1

(i)

Let the characteristic polynomial of $\mathbf{A}$ be $p(\lambda)$.

$$\begin{aligned}
p(\lambda) &= \det(\lambda\mathbf{I} - \mathbf{A}) \\
&= \det\left(\begin{bmatrix} \lambda - 1 & -c & -3 \\ -4 & \lambda - 1 & 0 \\ -3 & 0 & \lambda - 1 \end{bmatrix}\right) \\
&= (\lambda-1)^2(\lambda-1) - 4c(\lambda-1) - 9(\lambda-1) \\
&= (\lambda-1)\left((\lambda-1)^2 - 4c - 9\right) \\
&= (\lambda-1)\left(\lambda^2 - 2\lambda - 4c - 8\right) \\
&= (\lambda-1)((\lambda-6)(\lambda+4) + 16 - 4c) \\
&= (\lambda+4)(\lambda-1)(\lambda-6) + (16-4c)(\lambda-1)
\end{aligned}$$

Given that $\mathbf{A}$ has an eigenvalue of 6, $p(\lambda)$ must be in the form of:

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - 6)$$

Thus, $16 - 4c = 0 \iff c = \boxed{4}$ and the remaining eigenvalues are $\boxed{-4}$ and $\boxed{1}$.

(ii)

$$\ker(\mathbf{A} + 4\mathbf{I}) = \ker\left(\begin{bmatrix} 5 & 4 & 3 \\ 4 & 5 & 0 \\ 3 & 0 & 5 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & 0 & 5/3 \\ 0 & 1 & -4/3 \\ 0 & 0 & 0 \end{bmatrix}\right) = \left\{\mu_1 \begin{bmatrix} -5 \\ 4 \\ 3 \end{bmatrix} : \mu_1 \in \mathbb{R}\right\}$$

$$\ker(\mathbf{A} - \mathbf{I}) = \ker\left(\begin{bmatrix} 0 & 4 & 3 \\ 4 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3/4 \\ 0 & 0 & 0 \end{bmatrix}\right) = \left\{\mu_2 \begin{bmatrix} 0 \\ -3 \\ 4 \end{bmatrix} : \mu_2 \in \mathbb{R}\right\}$$

$$\ker(\mathbf{A} - 6\mathbf{I}) = \ker\left(\begin{bmatrix} -5 & 4 & 3 \\ 4 & -5 & 0 \\ 3 & 0 & -5 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & 0 & -5/3 \\ 0 & 1 & -4/3 \\ 0 & 0 & 0 \end{bmatrix}\right) = \left\{\mu_3 \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix} : \mu_3 \in \mathbb{R}\right\}$$

$$\mathbf{P} = \boxed{\begin{bmatrix} -5 & 0 & 5 \\ 4 & -3 & 4 \\ 3 & 4 & 3 \end{bmatrix}}$$

$$\mathbf{D} = \boxed{\begin{bmatrix} -4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}}$$

(iii)

$$
\begin{aligned}
\mathbf{Y}' &= \mathbf{AY} \\
\mathbf{Y}' &= \mathbf{PDP}^{-1}\mathbf{Y} \\
\mathbf{P}^{-1}\mathbf{Y}' &= \mathbf{DP}^{-1}\mathbf{Y} \\
\mathbf{U}' &= \mathbf{DU} \quad \square
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{U}' &= \mathbf{DU} \\
\begin{bmatrix} \mathrm{d}u_1/\mathrm{d}x \\ \mathrm{d}u_2/\mathrm{d}x \\ \mathrm{d}u_3/\mathrm{d}x \end{bmatrix} &= \begin{bmatrix} -4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \\
\begin{bmatrix} \mathrm{d}u_1/\mathrm{d}x \\ \mathrm{d}u_2/\mathrm{d}x \\ \mathrm{d}u_3/\mathrm{d}x \end{bmatrix} &= \begin{bmatrix} -4u_1 \\ u_2 \\ 6u_3 \end{bmatrix} \\
\begin{bmatrix} \mathrm{d}u_1/u_1 \\ \mathrm{d}u_2/u_2 \\ \mathrm{d}u_3/u_3 \end{bmatrix} &= \begin{bmatrix} -4\,\mathrm{d}x \\ \mathrm{d}x \\ 6\,\mathrm{d}x \end{bmatrix} \\
\begin{bmatrix} \ln|u_1| \\ \ln|u_2| \\ \ln|u_3| \end{bmatrix} &= \begin{bmatrix} -4x + c_1 \\ x + c_2 \\ 6x + c_3 \end{bmatrix} \\
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} &= \begin{bmatrix} \exp(-4x + c_1) \\ \exp(x + c_2) \\ \exp(6x + c_3) \end{bmatrix} \\
\mathbf{U} &= \begin{bmatrix} A_1 \exp(-x) \\ A_2 \exp(x) \\ A_3 \exp(6x) \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{U} &= \mathbf{P}^{-1}\mathbf{Y} \\
\mathbf{Y} &= \mathbf{PU} \\
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} &= \begin{bmatrix} -5 & 0 & 5 \\ 4 & -3 & 4 \\ 3 & 4 & 3 \end{bmatrix} \begin{bmatrix} A_1 \exp(-x) \\ A_2 \exp(x) \\ A_3 \exp(6x) \end{bmatrix}
\end{aligned}
$$

$$
\boxed{\begin{cases} y_1 = -5A_1e^{-x} + 5A_3e^{6x} \\ y_2 = 4A_1e^{-x} - 3A_2e^{x} + 4A_3e^{6x} \\ y_3 = 3A_1e^{-x} + 4A_2e^{x} + 3A_3e^{6x} \end{cases} , \; A_1, A_2, A_3 \in \mathbb{R}}
$$

2

(i)

Let the characteristic polynomial of $\mathbf{A}$ be $p(x)$.

$$
\begin{aligned}
p(x) &= (x-\alpha)(x-\beta)(x-\gamma) \\
&= (x-\alpha)\left(x^2-(\beta+\gamma)x+\beta\gamma\right) \\
&= x^3-(\beta+\gamma)x^2+\beta\gamma x-\alpha x^2+(\alpha\beta+\gamma\alpha)x-\alpha\beta\gamma \\
&= x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma
\end{aligned}
$$

Given $p(x)=x^3-x^2+kx+4$,

$$
\begin{cases}
\alpha+\beta+\gamma=1 \\
\alpha\beta+\beta\gamma+\gamma\alpha=k \\
\alpha\beta\gamma=-4
\end{cases}
$$

$$
\mathbf{A}=\mathbf{PDP}^{-1},\ \mathbf{P}=\begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\end{bmatrix},\ \mathbf{D}=\begin{bmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & \gamma\end{bmatrix}
$$

$$
\begin{aligned}
\mathbf{B} &= \mathbf{P}\begin{bmatrix}\alpha-\beta\gamma & 0 & 0\\ 0 & \beta-\gamma\alpha & 0\\ 0 & 0 & \gamma-\alpha\beta\end{bmatrix}\mathbf{P}^{-1} \\
&= \mathbf{P}\left(\begin{bmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & \gamma\end{bmatrix}-\begin{bmatrix}\beta\gamma & 0 & 0\\ 0 & \gamma\alpha & 0\\ 0 & 0 & \alpha\beta\end{bmatrix}\right)\mathbf{P}^{-1} \\
&= \mathbf{P}\begin{bmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & \gamma\end{bmatrix}\mathbf{P}^{-1}-\mathbf{P}\begin{bmatrix}-4/\alpha & 0 & 0\\ 0 & -4/\beta & 0\\ 0 & 0 & -4/\gamma\end{bmatrix}\mathbf{P}^{-1} \\
&= \mathbf{PDP}^{-1}+4\mathbf{PD}^{-1}\mathbf{P}^{-1} \\
&= \boxed{\mathbf{A}+4\mathbf{A}^{-1}}
\end{aligned}
$$

(ii)

Consider the three eigenvalues of $\mathbf{B}$:

$$
\alpha-\beta\gamma=\alpha-\frac{-4}{\alpha}=\frac{1}{\alpha}\left(\alpha^2+4\right)\neq 0,\ \forall\alpha\in\mathbb{R}
$$

$$
\beta-\gamma\alpha=\beta-\frac{-4}{\beta}=\frac{1}{\beta}\left(\beta^2+4\right)\neq 0,\ \forall\beta\in\mathbb{R}
$$

$$
\gamma-\alpha\beta=\gamma-\frac{-4}{\gamma}=\frac{1}{\gamma}\left(\gamma^2+4\right)\neq 0,\ \forall\gamma\in\mathbb{R}
$$

Since $\mathbf{B}$ has three non-zero eigenvalues, $\det(\mathbf{B})\neq 0 \implies \mathrm{col}(\mathbf{B})=\mathbb{R}^3$.

Since the set of three eigenvectors is linearly independent:

$$
\mathrm{span}\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}=\mathbb{R}^3
$$

Therefore, because $\mathrm{span}\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}=\mathrm{col}(\mathbf{B})=\mathrm{range}(T)$ and $|\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}|=3=\dim\left(\mathbb{R}^3\right)$, $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ forms a basis for the range of T. $\square$