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$$\{u_n\} : u_n = u_1 + (n-1)d, u_1, d \geqslant 0$$

$$\begin{aligned}\text{LHS} &= \frac{1}{u_1u_2} + \frac{1}{u_2u_3} + \frac{1}{u_3u_4} + \cdots + \frac{1}{u_{n-1}u_n} \\ &= \sum_{i=2}^{n} \frac{1}{u_{i-1}u_i}\end{aligned} \qquad \text{RHS} = \frac{n-1}{u_1u_n}$$

For the base case, $n = 2$.

$$\text{LHS} = \sum_{n=2}^{2} \frac{1}{u_{i-1}u_i} = \frac{1}{u_1u_2} \qquad \text{RHS} = \frac{1}{u_1u_2}$$

$$n = 2 \implies \text{LHS} = \text{RHS}$$

Assume for some $k \in \mathbb{Z}, k > 2$, $n = k \implies \text{LHS} = \text{RHS}$, such that:

$$\sum_{n=2}^{k} \frac{1}{u_{i-1}u_i} = \frac{k-1}{u_1u_k}$$

When $n = k + 1$.

$$\begin{aligned}\text{LHS} &= \sum_{n=2}^{k+1} \frac{1}{u_{i-1}u_i} \\ &= \sum_{n=2}^{k} \frac{1}{u_{i-1}u_i} + \frac{1}{u_ku_{k+1}} \\ &= \frac{k-1}{u_1u_k} + \frac{1}{u_ku_{k+1}} \\ &= \frac{(k-1)u_{k+1}}{u_1u_ku_{k+1}} + \frac{u_1}{u_1u_ku_{k+1}} \\ &= \frac{ku_{k+1} - u_{k+1} + u_1}{u_1u_ku_{k+1}} \\ &= \frac{(u_k + d)k - (u_1 + kd) + u_1}{u_1u_ku_{k+1}} \\ &= \frac{ku_k + kd - u_1 - kd + u_1}{u_1u_ku_{k+1}} \\ &= \frac{ku_k}{u_1u_ku_{k+1}} \\ &= \frac{k}{u_1u_{k+1}}\end{aligned} \qquad \begin{aligned}\text{RHS} &= \frac{(k+1)-1}{u_1u_{k+1}} \\ &= \frac{k}{u_1u_{k+1}}\end{aligned}$$

$$(n = k \implies \text{LHS} = \text{RHS}) \implies (n = k+1 \implies \text{LHS} = \text{RHS})$$

Therefore, by mathematical induction,

$$\frac{1}{u_1u_2} + \frac{1}{u_2u_3} + \frac{1}{u_3u_4} + \cdots + \frac{1}{u_{n-1}u_n} = \frac{n-1}{u_1u_n},\ n \geqslant 2 \quad \square$$

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$$\begin{aligned}
y &= x\sin x \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= x\cos x + \sin x \\
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \boxed{-x\sin x + 2\cos x} \\
\frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= -x\cos x - 3\sin x \\
\frac{\mathrm{d}^4y}{\mathrm{d}x^4} &= \boxed{x\sin x - 4\cos x} \\
\frac{\mathrm{d}^5y}{\mathrm{d}x^5} &= x\cos x + 5\sin x \\
\frac{\mathrm{d}^6y}{\mathrm{d}x^6} &= -x\sin x + 6\cos x \quad \square
\end{aligned}$$

By inspection, a conjecture is proposed such that:

$$\frac{\mathrm{d}^{2n}y}{\mathrm{d}x^{2n}} = (-1)^n(x\sin x - 2n\cos x),\ n \in \mathbb{Z}^+$$

For the base case, $n = 1$.

$$\begin{aligned}
\text{LHS} &= \frac{\mathrm{d}^2y}{\mathrm{d}x^2} \\
&= -x\sin x + 2\sin x
\end{aligned}
\qquad\qquad
\begin{aligned}
\text{RHS} &= (-1)(x\sin x - 2\cos x) \\
&= -x\sin x + 2\sin x
\end{aligned}$$

$$n = 1 \implies \text{LHS} = \text{RHS}$$

Assume for some $k \in \mathbb{Z}^+$, $n = k \implies \text{LHS} = \text{RHS}$, such that:

$$\frac{\mathrm{d}^{2k}y}{\mathrm{d}x^{2k}} = (-1)^k(x\sin x - 2k\cos x)$$

When $n = k + 1$.

$$\begin{aligned}
\text{LHS} &= \frac{\mathrm{d}^{2(k+1)}y}{\mathrm{d}x^{2(k+1)}} \\
&= \frac{\mathrm{d}^{2k+2}y}{\mathrm{d}x^{2k+2}} \\
&= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\mathrm{d}^{2k}y}{\mathrm{d}x^{2k}}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}\Big((-1)^k(x\sin x - 2k\cos x)\Big)\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\Big((-1)^k(x\cos x + (2k+1)\sin x)\Big) \\
&= (-1)^k(-x\sin x + (2k+2)\cos x)
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= (-1)^{k+1}(x\sin x - 2(k+1)\cos x) \\
&= (-1)^k(-1)(x\sin x - 2(k+1)\cos x) \\
&= (-1)^k(-x\sin x + (2k+2)\cos x)
\end{aligned}$$

$$(n = k \implies \text{LHS} = \text{RHS}) \implies (n = k + 1 \implies \text{LHS} = \text{RHS})$$

Therefore, by mathematical induction,

$$\boxed{\frac{\mathrm{d}^{2n}y}{\mathrm{d}x^{2n}} = (-1)^n(x\sin x - 2n\cos x),\ n \in \mathbb{Z}^+}$$

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$$\text{LHS} = \frac{\mathrm{d}^n}{\mathrm{d}\theta^n}(\sin a\theta) \qquad \text{RHS} = a^n \sin\left(a\theta + \frac{n\pi}{2}\right)$$

For the base case, $n = 1$.

$$\begin{aligned}
\text{LHS} &= \frac{\mathrm{d}}{\mathrm{d}\theta}(\sin a\theta) \\
&= a\cos a\theta
\end{aligned}
\qquad
\begin{aligned}
\text{RHS} &= a\sin\left(a\theta + \frac{\pi}{2}\right) \\
&= a\cos a\theta
\end{aligned}$$

$$n = 1 \implies \text{LHS} = \text{RHS}$$

Assume for some $k \in \mathbb{Z}^+$, $n = k \implies \text{LHS} = \text{RHS}$, such that:

$$\frac{\mathrm{d}^k}{\mathrm{d}\theta^k}(\sin a\theta) = a^k \sin\left(a\theta + \frac{k\pi}{2}\right)$$

When $n = k + 1$.

$$\text{LHS} = \frac{\mathrm{d}^{k+1}}{\mathrm{d}\theta^{k+1}}(\sin a\theta)$$
$$= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{d}^k}{\mathrm{d}\theta^k}(\sin a\theta)\right)$$
$$= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(a^k \sin\left(a\theta + \frac{k\pi}{2}\right)\right)$$
$$= a^k a \cos\left(a\theta + \frac{k\pi}{2} + \frac{\pi}{2} - \frac{\pi}{2}\right)$$
$$= a^{k+1} \sin\left(a\theta + \frac{(k+1)\pi}{2}\right)$$

$$\text{RHS} = a^{k+1} \sin\left(a\theta + \frac{(k+1)\pi}{2}\right)$$

$$(n = k \implies \text{LHS} = \text{RHS}) \implies (n = k + 1 \implies \text{LHS} = \text{RHS})$$

Therefore, by mathematical induction,

$$\boxed{\frac{\mathrm{d}^n}{\mathrm{d}\theta^n}(\sin a\theta) = a^n \sin\left(a\theta + \frac{n\pi}{2}\right),\ n \in \mathbb{Z}^+}$$

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For $n \in \mathbb{Z}^-$, $n!$ is undefined. Hence, consider $n \in \mathbb{Z}^+ \cup \{0\}$.

n	3^n	$n!$
0	1	1
1	3	1
2	9	2
3	27	6
4	81	24
5	243	120
6	729	720
7	2187	5040

Therefore, the smallest integer for which $3^n < n!$ is $n_0 = \boxed{7}$.

$$\text{LHS} = 3^n \qquad\qquad \text{RHS} = 3!$$

For the base case, $n = 7$.

$$\text{LHS} = 3^7$$
$$= 2187$$

$$\text{RHS} = 7! \quad = 5040$$

$$n = 1 \implies \text{LHS} < \text{RHS}$$

Assume for some $k \in \mathbb{Z}, k \geqslant 7$, $n = k \implies$ LHS $<$ RHS, such that:

$$3^k < k!$$

When $n = k + 1$.

$$\begin{aligned} \text{LHS} &= 3^{k+1} \\ &= 3\left(3^k\right) \end{aligned} \qquad\qquad \begin{aligned} \text{RHS} &= (k+1)! \\ &= (k+1)k! \end{aligned}$$

$$\begin{gathered} k \geqslant 7 \implies k + 1 \geqslant 8 > 3 \\ 3^k < k! \\ 3\left(3^k\right) < 3k! \\ 3\left(3^k\right) < (k+1)k! \\ \text{LHS} < \text{RHS} \\ (n = k \implies \text{LHS} < \text{RHS}) \implies (n = k + 1 \implies \text{LHS} < \text{RHS}) \end{gathered}$$

Therefore, by mathematical induction,

$$3^n < n!, \forall n \in \mathbb{Z}, n \geqslant 7 \quad \square$$

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$$\{u_n\} : u_{n+1} = \frac{u_n^2 + 4}{u_n + 2}, \, u_1 = 1, \, n \in \mathbb{Z}, \, n \geqslant 1$$

For base case, $n = 1$.

$$\begin{gathered} u_n = u_1 = 1 \\ n = 1 \implies 0 < u_n < 2 \end{gathered}$$

Assume for some $k \in \mathbb{Z}^+$, $n = k \implies 0 < u_n < 2$, such that:

$$0 < u_k < 2$$

$$\begin{aligned} 2 - u_{k+1} &= 2 - \frac{u_k^2 + 4}{u_k + 2} \\ &= 2 - \left(u_k - 2 + \frac{8}{u_k + 2}\right) \\ &= 4 - u_k - \frac{8}{u_k + 2} \end{aligned}$$

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$$\{u_n\} : u_{n+1} = \frac{u_n^2 + 3u_n}{u_n + 1}, \, u_1 = 1, \, n \in \mathbb{Z}, \, n \geqslant 1$$

Let $P(n) \iff u_n < 2n$. For base case, $n = 1$.

$$u_n = u_1 = 1 < 2 = 2(1) = 2n$$

$$P(1) \to \top$$