# PROBLEM SOLVING AND NUMBER THEORY
# TUTORIAL 3

1 Show that the number 1000! ends with 249 zeroes.

2 Let $a$, $b$, $c$ be integers satisfying $a^2+b^2=c^2$. Give two different proofs that $abc$ must be even.

3 Prove that there exist infinitely many primes $p$ such that $p=3 \bmod 4$.

4 If 0 = Sunday, 1 = Monday, 2 = Tuesday … 6 = Saturday, then January 1 of year $n$ occurs on the day of the week given by the following formula:

$$\left(n+\left\lfloor\frac{n-1}{4}\right\rfloor-\left\lfloor\frac{n-1}{100}\right\rfloor+\left\lfloor\frac{n-1}{400}\right\rfloor\right) \bmod 7$$

Use this formula to find the day for
(a) 1 January 2050,
(b) 1 January 2100.

Interpret the different components of this formula.

5 Let $x$ be a real number such that $x+x^{-1}$ is an integer. Prove that $x^n+x^{-n}$ is an integer for all positive integers $n$.

6 There are 2000 points on a circle and each point is given a number that is equal to the average of the numbers of its two nearest neighbours. Show that all the numbers must be equal.

7 The pigeonhole principle states that given $n$ items to be placed in $m$ containers, if $n>m$, then there will be at least one container containing more than one item, and likewise if $n<m$, then there will be at least one empty container.

Let $n$ be a positive integer. Show that if you have $n$ integers, then either one of them is a multiple of $n$ or a sum of several of them is a multiple of $n$. (Hint: Use pigeonhole principle)

8 Jay wants to express any positive fraction as a finite sum of fractions with numerator 1 and different denominators. He applies an algorithm, and he has the following expressions:

$$\frac{4}{49}=\frac{1}{13}+\frac{1}{213}+\frac{1}{67841}+\frac{1}{9204734721}$$
$$\frac{5}{121}=\frac{1}{25}+\frac{1}{757}+\frac{1}{763309}+\frac{1}{873960180913}+\frac{1}{1527612795642093418846225}$$
$$\frac{7}{260}=\frac{1}{38}+\frac{1}{1647}+\frac{1}{8136180}$$

(i) By looking at the fractions added in each step, describe Jay's algorithm using the expression for $\frac{4}{49}$.

(ii) Given a reduced fraction $\frac{p}{q}$ with $q$ the smallest integer possible, state the maximum number of steps needed for Jay's algorithm to terminate, with explanation.

(iii) None of the expressions are the shortest possible, for example $\frac{4}{49}$ is expressed as a sum of four fractions, but it can be expressed as a sum of a smaller number of fractions, all with numerator 1.

(a) State a possible shorter expression using one of the examples given.

(b) Describe your method in obtaining the shorter expression in (a), and if possible explain why your method produces a shorter expression compared to that obtained by Jay.