1

(i)

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -2 & a \\ 2 & -1 & 3 & -5 \\ -3 & -3 & 0 & 3 \end{bmatrix}$$
$$\to \begin{bmatrix} 6 & 18 & -12 & 6a \\ 6 & -3 & 9 & -15 \\ -6 & -6 & 0 & 6 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & -9 & 9 & -9 \\ 0 & 12 & -12 & 6a+6 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & -2 & 2 & -2 \\ 0 & 2 & -2 & a+1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & a-1 \end{bmatrix} = \text{rref}(\mathbf{A})$$

Given that the dimension of the null space of T is 2, the matrix **a** must have 2 non-pivot columns.

$$a - 1 = 0 \iff a = \boxed{1}$$

$$\text{nul}(T) = \text{nul}(\mathbf{A}) = \text{nul}(\text{rref}(\mathbf{A})) = \text{nul}\left(\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

$$x_4 = \mu,\ x_3 = \lambda,\ x_2 = \lambda - \mu,\ x_1 = -\lambda + \mu$$

$$\mathbf{x} = \begin{bmatrix} -\lambda + \mu \\ \lambda - \mu \\ \lambda \\ \mu \end{bmatrix} = \lambda \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Thus, a basis for the null space of T is:

$$\boxed{\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

(ii)

$$\dim(\text{range}(T)) = \text{rank}(\mathbf{A}) = 4 - \text{nullity}(\mathbf{A}) = 4 - 2 = 2$$

Thus, the range space of T is a plane. □

$$R : \mathbf{r} = \lambda \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \mu \begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix},\ \lambda, \mu \in \mathbb{R}$$

$$\mathbf{n} = \begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 7 \end{bmatrix}$$

$$\mathbf{r} \cdot \mathbf{n} = 0 \iff \mathbf{r} \cdot \begin{bmatrix} 9 \\ 6 \\ 7 \end{bmatrix} = 0$$

$$R : \boxed{9x + 6y + 7z = 0}$$

(iii)

$R \cup V$ is a vector space implies that $V \subset R$.

$$\mathbf{v} \cdot \mathbf{n} = 0 \iff \begin{bmatrix} 0 \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 6 \\ 7 \end{bmatrix} = 0$$

$$\boxed{6b + 7c = 0}$$

2

If $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ are linearly dependent, then:

$$\begin{aligned} a\mathbf{x}_1 + b\mathbf{x}_2 + c\mathbf{x}_3 &= \mathbf{0},\ a, b, c \in \mathbb{R}\backslash\{0\} \\ \mathbf{M}(a\mathbf{x}_1 + b\mathbf{x}_2 + c\mathbf{x}_3) &= \mathbf{M0} \\ \mathbf{M}a\mathbf{x}_1 + \mathbf{M}b\mathbf{x}_2 + \mathbf{M}c\mathbf{x}_3 &= \mathbf{0} \\ a\mathbf{M}\mathbf{x}_1 + b\mathbf{M}\mathbf{x}_2 + c\mathbf{M}\mathbf{x}_3 &= \mathbf{0},\ a, b, c \in \mathbb{R}\backslash\{0\} \end{aligned}$$

Therefore, $\mathbf{Mx}_1$, $\mathbf{Mx}_2$ and $\mathbf{Mx}_3$ are linearly dependent.

(i)

$$\det\left(\begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \mathbf{y}_3 \end{bmatrix}\right) = 0$$

Since the matrix $\begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \mathbf{y}_3 \end{bmatrix}$ is not full rank, the dimension of its column space is less than 3. This implies that the span of the vectors $\mathbf{y}_1$, $\mathbf{y}_2$ and $\mathbf{y}_3$ has a dimension that is less than 3, which is the number of vectors. Therefore, $\mathbf{y}_1$, $\mathbf{y}_2$ and $\mathbf{y}_3$ are linearly dependent.

(ii)

$$\mathbf{A} \begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \mathbf{y}_3 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 4 \\ 20 & 10 & 190 \\ 34 & 2 & 368 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 11 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the linear space spanned by the vectors $\mathbf{y}_1$, $\mathbf{y}_2$ and $\mathbf{y}_3$ is:

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 10 \\ 17 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} \right\}}$$

(iii)

$$\text{nul}(T) = \text{nul}(\mathbf{A}) = \text{nul}\left(\begin{bmatrix} -1 & 1 & 0 \\ 0 & 4 & 2 \\ 3 & 5 & 4 \end{bmatrix}\right) = \text{nul}\left(\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$x_3 = \lambda,\ x_2 = -\frac{1}{2}\lambda,\ x_1 = -\frac{1}{2}\lambda$$

$$\mathbf{x} = \begin{bmatrix} -\lambda/2 \\ -\lambda/2 \\ \lambda \end{bmatrix} = -\frac{\lambda}{2}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

A basis for the null space of T is:

$$\boxed{\left\{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\right\}}$$

(iv)

$$T(\mathbf{0}) = \mathbf{0} \neq \begin{bmatrix} 1 \\ 6 \\ 9 \end{bmatrix}$$

Since the zero vector is not contained by the set of solutions of the equation, it is not a vector space.

It is obvious that:

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 6 \\ 9 \end{bmatrix}$$

Thus,

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_p = k\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} k \\ 1+k \\ 1-2k \end{bmatrix},\ k \in \mathbb{R}$$

$$
\begin{aligned}
|\mathbf{x}| &= \left| \begin{bmatrix} k \\ 1+k \\ 1-2k \end{bmatrix} \right| \\
&= \sqrt{k^2 + (1+k)^2 + (1-2k)^2} \\
&= \sqrt{k^2 + 1 + 2k + k^2 + 1 - 4k + 4k^2} \\
&= \sqrt{2 - 2k + 6k^2} \\
&= \sqrt{2 + 6\left(-\frac{k}{3} + k^2\right)} \\
&= \sqrt{2 + 6\left(-\frac{1}{36} + \frac{1}{36} - \frac{k}{3} + k^2\right)} \\
&= \sqrt{2 - \frac{1}{6} + 6\left(\frac{1}{6} - k\right)^2} \\
&= \sqrt{\frac{11}{6} + \frac{1}{6}\left(1 - 6k\right)^2} \\
&= \frac{\sqrt{6}}{6}\sqrt{11 + \left(1 - 6k\right)^2}
\end{aligned}
$$

$$
\min(|\mathbf{x}|) = \frac{\sqrt{6}}{6}\sqrt{11} = \boxed{\frac{\sqrt{66}}{6}}
$$