Introduction Lesson Assignment

1 Determine if the following arguments are valid:

(a) Either John isn't stupid and he is lazy, or he's stupid.
John is stupid therefore John isn't lazy.

(b) The butler and the cook are not both innocent.
Either the butler is lying or the cook is innocent.
The butler is either lying or guilty or both.

2 (Transitivity of divisibility) For all integers a, b and c, if $a \mid b$ and $b \mid c$, then $a \mid c$.

3 Let $x \in \mathbb{Z}$. If $2 \mid x^2 - 1$, then $4 \mid x^2 - 1$.

4 Prove that for all real numbers a and b,

(i) $a^2 \geq 0$

(ii) $ab \leq \frac{1}{2}\left(a^2 + b^2\right)$

(iii) $ab \leq \frac{1}{2}\left(c^2 a^2 + c^{-2} b^2\right)$ for any non-zero real number c.

5 A famous mathematical inequality is the Cauchy-Schwarz (CS) Inequality which is stated as follows: For all real values $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$,

$$\left(a_1 b_1 + a_2 b_2 + \ldots + a_n b_n\right)^2 \leq \left(a_1^2 + a_2^2 + \ldots a_n^2\right)\left(b_1^2 + b_2^2 + \ldots b_n^2\right).$$

There are many proofs of the CS inequality. We will scaffold you through one of the proofs by guiding you through the intermediate steps.

(i) By considering the identity $(\alpha y - \beta x)^2 \geq 0$, show that for all $\alpha, \beta \in \mathbb{R}$, $x \in \mathbb{R}^+$ and $y \in \mathbb{R}^+$, $\dfrac{(\alpha + \beta)^2}{x + y} \leq \dfrac{\alpha^2}{x} + \dfrac{\beta^2}{y}$.

(ii) By repeated application of the result in part (i), show that for all $\alpha_i \in \mathbb{R}$, $x_i \in \mathbb{R}^+$,

$$\frac{(\alpha_1 + \alpha_2 + \ldots + \alpha_n)^2}{x_1 + x_2 + \ldots + x_n} \leq \frac{\alpha_1{}^2}{x_1} + \frac{\alpha_2{}^2}{x_2} + \ldots + \frac{\alpha_n{}^2}{x_n}.$$

(iii) Hence prove the Cauchy-Schwarz inequality for all non-zero real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, by expressing α_i and x_i in part (ii), in terms of a_i's and b_i's. (What happens when some of a_i's and b_i's are zero?)

Note: Equality for the inequality is achieved if and only if $a_i = kb_i$ for $i = 1, 2, 3, \ldots, n$ for a fixed real constant k. Proof of this will be given in a subsequent chapter.

6 *Independent research and self-study is critical to success in the H3 Maths course.* Provide an alternative direct proof of the CS inequality.