Solution for The Pure Mathematics Portion of 2020 GCE 'A' Levels Mathematics Paper 2

Gordon Chan

November 29, 2020

1

Let the curve be $y = ax^2 + bx + c$.

$$(1, -2) \implies a + b + c = -2$$

$$\begin{aligned} y' &= 2ax + b \\ y'(1) = 0 &\implies 2a + b = 0 \\ y'(2) = 5 &\implies 4a + b = 5 \end{aligned}$$

$$\begin{cases} a + b + c = -2 \\ 2a + b = 0 \\ 4a + b = 5 \end{cases} \implies \begin{cases} a = \frac{5}{2} \\ b = -5 \\ c = \frac{1}{2} \end{cases}$$

$$\therefore \boxed{y = \frac{5}{2}x^2 - 5x + \frac{1}{2}}$$

2

(a)

(i) (A)

The sequence increases indefinitely and diverges to positive infinity.

(i) (B)

The sequence stays constant at 5.

(ii)

$$\begin{aligned} u_5 &= 2u_4 - 5 \\ &= 2(2u_3 - 5) - 5 \\ &= 4u_3 - 15 \\ &= 4(2u_2 - 5) - 15 \\ &= 8u_2 - 35 \\ &= 8(2u_1 - 5) - 35 \\ &= 16p - 75 \end{aligned}$$

$$u_5 = 101 \implies 16p - 75 = 101 \implies p = \boxed{11}$$

(b)

(i)

$$\begin{aligned} v_4 &= v_4 \\ b + 2v_3 - 7 &= 2v_3 \\ b - 7 &= 0 \\ b &= \boxed{7} \end{aligned}$$

(ii)

$$\begin{aligned} v_5 &= v_3 + 2v_4 - 7 \\ &= v_3 + 4v_3 - 7 \\ &= 5v_3 - 7 \\ &= 5(a + 2b - 7) - 7 \\ &= 5(a + 14 - 7) - 7 \\ &= 5(a + 7) - 7 \\ &= 5a + 35 - 7 \\ &= \boxed{5a + 28} \end{aligned}$$

(c)

(i)

$$
\begin{aligned}
\sum_{r=1}^{n} a_r &= a_n + \sum_{r=1}^{n-1} a_r \\
a_n &= \sum_{r=1}^{n} a_r - \sum_{r=1}^{n-1} a_r \\
&= n^3 - 11n^2 + 4n - \left((n-1)^3 - 11(n-1)^2 + 4(n-1)n\right) \\
&= n^3 - 11n^2 + 4n - \left(n^3 - 3n^2 + 3n - 1 - 11\left(n^2 - 2n + 1\right) + 4n - 4\right) \\
&= n^3 - 11n^2 + 4n - \left(n^3 - 3n^2 + 7n - 5 - 11n^2 + 22n - 11\right) \\
&= n^3 - 11n^2 + 4n - \left(n^3 - 14n^2 + 29n - 16\right) \\
&= n^3 - 11n^2 + 4n - n^3 + 14n^2 - 29n + 16 \\
&= \boxed{3n^2 - 25n + 16}
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\sum_{r=1}^{m} a_r &= \sum_{r=1}^{3} a_r \\
m^3 - 11m^2 + 4m &= 3^3 - 11(3)^2 + 4(3) \\
m^3 - 11m^2 + 4m + 60 &= 0 \\
(m-3)\left(m^2 - 8m - 20\right) &= 0 \\
(m-3)(m+2)(m-10) &= 0 \\
m - 10 &= 0 \\
m &= \boxed{10}
\end{aligned}
$$

3

(i)

$$
\begin{aligned}
x = 3t^2 + 2 &\implies \frac{\mathrm{d}x}{\mathrm{d}t} = 6t \\
y = 6t - 1 &\implies \frac{\mathrm{d}y}{\mathrm{d}t} = 6 \\
\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} &= \frac{6}{6t} = \frac{1}{t}
\end{aligned}
$$

At $(x_0, y_0) = (14, 11)$, $y_0 = 11 \implies 11 = 6t_0 - 1 \implies t_0 = 2$.

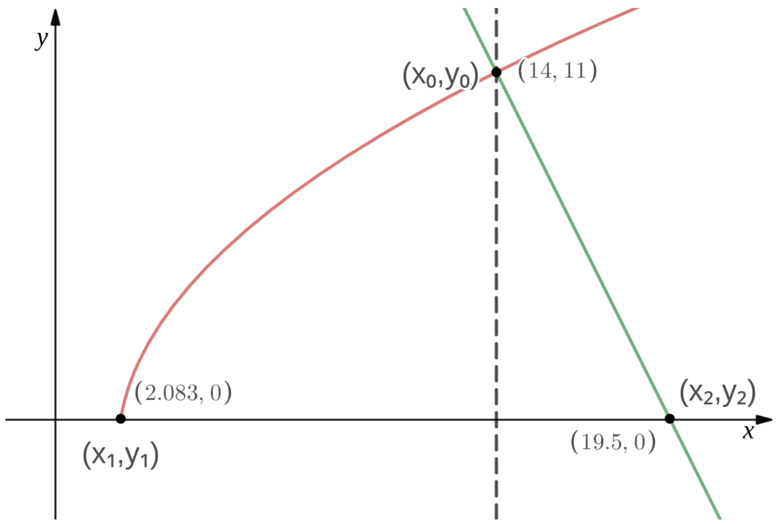
$$m = \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=t_o} = \frac{1}{t_0} = \frac{1}{2}$$

$$\begin{aligned} y - y_0 &= -\frac{1}{m}(x - x_0) \\ y - 11 &= -2(x - 14) \\ y - 11 &= -2x + 28 \end{aligned}$$

$$N : \boxed{2x + y = 39}$$

$$a = \boxed{2}, b = \boxed{1}, c = \boxed{39}$$

(ii)



$$\begin{aligned} y_1 &= 0 \\ 6t_1 - 1 &= 0 \\ 6t_1 &= 1 \\ t_1 &= \frac{1}{6} \\ x_1 &= 3\left(\frac{1}{6}\right)^2 + 2 \\ x_1 &= \frac{25}{12} \end{aligned}$$

$$\begin{aligned} y_2 &= 0 \\ 2x_2 + 0 &= 39 \\ x_2 &= \frac{39}{2} \end{aligned}$$

$$\begin{aligned} A &= \int_{x_1}^{x_0} y \,\mathrm{d}x + \frac{1}{2}(x_2 - x_0)y_0 \\ &= \int_{t_1}^{t_0} y \frac{\mathrm{d}x}{\mathrm{d}t} \,\mathrm{d}t + \frac{1}{2}(x_2 - x_0)y_0 \\ &= \int_{\frac{1}{6}}^{2} (6t-1)(6t) \,\mathrm{d}t + \frac{1}{2}\left(\frac{39}{2} - 14\right)(11) \\ &= 6\int_{\frac{1}{6}}^{2} \left(6t^2 - t\right) \mathrm{d}t + \frac{121}{4} \\ &= 6\left(2t^3 - \frac{1}{2}t^2\right)\Bigg|_{\frac{1}{6}}^{2} + \frac{121}{4} \\ &= 6\left(14 - \left(-\frac{1}{216}\right)\right) + \frac{121}{4} \\ &= 6\left(\frac{3025}{216}\right) + \frac{121}{4} \\ &= \frac{3025}{36} + \frac{121}{4} \\ &= \boxed{\frac{2057}{18} \text{ units}^2} \end{aligned}$$

(iii)

(a)

$$A' = (2 \times 3)A = 6\left(\frac{2057}{18}\right) = \boxed{\frac{2057}{3} \text{ units}^2}$$

(b)

$$C : \begin{cases} x = 3t^2 + 2 \\ y = 6t - 1 \end{cases} \quad , \; t \geqslant \frac{1}{6}$$

$$x = 3t^2 + 2 \implies t = \sqrt{\frac{x-2}{3}} \quad (t \geqslant 0)$$

$$t \geqslant \frac{1}{6} \implies x \geqslant \frac{25}{12}$$

$$y = 6t - 1$$

$$C: y = 6\sqrt{\frac{x-2}{3}} - 1,\ x \geqslant \frac{25}{12}$$

$$C \to D \implies (x, y) \to \left(\frac{x}{2}, \frac{y}{3}\right)$$

$$\frac{x}{2} \geqslant \frac{25}{12} \implies x \geqslant \frac{25}{6}$$

$$\frac{y}{3} = 6\sqrt{\frac{\frac{x}{2} - 2}{3}} - 1$$

$$\therefore D: \boxed{y = 18\sqrt{\frac{x-4}{6}} - 3,\ x \geqslant \frac{25}{6}}$$

4

(i)

$$h + a + h = 30 \implies h = \frac{30-a}{2}$$

$$H^2 + \left(\frac{a}{2}\right)^2 = h^2$$

$$\begin{aligned} H^2 &= \left(\frac{30-a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 \\ &= \frac{900 - 60a + a^2}{4} - \frac{a^2}{4} \\ &= 225 - 15a \quad \square \end{aligned}$$

(ii)

$$\begin{aligned} V &= \frac{1}{3}a^2 H \\ &= \frac{1}{3}\sqrt{a^4}\sqrt{225 - 15a} \\ V &= \frac{\sqrt{225a^4 - 15a^5}}{3} \\ \frac{\mathrm{d}V}{\mathrm{d}a} &= \frac{900a^3 - 75a^4}{6\sqrt{225a^4 - 15a^5}} \end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}a} &= 0 \\
900a^3 - 75a^4 &= 0 \\
900 - 75a &= 0 \\
a &= 12
\end{aligned}$$

$$V_{\max} = \frac{1}{3}(12)^2\sqrt{225 - 15(12)} = \boxed{144\sqrt{5}\,\mathrm{cm}}$$

(iii)

$$\begin{aligned}
A &= 4\left(\frac{1}{2}ah\right) \\
&= 2a\left(\frac{30-a}{2}\right) \\
A &= 30a - a^2 \\
\frac{\mathrm{d}A}{\mathrm{d}a} &= 30 - 2a \\
\frac{\mathrm{d}A}{\mathrm{d}a} &= 0 \\
30 - 2a &= 0 \\
a &= \boxed{15}
\end{aligned}$$

(iv)

The shape formed from the net in this case is a square with side lenth of $15\sqrt{2}\,\mathrm{cm}$.