H3 Mathematics
Problem Solving

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26 May 2020

Outline

T-shirt problem

Problem (1): H3 2018 Question 4

A clothes shop sells a particular make of T-shirt in four different colors. The shopkeeper has a large number of T-shirts of each color.

(a) A customer wishes to buy seven T-shirts.

  (i) In how many ways can he do this?

  (ii) In how many ways can he do this if he buys at least one of each color?

UP

OK. It is difficult to hold in my mind if I do not think of 4 physical color...

1. Colour 1: R
2. Colour 2: B
3. Colour 3: G
4. Colour 4: Y

UP

In part (a),

- we are not interested in arrangements (of any kind);
- we can divide the analysis into several cases, e.g., by the number of colors involved.

DP

Let us try something näive.

Heuristics

We use the heuristic of "dividing into cases".

We consider the number of ways the customer can do this by using only r colour, where $r = 1, 2, 3, 4$.

CP

r	Pattern	Number of ways
1	(1)	4
2	$(6,1)$	$4\times 3=12$
	$(5,2)$	$4\times 3=12$
	$(4,3)$	$4\times 3=12$
3	$(5,1,1)$	$4\times 3=12$
	$(4,2,1)$	$4\times 3\times 2=24$
	$(3,2,2)$	$4\times 3=12$
	$(3,3,1)$	$4\times 3=12$
4	$(4,1,1,1)$	4
	$(3,2,1,1)$	4×3
	$(2,2,2,1)$	4

CP

Solution.

(a)(i) Number of ways the customer can do this
$= 4 + (12 \times 3) + (12 \times 5) + (4 \times 5)$
$= 4 + 36 + 60 + 20$
$= 120.$ $\square$

CP

Solution.

(a)(ii) Number of ways the customer can do this if he buys at least one of each colour

$= 4 \times 5$

$= 20.$ $\square$

CE

Question

Are there any easier methods?

CE

Heuristics: Act it out

Imagine that the seven shirts are colorless, and to be placed in the distinct color boxes.

CE

(a)(i) Solution.

The number of ways the shirts can be purchased is

$$\binom{7+3}{3} = 120.$$

□

Think of choosing the dividers amongst 10 units (7 shirts and 3 potential dividers).

CE

For (a)(ii), we must ensure that each color must be included.

(a)(ii) Solution.

After putting in 4 shirts (one for each color) first, we are left with fewer shirts 3. Now applying the same principle of choosing dividers from $3+3=6$ units, the number of ways the customer can purchase the shirts is

$$\binom{3+3}{3} = 20.$$

$\square$

T-shirt problem

Problem (1): H3 2018 Question 4

(b) The shopkeeper places seven T-shirts in a line.

- (i) In how many ways can she do this?
- (ii) In how many ways can she do this if no two T-shirts of the same color are to be next to each other?
- (iii) Use the principle of inclusion and exclusion to find the number of ways in which she can do this if she has to use at least one T-shirt of each colour but with no other restrictions.

UP

In part (b), we are interested in the number of arrangement in a line.

DP & CP

For (b)(i), we can consider the number of possible colors that can be used to color the first shirt, then the second shirt, and so on.

Solution.

(b)(i) Number of ways the shopkeeper can do this
$= 4^7$
$= 16384$. $\square$

DP & CP

For (b)(ii), we note that

- the first T-shirt has 4 choices; and
- subsequent T-shirts each has 3 choices to avoid the one before it.

Hence the number of ways in which the shopkeeper can achieve this is

$$4 \times 3^6 = 2916.$$

DP

The plan has already been somewhat suggested:

Theorem (Principle of Inclusion and Exclusion)

Let E_i be finite sets. Then

$$\left|\bigcup_{i=1}^{n} S_i\right| = \sum_{i=1}^{n} |S_i| - \sum_{1\le i<j\le n} |S_i \cap S_j| + \sum_{1\le i<j<k\le n} |S_i \cap S_j \cap S_k| + \cdots + (-1)^n \left|\bigcap_{i=1}^{n} S_i\right|.$$

CP

Let S_i $(i = 1, 2, 3, 4)$ be the set of arrangements that does not contain the color i. Then the required set is

$$\bigcap_{i=1}^{4} S_i' = E - \bigcup_{i=1}^{4} S_i.$$

Now, by the PIE, we have

$$\left|\bigcup_{i=1}^{4} S_i\right| = 4 \times 3^7 - 6 \times 2^7 + 4 \times 1^7.$$

It follows that the number of ways the shopkeeper can achieve the required condition is

$$4^7 - (4 \times 3^7 - 6 \times 2^7 + 4 \times 1^7) = 8,400.$$

A differential equation

Problem (2): H3 2018 Question 7

The differential equation

$$y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + 1, \quad \text{for } x > 0 \tag{1}$$

has a solution curve S such that $\frac{d^2y}{dx^2}$ is non-zero for all points of S.

(i) By substituting $t = \frac{dy}{dx}$ into equation (1) and differentiating with respect to x, show that S has equation $y^2 = 4x$.

(ii) Show that a straight line is a tangent to the curve S if and only if it is itself a solution of the equation.

UP

The introduction of this new variable t hints towards the removal of one of the variables x or y.

DP

Follow the hint, starting with

$$t = \frac{dy}{dx}$$

and substituting this into the given equation.

CP

Substituting t into the equation yields:

$$yt = xt^2 + 1$$

Then differentiating w.r.t. x,

$$\frac{dy}{dx}t + y\frac{dt}{dx} = t^2 + x \cdot 2t\frac{dt}{dx}.$$

CP

Recall that $t = \dfrac{dy}{dx}$, we have

$$t^2 + y\frac{dt}{dx} = t^2 + x \cdot 2t\frac{dt}{dx},$$

which reduces to

$$y = 2xt.$$

CP

Now we have

$$y = 2x\frac{dy}{dx} \iff \frac{1}{x}\,dx = \frac{2}{y}\,dy.$$

Integrating both sides,

$$\ln(x) = 2\ln(y) + C,$$

i.e.,

$$y^2 = Ax,$$

where A is a constant to be determined.

CP

Now, we differentiate wrt x,

$$2y\frac{dy}{dx} = A$$

Substituting into the original DE,

$$\frac{A}{2} = x\left(\frac{A}{2y}\right)^2 + 1,$$

which is equivalent to

$$\frac{A}{2} = x \cdot \frac{A^2}{4(Ax)} + 1.$$

CP

Hence

$$\frac{A}{2} = \frac{A}{4} + 1 \implies A = 4.$$

Thus, the required solution curve S has equation

$$y^2 = 4x.$$

DP

We now turn to (ii), i.e., we want to prove that the following statements are equivalent:

1. A straight line is a tangent to the curve S.
2. It is itself a solution of the equation.

DP

There are two directions of proof:

- $(1) \implies (2)$;
- $(2) \implies (1)$.

Note that this is an instance of a *logical equivalence*:

$$(1) \iff (2).$$

DP

Let us begin by showing that if a straight line is a tangent to the curve S, then it is itself a solution of the differential equation. We must determine the equation of this tangent.

CP

Let us say that the particular point of tangency on the curve S ($y^2 = 4x$) is of the form

$$\left(\frac{a^2}{4}, a\right),$$

for some real number a.
It is important to find the gradient of the tangent at this point.

CP

Let this gradient be denoted by m.
Use the differential equation at $x = \frac{a^2}{4}$ and $y = a$ we have

$$am = \frac{a^2}{4}(m^2) + 1$$

which implies that

$$(am - 2)^2 = 0 \iff m = \frac{2}{a}.$$

CP

Hence the equation of the tangent is given by

$$y - a = \frac{2}{a}\left(x - \frac{a^2}{4}\right),$$

i.e.,

$$y = \frac{2}{a}x + \frac{a}{2}.$$

It remains to verify that this satisfies the original differential equation.

CP

We simplify the LHS:

$$y\frac{dy}{dx} = \left(\frac{2}{a}x + \frac{a}{2}\right)\left(\frac{2}{a}\right) = \frac{4}{a^2}x + 1.$$

Now the RHS is:

$$x\left(\frac{dy}{dx}\right)^2 + 1 = x\left(\frac{2}{a}\right)^2 + 1 = \frac{4}{a^2}x + 1.$$

We are done.

DP

We now turn to the converse of the result, i.e., assume that the straight line $y = mx + c$ satisfies the differential equation, and we must prove that this line is tangent to the curve S at some point.

CP

Since $y = mx + c$ satisfies the equation, we have

$$(mx + c)(m) = x(m)^2 + 1$$

for all x.
Hence $c = \frac{1}{m}$.
It remains to show that $y = mx + \frac{1}{m}$ is tangent to S at some point on S.

CP

Since $y^2 = 4x$, it follows that

$$2y\frac{dy}{dx} = 4.$$

Suppose that $\frac{dy}{dx} = m$. Then $y = \frac{2}{m}$. The equation of the tangent is $y = mx + \frac{1}{m}$, and so

$$\frac{2}{m} = mx + \frac{1}{m},$$

which implies that $x = \frac{1}{m^2}$.

CE

We can directly check that the point

$$\left(\frac{1}{m^2}, \frac{2}{m}\right)$$

is the required point of tangency, and the equation of the tangent is exactly $y = mx + c = mx + \frac{1}{m}$.

Check and expand/extend

Absolute value problem

For the purpose of stating the question, it is good to recall:

Definition (Absolute value function)

The *absolute value function* (also known as *modulus*) is defined as follows:

$$|x| := \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

Absolute value problem

Problem (3)

Sketch the graph whose equation is

$$|x| + |y| = 3.$$

UP

Following the definition of the absolute value function, we consider four possible combinations for the values of x and y:

- $x \geq 0$ and $y \geq 0$;
- $x < 0$ and $y \geq 0$;
- $x < 0$ and $y < 0$;
- $x \geq 0$ and $y < 0$.

DP

We shall consider rewriting the equation

$$|x| + |y| = 3$$

for each of the above regions.

CP

Carrying out the plan, we thus obtain:

$$
\begin{aligned}
x + y &= 3 \ (x \geq 0, y \geq 0) \\
-x + y &= 3 \ (x < 0, y \geq 0) \\
-x - y &= 3 \ (x < 0, y < 0) \\
x - y &= 3 \ (x \geq 0, y < 0).
\end{aligned}
$$

CP

Sketching within each region the graph is so much easier:

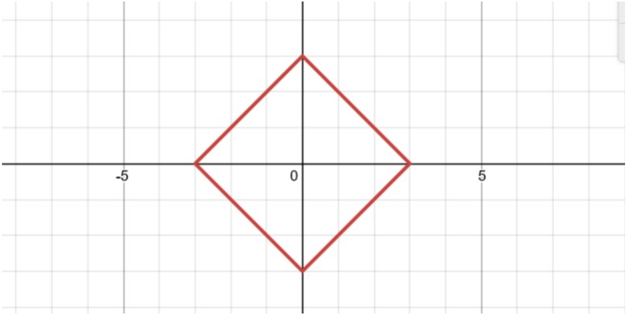


Figure: $|x| + |y| = 3$

Check and expand

- This stage is called 'Looking Back' in Pólya's original model. The idea is that we do not stop when the problem is 'solved' but we should look back at the solution to check and to see what we can learn from it.
- We have decided to call it 'Check and expand' to make the following features of looking back and looking forward clearer:
  - check the solution;
  - find out if the method can be used to solve other problems;
  - pose new problems from the original problem
  - suggest alternative solutions

Check and expand

3 facets of 'Expand':

- **Adapt**: To change certain features of the problem.
- **Extend**: To consider problems which are more 'difficult' or which have greater scope.
- **Generalise**: To consider problems which would include the given problem as a special example.

Check and expand

Consider the following 'expansion':

Problem (3) extended

Sketch the graph whose equation is

$$|x|^p + |y|^p = 3$$

for each of the following cases:

1. $0 < p < 1$;
2. $p \geq 1$.

CP

Consider the case when $p = 3 > 1$:

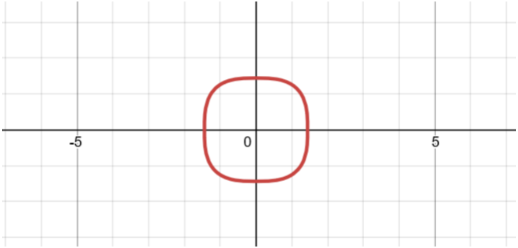


Figure: $|x|^3 + |y|^3 = 3$

CP

Consider the case when $0 < p = \frac{1}{2} < 1$:

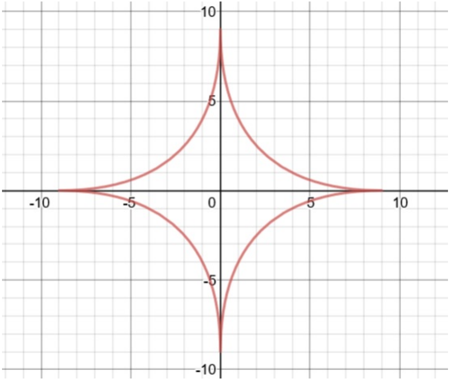


Figure: $|x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} = 3$

Check and expand

Consider yet another 'expansion':

Problem (3) extended

- You saw that the graph is a square with vertices $(-3, 0)$, $(0, 3)$, $(3, 0)$ and $(0, -3)$.
- Note that the implicit equation involves a single equation, owing to the expressiveness of the modulus function.
- Question: Is it possible to, in a single equation, express graphically an arbitrary quadrilateral?

CE

Arbitrary quadrilateral problem

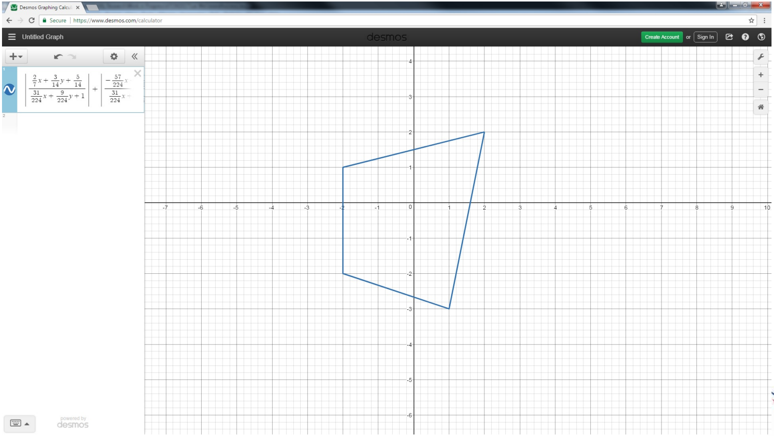
Find a single implicit equation that graphs the quadrilateral with the following vertices:

$$P = (-2, 1), Q = (2, 2),$$

$$R = (1, -3), S = (-2, 2).$$

CE

The required quadrilateral looks like this:



CE

It turns out that a solution can be given as

$$\left|\frac{\frac{2}{7}x+\frac{3}{14}y+\frac{5}{14}}{\frac{31}{224}x+\frac{9}{224}y+1}\right|+\left|\frac{-\frac{57}{224}x+\frac{57}{224}y}{\frac{31}{224}x+\frac{9}{224}y+1}\right|=1$$

Sum of all digits problem

Problem (4)

Find the sum of all digits of the numbers

$$1,\ 2,\ 3, \cdots,\ 999.$$

UP

Warning!

Are you thinking of

$$1 + 2 + \ldots + 999?$$

If so, read the question again!

DP

After understanding that we really want to add up all the digits appearing in the list, we plan to do so systematically:

- Single digit numbers;
- Double digit numbers;
- Triple digit numbers

We proceed to make a systematic list.

CP

For the single digit, the table reads:

$$\begin{array}{c}1\\2\\3\\\vdots\\9\end{array}$$

Thus, the sum of all the single digits is

$$S_1 = 1 + 2 + \cdots + 9 = \frac{9}{2}(1+9) = 45.$$

CP

For the double digits, we would have:

$$
\begin{array}{cc}
1 & 0 \\
\vdots & \vdots \\
1 & 9 \\
\hline
2 & 0 \\
\vdots & \vdots \\
2 & 9 \\
\hline
\vdots & \vdots \\
\hline
9 & 0 \\
\vdots & \vdots \\
9 & 9 \\
\hline
\end{array}
$$

CP

Counting carefully, the sum of all the digits for the double digit numbers is

$$
\begin{aligned}
S_2 &= \underbrace{10S_1}_{\text{Tens place}} + \underbrace{9S_1}_{\text{Units place}} \\
&= 19S_1.
\end{aligned}
$$

CP

For the triple digit numbers, we represent in this way.
Let $k = 1, 2, \ldots, 9$.

$$\begin{array}{ccc} k & 0 & 0 \\ \vdots & \vdots & \vdots \\ \vdots & 0 & 9 \\ \hline \vdots & 1 & 0 \\ \vdots & \vdots & \vdots \\ k & 9 & 9 \end{array}$$

CP

Summing over all the values of k, we obtain:

$$\begin{aligned} S_3 &= 10^2 S_1 + 9(S_1 + S_2) \\ &= 100 S_1 + 9(S_1 + 19 S_2) \\ &= 280 S_1. \end{aligned}$$

CP

Adding everything up, the sum of all the digits in $1, 2, \ldots, 999$ is

$$S_1 + 19S_1 + 280S_1 = 300S_1 = 13,500.$$

CE

Problem (4) extended

What is the sum of all the digits in

$$1, 2, \ldots, \underbrace{9\ldots9}_{n \text{ copies}}?$$

The solution can be recursively given by

$$S_{n+1} = 10^n S_1 + 9\sum_{k=1}^{n} S_k, \qquad S_1 = 45.$$

Schoenfeld's framework

Alan Schoenfeld's experience

- Alan Schoenfeld was a mathematics professor who discovered Pólya's model only after he obtained his PhD.
- Schoenfeld grappled with the apparent worth of Pólya's model and the real-world failure of its application in the classroom.
- His research led him to realize that there was more than just a direct application of the model and that in fact, other factors are crucial in successful problem solving.
- His research culminated in the construction of a framework for the analysis of complex problem-solving behaviour.

Four components

- **Cognitive resources**: the body of facts and procedures at ones disposal.
- **Heuristics**: 'rules of thumb' for making progress in difficult situations.
- **Control**: having to do with the efficiency with which individuals utilise the knowledge at their disposal. Sometimes, this is referred to as metacognition, which can be roughly translated as 'thinking about one's own thinking'.
- **Belief systems**: one's perspectives regarding the nature of a discipline and how one goes about working on it.

Control

1. These are questions to ask oneself to monitor one's thinking.
    - What (exactly) am I doing? [Describe it precisely.]
        - Be clear what I am doing NOW.
        - Why am I doing it? [Tell how it fits into the solution.]
    - Be clear what I am doing in the context of the BIG picture – the solution.
        - Be clear what I am going to do NEXT.

Control

2. Stop and reassess your options when you
   - cannot answer the questions satisfactorily [probably you are on the wrong track]; OR
   - are stuck in what you are doing [the track may not be right or it is right but it is at that moment too difficult for you].

Control

3. Decide if you want to
   - carry on with the plan,
   - abandon the plan, OR
   - put on hold and try another plan.

Average problem

Problem (5)

A worker in a garment factory is rated according to the average number of dresses that she sews in a day. Workers in the factory do not have to work every day and often work for different number of days per month. Fann had a higher average than Zoe for the month of January. She also had a higher average than Zoe for the month of February. Can we say that Fann is a better worker than Zoe over the two months?

UP

Here are some questions you ask yourself:

- How do you work out the average for each month?
- How do you compare which is the better worker over the 2 months? Using total or average? Or no difference? Or other means?

DP

We make use of a table to organize the relevant information.

Fann				
$\bar{x}$	Jan: m	$\bar{y}$	Feb: n	Overall: $\bar{w} := \frac{m\bar{x}+n\bar{y}}{m+n}$
Zoe				
$\bar{x}'$	Jan: m'	$\bar{y}'$	Feb: n'	Overall: $\bar{w}' = \frac{m'\bar{x}'+n'\bar{y}'}{m'+n'}$

Fill in some numbers for the first four columns, and **check**.

Nice numbers

Problem (6)

A 'nice' number is a number that can be expressed as the sum of a string of two or more consecutive positive integers. Determine which of the numbers from 50 to 70 inclusive are 'nice'.

UP

To understand the problem, try out some small numbers:

$$\begin{array}{llll} 1 =, & 2 =, & 3 = 1+2, & 4 =, \\ 5 = 2+3, & 6 = 1+2+3, & 7 = 3+4, & 8 =, \\ 9 = 4+5, & 10 = 1+2+3+4, & 11 = 5+6, & 12 = 3+4+5, \\ 13 = 6+7, & 14 = 2+3+4+5, & 15 = 7+8, & 16 =, \end{array}$$

What can you conjecture about nice/non-nice numbers?

UP

We conjecture that

Conjecture

A positive integer is nice if and only if it is not a power of 2.

DP

We need to make a few observations about the number of ways in which a number can be written as a sum of consecutive integers.

Example

Consider the decomposition of 15 into sum of consecutive integers:

$$(15), (7, 8), (4, 5, 6), (1, 2, 3, 4, 5),$$

The first bracket is actually a trivial one – we do not count that!

Observe

In each set, what is the common difference? How many terms are there? What can you observe?

DP & CP

I want you to make use of the above observation to come up with the solution.
You may make use of the following hint:

$$-m + (-m+1) + \ldots + 0 + 1 + \ldots + m = 0$$

holds for any positive integer m.

CE

Problem (6) extended

A positive integer is called 'squarish' if it is the sum of consecutive squares. Which of the numbers from 1 to 100 are 'squarish'?

Overview and summary

- What is a problem?
- Draw the model of Pólya's problem solving strategy.
- What are heuristics? Name some of them.
- Name the 4 components of Schoenfeld's framework.
- How do you know if you are 'stuck'?
- What do you do when you are 'stuck'?

The journey has just started