H3 MATH
2021 / YR 6

ANALYSIS
TUTORIAL 2: INEQUALITIES

1 If a, b, c are sides of a triangle, show that $\dfrac{a}{b+c-a}+\dfrac{b}{a+c-b}+\dfrac{c}{a+b-c}\geq 3$.

2 (2016 H3 Mathematics A-Level Q3)

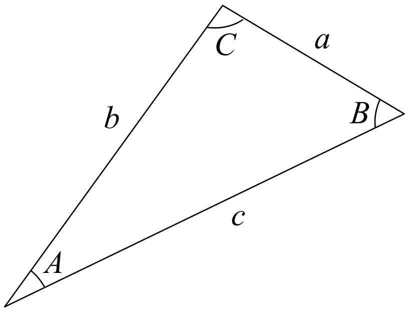
(i) For some positive integer n, let $x_1 \leq x_2 \leq \ldots \leq x_n$ and $y_1 \leq y_2 \leq \ldots \leq y_n$ be real numbers. By considering the sum of all n^2 terms of the form

$$\left(x_i - x_j\right)\left(y_i - y_j\right),$$

prove that

$$\sum_{i=1}^{n} x_i y_i \geq \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right).$$ [5]

(ii) Let a triangle have angles A, B, C and let the lengths of the opposite sides be a, b, and c.



By applying the result of part **(i)**, prove that $aA + bB + cC \geq \dfrac{1}{3}\pi\left(a+b+c\right)$. [4]

(iii) Let a, b, c be three positive numbers such that $a^2+b^2+c^2=1$. By applying the result of part **(i)** with $\left\{x_i\right\}=\left\{\dfrac{a+b}{c}, \dfrac{c+a}{b}, \dfrac{b+c}{a}\right\}$, find the minimum possible value of $\dfrac{(a+b)\left(a^2+b^2\right)}{c}+\dfrac{(c+a)\left(c^2+a^2\right)}{b}+\dfrac{(b+c)\left(b^2+c^2\right)}{a}$. [7]

3 By considering $a_i = \sqrt[n]{x_i}$ for $i = 1, \ldots, n$, show that for all positive real numbers $x_1, x_2, \ldots, x_n$ such that $x_1 x_2 \cdots x_n = 1$, the following inequality holds:

$$\frac{1}{n-1+x_1}+\frac{1}{n-1+x_2}+\ldots+\frac{1}{n-1+x_n}\leq 1.$$

4 **(a)** For all positive real numbers x, y, z, prove that

$$\left(\frac{x}{y}\right)^2+\left(\frac{y}{z}\right)^2+\left(\frac{z}{x}\right)^2 \geq \frac{x}{z}+\frac{y}{x}+\frac{z}{y} \geq 3.$$

(b) **(i)** Let $\mathbf{a}=\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ be two non-zero vectors. By considering the scalar product of **a** and **b**, or otherwise, prove that

$$\left(\sum_{i=1}^{3} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{3} a_i^2\right)\left(\sum_{i=1}^{3} b_i^2\right)$$

and state the necessary condition for equality to hold.

(ii) Hence, for all positive real numbers x, y, z, prove that

$$x+y+z \leq 2\left(\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\right).$$

5 Prove by induction, the AM – GM Inequality for general n,

i.e. $\frac{1}{n}\sum_{i=1}^{n} x_i \geq \sqrt[n]{x_1 x_2 \cdots x_n}$ for $x_1, x_2, \ldots, x_n \in \mathbb{R}_0^+$ for all $n \in \mathbb{Z}^+$, by showing

(i) P(1) and P(2) are both true **(ii)** $\mathrm{P}(k)$ is true $\Rightarrow \mathrm{P}(2k)$ is true

(iii) $\mathrm{P}(k)$ is true $\Rightarrow \mathrm{P}(k-1)$ is true

6 (Nesbitt's Inequality) For positive real numbers a, b, c, prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}$$

using **(i)** AM – GM Inequality, **(ii)** Cauchy-Schwarz Inequality.

(Hint: Consider the equivalent problem $\frac{a}{b+c}+1+\frac{b}{c+a}+1+\frac{c}{a+b}+1 \geq \frac{9}{2}$ instead)

7 (Carlson's Inequality) Consider n arbitrary real numbers x_1, x_2, x_3, ... , x_n.

Show that $\left(x_1+x_2+x_3+\ldots+x_n\right)^2 \leq \frac{\pi}{6}\left(x_1^2+4x_2^2+9x_3^2+\ldots+n^2x_n^2\right)$.

(You may use the well-known result $\sum_{r=1}^{\infty}\frac{1}{r^2}=\frac{\pi}{6}$ without proof)

8 Suppose $x,\ y,\ z>0$ and $x+y+z=1$. Show that $\frac{1}{x}+\frac{4}{y}+\frac{9}{z} \geq 36$.