ANALYSIS
TUTORIAL 5: INTEGRATION

1 Comment on the student's solution when he attempts to evaluate the following integral thus:

$$\int_0^2 \frac{1}{(x-1)^2}\,\mathrm{d}x = \left[\frac{1}{-(x-1)}\right]_0^2 = -1-1=-2$$

Are you able to generalise your observation, and thus state a <u>sufficient</u> condition for the following statement to be true?

If $\mathrm{F}'(x)=\mathrm{f}(x)$, and both $\mathrm{F}(a)$ and $\mathrm{F}(b)$ are finite, then $\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \mathrm{F}(b)-\mathrm{F}(a)$.

2 **(Math S J88)**

Let $I=\int_0^{2\pi}\frac{1}{2-\cos x}\,\mathrm{d}x$. Explain the error in the following argument:

"Since $|\cos x|\leqslant 1$, it follows that $\frac{1}{2-\cos x}>0$, and, interpreting the integral as an area, it follows that I is positive. However, putting $t=\tan\frac{x}{2}$,

$$I=\int_{\tan 0}^{\tan\pi}\frac{\frac{2}{1+t^2}}{2-\frac{1-t^2}{1+t^2}}\,\mathrm{d}t = 2\int_0^0 \frac{1}{1+3t^2}\,\mathrm{d}t = 0.$$

Thus the positive number I is equal to zero." [3]

Prove that $I=2\int_0^{\pi}\frac{1}{2-\cos x}\,\mathrm{d}x$, and deduce that $I=\frac{2\pi\sqrt{3}}{3}$. [6]

3 **(Math S N89)**

Without using GC, evaluate

(i) $\int_0^{\frac{\pi}{2}} \sin x\cos 2x\sin 3x\,\mathrm{d}x$, [7]

(ii) $\int_1^2 \frac{1}{x+\sqrt{x}}\,\mathrm{d}x$. [4]

4 Let $H=\int_0^a \mathrm{e}^x\sin x\,\mathrm{d}x,\ 0\le a\le\frac{\pi}{2}$. By applying integration by parts twice, evaluate H in terms of a.

5 Evaluate the following integrals:

(a) $\int 3^{\sqrt{2x+1}}\ \mathrm{d}x$, **(b)** $\int \frac{1}{x\sqrt{1+x^n}}\ \mathrm{d}x$, where $n \neq 0$.

6 **(GCE A-Level Math S-Paper Nov 2002 Q4(b))**
Prove that, for all values of θ,

$$5-4\cos 2\theta+3\sin 2\theta=\cos^2\theta\left(1+3\tan\theta\right)^2.$$

Hence or otherwise, find $\int \frac{1}{5-4\cos 2\theta+3\sin 2\theta}\ \mathrm{d}\theta$.

7 **(a)** Let $I=\int_0^a \frac{\mathrm{f}(x)}{\mathrm{f}(x)+\mathrm{f}(a-x)}\mathrm{d}x$.

Use a substitution to show that

$$I=\int_0^a \frac{\mathrm{f}(a-x)}{\mathrm{f}(x)+\mathrm{f}(a-x)}\mathrm{d}x$$

and hence evaluate I in terms of a.

Use this result to evaluate the integrals

$$\int_0^1 \frac{\ln(x+1)}{\ln\left(2+x-x^2\right)}\ \mathrm{d}x \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin\left(x+\frac{\pi}{4}\right)}\ \mathrm{d}x.$$

(b) Using a suitable substitution, evaluate $\int_{\frac{1}{2}}^{2} \frac{\sin x}{x\left(\sin x+\sin\frac{1}{x}\right)}\ \mathrm{d}x$.

8 **(i)** Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right)\right)=\frac{1}{1+\cos^2 x}$.

(ii) Use **(i)** to show that $\int_0^{\pi} \frac{x}{1+\cos^2 x}\ \mathrm{d}x=\frac{\pi^2}{2\sqrt{2}}$.

9 Given n to be an integral greater than 1, show that

$$\int_0^{\infty} \mathrm{e}^{-x}x^{n-1}\ \mathrm{d}x=\left(n-1\right)!$$

(For information: this is the probability density function of the gamma distribution)

10 (In this question all indices n are integers)

Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx, \ n \ge 0.$

(i) Show that $I_n = \dfrac{n-1}{n} I_{n-2}, \ n \ge 2.$

(ii) Use **(i)** to show that $I_{2n+1} = \dfrac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)}, \ n \ge 1.$

(iii) Use **(i)** to show that $I_{2n} = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \dfrac{\pi}{2}, \ n \ge 1.$

(iv) Use **(i)** or **(iii)** to show that $\dfrac{I_{2n+2}}{I_{2n}} = \dfrac{2n+1}{2n+2}, \ n \ge 1.$

(v) By considering the integral and comparing $\sin^{k+1} x$ with $\sin^k x$, show that $I_{2n+2} \le I_{2n+1} \le I_{2n}, \ n \ge 1.$

(vi) Use **(iv)** and **(v)** to show that $\dfrac{2n+1}{2n+2} \le \dfrac{I_{2n+1}}{I_{2n}} \le 1, \ n \ge 1.$

Hence deduce that $\lim\limits_{n \to \infty} \dfrac{I_{2n+1}}{I_{2n}} = 1.$

(vii) Use **(ii), (iii)** and **(vi)** to show that $\dfrac{2}{1} \cdot \dfrac{2}{3} \cdot \dfrac{4}{3} \cdot \dfrac{4}{5} \cdot \dfrac{6}{5} \cdot \dfrac{6}{7} \cdots = \dfrac{\pi}{2}.$