1

(i)

$$\begin{aligned}
\text{LHS} &= (\mathbf{u}\cdot\mathbf{v} + |\mathbf{u}\times\mathbf{v}|)^2 \\
&= (|\mathbf{u}||\mathbf{v}|\cos\theta + ||\mathbf{u}||\mathbf{v}|\sin\theta\hat{\mathbf{n}}|)^2 \\
&= (\cos\theta + \sin\theta)^2 \\
&= \cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta \\
&= 1 + \sin 2\theta \\
&= \text{RHS} \quad \square
\end{aligned}$$

(ii)

$$\begin{aligned}
(\mathbf{u}+\mathbf{w})\cdot\mathbf{v} &= (\mathbf{u}+\mathbf{w})\cdot\mathbf{v} \\
k\mathbf{v}\cdot\mathbf{v} &= \mathbf{u}\cdot\mathbf{v} + \mathbf{w}\cdot\mathbf{v} \\
k|\mathbf{v}|^2 &= |\mathbf{u}||\mathbf{v}|\cos\theta + |\mathbf{w}||\mathbf{v}|\cos\theta \\
k &= \boxed{2\cos\theta}
\end{aligned}$$

(iii)

$$k \in \boxed{(0,2)}$$

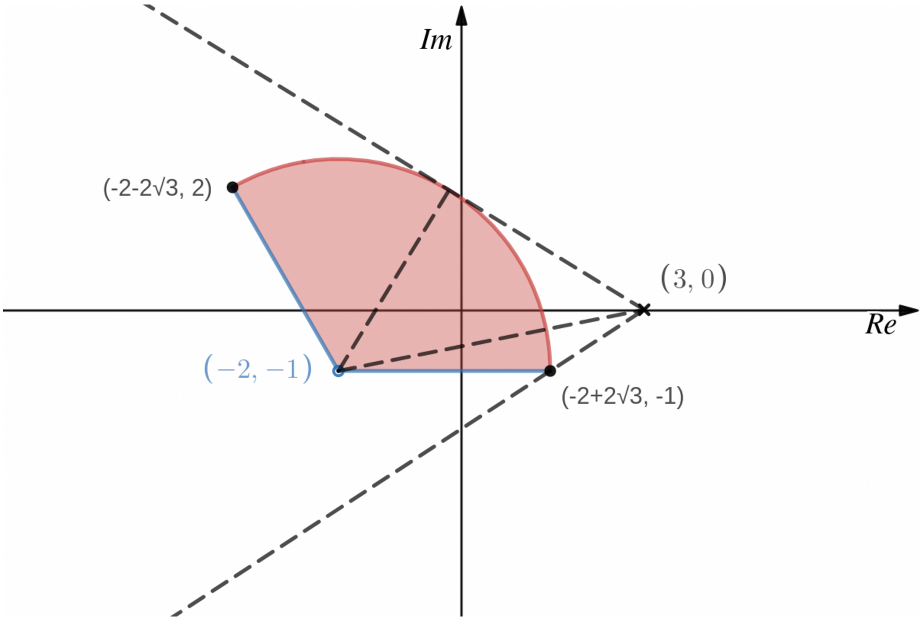
2

(i)

Let $z = x + iy$.

$$\begin{aligned}
\sqrt{3}|z-i| &= \sqrt{2}|z-1-2i| \\
\sqrt{3}|x+iy-i| &= \sqrt{2}|x+iy-1-2i| \\
3|x+i(y-1)|^2 &= 2|x-1+i(y-2)|^2 \\
3\Big(x^2+(y-1)^2\Big) &= 2\Big((x-1)^2+(y-2)^2\Big) \\
3\big(x^2+y^2-2y+1\big) &= 2\big(x^2-2x+1+y^2-4y+4\big) \\
3\big(x^2+y^2-2y+1\big) &= 2\big(x^2+y^2-2x-4y+5\big) \\
3x^2+3y^2-6y+3 &= 2x^2+2y^2-4x-8y+10 \\
x^2+y^2+4x+2y &= 7 \\
x^2+4x+4+y^2+2y+1 &= 7+4+1 \\
(x+2)^2+(y+1)^2 &= \sqrt{12} \\
(x-(-2))^2+(y-(-1))^2 &= \left(\sqrt{12}\right)^2 \quad \square
\end{aligned}$$

(ii)



$$\arg(w-3) \in \boxed{\left[\cot^{-1}\left(5-\sqrt{12}\right)-\pi, \cot^{-1} 5 - \csc^{-1}\frac{\sqrt{78}}{6}+\pi\right]}$$

3

(i)

$$\boxed{I(c, 0)}$$

(ii)

$$C_1 : \boxed{x^2 = 4cy}$$

(iii)

$$x^2 = 4c\left(\frac{c}{2}\right)$$
$$x^2 = 2c^2$$
$$x = \pm c\sqrt{2}$$

$$\begin{aligned}
AB &= 40 \\
c\sqrt{2} - \left(-c\sqrt{2}\right) &= 40 \\
2c\sqrt{2} &= 40 \\
c &= \frac{20}{\sqrt{2}} \\
c &= \boxed{10\sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
C_1 : x^2 &= 40\sqrt{2}y \\
C_2 : x^2 &= 40\sqrt{2}\left(-\left(y - 10\sqrt{2}\right)\right) \\
&= 40\sqrt{2}\left(10\sqrt{2} - y\right) \\
&= 800 - 40\sqrt{2}y \quad \square
\end{aligned}$$

(iv)

$$\begin{aligned}
x^2 &= 800 - 40\sqrt{2}y \\
40\sqrt{2}y &= 800 - x^2 \\
y &= \frac{800 - x^2}{40\sqrt{2}} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-2x}{40\sqrt{2}} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x}{20\sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
A &= \int 2\pi x \,\mathrm{d}s \\
&= 2\pi \int_3^{20} x\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \\
&= 2\pi \int_3^{20} x\sqrt{1 + \frac{x^2}{800}}\,\mathrm{d}x \\
&= 800\pi \int_3^{20} \frac{x}{400}\sqrt{1 + \frac{x^2}{800}}\,\mathrm{d}x \\
&= 800\pi\, \frac{2}{3}\left(1 + \frac{x^2}{800}\right)^{\frac{3}{2}}\Bigg|_3^{20} \\
&= \boxed{1374.25\,\mathrm{cm}^2 \text{ (2 dp)}}
\end{aligned}$$

(v)

$$
\begin{aligned}
V &= 2\int_0^{20} 2\pi x\Big(y-5\sqrt{2}\Big)\,\mathrm{d}x - \int_0^{3} 2\pi x\Big(10\sqrt{2}-y\Big)\,\mathrm{d}x \\
&= 4\pi\int_0^{20} x\left(\frac{800-x^2}{40\sqrt{2}} - 5\sqrt{2}\right)\mathrm{d}x - 2\pi\int_0^{3} x\left(10\sqrt{2}-\frac{800-x^2}{40\sqrt{2}}\right)\mathrm{d}x \\
&= \boxed{8883.52\,\mathrm{cm}^3\ (2\ \mathrm{dp})}
\end{aligned}
$$

4

(i)

$$
x_{t+1} = \frac{\Delta x}{2} + b = \frac{x_t - x_{\text{intended}}}{2} + b = \frac{x_t - 0^\circ}{2} + b = \frac{1}{2}x_t + b \quad \square
$$

(ii)

$$
\begin{aligned}
x_{t+1} + a &= B(x_t + a) \\
x_{t+1} + a &= Bx_t + aB \\
x_{t+1} &= Bx_t + aB - a \\
x_{t+1} &= Bx_t + a(B-1)
\end{aligned}
$$

$$
\begin{cases}
B = \frac{1}{2} \\
a(B-1) = b \implies a = -2b
\end{cases}
$$

$$
\begin{aligned}
x_{t+1} &= \frac{1}{2}x_t + b \\
x_{t+1} - 2b &= \frac{1}{2}(x_t - 2b) \\
x_t - 2b &= \left(\frac{1}{2}\right)^t (x_0 - 2b) \\
x_t &= \boxed{\frac{x_0 - 2b}{2^t} + 2b}
\end{aligned}
$$

(iii)

$$
\lim_{t\to\infty} x_t = \lim_{t\to\infty}\left(\frac{x_0 - 2b}{2^t} + 2b\right) = 2b \neq 0^\circ
$$

(iv)

$$
\begin{aligned}
x_{t+1} &= 2x_t + b \\
x_{t+1} + b &= 2(x_t + b) \\
x_t + b &= 2^t(x_0 + b) \\
x_t &= \boxed{2^t(x_0 + b) - b}
\end{aligned}
$$

$$
\lim_{t\to\infty} x_t = \lim_{t\to\infty}\big(2^t(x_0 + b) - b\big) = \infty \neq 0^\circ
$$

(iv)

Peter's boat will spiral clockwise with an increasing rate.

(vi)

First, Peter make sure that $x_0 = 0°$. Then, he can manually steer the boat such that it changes the direction of the boat by a constant b degrees towards the West.