4

Let $P(n): \sum\limits_{r=1}^{n} \sin(2rx)\sin x = \sin(nx)\sin(n+1)x$.

$$n = 1 \implies \sum_{r=1}^{1} \sin(2rx)\sin x = \sin(2x)\sin x = \sin(1x)\sin(1+1)x \implies P(1) \to \top$$

Assume $(\exists k \in \mathbb{Z}^{+})\, P(n) \to \top \implies \sum\limits_{r=1}^{k} \sin(2rx)\sin x = \sin(kx)\sin(k+1)x$.

$$\begin{aligned}
n = k+1 \implies \quad & \sum_{r=1}^{n} \sin(2rx)\sin x \\
&= \sum_{r=1}^{k+1} \sin(2rx)\sin x \\
&= \sum_{r=1}^{k} \sin(2rx)\sin x + \sin(2(k+1)x)\sin x \\
&= \sin(kx)\sin(k+1)x + \sin(2(k+1)x)\sin x \\
&= \sin(kx)\sin(kx+x) + \sin(2kx+2x)\sin x \\
&= -\frac{1}{2}\left(-2\sin\frac{1}{2}(2kx)\sin\frac{1}{2}(2kx+2x) - 2\sin\frac{1}{2}(4kx+4x)\sin\frac{1}{2}(2x)\right) \\
&= -\frac{1}{2}\left(\cos(2kx+x) - \cos x + \cos(2kx+3x) - \cos(2kx+x)\right) \\
&= -\frac{1}{2}\left(\cos(2kx+3x) - \cos x\right) \\
&= -\frac{1}{2}\left(-2\sin\frac{1}{2}(2kx+4x)\sin\frac{1}{2}(2kx+2x)\right) \\
&= \sin(kx+2x)\sin(kx+x) \\
&= \sin(k+1)x\sin((k+1)+1)x \\
&= \sin(nx)\sin(n+1)x \implies P(n+1) \to \top
\end{aligned}$$

$$((P(1) \to \top) \land (P(k) \to \top \implies P(k+1) \to \top)) \implies \left(\forall n \in \mathbb{Z}^{+}\right) P(n) \to \top$$

$$\therefore \sum_{r=1}^{n} \sin(2rx)\sin x = \sin(nx)\sin(n+1)x,\ \forall n \in \mathbb{Z}^{+} \quad \square$$

$$
\begin{aligned}
\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin 3x \sin 4x}{\sin x}\,\mathrm{d}x &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin 3x \sin (3+1)x}{\sin x}\,\mathrm{d}x \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sum\limits_{r=1}^{3} \sin(2rx)\sin x}{\sin x}\,\mathrm{d}x \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin(2x)\sin x + \sin(4x)\sin x + \sin(6x)\sin x}{\sin x}\,\mathrm{d}x \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\sin(2x) + \sin(4x) + \sin(6x)\right)\mathrm{d}x \\
&= \left(-\frac{1}{2}\cos 2x - \frac{1}{4}\cos 4x - \frac{1}{6}\cos 6x\right)\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
&= -\frac{1}{2}(-1) - \frac{1}{4}(0) - \frac{1}{6}(-1) \\
&= \boxed{\frac{2}{3}}
\end{aligned}
$$

5

Let $P(n) : 7 \mid 10^{3n} + 13^{n+1}$.

$$n = 1 \implies 10^{3n} + 13^{n+1} = 10^3 + 13^2 = 1169 = 7(169) \implies P(1) \to \top$$

Assume $(\exists k \in \mathbb{Z}^+)\, P(k) \to \top \implies 7 \mid 10^{3k} + 13^{k+1}$.

$$
\begin{aligned}
10^{3k} + 13^{k+1} &= 7m,\ m \in \mathbb{Z} \\
13^{k+1} &= 7m - 10^{3k}
\end{aligned}
$$

$$
\begin{aligned}
n = k+1 \implies 10^{3n} + 13^{n+1} &= 10^{3(k+1)} + 13^{(k+1)+1} \\
&= 10^{3k+3} + 13^{k+2} \\
&= 1000 \cdot 10^{3k} + 13 \cdot 13^{k+1} \\
&= 1000\left(10^{3k}\right) + 13\left(7m - 10^{3k}\right) \\
&= 1000\left(10^{3k}\right) + 13(7m) - 13\left(10^{3k}\right) \\
&= 987\left(10^{3k}\right) + 7(13m) \\
&= 7 \cdot 141\left(10^{3k}\right) + 7(13m) \\
&= 7\left(141\left(10^{3k}\right) + 13m\right) \\
&= 7M,\ M \in \mathbb{Z} \implies P(k+1) \to \top
\end{aligned}
$$

$$((P(1) \to \top) \wedge (P(k) \to \top \implies P(k+1) \to \top)) \implies \left(\forall n \in \mathbb{Z}^+\right) P(n) \to \top$$

$$\therefore 7 \mid 10^{3n} + 13^{n+1},\ \forall n \in \mathbb{Z}^+ \quad \square$$