$\sum\limits_{j=1}^{\chi} \left\lfloor \sqrt{j} \right\rfloor$

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Let $\Xi(\chi) = \sum\limits_{j=1}^{\chi} \left\lfloor \sqrt{j} \right\rfloor$.

$$
\begin{aligned}
\Xi\left(n^2-1\right) &= 1(0) + 3(1) + 5(2) + \cdots + (2n-1)(n-1) \\
&= \sum_{a=1}^{n} (2a-1)(a-1) \\
&= \sum_{a=1}^{n} \left(2a^2 - 3a + 1\right) \\
&= 2\sum_{x=1}^{n} x^2 - 3\sum_{y=1}^{n} y + \sum_{z=1}^{n} 1 \\
&= 2\left(\frac{n(n+1)(2n+1)}{6}\right) - 3\left(\frac{n(n+1)}{2}\right) + n \\
&= \frac{4n}{3}(n+1)(2n+1) - \frac{9n}{6}(n+1) + n \\
&= \frac{n}{6}\left(2\left(2n^2+3n+1\right) - 9(n+1) + 6\right) \\
&= \frac{n}{6}\left(4n^2 + 6n + 2 - 9n - 9 + 6\right) \\
&= \frac{n}{6}\left(4n^2 - 3n - 1\right)
\end{aligned}
$$

$$\begin{aligned}\Xi\left(n^2-1+k\right) &= \Xi\left(n^2-1\right)+kn\\ &= \frac{n}{6}\left(4n^2-3n-1\right)+kn\\ &= \frac{n}{6}\left(4n^2-3n-1+6k\right)\end{aligned}$$

$$\chi = n^2-1+k \implies n = \lfloor\sqrt{\chi}\rfloor$$

$$\chi = \lfloor\sqrt{\chi}\rfloor^2-1+k \implies k=\chi-\lfloor\sqrt{\chi}\rfloor^2+1$$

$$\chi = n^2-1+k \implies \chi=\lfloor\sqrt{\chi}\rfloor^2-1+\left(\chi-\lfloor\sqrt{\chi}\rfloor^2+1\right)$$

$$\begin{aligned}\Xi(\chi) &= \Xi\left(\lfloor\sqrt{\chi}\rfloor^2-1+\left(\chi-\lfloor\sqrt{\chi}\rfloor^2+1\right)\right)\\ &= \frac{\lfloor\sqrt{\chi}\rfloor}{6}\left(4\lfloor\sqrt{\chi}\rfloor^2-3\lfloor\sqrt{\chi}\rfloor-1+6\left(\chi-\lfloor\sqrt{\chi}\rfloor^2+1\right)\right)\\ &= \frac{\lfloor\sqrt{\chi}\rfloor}{6}\left(4\lfloor\sqrt{\chi}\rfloor^2-3\lfloor\sqrt{\chi}\rfloor-1+6\chi-6\lfloor\sqrt{\chi}\rfloor^2+6\right)\\ &= \frac{\lfloor\sqrt{\chi}\rfloor}{6}\left(6\chi-2\lfloor\sqrt{\chi}\rfloor^2-3\lfloor\sqrt{\chi}\rfloor+5\right)\end{aligned}$$

$$\therefore \boxed{\sum_{j=1}^{\chi}\left\lfloor\sqrt{j}\right\rfloor = \frac{\lfloor\sqrt{\chi}\rfloor}{6}\left(6\chi-2\lfloor\sqrt{\chi}\rfloor^2-3\lfloor\sqrt{\chi}\rfloor+5\right)}$$

$$\begin{aligned}\sum_{j=1}^{16}\left\lfloor\sqrt{j}\right\rfloor &= \frac{\lfloor\sqrt{16}\rfloor}{6}\left(6(16)-2\left\lfloor\sqrt{16}\right\rfloor^2-3\left\lfloor\sqrt{16}\right\rfloor+5\right)\\ &= \frac{\lfloor 4\rfloor}{6}\left(6(16)-2\lfloor 4\rfloor^2-3\lfloor 4\rfloor+5\right)\\ &= \frac{4}{6}\left(6(16)-2(4)^2-3(4)+5\right)\\ &= \frac{2}{3}(96-32-12+5)\\ &= \frac{2}{3}(57)\\ &= \boxed{38}\end{aligned}$$