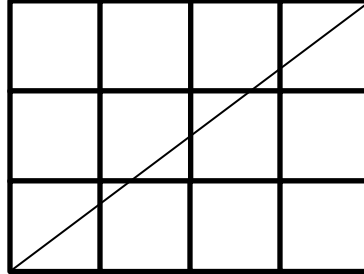# PROBLEM SOLVING AND NUMBER THEORY
TUTORIAL 1

1   A rectangle 4 by 3 has six squares passed through by a diagonal as shown in the figure below:

Find the number of squares passed through by a diagonal for a rectangle of size m by n.

2   In the sliding puzzle with 15 number tiles and 1 blank space, one can form different arrangements by sliding adjacent tiles either horizontally or vertically into the blank space.

Suppose the starting arrangement is the left arrangement shown below. Can it ever be converted into the right arrangement shown below?
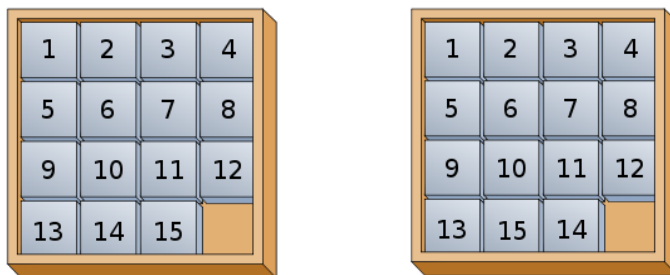
3   Let $u_0, u_1, u_2, \ldots$ be a sequence of positive integers such that u_0 is arbitrary and for any non-negative integer n,

$$u_{n+1} = \begin{cases} \dfrac{1}{2}u_n & \text{if } u_n \text{ is even} \\ u_n + a & \text{if } u_n \text{ is odd} \end{cases}$$

where a is some fixed odd positive integer. Prove that the sequence is periodic from a certain step.

4   The numbers 1 to n are written around a circle in some order. What is the smallest and largest possible sum of the absolute differences of adjacent numbers?

5   (The Game of NIM) Determine the best strategy for each player in the following two-player game. There are three piles, each of which contains some number of coins. Players alternate turns, each turn consisting of removing any (non-zero) number of coins from a single pile. The goal is to be the person to remove the last coin(s).

6   Find all possible values for the real number x such that $x\lfloor x\lfloor x\lfloor x\rfloor\rfloor\rfloor = 50$.

(For real number x, $\lfloor x\rfloor$ is equal to the greatest integer less than or equal to x. For example, $\lfloor \sqrt{2}\rfloor = 1$, $\lfloor -3.9\rfloor = -4$ and $\lfloor 10\rfloor = 10$.)