H3 MATH
2021 / YR 6

ANALYSIS
CHAPTER 1: FUNCTIONS AND GRAPHS

Learning Objectives:

By the end of this chapter, students should be able to:

- Understand the definitions and the use of functions and its related graphs and features
- Analyze and use the properties of various functions and graphs to tackle problems across topics

We cannot solve our problems with the same thinking we used when we created them.

— Albert Einstein

Pre-requisites

- Functions (from H2 Mathematics)
- Injections, Surjections, Bijections (from Chapter 1 Combinatorics)
- All H2 Mathematics content in general as they will be used from time to time

Setting the Context

Functions describe the relationships between variables and arise out of many real-life situations. In this chapter, we will leverage on the learning of functions from H2 Mathematics and introduce more specific functions and see how they are used in higher mathematics.

§1 Re-Introduction

See Chapter 3 Functions from H2 Mathematics for

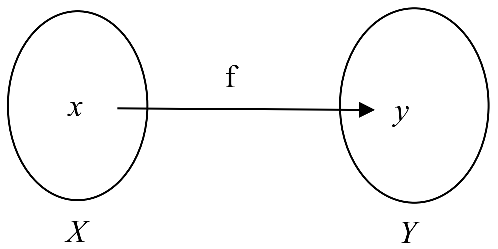
- Definition and representation of functions
- Graphing
- 1-1 Functions (when coupled with being Surjective, results in Bijective functions) and existence of Inverse functions
- Composite functions and its existence
- Specific basic functions such as Polynomial functions, Trigonometric functions, Exponential functions, Logarithmic functions, Rational functions (functions of the form $\dfrac{\mathrm{f}(x)}{\mathrm{g}(x)}$ where f and g are polynomial functions.

Also see Chapter 1 from Combinatorics for the definition of Injective mappings and Surjective mappings which are basically mappings for Injective functions and Surjective functions respectively.

§1.1 Additional Concepts

A **function** f is a rule or relation which assigns each and every element of $x \in X$ to **one and only one** element $y \in Y$ **.** We write this as $\mathrm{f}: X \mapsto Y$ and read it as 'f maps *X* to *Y*'.

The set of *x*-values (called inputs) in *X* is the **domain** of f, denoted as $\mathrm{D_f}$. $\mathrm{f}:\mathbb{R} \mapsto [0,\ \infty)\mathbb{R}$
The set of *y*-values (called outputs or images) in *Y* is the **codomain** of f, and *y* is written as $y = \mathrm{f}(x)$.



Essentially, a function is a *rule* that specifies how an input would be transformed. For example, $y = \mathrm{f}(x) = x^2$

The arrow diagram above illustrates the mapping.

In H2 Mathematics, we simplify the definition involved to only map elements from the domain to its range. For example, we can define the function such that $\mathrm{f}(x) = x^2$, a simple quadratic function with the domain being and the range $[0,\ \infty)$. However, this will mean that for each and every function, we need to find its range before being able to define it completely. Moreover, we can observe that any possible outcome of polynomial function with real coefficients will always map the real domain to elements in the real domain.

Thus for the same function f, we write $\mathrm{f}:\mathbb{R} \mapsto \mathbb{R}$ where $\mathrm{f}(x) = x^2$, and the $\mathbb{R}$ involved in the function mapping that differs from the initial definition simply refers to the codomain of the function, which is the target set of the function into which all of the output of the function is constrained to fall. In short, it is the set *Y* in the notation $\mathrm{f}: X \mapsto Y$.

§1.2 Injective Functions and Surjective Functions

Given a function $\mathrm{f}: X \mapsto Y$,

Injective Function
The function f is injective (one-to-one) if each element of the codomain is mapped to by the function f from at most one element in the domain. In mathematical terms, it is written as follows:

$$\forall x_1,\ x_2 \in X, \mathrm{f}(x_1) = \mathrm{f}(x_2) \Rightarrow x_1 = x_2.$$

Surjective Function
The function f is surjective (onto) if each element of the codomain is mapped to by the function f from at least one element in the domain (which also implies that the range and the codomain are equal, see §1.1). In mathematical terms, it is written as follows:

$$\forall y \in Y,\ \exists x \in X \text{ such that } y = \mathrm{f}(x).$$

Bijective Function

The function f is bijective if each element of the codomain is mapped to by exactly one element from the domain. Equivalently, the function f is bijective if and only if the function f is both injective and surjective. Note that not all injective functions are surjective, and vice-versa.

Example 1 (Complex number notation)

A non-zero complex number can be written in two forms: the cartesian form $x + y\mathrm{i}$ with x the real part and yi the imaginary part, and the polar form $re^{\mathrm{i}\theta}$ with r being the modulus and θ being the principal argument. Let f be the function that maps the cartesian form of each non-zero complex number to its polar form, i.e.

$$\mathrm{f}: \mathbb{R}^2 \setminus (0,0) \mapsto \mathbb{R}^+ \times (-\pi, \pi] \text{ such that } \mathrm{f}(x, y) = (r, \theta)$$

(a) State the rule of the function f by expressing r and θ in terms of x and y.

(b) Express x and y in terms of r and θ, and explain why
 (i) f is injective, (ii) f is surjective.

[Solution]

(a) $r = \sqrt{x^2 + y^2}$,

$$\theta = \begin{cases} \tan^{-1}\left(\dfrac{y}{x}\right) & \text{if } x > 0 \text{ (i.e. 1st or 4th quadrant),} \\ \tan^{-1}\left(\dfrac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \text{ (i.e. 2nd quadrant),} \\ \tan^{-1}\left(\dfrac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0 \text{ (i.e. 3nd quadrant)} \\ \dfrac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\dfrac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \end{cases}$$

(Note that θ is undefined when the complex number is 0 i.e. there is no meaningful polar form for the number 0, hence the function f is defined for non-zero complex number mappings)

(b) $x = r\cos\theta,\ y = r\sin\theta$ (proof from (a) is left as an exercise, lengthy)

 (i) Given $(r_1, \theta_1) = (r_2, \theta_2)$,

 $x_1 = r_1 \cos\theta_1 = r_2 \cos\theta_2 = x_2$

 $y_1 = r_1 \sin\theta_1 = r_2 \sin\theta_2 = y_2$

 Therefore f is injective.

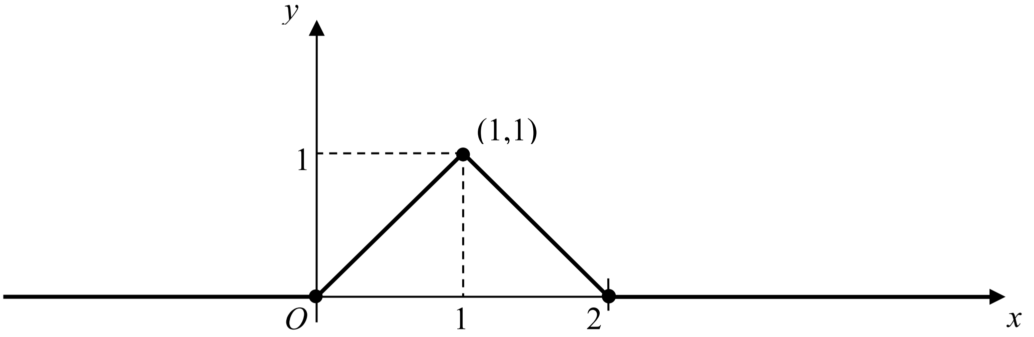
 (ii) For every image $(r, \theta) \in \mathbb{R}^+ \times (-\pi, \pi]$ in the codomain of the function f, there will be a preimage $(r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \setminus (0,0)$ and hence f is surjective.

§2 Piecewise Functions

A function that has its domain divided into **separate partitions and each partition of the domain** given a different formula or rule is known as a piecewise function, i.e. a function defined "piece-wise". Note that a piecewise function need not be continuous and special care is to be taken at the end points of each partition.

The following function f is an example of a piecewise function.

$$\mathrm{f}(x) = \begin{cases} 2-x & 1 < x \le 2 \\ x & 0 < x \le 1 \\ 0 & x \le 0 \text{ or } x > 2 \end{cases}$$



§2.1 Absolute Value and Absolute Value Functions

The **absolute value** of x, denoted by $|x|$, is the distance of x from 0 on the real number line. Distances are always positive or 0, so we have

$$|x| \ge 0 \qquad \text{for every number } x.$$

For example,

$$|3| = 3, \quad |-3| = 3, \quad |\sqrt{2}-1| = \sqrt{2}-1, \quad |2-\pi| = \pi - 2$$

Hence **Absolute Value Functions** are an example of a piecewise function; the most basic absolute value function is the following:

$$|x| = \begin{cases} -x & x < 0 \\ x & x \ge 0 \end{cases}$$

Also we have the following properties: Let a be a constant with $a > 0$.

(i) $|x| = a \Leftrightarrow x = \pm a$ (ii) $|x| < a \Leftrightarrow -a < x < a$ (iii) $|x| > a \Leftrightarrow x < -a$ or $x > a$

Example 2

Show that the function $f(x) = -\frac{1}{2}\left(2|x-1| - |x-2| - |x|\right)$, $x \in \mathbb{R}$ is equivalent to following piecewise function,

$$g(x) = \begin{cases} 2-x & 1 < x \leq 2 \\ x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

i.e. $f(x) = g(x)$ for all $x \in \mathbb{R}$.

[Solution]

We just need to verify that that the two functions are equivalent separately for the **4 partitions of the domain**. Note that the last partition is actually comprised of the union of the sets $(-\infty, 0]$ and $(2, \infty)$.

When $x < 0$, we have $|x| = -x$, $|x-1| = 1-x$, $|x-2| = 2-x$,

$$\begin{aligned} f(x) &= -\frac{1}{2}\left(2|x-1| - |x-2| - |x|\right) \\ &= -\frac{1}{2}\left(2(1-x) - (2-x) - (-x)\right) = 0 = g(x) \end{aligned}$$

When $0 \leq x \leq 1$, we have $|x| = x$, $|x-1| = 1-x$, $|x-2| = 2-x$,

$$\begin{aligned} f(x) &= -\frac{1}{2}\left(2|x-1| - |x-2| - |x|\right) \\ &= -\frac{1}{2}\left(2(1-x) - (2-x) - (x)\right) = x = g(x) \end{aligned}$$

When $1 < x \leq 2$, we have $|x| = x$, $|x-1| = x-1$, $|x-2| = 2-x$,

$$\begin{aligned} f(x) &= -\frac{1}{2}\left(2|x-1| - |x-2| - |x|\right) \\ &= -\frac{1}{2}\left(2(x-1) - (2-x) - (x)\right) = 2-x = g(x) \end{aligned}$$

When $x > 2$, we have $|x| = x$, $|x-1| = x-1$, $|x-2| = x-2$,

$$\begin{aligned} f(x) &= -\frac{1}{2}\left(2|x-1| - |x-2| - |x|\right) \\ &= -\frac{1}{2}\left(2(x-1) - (x-2) - (x)\right) = 0 = g(x) \end{aligned}$$

Since $f(x) = g(x)$ for all partitions of the domain, we have $f(x) = g(x)$ for all $x \in \mathbb{R}$.

§2.2 The Greatest Integer Function

We define the **greatest integer function** $\mathrm{f}(x)=\lfloor x \rfloor$ as the greatest integer smaller than or equals to x. In other words, if x is a real number such that $n \le x < n+1$, then $\lfloor x \rfloor = n$.

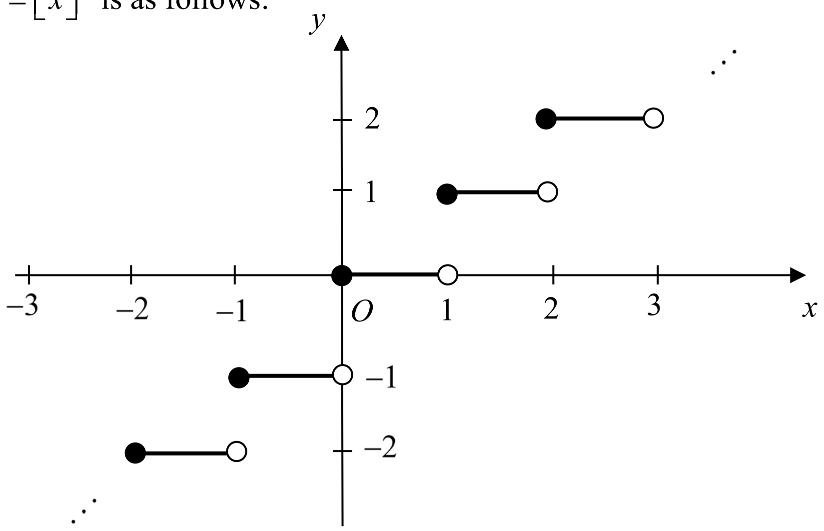
For example:
$\lfloor 2.1 \rfloor = 2$ since $2 \le 2.1 < 3$, $\lfloor 1 \rfloor = 1$ since $1 \le 1 < 2$,
$\lfloor 0.8 \rfloor = 0$ since $0 \le 0.8 < 1$, $\lfloor -1.5 \rfloor = -2$ $[-1.5] = -2$ since $-2 \le -1.5 < -1$,
$\lfloor -7 \rfloor = -7$ etc.

We therefore have, for $x \in \mathbb{R}$, $$\lfloor x \rfloor = n. \Leftrightarrow n \le x < n+1,\ n \in \mathbb{Z}.$$

Remarks:
1. If $x \in \mathbb{Z}$, then $n = \lfloor x \rfloor = x$.
2. If $x \notin \mathbb{Z}$, then $n = \lfloor x \rfloor < x$.

$$\lfloor x \rfloor = \begin{cases} \uparrow & \uparrow \\ -2, & -2 \le x < -1 \\ -1, & -1 \le x < 0 \\ 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & 2 \le x < 3 \\ \downarrow & \downarrow \end{cases}$$

The graph of $y = \lfloor x \rfloor$ is as follows:



Note that the greatest integer function is a special type of step functions (piecewise constant function). The greatest integer function is also known as the **floor function**.

Example 3
Prove that for $x \in \mathbb{R}$ and $n \in \mathbb{Z}$, $\lfloor x+n \rfloor = \lfloor x \rfloor + n$.

[Solution]
Let $\lfloor x \rfloor = m$.
By definition, $m \in \mathbb{Z}$ and $m \le x < m+1$.
Adding n to $m \le x < m+1$ gives
$\underbrace{m+n}_{\in\mathbb{Z}} \le x+n < \underbrace{(m+n)+1}_{\in\mathbb{Z}}$.
Since $m+n \in \mathbb{Z}$, we must have
$$\begin{aligned}\lfloor x+n \rfloor &= m+n \\ &= \lfloor x \rfloor + n\end{aligned}$$
as desired.

Example 4
Prove or disprove that
(i) $\lfloor x-y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$ for all $x, y \in \mathbb{R}$.
(ii) $\lfloor nx \rfloor = n\lfloor x \rfloor$ for all $x \in \mathbb{R}$, $n \in \mathbb{Z}^+$.

[Solution]
(i) We disprove this using a counter-example by letting $x = 0.1$, $y = 0.2$, with
$\lfloor x-y \rfloor = \lfloor 0.1-0.2 \rfloor = \lfloor -0.1 \rfloor = -1$ and $\lfloor x \rfloor - \lfloor y \rfloor = \lfloor 0.1 \rfloor - \lfloor 0.2 \rfloor = 0-0=0$
$\lfloor x-y \rfloor \ne \lfloor x \rfloor - \lfloor y \rfloor$ in general.
In fact, $\lfloor x+y \rfloor \ne \lfloor x \rfloor + \lfloor y \rfloor$ in general as well. Put $x = y = 0.5$ to see why.

(ii) Counter-example: Take $n = 2$, $x = 0.5$.

Note that likewise, there is also a **ceiling function** $f(x) = \lceil x \rceil$ which is the direct opposite of the floor function. The ceiling function maps all real numbers in the domain to the smallest integer not smaller that it.

$$\lceil x \rceil = \begin{cases} \lfloor x \rfloor + 1, & x \notin \mathbb{Z} \\ \lfloor x \rfloor, & x \in \mathbb{Z} \end{cases}$$

§3 Symmetrical Functions

There are some special functions which have some form of geometric symmetry. We will introduce some of them, namely even functions, odd functions and periodic functions.

§3.1 Even Functions

If $\mathrm{f}(-x)=\mathrm{f}(x)$ for every x in the domain, then f is called an even function. The graph of an even function is **symmetric about the *y*-axis.**

The following functions are examples of even functions:

- $\mathrm{f}(x)=x^{2n}$, where $n\in\mathbb{Z}_0^+$.
- $\mathrm{f}(x)=k\left(\cos(nx)\right)$, where k, n are constants
- $\mathrm{f}(x)=\sum_{i=1}^{n}\mathrm{f}_i(x)$ where all $\mathrm{f}_i(x)$ are even functions
- $\mathrm{f}(x)=\prod_{i=1}^{n}\mathrm{f}_i(x)$ where all $\mathrm{f}_i(x)$ are even functions
- $\mathrm{f}(x)=|x|$ for all $x\in\mathbb{R}$

The first refers to even powered single term polynomial, the second the cosine function, the third says that the sum of finite number of even functions are even and the fourth says that the product of even functions are even.

Exercise: Can you verify that the functions above are all even?

Example 5
Show that if f is an even function, then $-\mathrm{f}$ is also an even function.

[Solution]

$$\begin{aligned}(-\mathrm{f})(-x)&=-\mathrm{f}(-x) && \text{(by definition of } -\mathrm{f})\\ &=-\mathrm{f}(x) && \text{(f is an even function)}\\ &=(-\mathrm{f})(x) && \text{(by definition of } -\mathrm{f})\end{aligned}$$

Therefore $-\mathrm{f}$ is also an even function when f is an even function.

Example 6

Show that $\mathrm{f}(x)=\sum_{i=1}^{n}\mathrm{f}_i(x)$, where all $\mathrm{f}_i(x)$ are even functions, is an even function.

[Solution]

$$\begin{aligned}\mathrm{f}(-x)&=\sum_{i=1}^{n}\mathrm{f}_i(-x)\\ &=\sum_{i=1}^{n}\mathrm{f}_i(x) && \left(\because\ \mathrm{f}_i(x)\text{ are even functions, i.e. }\mathrm{f}_i(-x)=\mathrm{f}_i(x)\right)\\ &=\mathrm{f}(x)\ \text{(shown)} && \text{Thus f is an even function.}\end{aligned}$$

§3.2 Odd Functions

If $f(-x) = -f(x)$ for every x in the domain, then f is called an odd function. The graph of an odd function is **symmetric about the origin,** i.e., we can obtain the entire graph of $y = f(x)$ from itself by rotating through 180° about the origin.

The following functions are examples of odd functions:

- $f(x) = x^{2n+1}$, where $n \in \mathbb{Z}_0^+$.
- $f(x) = k(\sin(nx))$, where k, n are constants
- $f(x) = \sum_{i=1}^{n} f_i(x)$ where all $f_i(x)$ are odd functions
- $f(x) = \prod_{i=1}^{2n+1} f_i(x)$ where all $f_i(x)$ are odd functions

The first refers to odd powered single term polynomial, the second the sine function, the third says that the sum of finite number of odd functions are odd and the fourth says that the product of finite odd number of odd functions are odd.

Exercise: Can you verify that the functions above are all odd?

Example 7
Show that if f is an odd function, then $-f$ is also an odd function.

[Solution]

$$\begin{aligned}(-f)(-x) &= -f(-x) && \text{(by definition of } -f) \\ &= -(-f(x)) && \text{(f is an odd function)} \\ &= -(-f)(x) && \text{(by definition of } -f)\end{aligned}$$

Therefore $-f$ is also an odd function when f is an odd function.

Example 8

Show that $f(x) = \prod_{i=1}^{2n+1} f_i(x)$, where all $f_i(x)$ are odd functions, is an odd function.

[Solution]

$$\begin{aligned}f(-x) &= \prod_{i=1}^{2n+1} f_i(-x) \\ &= \prod_{i=1}^{2n+1} [-f_i(x)] \qquad (\because f_i(x) \text{ are odd functions, i.e. } f_i(-x) = -f_i(x)\ ) \\ &= (-1)^{2n+1} \prod_{i=1}^{2n+1} f_i(x) \\ &= -\prod_{i=1}^{2n+1} f_i(x) \\ &= -f(x) \qquad \text{(shown)}\end{aligned}$$

Thus f is an odd function.

Example 9
Show that the derivative f' of a differentiable even function f is odd and vice versa.

[Solution]

$$f(-x) = f(x)$$

$$\frac{d[f(-x)]}{dx} = \frac{d[f(x)]}{dx}$$

$$-f'(-x) = f'(x) \quad \text{(by Chain Rule)}$$

$$f'(-x) = -f'(x) \quad \text{(shown)}$$

Therefore f' of a differentiable even function is odd.

Also,

$$f(-x) = -f(x)$$

$$\frac{d[f(-x)]}{dx} = \frac{d[-f(x)]}{dx}$$

$$-f'(-x) = -f'(x) \quad \text{(by Chain Rule)}$$

$$f'(-x) = f'(x) \quad \text{(shown)}$$

Therefore f' of a differentiable odd function is even.

§3.3 Periodic Functions

If $f(x+p) = f(x)$ for all x defined in the domain of the function, where p is a positive constant, then f is called a **periodic function** and **the smallest such number *p*** is called the **period**.

In addition, if the domain is $\mathbb{R}$ (sufficient but not necessary condition) then the function will have translational symmetry (i.e. translation of p units left or right have no effect on the graph).

The following table shows some basic periodic functions with their corresponding period.

Function f(x)	Period
$\sin x$	2π
$\cos x$	2π
$\tan x$	π

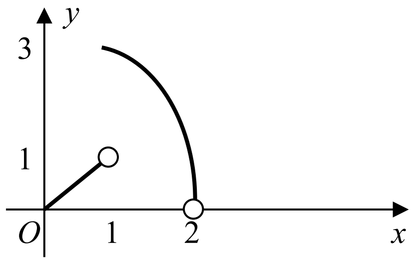
These Trigonometric functions are examples of **simple periodic functions**.

Example 10 (Compound Periodic Function)

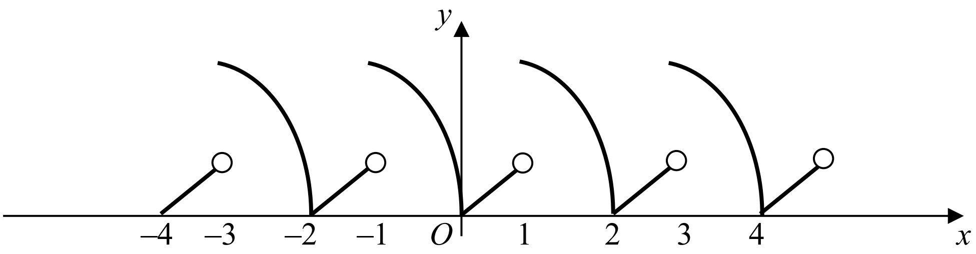
A function $\mathrm{f}(x)$ is defined as $\mathrm{f}(x)=\begin{cases} x, & 0 \le x < 1 \\ 4-x^2, & 1 \le x < 2 \end{cases}$

and such that $\mathrm{f}(x)=\mathrm{f}(x+2)$ for all $x \in \mathbb{R}$.

For $0 \le x < 2$ (called the **fundamental domain**), a section of the graph $y=\mathrm{f}(x)$ is shown below.



Since $\mathrm{f}(x)=\mathrm{f}(x+2)$ for all $x \in \mathbb{R}$, the function is periodic with period 2. So the above pattern repeats itself at regular intervals of 2 units.



(i) Compute f(–10), f(33), f(49.5), f(2013), f(2014), $\dfrac{\mathrm{f}(2n+1)}{\mathrm{f}(4n+3)}$ where $n \in \mathbb{Z}$.

(ii) Find the possible values of x for which $\mathrm{f}(x)=2$.

[Solution]

f(–10) = f(–10 + 2) = f(–8) = f(–8 + 2) = f(–8 + 2) = f(–6) = … = f(0) = 0.

f(33) = f(32 + 1) = f(30 + 1) = f(28 + 1) = … = f(1) = 4 – 1^2 = 3.

f(49.5) = f(2×24 + 1.5) = f(1.5) = 4 – 1.5^2 = 1.75.

f(2013) = f(2×1006 + 1) = f(1) = 3.

f(2014) = f(2×1007) = f(0) = 0.

$\dfrac{\mathrm{f}(2n+1)}{\mathrm{f}(4n+3)}=\dfrac{\mathrm{f}(1)}{\mathrm{f}(3)}=\dfrac{\mathrm{f}(1)}{\mathrm{f}(1)}=1.$

For $0 \le x < 2$, $\mathrm{f}(x)=2 \Rightarrow 4-x^2=2 \Rightarrow x^2=2 \Rightarrow x=\pm\sqrt{2}$. But since $0 \le x < 2$, $x=\sqrt{2}$.

By periodicity, $\mathrm{f}(x)=2 \Rightarrow x=2n+\sqrt{2}, n \in \mathbb{Z}$.

§4 Strictly Monotonic Functions / Increasing and Decreasing Functions

A function f is said to be **strictly monotonic increasing** if for all $x_1, x_2 \in D_f$

$$x_2 > x_1 \Leftrightarrow f(x_2) > f(x_1) \quad \text{(see Fig. 1).}$$

For example, $f(x) = x$, e^x, $\ln x$ $(x > 0)$, $\sqrt{x}$, x^2 $(x \geq 0)$ are strictly monotonic increasing functions.

Similarly, a function f is said to be **strictly monotonic decreasing** if for all $x_1, x_2 \in D_f$

$$x_2 > x_1 \Leftrightarrow f(x_2) < f(x_1) \quad \text{(see Fig. 2).}$$

For example, $f(x) = -x$, e^{-x}, $-\ln x$ $(x > 0)$, $-\sqrt{x}$, x^2 $(x \leq 0)$ are strictly monotonic decreasing functions.

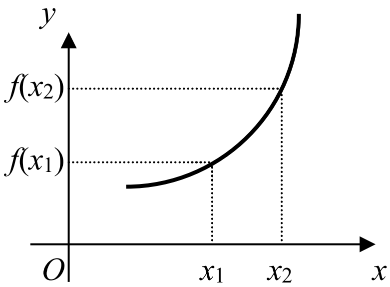


Figure 1

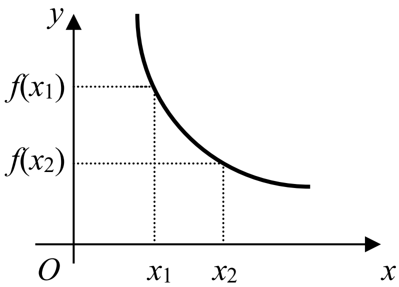


Figure 2

Note: For **differentiable functions**, we can show the function is strictly increasing on an interval I by showing that $f'(x) > 0$ $\forall x \in I$, and correspondingly strictly decreasing on an interval I by showing that $f'(x) < 0$ $\forall x \in I$.

Example 11

Given the function $f : \mathbb{R}^+ \to \mathbb{R}$ such that $f(x) = x - \frac{1}{x}$, Show that f is strictly monotonic increasing.

[Solution]

<u>Method 1: Algebraic Method</u>

Now $x_2 > x_1 \Leftrightarrow$ $f(x_2) - f(x_1) = \left(x_2 - \frac{1}{x_2}\right) - \left(x_1 - \frac{1}{x_1}\right)$

$$= (x_2 - x_1) + \left(\frac{1}{x_1} - \frac{1}{x_2}\right) > 0 \text{ since } x_2 - x_1 > 0 \text{ and } \frac{1}{x_1} - \frac{1}{x_2} > 0$$

$$\Leftrightarrow f(x_2) > f(x_1)$$

$\Rightarrow$ f is strictly monotonic increasing on $\mathbb{R}^+$.

<u>Method 2: Analytical Method</u>

Let $y = x - \frac{1}{x}$. Clearly the function $x - \frac{1}{x}$ is continuous and differentiable for all $x \in \mathbb{R}^+$.

So, differentiating y with respect to x, we get $\frac{dy}{dx} = 1 + \frac{1}{x^2} > 0$ for all $x \in \mathbb{R}^+$.

Hence f is strictly monotonic increasing on $\mathbb{R}^+$.

Example 12

A function f is said to be 1–1 if $x_1 = x_2 \Leftrightarrow \mathrm{f}(x_1) = \mathrm{f}(x_2)$ for all $x_1, x_2 \in D_\mathrm{f}$. Prove that

(i) $\mathrm{f}(x) = \dfrac{1}{x^4+1}$ where $x \geq 0$ is 1–1.

(ii) if f is strictly monotonic increasing (decreasing), then f is 1–1. Is the converse true?

[Solution]

<table>
<tr>
<td>(i) $\mathrm{f}(x_1) = \mathrm{f}(x_2) \Leftrightarrow \dfrac{1}{x_1^4+1} = \dfrac{1}{x_2^4+1} \Leftrightarrow x_1^4 - x_2^4 = 0$
$\Leftrightarrow (x_1^2 + x_2^2)(x_1^2 - x_2^2) = 0 \Leftrightarrow x_1^2 + x_2^2 = 0$ or $x_1^2 - x_2^2 = 0$
$\Leftrightarrow x_1 = x_2 = 0$ or $x_1 = \pm x_2$
But since $x_1,\ x_2 \geq 0,\ x_1 \neq -x_2$ except for the case $x_1 = x_2 = 0$.
Hence $x_1 = x_2$ must hold and this proves that f is 1–1.

Note: The contrapositive statement of the above is:
f is 1–1 if $x_1 \neq x_2 \Leftrightarrow \mathrm{f}(x_1) \neq \mathrm{f}(x_2)$ for all $x_1, x_2 \in \mathrm{D_f}$.</td>
<td>We want to show
$\mathrm{f}(x_1) = \mathrm{f}(x_2) \Leftrightarrow x_1 = x_2$

Thus $x_1 = x_2$ in every case</td>
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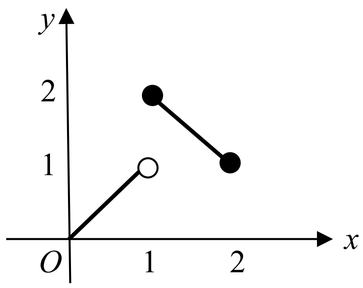
<tr>
<td>(ii) Suppose f is strictly monotonic increasing. Then we have
$x_2 \underset{\neq}{>} x_1 \Leftrightarrow \mathrm{f}(x_2) \underset{\neq}{>} \mathrm{f}(x_1)$.
This implies f is 1–1 by the contrapositive statement in (i).
The converse is not true. That is, f is 1–1 does not imply that f is monotonic increasing or decreasing. Consider the piecewise function:
$\mathrm{f}(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ -x+3 & \text{if } 1 \leq x \leq 2 \end{cases}$.

We can easily graph the function as follows:


Clearly, f is 1–1 but is neither strictly monotonic increasing nor decreasing.</td>
<td>Recall that the contrapositive of the statement ‘ $P \Rightarrow Q$ ’ is ‘ $Q' \Rightarrow P'$ ’

<u>Extension</u>
What if we relax the condition ‘f is strictly monotonic increasing’ to just ‘f is monotonic increasing’? Is f still 1–1?

[No. For example,
$\mathrm{f}(x) = \begin{cases} x, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \\ x-1, & x \geq 2 \end{cases}$
f(1) = f(1.5) but 1 ≠ 0.5]</td>
</tr>
</table>

§4.1 Increasing or Decreasing Functions

A function f is **increasing** on an interval I if

$$\mathrm{f}(x_1) \le \mathrm{f}(x_2) \qquad \text{for all } x_1 < x_2 \text{ in } I$$

A function f is **decreasing** on an interval I if

$$\mathrm{f}(x_1) \ge \mathrm{f}(x_2) \qquad \text{for all } x_1 < x_2 \text{ in } I$$

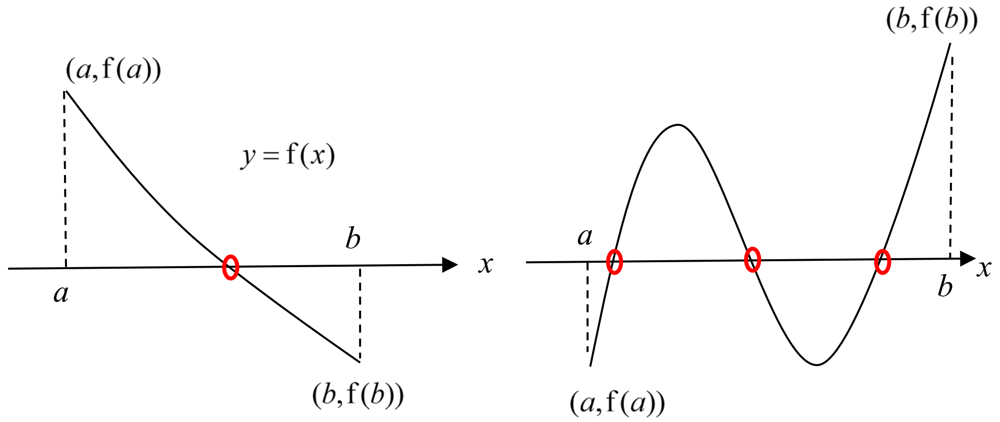
Note: For **differentiable functions**, we can show the function is increasing on an interval I by showing that $\mathrm{f}'(x) \ge 0 \ \forall x \in I$, and correspondingly decreasing on an interval I by showing that $\mathrm{f}'(x) \le 0 \ \forall x \in I$.

§4.2 Location of roots of an equation

Often, it is difficult to solve for the exact value of the roots of an equation, eg, $\mathrm{e}^x + x = 3$. In which case, we may be interested to approximate the location of the root in an interval, where upon it could be further improved on using various numerical method.

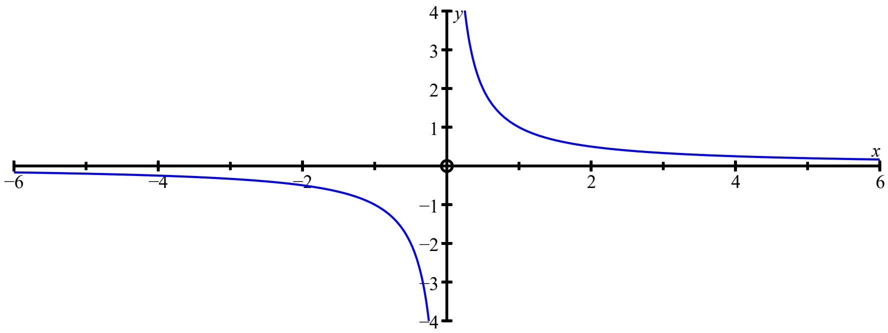
Suppose the graph of $y = \mathrm{f}(x)$ is **continuous** in the interval $[a, b]$.

(i) If $\mathrm{f}(a)$ and $\mathrm{f}(b)$ have **opposite** signs (i.e. $\mathrm{f}(a) \cdot \mathrm{f}(b) < 0$), then there is **at least** one real root in the interval $[a, b]$.
In fact, there is an odd number of real roots, some of which may be equal.



Note
If $\mathrm{f}(a)$ and $\mathrm{f}(b)$ have **opposite** signs and the function $y = \mathrm{f}(x)$ is either **strictly increasing or decreasing** in the interval $[a, b]$, then the equation $\mathrm{f}(x) = 0$ has **exactly one real root** in the interval $[a, b]$.

If $\mathrm{f}(a) \cdot \mathrm{f}(b) < 0$ but $y = \mathrm{f}(x)$ is **not continuous** in the interval $[a, b]$, then the existence of a root of $\mathrm{f}(x) = 0$ cannot be ensured. For e.g. $y = \mathrm{f}(x) = \dfrac{1}{x}$



$f(-1)=-1<0$ and $f(1)=1>0$ but $f(x)=0$ has no real root in $[-1,1]$ as f is not continuous in $[-1,1]$.

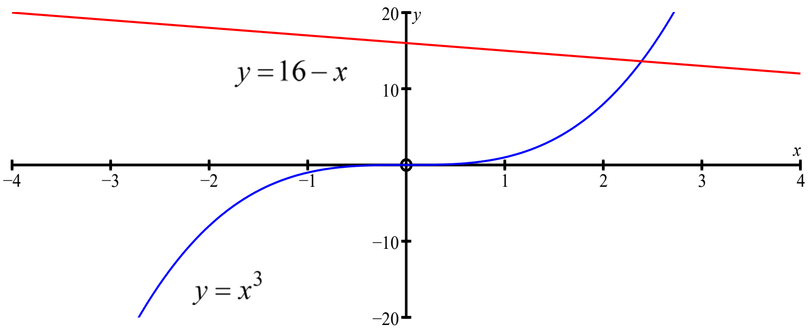
(ii) If $f(a)$ and $f(b)$ have the **same** sign (i.e. $f(a)\cdot f(b)>0$), then there is **no** real root or even number in the interval $[a,b]$.

Example 13
Show that $x^3+x-16=0$ has exactly one real root α and find the integer N such that $N<\alpha<N+1$.

[Solution]
Let $f(x)=x^3+x-16$
$f'(x)=3x^2+1>0 \quad \forall x\in\mathbb{R}$

Since f is continuous and strictly increasing from $-\infty$ to ∞, $y=f(x)$ cuts the x-axis once so there is exactly one real root.

Alternative: Graphical Method
$x^3+x-16=0 \Leftrightarrow x^3=16-x$



The no. of real roots of $x^3+x-16=0$ comes from the number of intersections of the graphs of $y=x^3$ and $y=16-x$. Since there is only 1 intersection, $x^3+x-16=0$ has exactly one real root.
$f(2)=2^3+2-16=-6$
$f(3)=3^3+3-16=14$
Since f is continuous and $f(2)\cdot f(3)<0$, $2<\alpha<3$. $\therefore$ the integer N is 2.

Example 14 (N80/2/15 modified)

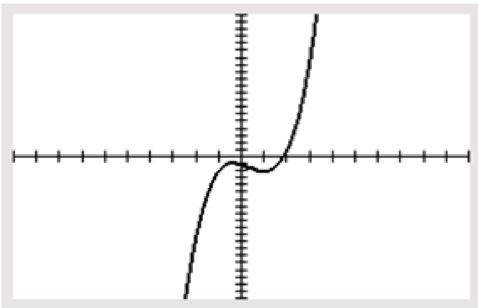
Find the exact coordinates of the turning points on the graph of $y = \mathrm{f}(x)$ where $\mathrm{f}(x) = x^3 - x^2 - x - 1$. Deduce that the equation $\mathrm{f}(x) = 0$ has only one real root α, and prove that α lies between 1 and 2.

[Solution]

$$\mathrm{f}(x) = x^3 - x^2 - x - 1$$
$$\mathrm{f}'(x) = 3x^2 - 2x - 1 = (3x+1)(x-1)$$
$$\mathrm{f}'(x) = 0 \Rightarrow x = -\frac{1}{3} \text{ or } x = 1$$
$$\mathrm{f}(-\frac{1}{3}) = \left(-\frac{1}{3}\right)^3 - \left(-\frac{1}{3}\right)^2 - \left(-\frac{1}{3}\right) - 1 = -\frac{22}{27}$$
$$\mathrm{f}(1) = 1 - 1 - 1 - 1 = -2$$

The turning points are $\left(-\frac{1}{3}, -\frac{22}{27}\right)$ and $(1, -2)$.

Since the turning points are below the x-axis, the graph would look like this for the cubic curve.



Therefore the equation $\mathrm{f}(x) = 0$ has only one real root α.

$$\mathrm{f}(1) = -2 < 0$$
$$\mathrm{f}(2) = 1 > 0$$

Since f is continuous and $\mathrm{f}(1) \cdot \mathrm{f}(2) < 0$, α lies between 1 and 2.

§5 Convex and Concave Functions

§5.1 Convex Functions (Concave Up)

A function f is convex (concave upwards) if for **any** two points x_1 and x_2 in the domain of the function and $0 \le t \le 1$, we have

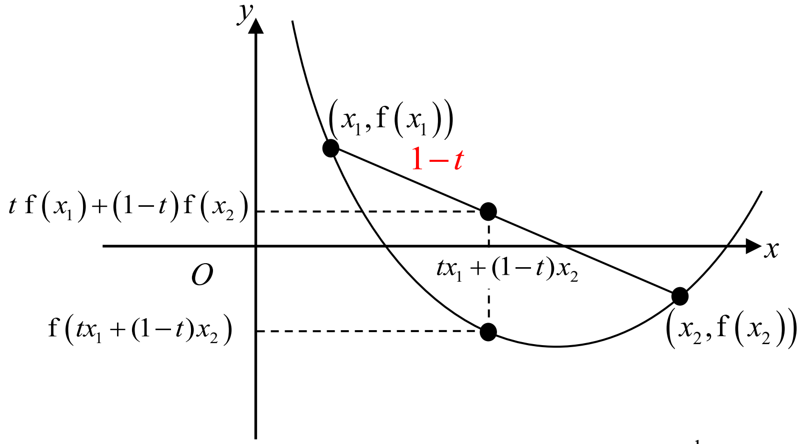
$$\mathrm{f}\left(tx_1+(1-t)x_2\right) \le t\,\mathrm{f}\left(x_1\right)+\left(1-t\right)\mathrm{f}\left(x_2\right)$$

Note that equality holds when $x_1 = x_2$.

A function f is **strictly** convex if for **any** two points x_1 and x_2 in the domain of the function with $x_1 \ne x_2$ and $0 < t < 1$, we have

$$\mathrm{f}\left(tx_1+(1-t)x_2\right) < t\,\mathrm{f}\left(x_1\right)+\left(1-t\right)\mathrm{f}\left(x_2\right)$$

t



Also, recall that for a continuous convex function that has a 2nd derivative, $\mathrm{f}''\left(x\right) > 0$. It can be shown that using Taylor's Theorem that this result is equivalent to the definition given.

[Proof]

The y-coordinate of the line segment joining $\left(x_1, \mathrm{f}\left(x_1\right)\right)$ and $\left(x_2, \mathrm{f}\left(x_2\right)\right)$ when $x = tx_1+(1-t)x_2$ is given by

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$

$$\frac{y-\mathrm{f}\left(x_2\right)}{\left[tx_1+(1-t)x_2\right]-x_2} = \frac{\mathrm{f}\left(x_1\right)-\mathrm{f}\left(x_2\right)}{x_1-x_2}$$

$$\begin{aligned} y &= \mathrm{f}\left(x_2\right)+\frac{\mathrm{f}\left(x_1\right)-\mathrm{f}\left(x_2\right)}{x_1-x_2}t\left(x_1-x_2\right) \\ &= t\mathrm{f}\left(x_1\right)+\left(1-t\right)\mathrm{f}\left(x_2\right) \end{aligned}$$

Since curve is strictly convex, the y-value of the curve is smaller than the y-value of the line segment, i.e.

$$\mathrm{f}\left(tx_1+(1-t)x_2\right) < t\,\mathrm{f}\left(x_1\right)+\left(1-t\right)\mathrm{f}\left(x_2\right)$$

Example 15

Show that the function $f(x)=x^2$ is convex without using differentiation.

[Solution]

For $0\le t\le 1$ and when $x=tx_1+(1-t)x_2$,

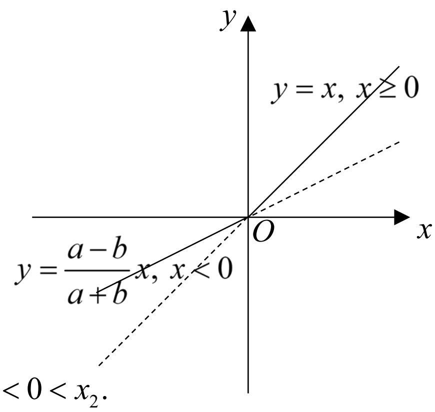
$$\text{LHS}=f\left(tx_1+(1-t)x_2\right)=\left(tx_1+(1-t)x_2\right)^2$$
$$\text{RHS}=t\,f\left(x_1\right)+(1-t)\,f\left(x_2\right)=t\left(x_1\right)^2+(1-t)\left(x_2\right)^2$$
$$\begin{aligned}\text{LHS}-\text{RHS}&=\left(tx_1+(1-t)x_2\right)^2-\left(t\left(x_1\right)^2+(1-t)\left(x_2\right)^2\right)\\&=t^2\left(x_1\right)^2+2t(1-t)x_1x_2+(1-t)^2\left(x_2\right)^2-t\left(x_1\right)^2-(1-t)\left(x_2\right)^2\\&=-t(1-t)\left(\left(x_1\right)^2-2x_1x_2+\left(x_2\right)^2\right)\\&=-t(1-t)\left(x_1-x_2\right)^2\\&\le 0\end{aligned}$$

$\therefore f\left(tx_1+(1-t)x_2\right)\le t\,f\left(x_1\right)+(1-t)\,f\left(x_2\right)$ and hence f is a convex function.

Example 16

Given $a>b>0,$ show algebraically that the function $f(x)=\dfrac{ax+b|x|}{a+b}$ is a convex function.

$$f(x)=\frac{ax+b|x|}{a+b}=\begin{cases}x & \text{if } x\ge 0,\\ \left(\dfrac{a-b}{a+b}\right)x & \text{if } x<0.\end{cases}$$



If $0\le x_1\le x_2$ or $x_1\le x_2\le 0$ then both $\left(x_1,f\left(x_1\right)\right)$ and $\left(x_2,f\left(x_2\right)\right)$ lie on the same linear line and hence convexity holds.

Thus, we are only left to show that convexity holds for $x_1<0<x_2$.

For $0\le t\le 1,$

$$\begin{aligned}f\left(tx_1+(1-t)x_2\right)&=\max\left(\left(\frac{a-b}{a+b}\right)\left(tx_1+(1-t)x_2\right),\left(tx_1+(1-t)x_2\right)\right)\\&\le t\left(\frac{a-b}{a+b}\right)x_1+\left(1-t\right)x_2 \qquad \left(\text{since } x_1<0<x_2 \text{ and } 0<\frac{a-b}{a+b}<1\right)\\&=t\,f\left(x_1\right)+\left(1-t\right)f\left(x_2\right)\end{aligned}$$

Therefore, f is a convex function.

§5.2 Concave Functions (Concave Down)

A function f is concave (concave downwards) if for **any** two points x_1 and x_2 in the domain of the function and $0 \le t \le 1$, we have

$$\mathrm{f}\left(tx_1+(1-t)x_2\right) \ge t\,\mathrm{f}\left(x_1\right)+\left(1-t\right)\mathrm{f}\left(x_2\right)$$

Notice the switch in direction of inequality sign, as compared to that of convex function.

A function f is **strictly** concave if for **any** two points x_1 and x_2 in the domain of the function with $x_1 \ne x_2$ and $0 < t < 1$, we have

$$\mathrm{f}\left(tx_1+(1-t)x_2\right) > t\,\mathrm{f}\left(x_1\right)+\left(1-t\right)\mathrm{f}\left(x_2\right)$$

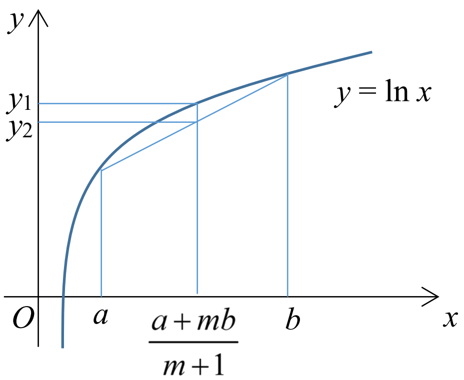
Similarly, for a continuous concave function that has a 2nd derivative, $\mathrm{f}''\left(x\right) < 0$.

Example 17 (2015 RI H3 Prelim Q1(b)(i))
Sketch the graph of $y = \ln x$ and hence explain why

$$\frac{\ln a + m\ln b}{m+1} \le \ln\left(\frac{a+mb}{m+1}\right)$$

where *a* and *b* are positive real numbers and *m* is a positive integer.

[Solution]



From the graph, $y = \ln x$ is concave. Thus

$$y_2 \le y_1 \Rightarrow \frac{\ln a + m\ln b}{m+1} \le \ln\left(\frac{a+mb}{m+1}\right)$$ with equality when $a = b$.