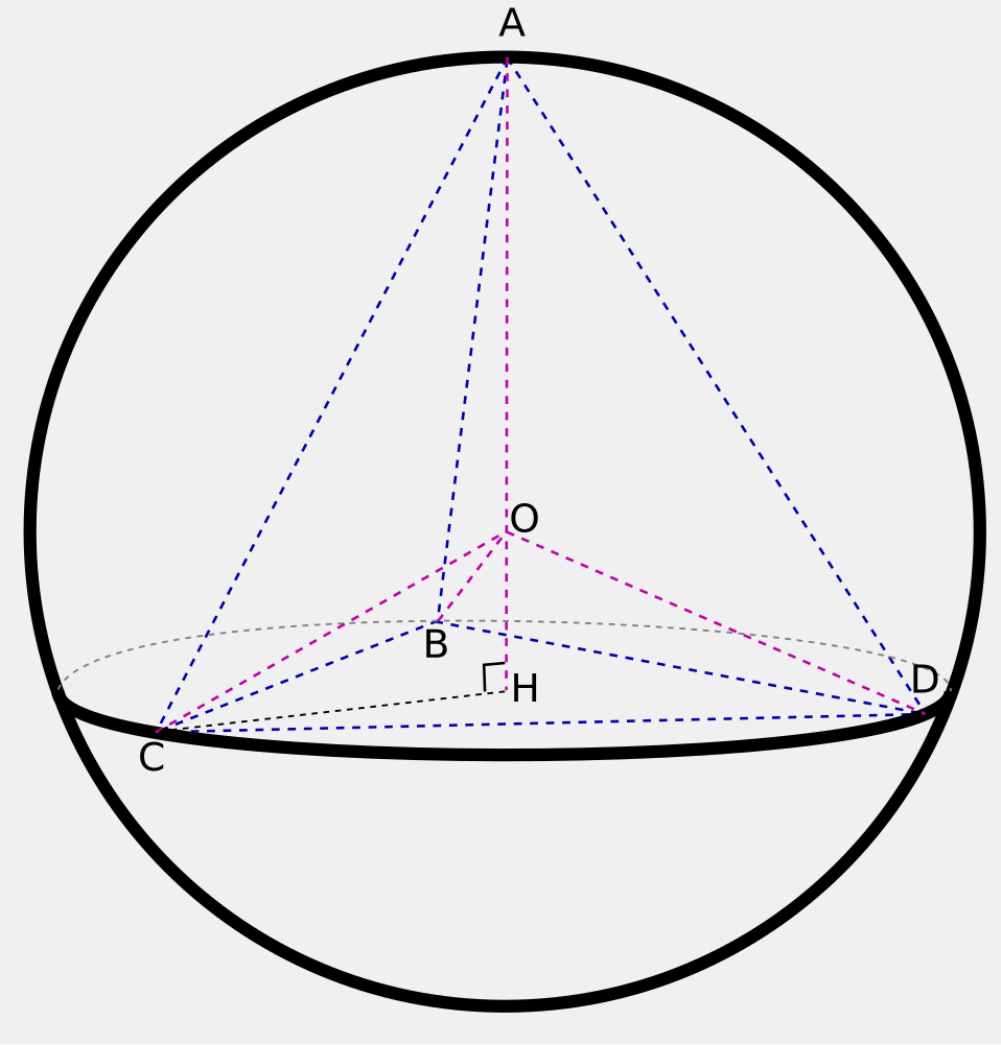
Vertex-Centre-Vertex Angle of Tetrahedra

Gordon Chan

April 10, 2021



1 Derivation of the Half Angle Formula

The half angle formula is used to make the angle computed in the latter part to be presented in a more simplified form.
By the unit circle identity and the double angle formula,

$$\begin{aligned} 1 &= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \\ + \quad \left(\cos 2\left(\frac{\theta}{2}\right) &= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) \end{aligned}$$

$$\begin{aligned} 1 + \cos\theta &= 2\cos^2\frac{\theta}{2} \\ \cos^2\frac{\theta}{2} &= \frac{1+\cos\theta}{2} \\ \cos\frac{\theta}{2} &= \pm\sqrt{\frac{1+\cos\theta}{2}} \end{aligned}$$

2 Computation of the Vertex-Centre-Vertex Angle of a Tetrahedron

Let the side length of the tetrahedron be a.
By similar triangles,

$$\begin{aligned}\frac{h}{r} &= \frac{\frac{a}{2}}{a} \\ \frac{h}{r} &= \frac{1}{2} \\ h &= \frac{r}{2}\end{aligned}$$

By the Pythagoras' theorem,

$$\begin{aligned}(r+h)^2 + \left(\frac{a}{2}\right)^2 &= a^2 \\ \left(r + \frac{r}{2}\right)^2 &= a^2 - \left(\frac{a}{2}\right)^2 \\ \left(\frac{3r}{2}\right)^2 &= a^2 - \frac{a^2}{4} \\ \frac{9r^2}{4} &= \frac{3a^2}{4} \\ r^2 &= \frac{a^2}{3} \\ r &= \frac{a}{\sqrt{3}}\end{aligned}$$

By trigonometry,

$$\begin{aligned}\sin\varphi &= \frac{r}{a} \\ &= \frac{\frac{a}{\sqrt{3}}}{a} \\ &= \frac{1}{\sqrt{3}}\end{aligned}$$

By angular sum of triangle,

$$\begin{aligned}
\theta + \varphi + \varphi &= \pi \\
\pi - \theta &= 2\varphi \\
\frac{(\pi - \theta)}{2} &= \varphi \\
\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) &= \sin\phi \\
\cos\frac{\theta}{2} &= \frac{1}{\sqrt{3}} \\
\cos^2\frac{\theta}{2} &= \frac{1}{3} \\
\frac{1+\cos\theta}{2} &= \frac{1}{3} \\
1 + \cos\theta &= \frac{2}{3} \\
\cos\theta &= -\frac{1}{3} \\
\theta &= \boxed{\cos^{-1}\left(-\frac{1}{3}\right)} \\
&= 1.9106332362\ldots \\
&= 109.47122063\ldots^\circ \\
&\approx 109.5^\circ
\end{aligned}$$

3 Related: Structural Formula of a Methane Molecule

Ew chemistry.