MOE H3 Math   Numbers and Proofs

Problem Set 2

1. Is each of the following statements true or false? Give a proof if it is true, and give a counter-example if it is false.
   (a) For each pair of real numbers x and y, if $x + y$ is irrational, then x is irrational and y is irrational.
   (b) For each pair of real numbers x and y, if $x + y$ is irrational, then x is irrational or y is irrational.

2. Determine whether each of the following real numbers is rational or irrational. Justify your answers.

   (a) $\sqrt{3} + \sqrt{5}$;   (b) $\sqrt{2} + \sqrt{8}$;   (c) $\dfrac{1 + \sqrt{2}}{1 + \sqrt{3}}$

   (You may need to use the result in Q6 below.)

3. Use proof by contradiction to show that the sum of square of two odd integers is not divisible by 4.

4. Prove that there are no integers a and n with $n \geq 2$ and $a^2 + 1 = 2^n$.

5. Let a and b be integers, not both 0. Show that $\gcd(a, b)$ is the smallest possible positive linear combination of a and b. (i.e. There is no positive integer $c < \gcd(a, b)$ such that $c = ax + by$ for some integers x and y.)

6. Unique Factorization Theorem states that:
   Every integer $n > 1$ has a unique standard factored form. i.e. there is exactly one way to express
   $$n = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$$
   where $p_1 < p_2 < \cdots < p_t$ are distinct primes and $k_1, k_2, \cdots, k_t$ are some positive integers.

   [We shall look at the proof of this theorem in lecture 5]

   Use the Unique Factorization Theorem to prove that, if a positive integer n is not a perfect square, then $\sqrt{n}$ is irrational.

Hints

1. You probably want to consider the contrapositive of the statements.

2. If you claim that a certain number is irrational, prove it by contradiction. You need to do some algebraic manipulation to the numbers.

3. Make sure you start with the correct negation, which is an existential statement.

4. Prove by contradiction. The negation of the statement is simply by removing the 'no' from the original statement. Also need to consider two cases on a by parity.

5. Prove by contradiction. Start with the negation of: 'There is no positive integer $c < \gcd(a, b)$ such that $c = ax + by$ for some integers x and y'.

6. You may use the fact that n is a perfect square if and only if its standard factored form is $n = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$ where all the k_i's are even.