3

$$x = \tan\theta \implies \frac{\mathrm{d}x}{\mathrm{d}\theta} = \sec^2\theta = 1 + \tan^2\theta = 1 + x^2$$

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\mathrm{d}y}{\mathrm{d}x}\left(1+x^2\right) \implies \left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta} \quad \square$$

$$\begin{aligned}
\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}\theta} \\
\left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}^2y}{\mathrm{d}\theta^2}\frac{\mathrm{d}\theta}{\mathrm{d}x} \\
\left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}^2y}{\mathrm{d}\theta^2}\left(1+x^2\right)^{-1} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} \\
\left(1+x^2\right)^3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}^2y}{\mathrm{d}\theta^2}\left(1+x^2\right) - 2x\frac{\mathrm{d}y}{\mathrm{d}x}\left(1+x^2\right)^2 \\
\left(1+x^2\right)^3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}\theta^2} - 2x\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}\theta}
\end{aligned}$$

$$\begin{aligned}
\left(1+x^2\right)^3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\left(1+x^2\right)^2(1+x)\frac{\mathrm{d}y}{\mathrm{d}x} - 3\left(1+x^2\right)y &= x^2 + 6x - 1 \\
\left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}\theta^2} - 2x\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}\theta} + 2\left(1+x^2\right)(1+x)\frac{\mathrm{d}y}{\mathrm{d}\theta} - 3\left(1+x^2\right)y &= x^2 + 6x - 1 \\
\left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}\theta^2} + 2\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}\theta} - 3\left(1+x^2\right)y &= x^2 + 6x - 1 \\
\frac{\mathrm{d}^2y}{\mathrm{d}\theta^2} + 2\frac{\mathrm{d}y}{\mathrm{d}\theta} - 3y &= \frac{x^2+6x-1}{1+x^2}
\end{aligned}$$

$$\begin{aligned}
\frac{x^2+6x-1}{1+x^2} &= \frac{\tan^2\theta + 6\tan\theta - 1}{\sec^2\theta} \\
&= \cos^2\theta\left(\frac{\sin^2\theta}{\cos^2\theta} + 6\frac{\sin\theta}{\cos\theta} - 1\right) \\
&= \sin^2\theta + 6\sin\theta\cos\theta - \cos^2\theta \\
&= 3(2\sin\theta\cos\theta) - \left(\cos^2\theta - \sin^2\theta\right) \\
&= 3\sin 2\theta - \cos 2\theta
\end{aligned}$$

$$\therefore \frac{\mathrm{d}^2y}{\mathrm{d}\theta^2} + 2\frac{\mathrm{d}y}{\mathrm{d}\theta} - 3y = 3\sin 2\theta - \cos 2\theta \quad \square$$

$$\boxed{a = 2,\ b = -3,\ c = 3,\ d = -1}$$

$$s^2 + 2s - 3 = 0 \implies s_1 = -3,\ s_2 = 1$$

$$y_c = Ae^{-3\theta} + Be^{\theta}$$

$$\begin{aligned} y_p &= p \sin 2\theta + q \cos 2\theta \\ \frac{\mathrm{d}y_p}{\mathrm{d}\theta} &= 2p \cos 2\theta - 2q \sin 2\theta \\ \frac{\mathrm{d}^2 y_p}{\mathrm{d}\theta^2} &= -4p \sin 2\theta - 4q \cos 2\theta \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}^2 y_p}{\mathrm{d}\theta^2} + 2\frac{\mathrm{d}y_p}{\mathrm{d}\theta} - 3y_p &= 3 \sin 2\theta - \cos 2\theta \\ -4p \sin 2\theta - 4q \cos 2\theta + 4p \cos 2\theta - 4q \sin 2\theta - 3p \sin 2\theta - 3q \cos 2\theta &= 3 \sin 2\theta - \cos 2\theta \\ -(4p + 7q) \sin 2\theta + (4p - 7q) \cos 2\theta &= 3 \sin 2\theta - \cos 2\theta \end{aligned}$$

$$\begin{cases} 4p + 7q = -3 \\ 4p - 7q = -1 \end{cases} \implies p = -\frac{1}{2},\ q = -\frac{1}{7}$$

$$y_p = -\frac{1}{2} \sin 2\theta - \frac{1}{7} \cos 2\theta$$

$$y = y_c + y_p = Ae^{-3\theta} + Be^{\theta} - \frac{1}{2} \sin 2\theta - \frac{1}{7} \cos 2\theta$$

$$\therefore \boxed{y = Ae^{-3\tan^{-1} x} + Be^{\tan^{-1} x} - \frac{\sqrt{53}}{14} \sin\left(2\tan^{-1} x + \tan^{-1} \frac{2}{7}\right)}$$