Nanyang Technological University ◇ National Institute of Education
H3 Mathematics: Problem Solving

Homework (3)

NIE/MME/HWK/2021

> I can't change the direction of the wind but I can adjust my sails to always reach my destination.
>
> *Jimmy Dean*

In this lesson, we have learnt about induction as a powerful proof principle, and walked through the problem solving cycle again. For the exercises below, be cognizant of the problem solving strategy that you are employing, and grow in your awareness of the different heuristics you have used.

1. (H3 Specimen Paper Question 7)

   The figures below show, respectively, a square board of 4 unit squares with one unit square covered, and a triomino consisting of 3 unit squares.

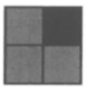
   

   A square board with 4 unit squares.
   One unit square is covered.

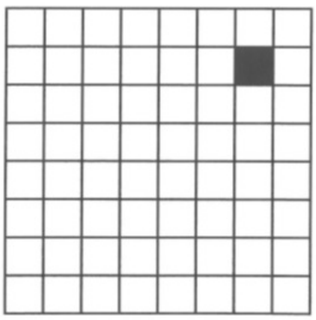
   

   A triomino

   Irrespective of which unit square is covered, a triomino can cover the remaining 3 unit squares of the square board as shown.

   Consider a square board made up of 4^n squares, where $n \geq 1$, with one of the unit squares covered. An example of such a square, with $n = 3$, is shown below.

   (i) Explain how, irrespective of unit square is initially covered, a triomino can be placed on the board in such a way that each quarter of the board now has one unit square covered.

(ii) Use mathematical induction to prove that, irrespective of which unit square is initially covered, the remaining squares can be covered by triminoes. State the number of triminoes required.

2. (H3 2018 Question 4)
A clothes shop sells a particular make of T-shirt in four different colors. The shopkeeper has a large number of T-shirts of each color.

   (i) A customer wishes to buy seven T-shirts.
      (a) In how many ways can he do this?
      (b) In how many ways can he do this if he buys at least one of each color?

   (ii) The shopkeeper places seven T-shirts in a line.
      (a) In how many ways can she do this?
      (b) In how many ways can she do this if no two T-shirts of the same color are to be next to each other?
      (c) Use the principle of inclusion and exclusion to find the number of ways in which she can do this if she has to use at least one T-shirt of each colour but with no other restrictions.

3. (H3 2019 Question 2)
There are four types of DNA base, A, C, G and T.

   (i) Eight DNA bases are to be selected. In how many ways can this be done
      (a) if there are no other restrictions,
      (b) if each type of base must used at least once?

   (ii) A sequence of eight DNA bases is to be formed. In how many ways can this be done
      (a) if there are no other restrictions,
      (b) if no two adjacent bases are of the same type,
      (c) if each type of base must be used at least once?

4. (H3 2017 Question 5)

   (i) Explain why the number of ways to distribute r distinct objects, where $r \geq 2$, into 2 **distinct** boxes such that neither is empty is $2^r - 2$.

   (ii) Let $S(r, n)$ denote the number of ways to distribute r objects into n **identical** boxes such that no box is empty.

   (a) Explain why, for $r \geq 3$,

$$S(r, 3) = 2^{r-2} - 1 + 3S(r-1, 3).$$

   (b) Prove that, for $r \geq 3$,

$$S(r, 3) = \begin{cases} 0 \pmod 6 & \text{if } r \text{ is even,} \\ 1 \pmod 6 & \text{if } r \text{ is odd,} \end{cases}$$

5. (H3 2018 Question 7)
   The differential equation

$$y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + 1, \qquad \text{for } x > 0 \tag{1}$$

   has a solution curve S such that $\frac{d^2y}{dx^2}$ is non-zero for all points of S.

   (i) By substituting $t = \frac{dy}{dx}$ into equation (1) and differentiating with respect to x, show that S has equation $y^2 = 4x$.

   (ii) Show that a straight line is a tangent to the curve S if and only if it is itself a solution of the equation.

6. (H3 2017 Question 5)

   (i) Use the substitution $t = \frac{du}{dx}$ to find the general solution of the equation

$$\frac{d^2u}{dx^2} = \frac{du}{dx}.$$

   (ii) Show that the differential equation

$$\frac{dy}{dx} = f(x)y^2 + g(x)y$$

   can be transformed into the equation

$$f(x)\frac{d^2u}{dx^2} - (f'(x) + f(x)g(x))\frac{du}{dx} = 0$$

   by the substitution

$$u = e^{-\int f(x)\ dx}.$$

   (iii) A solution curve of the differential equation

$$\frac{dy}{dx} = e^{-2x}y^2 + 3y$$

   pass through the point $\left(0, -\frac{1}{4}\right)$. Find the equation of the curve.