**Nanyang Technological University ◇ National Institute of Education**
**H3 Mathematics: Problem Solving**

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**Homework (1)** NIE/MME/HWK/2021

> To know what you know and what you do not know, that is true knowledge.
>
> *Confucius*

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When solving each of the following problems, be mindful of the Pólya problem solving cycle that you may engage in. This is especially the case when you are 'stuck in' the initial stages of solving the problem. After you have managed to solve the problem, do not forget to check and extend. Is an awareness of the problem solving cycle aiding your problem solving enterprise?

1. (H3 2020 Question 1)

   (i) For any positive integer $n$ and positive numbers $x$ and $y$, prove that
   $$((n-1)x+y)^n \geq n^n x^{n-1} y.$$

   (ii) Hence, for any positive numbers $a$, $b$ and $c$ such that $abc = 1$, prove that
   $$(1+a)^2(1+b)^3(1+c)^4 \geq 256.$$

2. (H3 2019 Question 3)
   A sequence is defined by
   $$x_1 = 1 \text{ and } x_{i+1} = \left(\frac{i+a}{i+1}\right) x_i, \ i \geq 1.$$

   (i) Assume that $a \geq 0$.

   (a) Prove that $x_i \geq \frac{1}{i}$, for all positive integers $i$.

   (b) Prove that
   $$\sum_{i=n+1}^{2n} x_i \geq \frac{1}{2},$$
   for all positive integers $n$.

   (c) Hence prove that $\sum_{i=1}^{\infty} x_i$ is unbounded.

   (ii) Assume that $a < 0$,

   (a) Prove that
   $$a \sum_{i=m}^{n} x_i = (n+1)x_{n+1} - m x_m$$
   for all positive integers $m$ and $n$ such that $n > m$.

(b) For any sufficiently large integers $m$ and $n$, prove that $x_m x_n \geq 0$.

3. (H3 2017 Question 4)
   Let $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n(x)\, \mathrm{d}x$.

   (i) For $n > 1$, prove that $I_n + I_{n-2} = \dfrac{1}{n-1}$.

   (ii) Justify the statement that $\tan x \leq \dfrac{4}{\pi}x$ on $\left[0, \dfrac{\pi}{4}\right]$.

   (iii) Hence, prove that $I_n$ tends to zero as $n$ tends to infinity.

   (iv) Find the sum of the infinite series
   $$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

4. Determine whether $10^{5^{10^{5^{10}}}} + 5^{10^{5^{10^{5}}}}$ is divisible by 11.

5. (H3 2018 Question 3)
   A triangle has sides of lengths $a$, $b$ and $c$ units. In each of the following cases, prove that there is a triangle having sides of the given lengths.

   (i) $\dfrac{a}{1+a}$, $\dfrac{b}{1+b}$ and $\dfrac{c}{1+c}$ units.

   (ii) $\sqrt{a}$, $\sqrt{b}$ and $\sqrt{c}$ units.

   (iii) $\sqrt{a(b+c-a)}$, $\sqrt{b(c+a-b)}$ and $\sqrt{c(a+b-c)}$ units.

6. 2. From Terence Tao's (Fields medalist) blog: `https://terrytao.wordpress.com/2008/12/09/an-airport-inspired-puzzle/`:
   I was recently at an international airport, trying to get from one end of a very long terminal to another. It inspired in me the following simple maths puzzle, which I thought I would share here: Suppose you are trying to get from one end $A$ of a terminal to the other end $B$. (For simplicity, assume the terminal is a one-dimensional line segment.) Some portions of the terminal have moving walkways, other portions do not. Your walking speed is a constant, but while on a walkway, it is boosted by the speed of the walkway. Your objective is to get from $A$ to $B$ in the shortest time possible. Suppose you need to pause for some period of time, say to tie your shoe. Is it more efficient to do so while on a walkway, or off the walkway? Assume the period of time required is the same in both cases.