: The practical paradigm

"As we look at the nature of science we see two quite distinct strands. The knowledge, the important content and concepts of science and their interrelationships, and also the processes which a scientist uses in his working life. In teaching science we should be concerned both with introducing students to the important body of scientific knowledge, that they might understand and enjoy it, and also with familiarizing students with the way a problem-solving scientist works, that they too might develop such habits and use them in their own lives."

– (Woolnough and Allsop, 1985 p.32)

Discussion: The practical paradigm

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– (Woolnough and Allsop, 1985 p.32)

Dots problem

Problem (4): H3 2018 Question 8

For any positive real number x, $\mathrm{n}(x)$ is defined as the nearest integer to x, with halves rounded up.
For example, $\mathrm{n}(3.5) = 4$, and $\mathrm{n}(\pi) = 3$.

(i) Show that $\sum_{r=1}^{3} \mathrm{n}\left(\frac{11}{7}r\right) = 10$.

UP

Recall that the floor function is given by

$$\lfloor x \rfloor := \max\{m \in \mathbb{Z} \mid m \leq x\}, \qquad x \in \mathbb{R},$$

i.e., $\lfloor x \rfloor$ is the greatest integer less than or equal to x. The rounding up function n can be defined formally by

$$\mathrm{n}(x) := \lfloor x + \frac{1}{2} \rfloor, \qquad x \in \mathbb{R}.$$

Dots problem

Proof.

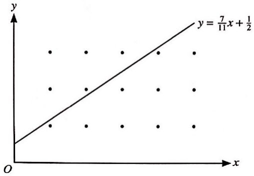
(i) Making a systematic list, we have

r	$\frac{11}{7}r$	$\mathrm{n}\left(\frac{11}{7}r\right)$
1	1.57143	2
2	3.14286	3
3	4.71429	5

Hence $\sum_{r=1}^{3} \mathrm{n}\left(\frac{11}{7}r\right) = 2 + 3 + 5 = 10.$ $\square$

Dots problem

The diagram shows the line $y = \frac{7}{11}x + \frac{1}{2}$ and the integer (x, y) such that $1 \leq x \leq 5$, $1 \leq y \leq 3$.



(ii) Find $\sum_{r=1}^{5} \text{n}\left(\frac{7}{11}r\right)$ and explain the connection between your answer and the points underneath the line $y = \frac{7}{11}x + \frac{1}{2}$.

Dots problem

Proof.

Solution. Again, making a systematic list:

r	$\frac{7}{11}r$	$n\left(\frac{7}{11}r\right)$
1	0.63636	1
2	1.27273	1
3	1.90909	2
4	2.54545	3
5	3.18182	3

Hence $\sum_{r=1}^{5} n\left(\frac{7}{11}r\right) = 1+1+2+3+3 = 10.$ □

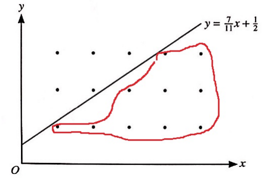
UP

Because $\mathrm{n}(x) := \lfloor x + \frac{1}{2} \rfloor$ for any given real number x, for any real number $a > 0$ and any integer $r \geq 1$,

$$\mathrm{n}(ar)$$

represents the number of lattice points in $\mathbb{Z} \times \mathbb{Z}$ whose x-coordinate reads r and which are underneath the line $y = ax + \frac{1}{2}$.

Dots problem



Solution.

Thus, $\sum_{r=1}^{5} n\left(\frac{7}{11}r\right) = 1 + 1 + 2 + 3 + 3 = 10$ equals the number of lattice points from $x = 1$ to $x = 5$ which are underneath the line $y = \frac{7}{11}x + \frac{1}{2}$. $\square$

Dots problem

(iii) The line $y = \frac{7}{11}x + \frac{1}{2}$ is rotated through 180° about $(3, 2)$. Find the equation of the new line in the form $x = my + c$ and hence comment on the connection between

$$\sum_{r=1}^{3} \mathrm{n}\left(\frac{11}{7}r\right) = \sum_{r=1}^{5} \mathrm{n}\left(\frac{7}{11}r\right).$$

Dots problem

Let (x, y) be an arbitrary point on the line $y = \frac{7}{11}x + \frac{1}{2}$. First translate the point (and hence the line) by the vector

$$\begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

to yield $(x - 3, y - 2)$.

Dots problem

Now, rotate the position vector $\begin{pmatrix} x-3 \\ y-2 \end{pmatrix}$ through 180° to yield

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x-3 \\ y-2 \end{pmatrix} = \begin{pmatrix} 3-x \\ 2-y \end{pmatrix}.$$

Dots problem

Then translate the point $(3 - x, 2 - y)$ by the vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ to yield

$$(\underbrace{6 - x}_{X}, \underbrace{4 - y}_{Y}).$$

It follows that

$$x = 6 - X, \text{ and } y = 4 - Y.$$

Dots problem

But $y = \frac{7}{11}x + \frac{1}{2}$

$$4 - Y = \frac{7}{11}x + \frac{1}{2} = \frac{7}{11}(6 - X) + \frac{1}{2}.$$

Simplifying yields

$$X = \frac{11}{7}Y + \frac{1}{2}.$$

So the required equation of the new line is

$$x = \frac{11}{7}y + \frac{1}{2}.$$

Dots problem

Rotating the dots beneath the line $y = \frac{7}{11}x + \frac{1}{2}$ from $x = 1$ to $x = 5$ yields those dots above the line $x = \frac{11}{7}y + \frac{1}{2}$ from $y = 1$ to $y = 3$.

Thus, the number of dots remains unchanged after the rotation, i.e.,

$$\sum_{r=1}^{3} \mathrm{n}\left(\frac{11}{7}r\right) = \sum_{r=1}^{5} \mathrm{n}\left(\frac{7}{11}r\right).$$

Dots problem

(iv) Let p and q be odd integers greater than 1 and consider the integer points (x, y) such that $1 \leq x \leq \dfrac{p-1}{2}$, $1 \leq y \leq \dfrac{q-1}{2}$. Let N be the number of points which lie in between the lines $y = \dfrac{q}{p}x + \dfrac{1}{2}$ and $x = \dfrac{p}{q}y + \dfrac{1}{2}$.

Explain why $N + \left(\dfrac{p-1}{2}\right)\left(\dfrac{q-1}{2}\right) \equiv 0 \pmod 2$.

Dots problem

- Rotating the line $y = \frac{q}{p}x + \frac{1}{2}$ about the point $\left(\frac{q+1}{4}, \frac{p+1}{4}\right)$ yields the line $x = \frac{p}{q}y + \frac{1}{2}$.
- The number of dots beneath the line $y = \frac{q}{p}x + \frac{1}{2}$ between $x = 1$ and $x = \frac{q+1}{2}$ is equal to that above the line $x = \frac{p}{q}y + \frac{1}{2}$ between $y = 1$ and $y = \frac{p+1}{2}$, i.e.,

$$\sum_{r=1}^{\frac{p-1}{2}} \mathrm{n}\left(\frac{q}{p}\right) = \sum_{r=1}^{\frac{q-1}{2}} \mathrm{n}\left(\frac{p}{q}\right).$$

Dots problem

Adding all the points from the two sets (beneath and above the lines) together, we have

$$N + \left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right),$$

and since the two set of points are equal in numbers, we have

$$N + \left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right) = 0 \pmod{2}.$$

Dots problem

Remark

This problem gives a geometric proof of a lemma which can be used to prove the celebrated Quadratic Reciprocity Theorem.

Groundhog problem revisited

Groundhog problem revisited

Problem (5)

A groundhog has made an infinite number of holes 1 metre apart in a straight line in both directions on an infinite plane. Every day it travels a fixed number of holes in one direction. A farmer would like to catch the groundhog by shining a torch into one of the holes at midnight when it is asleep. What strategy can the farmer use to ensure that he catches the groundhog eventually?

DP

We enumerate the possible pairs

$$(a, v),$$

where

- $a \in \mathbb{Z}$ is the initial position; and
- $v \in \mathbb{Z}$ is the velocity.

That is, we define an injection ψ

$$(a, v) \mapsto \lceil (a, v) \rceil$$

from $\mathbb{Z} \times (\mathbb{Z} - \{0\}) \longrightarrow \mathbb{Z}$.

DP

Then on the nth day, inspect the hole number

$$\pi_1(\psi^{-1}(n)) + n \cdot \pi_2(\psi^{-1}(n)),$$

where

- π_1 is the first projection; and
- π_2 is the second projection.

CP

How do we get the precise definition of ψ?

CE

Can we do better?

- What if the holes are located in $\mathbb{Z} \times \mathbb{Z}$?
- What if the ground hog is a constantly accelerating ground hog? polynomial groundhog?

If time ...

We do only the first two parts.

Problem (6)

For any non-negative integer n, the function P_n is defined by

$$P_n(t) = \sum_{i=0}^{n} \frac{t^i}{i!}.$$

(i) Use mathematical induction to prove that

$$\int_0^t x^n e^{-x} \mathrm{d}x = n!(1 - e^{-t} P_n(t)).$$

If time ...

Formalize by stating the statement we want to prove:

$$P(n) : \int_0^t x^n e^{-x}\,\mathrm{d}x = n!(1 - e^{-t} P_n(t)), \ \ n = 0, 1, \cdots$$

Base case

When $n = 0$, we must prove that

$$\int_0^t x^0 e^{-x} \mathrm{d}x = 0!(1 - e^{-t} P_0(t)).$$

This part is straightforward.

Base case

The working is direct:

$$\begin{aligned}\int_0^t x^0 e^{-x}\,\mathrm{d}x &= \int_0^t e^{-x}\,\mathrm{d}x\\ &= \left[-e^{-x}\right]_0^t\\ &= -e^{-t}+1\\ &= \underbrace{0!}_{=1}(1-e^{-1}\underbrace{P_0(t)}_{=1}).\end{aligned}$$

Inductive step

Assume that $P(n)$ holds, and to prove that

$$P(n+1) : \int_0^t x^{n+1} e^{-x}\, \mathrm{d}x = (n+1)!(1 - e^{-t} P_{n+1}(t))$$

holds.

Inductive step

Integrating by parts, let

$$u = x^{n+1}, \quad \text{and } \frac{dv}{dx} = e^{-x};$$

we have...

Inductive step

$$\int_0^t x^{n+1}e^{-x}\,\mathrm{d}x$$

$$= \left[-x^{n+1}e^{-x}\right]_0^t - \int_0^t (n+1)x^n(-e^{-x})\mathrm{d}x$$

$$=(-t^{n+1}e^{-t}) + (n+1)\int_0^t x^n e^{-x}\mathrm{d}x$$

$$=(n+1)!(1-e^{-t}(P_n(t)+\frac{t^{n+1}}{(n+1)!})$$

$$=(n+1)!(1-e^{-t}P_{n+1}(t)).$$

(ii) State the value of

$$\int_0^\infty x^n e^{-x}\,\mathrm{d}x,$$

and briefly justify your answer.

Proof.

$$\int_0^\infty x^n e^{-x}\mathrm{d}x = \lim_{t\to\infty} n!(1 - e^{-t}P_n(t)) = n!$$

since

$$\lim_{t\to\infty} \frac{P_n(t)}{e^t} = 0.$$

□

Homework (3)

- Try all the Homework (3) problems.
- We shall discuss H3 2018 Question 4 and Question 7.