9

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}x} = f(x) \\ (x, y) = (x_0, y_0) \end{cases}$$

(i)

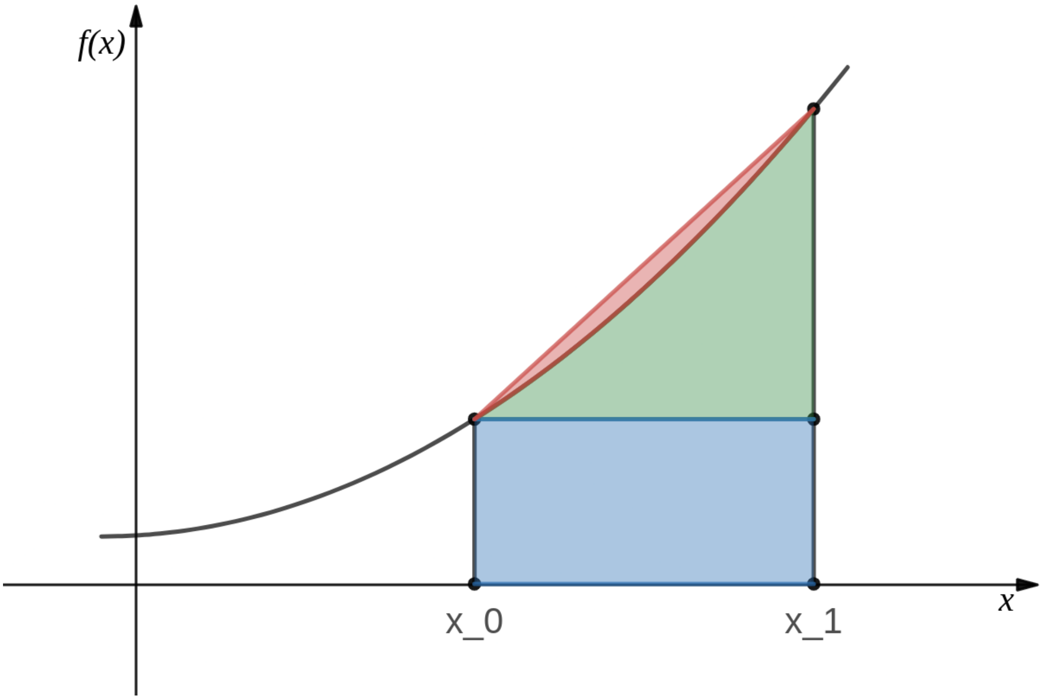
Let the estimated value of y at $x = x_1$ be y_1.
By the Euler method,

$$\begin{aligned} y_1 &= y_0 + \Delta x \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0} \\ &= \boxed{y_0 + (x_1 - x_0) f(x_0)} \end{aligned}$$

By the improved Euler method,

$$\begin{aligned} y_1 &= y_0 + \Delta x \frac{\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0} + \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_1}}{2} \\ &= \boxed{y_0 + (x_1 - x_0)\frac{f(x_0) + f(x_1)}{2}} \end{aligned}$$

(ii)



For the Euler method,

$$\begin{aligned}\Delta y &= y(1) - y_1 \\ &= \left(y_0 + \int_0^1 f(x)\,\mathrm{d}x\right) - (y_0 + (x_1 - x_0) f(x_0)) \\ &= \textcolor{green}{B} + \textcolor{blue}{C} - \textcolor{blue}{C} \\ &= \textcolor{green}{B} \\ &> 0\end{aligned}$$

For the improved Euler method,

$$\begin{aligned}\Delta y &= y(1) - y_1 \\ &= \left(y_0 + \int_0^1 f(x)\,\mathrm{d}x\right) - \left(y_0 + (x_1 - x_0)\frac{f(x_0) + f(x_1)}{2}\right) \\ &= \textcolor{green}{B} + \textcolor{blue}{C} - \textcolor{red}{A} - \textcolor{green}{B} - \textcolor{blue}{C} \\ &= -\textcolor{red}{A} \\ &< 0\end{aligned}$$

(iii)

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= a + bx + cx^2 \\
y &= d + ax + \frac{b}{2}x^2 + \frac{c}{3}x^3 \\
y_0 &= d + ax_0 + \frac{b}{2}x_0^2 + \frac{c}{3}x_0^3 \\
y_0 &= d + a(0) + \frac{b}{2}(0)^2 + \frac{c}{3}(0)^3 \\
d &= y_0
\end{aligned}
$$

$$
\begin{aligned}
y &= y_0 + ax + \frac{b}{2}x^2 + \frac{c}{3}x^3 \\
y(h) &= y_0 + ah + \frac{b}{2}h^2 + \frac{c}{3}h^3 \\
&= y_0 + \frac{h}{6}\left(6a + 3bh + 2ch^2\right)
\end{aligned}
$$

$$
\begin{aligned}
y_1 &= y_0 + (x_1 - x_0)\frac{f(x_0) + f(x_1)}{2} \\
y_1 &= y_0 + (h - 0)\frac{f(0) + f(h)}{2} \\
&= y_0 + h\frac{a + a + bh + ch^2}{2} \\
&= y_0 + \frac{h}{6}\left(6a + 3bh + 3ch^2\right)
\end{aligned}
$$

$$
\begin{aligned}
\Delta y &= y(h) - y_1 \\
&= \left(y_0 + \frac{h}{6}\left(6a + 3bh + 2ch^2\right)\right) - \left(y_0 + \frac{h}{6}\left(6a + 3bh + 3ch^2\right)\right) \\
&= y_0 + \frac{h}{6}\left(6a + 3bh + 2ch^2\right) - y_0 - \frac{h}{6}\left(6a + 3bh + 3ch^2\right) \\
&= \frac{h}{6}\left(6a + 3bh + 2ch^2 - 6a - 3bh - 3ch^2\right) \\
&= \frac{h}{6}\left(-ch^2\right) \\
&= \boxed{-\frac{ch^3}{6}}
\end{aligned}
$$