2019 Year 6 H3 Math Prelim Exam Solutions

Qn	Solution
1(i)	$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{(c+t)x}{1-t^2}$ $\frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{(c+t)}{1-t^2}$ $\int \frac{1}{x}\,\mathrm{d}x=\int \frac{c}{1-t^2}-\frac{1}{2}\frac{-2t}{1-t^2}\,\mathrm{d}t$ $\ln\lvert x\rvert-d=\frac{c}{2}\ln\left\lvert\frac{1+t}{1-t}\right\rvert-\frac{1}{2}\ln\lvert 1-t^2\rvert$ $\ln\left\lvert\frac{x}{k}\right\rvert=\ln\left\lvert\frac{1+t}{1-t}\right\rvert^{\frac{c}{2}}-\ln\lvert(1+t)(1-t)\rvert^{\frac{1}{2}}$ $\left\lvert\frac{x}{k}\right\rvert=\frac{\lvert 1+t\rvert^{\frac{c-1}{2}}}{\lvert 1-t\rvert^{\frac{c+1}{2}}}$ $x=\frac{p\lvert 1+t\rvert^{\frac{c-1}{2}}}{\lvert 1+t\rvert^{\frac{c+1}{2}}}$, where p is an arbitrary constant.
(ii)	$y=tx$ $\frac{\mathrm{d}y}{\mathrm{d}x}=t+x\frac{\mathrm{d}t}{\mathrm{d}x}$ $t+x\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{ax+b(tx)}{bx+a(tx)}$ $t+x\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{a+b(t)}{b+at}$ $x\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{a+b(t)}{b+at}-t$ $x\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{a-at^2}{b+at}$ $\frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{b+at}{a(1-t^2)}$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\left(\frac{b}{a}+t\right)x}{1-t^2}$ and the resulting equation is equivalent to equation (A) with $c=\frac{b}{a}$.

Qn	Solution
1(iii)	$\left\|\frac{x}{k}\right\|=\frac{\left\|1+\frac{y}{x}\right\|^{\frac{\frac{b}{a}-1}{2}}}{\left\|1-\frac{y}{x}\right\|^{\frac{\frac{b}{a}+1}{2}}}$ $\|x\|\|x-y\|^{\frac{\frac{b}{a}+1}{2}}\|x\|^{-\frac{\frac{b}{a}+1}{2}}=\|k\|\|x+y\|^{\frac{\frac{b}{a}-1}{2}}\|x\|^{-\frac{\frac{b}{a}-1}{2}}$ $\|x\|\|x-y\|^{\frac{\frac{b}{a}+1}{2}}=\|k\|\|x+y\|^{\frac{\frac{b}{a}-1}{2}}\|x\|^{\frac{\frac{b}{a}+1}{2}-\frac{\frac{b}{a}-1}{2}}$ $\|x\|\|x-y\|^{\frac{b+a}{2a}}=\|k\|\|x+y\|^{\frac{b-a}{2a}}\|x\|$ $\|x\|\|x-y\|^{b+a}=\|k\|^{2a}\|x+y\|^{b-a}\|x\|$ $\|x-y\|^{b+a}=D\|x+y\|^{b-a}$
2(i)	Using the substitution $y=a+b-x,$ we have $\int_a^b \mathrm{f}(x)\ \mathrm{d}x=\int_b^a -\mathrm{f}(a+b-y)\ \mathrm{d}y$ $=\int_a^b \mathrm{f}(a+b-y)\ \mathrm{d}y$ $=\int_a^b \mathrm{f}(a+b-x)\ \mathrm{d}x$ (shown)
(ii)	$\int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\ \mathrm{d}\theta$ $=\int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)\ \mathrm{d}\theta$ $=\int_0^{\frac{\pi}{4}} \ln\left(1+\frac{\tan\left(\frac{\pi}{4}\right)-\tan\theta}{1+\tan\left(\frac{\pi}{4}\right)\tan\theta}\right)\ \mathrm{d}\theta$ $=\int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)\ \mathrm{d}\theta$ $=\int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan\theta}\right)\ \mathrm{d}\theta$ (shown)

Qn	Solution
2(iii)	$\int_0^1 \frac{\ln(1+x)}{1+x^2}\ \mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\ \mathrm{d}\theta$ by using the substitution $x = \tan\theta$. Also, we have $\int_0^1 \frac{\ln(1+x)}{1+x^2}\ \mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan\theta}\right)\ \mathrm{d}\theta.$ Summing up both equalities, we have $2\int_0^1 \frac{\ln(1+x)}{1+x^2}\ \mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\ \mathrm{d}\theta + \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan\theta}\right)\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta) + \ln 2 - \ln(1+\tan\theta)\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} \ln 2\ \mathrm{d}\theta = \frac{\pi}{4}\ln 2$ $\therefore \int_0^1 \frac{\ln(1+x)}{1+x^2}\ \mathrm{d}x = \frac{\pi}{8}\ln 2$
(iv)	$\int_0^{\frac{\pi}{4}} \frac{\cos x}{\cos\left(x-\frac{\pi}{4}\right)}\ \mathrm{d}x = \int_0^{\frac{\pi}{4}} \frac{\cos\left(\frac{\pi}{4}-x\right)}{\cos\left(\left(\frac{\pi}{4}-x\right)-\frac{\pi}{4}\right)}\ \mathrm{d}x$ $= \int_0^{\frac{\pi}{4}} \frac{\cos\left(\frac{\pi}{4}\right)\cos x + \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)}{\cos(-x)}\ \mathrm{d}x$ $= \int_0^{\frac{\pi}{4}} \frac{\cos\left(\frac{\pi}{4}\right)\cos x + \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)}{\cos(-x)}\ \mathrm{d}x$ $= \int_0^{\frac{\pi}{4}} \frac{\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x}{\cos x}\ \mathrm{d}x$ $= \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\tan x\ \mathrm{d}x$ $= \left[\frac{x}{\sqrt{2}} + \frac{1}{\sqrt{2}}\ln\lvert\sec x\rvert\right]_0^{\frac{\pi}{4}}$ $= \left(\frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}}\ln\left\lvert\sec\left(\frac{\pi}{4}\right)\right\rvert\right) - 0$ $= \frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}}\ln\left\lvert\sqrt{2}\right\rvert$ $= \frac{\pi}{4\sqrt{2}} + \frac{1}{2\sqrt{2}}\ln 2$

Qn	Solution
3(a)	$\frac{1}{y+z},\ \frac{1}{x+z},\ \frac{1}{x+y}$ are three consecutive numbers of an AP $\Leftrightarrow \frac{1}{x+z}-\frac{1}{y+z}=\frac{1}{x+y}-\frac{1}{x+z}$ $\Leftrightarrow \frac{(y+z)-(x+z)}{(x+z)(y+z)}=\frac{(x+z)-(x+y)}{(x+y)(x+z)}$ $\Leftrightarrow \frac{y-x}{(x+z)(y+z)}=\frac{z-y}{(x+y)(x+z)}$ $\Leftrightarrow \frac{(y-x)(x+y)}{(x+y)(x+z)(y+z)}=\frac{(z-y)(y+z)}{(x+y)(x+z)(y+z)}$ $\Leftrightarrow \frac{y^2-x^2}{(x+y)(x+z)(y+z)}=\frac{z^2-y^2}{(x+y)(x+z)(y+z)}$ $\Leftrightarrow y^2-x^2=z^2-y^2$ $\Leftrightarrow x^2,\ y^2,\ z^2$ are three consecutive numbers of an AP
(b) **(i)**	$\frac{a_{n+2}+a_n}{a_{n+1}}=\frac{\frac{a_{n+1}^2+2}{a_n}+a_n}{a_{n+1}}=\frac{a_{n+1}^2+2+a_n^2}{a_{n+1}a_n}=\frac{a_{n+1}+\frac{a_n^2+2}{a_{n+1}}}{a_n}=\frac{a_{n+1}+a_{n-1}}{a_n},\ n\geq 2$
(b) **(ii)**	$a_3=\frac{a_2^2+2}{a_1}=\frac{1^2+2}{1}=3$ $\frac{a_{n+1}+a_{n-1}}{a_n}=\ldots=\frac{a_3+a_1}{a_2}=\frac{3+1}{1}=4$ $a_{n+1}=4a_n-a_{n-1},\ n\geq 2.$ Therefore a_{n+1} has the same parity as a_{n-1} since $4a_n$ is even. Since both a_1 and a_2 are both odd, by induction, a_n is an odd integer for all $n\in\mathbb{Z}^+$. Alternatively, let P(n) be the statement that a_n is odd for $n\in\mathbb{Z}^+$. P(1) and P(2) are both true trivially by definition. Suppose a_k and a_{k+1} are both odd for some positive integer k. Then $a_{k+2}=\frac{a_{k+1}^2+2}{a_k}=\frac{(\text{odd})^2+2}{\text{odd}}=\frac{\text{odd}}{\text{odd}}\neq\text{even}$ Hence $\text{P}(k),\text{P}(k+1)$ both true $\Rightarrow \text{P}(k+2)$ is true. Therefore, since P(1) and P(2) are both true, and $\text{P}(k),\text{P}(k+1)$ both true $\Rightarrow \text{P}(k+2)$ is true, by the Principle of Mathematical Induction, a_n is an odd integer for all $n\in\mathbb{Z}^+$.

Qn	Solution
3(c)	Suppose there is an arithmetic progression $a_1, a_2, \ldots$ with common difference d that has $1, \sqrt{2}, \sqrt{3}$ among its terms. Then there exist distinct positive integers m, n and p such that $a_m = 1, a_n = \sqrt{2}, a_p = \sqrt{3}.$ Thus we have $\sqrt{2}-1 = a_n - a_m = (n-m)d$ and $\sqrt{3}-\sqrt{2} = a_p - a_n = (p-n)d$, so $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1} = \dfrac{n-m}{p-n}$ is rational. Since $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1} = \left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{2}+1\right) = \sqrt{6}-2+\sqrt{3}-\sqrt{2}$ thus $a = \sqrt{6}+\sqrt{3}-\sqrt{2}$ is a rational number. Then, $a+\sqrt{2} = \sqrt{6}+\sqrt{3}$ $\left(a+\sqrt{2}\right)^2 = \left(\sqrt{6}+\sqrt{3}\right)^2$ $a^2+2a\sqrt{2}+2 = 6+2\sqrt{18}+3$ $2a\sqrt{2}-6\sqrt{2} = 7-a^2$ $(2a-6)\sqrt{2} = 7-a^2$ Since $a=3$ does not satisfy the equality, we can divide throughout by $(2a-6)$ so $\sqrt{2} = \dfrac{7-a^2}{2a-6} \in \mathbb{Q}$ which is a contradiction. Hence the supposition does not hold and the result required is shown.
4(i)	Applying Cauchy-Schwarz Inequality $\left(\sum_{i=1}^{n+1} a_i^2\right)\left(\sum_{i=1}^{n+1} b_i^2\right) \geq \left(\sum_{i=1}^{n+1} a_i b_i\right)^2$ with $a_i = x_{i-1}-x_i$ for $i = 1, 2, \ldots, n$, $a_{n+1} = x_n$ and $b_i = 1$ for $i = 1, 2, \ldots, n+1$, we have $\left(\left(\sum_{i=1}^{n}\left(x_{i-1}-x_i\right)^2\right)+x_n^2\right)\left(\sum_{i=1}^{n+1} 1^2\right) \geq \left(\left(\sum_{i=1}^{n}\left(x_{i-1}-x_i\right)\right)+x_n\right)^2 = 1$ $\left(\left(\sum_{i=1}^{n}\left(x_{i-1}-x_i\right)^2\right)+x_n^2\right) \geq \dfrac{1}{n+1}$

Qn	Solution
4(ii)	Since equality holds, we have $a_i = mb_i = m$ (constant). Thus, $a_1 = a_2 = ... = a_{n+1}$. Since $\sum_{i=1}^{n+1} a_i = 1$, therefore we have $a_i = \frac{1}{n+1}$ for $i = 1,\ 2,\ ...,\ n+1$, Thus, $x_n = \frac{1}{n+1}$, $x_{n-1} = a_n + x_n = \frac{2}{n+1}$, $x_{n-2} = a_{n-1} + x_{n-1} = \frac{3}{n+1}$, and so on so forth. Generalizing, we have $x_k = \frac{n+1-k}{n+1}, \quad k = 0,\ 1,\ 2,\ ...,\ n.$
5(i)	Number of ways $= \binom{24+3-1}{3-1} = 325$
(ii)	The problem is $x_1 + x_2 + x_3 = 24$ $0 \le x_i \le 10,\ i = 1, 2, 3$ with $x_1,\ x_2,\ x_3$ being the number of cards in the red, green and blue boxes respectively. Method 1 Let A_i denote the event where $0 \le x_i \le 10$. Number of ways $\left\|A_1 \cap A_2 \cap A_3\right\|$ $= \left\|S\right\| - \left\|A_1' \cup A_2' \cup A_3'\right\|$ $= \left\|S\right\| - \left(\sum_{i=1}^{3} \left\|A_i'\right\| - \sum_{\substack{i \ne j \\ i,j \in \{1,2,3\}}} \left\|A_i' \cap A_j'\right\| + \left\|A_1' \cap A_2' \cap A_3'\right\| \right)$ $= 325 - \left(3\binom{13+3-1}{3-1} - 3\binom{2+3-1}{3-1} + 0 \right)$ $= 325 - \left(3(105) - 3(6) + 0\right) = 28$ (Continued)

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>5(ii)</td><td>

<u>Method 2</u>
$x_i \geq 4$ since $x_i \leq 3$ for any *i* will result in
$x_1 + x_2 + x_3 < 24$ for $0 \leq x_i \leq 10,\ i = 1, 2, 3.$

<table>
<tr><th>x_1</th><th>$x_2 + x_3$</th><th>$(x_2,\ x_3)$ or corresponding cases for x_2</th><th>Number of Ways</th></tr>
<tr><td>4</td><td>20</td><td>$(10,\ 10)$</td><td>1</td></tr>
<tr><td>5</td><td>19</td><td>$(9,\ 10),\ (10,\ 9)$</td><td>2</td></tr>
<tr><td>6</td><td>18</td><td>$8 \leq x_2 \leq 10$</td><td>3</td></tr>
<tr><td>7</td><td>17</td><td>$7 \leq x_2 \leq 10$</td><td>4</td></tr>
<tr><td>8</td><td>16</td><td>$6 \leq x_2 \leq 10$</td><td>5</td></tr>
<tr><td>9</td><td>15</td><td>$5 \leq x_2 \leq 10$</td><td>6</td></tr>
<tr><td>10</td><td>14</td><td>$4 \leq x_2 \leq 10$</td><td>7</td></tr>
</table>

Therefore number of ways = 1 + 2 + … + 7 = 28

<u>Method 3</u>
Number of ways
= (none with 10 units) + (one with 10 units)
+ (two with 10 units)

$$= \left(\underbrace{1}_{(8,8,8)} + \underbrace{\frac{3!}{2!}}_{(6,9,9)^*} + \underbrace{3!}_{(7,8,9)^*} \right) + \underbrace{5}_{(5,9)\ \ldots\ (9,5)} \times {}^3C_1 + \underbrace{{}^3C_2}_{(4,10,10)^*}$$

$$= 10 + 15 + 3 = 28$$

</td></tr>
<tr><td>(iii)</td><td>

Sue can take either no work, or up to 4 units of work, since it is not possible to have Sue take 5 or more units of work as the total amount of work is 24 units (5 + 10 + 10 =25 > 24).

If Sue takes *k* units of work where *k* = 0 to 4, then the other 2 must take at least 2*k* units of work from the remaining $(24 - k)$ units of work,

no. of ways $= \binom{24 - k - 4k + 2 - 1}{2 - 1} = 25 - 5k$

since *k* units of work is assigned to Sue, and we place 2*k* units of work each into the other two boxes.

Number of ways supervisor can distribute the workload $= \sum_{k=0}^{4} (25 - 5k) = 75$

</td></tr>
<tr><td>(iv)</td><td>$P(r,\ n) = P(r - 1,\ n - 1) + P(r - n,\ n)$</td></tr>
</table>

Qn	Solution
5(v)	$P(r,\ n)$ $= P(r-1,\ n-1)+P(r-n,\ n)$ (applying **(iv)** on $P(r,\ n)$) $=\left[P(r-2,\ n-2)+P\left((r-1)-(n-1),\ n-1\right)\right]$ $+P(r-n,\ n)$ (applying **(iv)** on $P(r-1,\ n-1)$) $=P(r-2,\ n-2)+\left[P(r-n,\ n-1)+P(r-n,\ n)\right]$ (rearrangement) $=P\left(r-(n-1),\ 1\right)$ $+\left[P\left(r-n,\ n-(n-2)\right)+\ldots+P(r-n,\ n)\right]$ (applying **(iv)** on $P(r-2,\ n-2),\ldots,$ until $P(r-n+1,\ 1)$) $=P\left(r-n+1,\ 1\right)+\left[P\left(r-n,\ 2\right)+\ldots+P(r-n,\ n)\right]$ $=P\left(r-n,\ 1\right)+\left[P\left(r-n,\ 2\right)+\ldots+P(r-n,\ n)\right]$ ($\because\ P\left(r-n+1,\ 1\right)=P\left(r-n,\ 1\right)=1$) $=\sum_{k=1}^{n}P(r-n,\ k)$ (shown)
(vi)	Method 1 (Using **(v)**) $P(10,\ 3)=\sum_{k=1}^{3}P(7,\ k)$ $=P(7,\ 1)+P(7,\ 2)+P(7,\ 3)$ $=1+\sum_{k=1}^{2}P(5,\ k)+\sum_{k=1}^{3}P(4,\ k)$ $=1+P(5,\ 1)+P(5,\ 2)+P(4,\ 1)+P(4,\ 2)+P(4,\ 3)$ $=1+1+P(5,\ 2)+1+P(4,\ 2)+1$ $=4+\sum_{k=1}^{2}P(3,\ k)+\sum_{k=1}^{2}P(2,\ k)$ $=4+P(3,\ 1)+P(3,\ 2)+P(2,\ 1)+P(2,\ 2)$ $=4+1+1+1+1=8$ Method 2 (Using **(iv)**) $P(10,\ 3)$ $=P(9,\ 2)+P(7,\ 3)$ $=\left(P(8,\ 1)+P(7,\ 2)\right)+\left(P(6,\ 2)+P(4,\ 3)\right)$ $=1+\left(P(6,\ 1)+P(5,\ 2)\right)+\left(P(5,\ 1)+P(4,\ 2)\right)+1$ $=1+1+\left(P(4,\ 1)+P(3,\ 2)\right)+1+\left(P(3,\ 1)+P(2,\ 2)\right)+1$ $=8$
6(i) **(a)**	$T_1=1,\ T_2=2.$
(i) **(b)**	$T_{n+2}=T_{n+1}+T_n$ for $n\geq 1.$

Qn	Solution
6(ii)	Consider the “odd” tile / last tile in a tiling of P_{n+1}. It can only be covered by a 1×1 or a 1×2 tile. If it is covered by a 1×1 tile, the rest form a tiling of Q_n. If it is covered by a 1×2 tile, the rest form a tiling of P_n. Thus $P_{n+1}=P_n+Q_n$.
(iii)	Consider the last column of 2 tiles in a tiling of Q_{n+1}. The following cases are possible: • 2×2 tile: The rest form a tiling of Q_{n-1}. • 1×2 tile (vertical): The rest form a tiling of Q_n • Two 1×1 tiles: The rest form a tiling of Q_n. • Two 1×2 tiles (horizontal): The rest form a tiling of Q_{n-1}. • One 1×1 tile and one 1×2 tile (horizontal): The rest form a tiling of P_n. Note that this case counts twice (depending on which tile covers the top line and which tile covers the bottom line). Thus $Q_{n+1}=2Q_n+2Q_{n-1}+2P_n$ $=2P_{n+1}+2Q_{n-1}$ using the result from (**ii**). Alternative Tiling of Q_{n+1} can be obtained from • Q_{n-1} using a 2×2 tile or two horizontal 1×2 tiles. • Q_n using one vertical 1×2 tile • P_{n+1} using one 1×1 tile The first two cases involve non-1×1 tiles on the last column, and the last case involves 1×1 tiles on the last column. However, the last case we must consider the double counting of the subcase where tiles from the last column are both 1×1 tiles, and the number of ways to do so is Q_n. Thus $Q_{n+1}=2Q_{n-1}+Q_n+\left(2P_{n+1}-Q_n\right)=2P_{n+1}+2Q_{n-1}$
(iv)	Add $P_{n+1}+2P_{n-1}$ to both sides of (**iii**): $P_{n+1}+2P_{n-1}+Q_{n+1}=P_{n+1}+2P_{n-1}+2P_{n+1}+2Q_{n-1}$ $P_{n+2}+2P_{n-1}=3P_{n+1}+2P_n$ (using result from (**ii**))
(v)	Number of tilings of $2\times n$ path is Q_n So $Q_n=P_{n+1}-P_n$ $=\frac{2}{7}(-1)^n+\frac{1+2\sqrt{2}}{14}\left(2+\sqrt{2}\right)^n\left(2+\sqrt{2}-1\right)+\frac{1-2\sqrt{2}}{14}\left(2-\sqrt{2}\right)^n\left(2-\sqrt{2}-1\right)$ $=\frac{2}{7}(-1)^n+\frac{5+3\sqrt{2}}{14}\left(2+\sqrt{2}\right)^n+\frac{5-3\sqrt{2}}{14}\left(2-\sqrt{2}\right)^n$

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>7(a)</td><td>
We show by proving the contrapositive statement:
if $7 \nmid a$ or $7 \nmid b$ then $7 \nmid (a^2+b^2)$.

For positive integers n not divisible by 7,

<table>
<tr><th>n (mod 7)</th><th>n^2 (mod 7)</th></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>4</td></tr>
<tr><td>3</td><td>2</td></tr>
<tr><td>4</td><td>2</td></tr>
<tr><td>5</td><td>4</td></tr>
<tr><td>6</td><td>1</td></tr>
</table>

Therefore $(a^2+b^2)(\bmod 7)$ can only take values
1 (from 4 + 4), 2 (from 1 + 1), 3 (from 1 + 2),
4 (from 2 + 2), 5 (from 1 + 4) 6 (from 2 + 4) but never 0.

Hence the contrapositive statement is shown.
</td></tr>
<tr><td>(b)</td><td>
When n is even, n^4+4^n is even and hence not a prime.

When n is odd, i.e. $n=2k+1$ with integer $k \geq 1$,
(since $n>1$)
$n^4+4^n = \left(n^2+2^n\right)^2 - 2\left(n^2\right)\left(2^n\right)$
$= \left(n^2+2^n\right)^2 - \left(n^2\right)\left(2^{2k+2}\right)$
$= \left(n^2+2^n\right)^2 - \left(2^{k+1}n\right)^2$
$= \left(n^2+2^n-2^{k+1}n\right)\left(n^2+2^n+2^{k+1}n\right)$

The smaller factor is $\left(n^2+2^n-2^{k+1}n\right)$, and
$n^2+2^n-2^{k+1}n = n^2-2\left(2^k\right)n+2^{2k}-2^{2k}+2^{2k+1}$
$= \left(n-2^k\right)^2+2^{2k} > 1$
so n^4+4^n cannot be prime.
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</table>

Qn	**Solution**
7(c)	Method 1: Consider the values of 16^n, $10n$ and -1 (mod 25)

n	16^n (mod 25)	$10n$ (mod 25)	-1	Sum (mod 25)
1	16	10	-1	0
2	6	20	-1	0
3	21	5	-1	0
4	11	15	-1	0
5	1	0	-1	0
6	16	10	-1	0
7	6	20	-1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Since the table repeats cyclically for every 5 values of n, $16^n+10n-1$ is divisible by 25, for $n\in\mathbb{Z}^+$.

Method 2: Mathematical Induction

Let P(n) be the statement that $16^n+10n-1$ is divisible by 25, for $n\in\mathbb{Z}^+$.

P(1) is true since $16^1+10(1)-1=25$ which is divisible by 25.

Suppose P(k) is true for some positive integer k.

Then for $\mathrm{P}(k+1)$,

$$16^{k+1}+10(k+1)-1$$
$$=16\left(\left(16^k\right)+10k-1\right)-160k+16+10(k+1)-1$$
$$=16\left(\left(16^k\right)+10k-1\right)-150k+25$$

which is divisible by 25.

Hence $\mathrm{P}(k)$ is true $\Rightarrow$ $\mathrm{P}(k+1)$ is true.

Therefore, since P(1) is true, and $\mathrm{P}(k)$ is true $\Rightarrow$ $\mathrm{P}(k+1)$ is true, by the Principle of Mathematical Induction, $16^n+10n-1$ is divisible by 25, for $n\in\mathbb{Z}^+$.

Qn	**Solution**
8(i)	$L_1(x_1)=1,$ $L_1(x_2)=0,\ \ L_1(x_3)=0.$
(ii)	$L_2(x)=\dfrac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}$ $L_3(x)=\dfrac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$
(iii)	Suppose there exist two quadratic polynomials p_1, p_2 (of degree less or equal to $n-1=3-1=2)$, with $p_1(x_i)=p_2(x_i)=y_i,$ for $i=1,\ 2,\ 3.$ Then the difference polynomial $q=p_1-p_2$ is a polynomial of degree less or equal to $n-1=3-1=2$ that satisfy $h(x_i)=0,$ for $i=1,\ 2,\ 3.$ Since the number of zeroes of a nonzero polynomial is equal to its degree, it follows that $q(x)=p_1(x)-p_2(x)=0,\ \ x\in\mathbb{R},$ i.e. $p_1(x)=p_2(x),$ for all $x\in\mathbb{R}.$
(iv)	$P(x)=y_1L_1(x)+y_2L_2(x)+y_3L_3(x),\ \ x\in\mathbb{R}.$ This is because since $L_i(x_i)=1,\ \ L_i(x_j)=0$ for $i\neq j,\ \ i,j=1,\ 2,\ 3$ from the earlier parts, we have $P(x_i)=y_i,$ for $i=1,\ 2,\ 3.$
(v)	Let $X=Y=\{1,\ 2,\ 3\}$ and a function $\mathrm{f}:X\to X,$ satisfying $\mathrm{f}(x_i)=y_i,\ \ y_i\neq i$ for $i=1,\ 2,\ 3,$ where $x_i=i,$ and $y_i\in Y=X,$ for $i=1,\ 2,\ 3.$ Note that the above translates to a derangement problem, where the number of such possible mappings is given by $D_3=3!\left(1-\dfrac{1}{1!}+\dfrac{1}{2!}-\dfrac{1}{3!}\right)=2.$ In fact, the 2 derangements are given by Derangement 1: $\mathrm{f}(1)=2,\ \ \mathrm{f}(2)=3,\ \ \mathrm{f}(3)=1,$ Derangement 2: $\mathrm{f}(1)=3,\ \ \mathrm{f}(2)=1,\ \ \mathrm{f}(3)=2.$ We now show that we can extend this derangement property of function f defined on $\{1,\ 2,\ 3\}$ to a quadratic polynomial Q, which is defined on the real line by finding explicitly 2 quadratic polynomials Q_1, Q_2 such that they satisfy $Q_1(1)=2,\ \ Q_1(2)=3,\ \ Q_1(3)=1,$ $Q_2(1)=3,\ \ Q_2(2)=1,\ \ Q_2(3)=2.$ To do so, we can use either of the following two methods:

Qn	Solution
8(v)	Method 1 We can find two quadratic polynomials $Q_1,\ Q_2$ such that they satisfy $Q_1(1)=2,\ Q_1(2)=3,\ Q_1(3)=1,$ $Q_2(1)=3,\ Q_2(2)=1,\ Q_2(3)=2.$ from parts **(iii)** and the interpolatory property **(iv)**, we get $Q_1(x)=2L_1(x)+3L_2(x)+1L_3(x),$ $Q_2(x)=3L_1(x)+1L_2(x)+2L_3(x),$ where $L_1(x)=\frac{(x-2)(x-3)}{(1-2)(1-3)}=\frac{1}{2}(x-2)(x-3),$ $L_2(x)=\frac{(x-1)(x-3)}{(2-1)(2-3)}=-(x-1)(x-3),$ $L_3(x)=\frac{(x-1)(x-2)}{(3-1)(3-2)}=\frac{1}{2}(x-1)(x-2).$ Method 2 Let $Q_1(x)=a_2x^2+a_1x+a_0$, and $Q_2(x)=b_2x^2+b_1x+b_0$. We solve for coefficients a_2,a_1,a_0 and b_2,b_1,b_0 such that $Q_1(1)=2,\ Q_1(2)=3,\ Q_1(3)=1,$ or $Q_2(1)=3,\ Q_2(2)=1,\ Q_2(3)=2.$ We obtain the resulting system of linear equations $a_2(1)^2+a_1(1)+a_0=2,$ $a_2(2)^2+a_1(2)+a_0=3,$ $a_2(3)^2+a_1(3)+a_0=1.$ $b_2(1)^2+b_1(1)+b_0=3,$ $b_2(2)^2+b_1(2)+b_0=1,$ $b_2(3)^2+b_1(3)+b_0=2.$ $a_2=-\frac{3}{2},\ a_1=\frac{11}{2},\ a_0=-2,$ $b_2=\frac{3}{2},\ b_1=-\frac{13}{2},\ b_0=8,$ Therefore, $Q_1(x)=-\frac{3}{2}x^2+\frac{11}{2}x-2,\ Q_2(x)=\frac{3}{2}x^2-\frac{13}{2}x+8.$

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>9(i)
(a)</td><td>$p+1$</td></tr>
<tr><td>(i)
(b)</td><td>$S(pq)=S(p)S(q)=\left(p+1\right)\left(q+1\right)$</td></tr>
<tr><td>(i)
(c)</td><td>$S(p^m q^n)=S(p^m)S(q^n)=\left(1+p+\ldots+p^m\right)\left(1+q+\ldots+q^n\right)=\dfrac{\left(p^{m+1}-1\right)\left(q^{n+1}-1\right)}{\left(p-1\right)\left(q-1\right)}$</td></tr>
<tr><td>(ii)</td><td>$220=2^2\times5\times11$. The proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110, and
Sum of all proper divisors of 220 $=1+2+4+5+10+11+20+22+44+55+110$
$=284$

$284=2^2\times71$. The proper divisors of 2924 are 1, 2, 4, 71 and 142, and
Sum of all proper divisors of 284 $=1+2+4+71+142=220$</td></tr>
<tr><td>(iv)</td><td>$a(pq+r)=M+N=S\left(M\right)=S\left(apq\right)=S\left(a\right)S\left(p\right)S\left(q\right)=\left(S(a)\right)\left(p+1\right)\left(q+1\right)$</td></tr>
<tr><td>(v)</td><td>$S\left(apq\right)=S\left(ar\right)$
$\left(p+1\right)\left(q+1\right)S\left(a\right)=\left(r+1\right)S\left(a\right)$
$r+1=\left(p+1\right)\left(q+1\right)$
$$\begin{aligned}\left(2a-S(a)\right)\left(p+1\right)\left(q+1\right)&=2a\left(p+1\right)\left(q+1\right)-S(a)\left(p+1\right)\left(q+1\right)\\&=2a\left(p+1\right)\left(q+1\right)-a(pq+r)\\&=2a\left(p+1\right)\left(q+1\right)-a\left(pq+\left(p+1\right)\left(q+1\right)\right)\\&=2a\left(pq+p+q+1\right)-a(2pq+p+q)\\&=\left(2apq+2ap+2aq+2a\right)-(2apq+ap+aq)\\&=\left(ap+aq+2a\right)=a\left(p+q+2\right)\text{ (shown)}\end{aligned}$$</td></tr>
<tr><td>(vi)</td><td>$a=4,\ S(4)=1+2+4=7.$
$$\left(p+1-\frac{a}{2a-S(a)}\right)\left(q+1-\frac{a}{2a-S(a)}\right)=\left(\frac{a}{2a-S(a)}\right)^2$$becomes
$$\left(p+1-\frac{4}{2\left(4\right)-7}\right)\left(q+1-\frac{4}{2\left(4\right)-7}\right)=\left(\frac{4}{2\left(4\right)-7}\right)^2$$$\left(p-3\right)\left(q-3\right)=16$

WLOG let $p\le q$,
<table>
<tr><th>$p-3$</th><th>$q-3$</th><th>p</th><th>q</th><th>r</th></tr>
<tr><td>1</td><td>16</td><td>4</td><td>19</td><td>99</td></tr>
<tr><td>2</td><td>8</td><td>5</td><td>11</td><td>71</td></tr>
<tr><td>4</td><td>4</td><td>7</td><td>7</td><td>63</td></tr>
</table>
So the only solution is $p=5,\ q=11,\ r=71,$ and
$M=4\times5\times11=220,\ N=4\times71=284.$</td></tr>
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