Second Order Differential Equations Tutorial

Problem 8.

Solution.

The damped vibrating has equation

$$m\ddot{y} + \lambda\dot{y} + ky = 0 \tag{1}$$

with $m = 1$, $k = 25$ and $\lambda = 10$. It has the characteristic equation

$$s^2 + 10s + 25 = 0 \tag{2}$$

with characteristic root

$$s = -5 \tag{3}$$

The solution is in the form of

$$\boxed{y(t) = e^{-5t}(c_1 + c_2 t)} \tag{4}$$

with initial conditions $y(0) = 0$ and $\dot{y}(0) = 0$.

The initial conditions are satisfied when $c_1 = 1$ and $c_2 = 5$.

$$y(t) = e^{-5t}(1 + 5t) \tag{5}$$

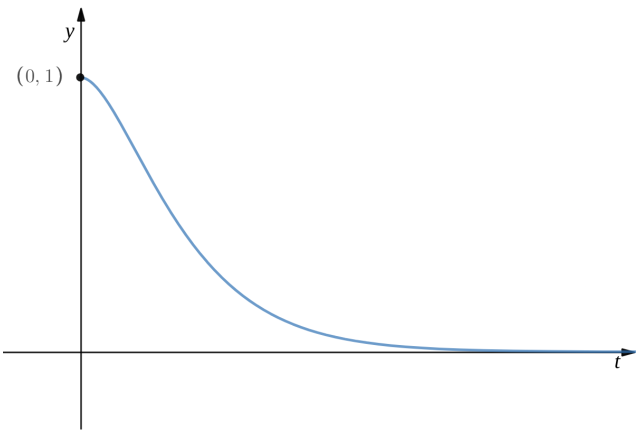

Figure 1: Graph of the equation of motion.

This motion is suitable to be used to close a door because the door slows down as it approaches the door.

Problem 9.

Solution.

The tip of the tuning fork has equation

$$m\ddot{x} + k\dot{x} + m\omega^2 x = 0 \tag{6}$$

with $m > 0$, $k > 0$, $\omega > 0$ and $k^2 \approx 0$. It has the characteristic equation

$$ms^2 + ks + m\omega^2 = 0 \tag{7}$$

with characteristic root

$$s = \frac{-k \pm \sqrt{k^2 - 4m^2\omega^2}}{2m} \approx \frac{-k \pm \sqrt{-(2m\omega)^2}}{2m} = -\frac{k}{2m} \pm i\omega \tag{8}$$

The solution is in the form of

$$x(t) = e^{-\frac{k}{2m}t}\left(c_1 \cos(\omega t) + c_2 \sin(\omega t)\right) \tag{9}$$

with initial conditions $x(0) = 0$ and $\dot{x}(0) = v$.

The initial conditions are satisfied when $c_1 = 0$ and $c_2 = \frac{v}{\omega}$.

$$\boxed{x(t) = \frac{v}{\omega} e^{-\frac{k}{2m}t} \sin(\omega t)} \tag{10}$$

The period of the vibrations over time is constant, given by

$$\Delta t = \frac{2\pi}{\omega} \tag{11}$$

Let the time at the n^{th} time when the vibrations is at their maximum displacement be t_n. The time at the first amplitude is

$$t_1 = \frac{\pi}{2\omega} \tag{12}$$

which follows

$$t_n = t_1 + n\Delta t = \frac{\pi}{2\omega} + n\frac{2\pi}{\omega} \tag{13}$$

Let the n^{th} amplitude of successive vibrations be A_n, such that

$$A_n = \frac{v}{\omega} e^{-\frac{k}{2m}t_n} = \frac{v}{\omega} e^{-\frac{k}{2m}\left(\frac{\pi}{2\omega} + n\frac{2\pi}{\omega}\right)} = \frac{v}{\omega} e^{-\frac{k\pi}{4m\omega}} \left(e^{-\frac{k\pi}{m\omega}}\right)^n \tag{14}$$

Since A_n is in the form of ar^n, it follows a geometric progression.

When $k^2 > 4m^2\omega^2$, $k^2 - 4m^2\omega^2 > 0$. This means that the characteristic roots are real and distinct, resulting in an overdamping case. In this case, x approaches 0 slowly as time progresses, having a horizontal asymptote of $x = 0$.

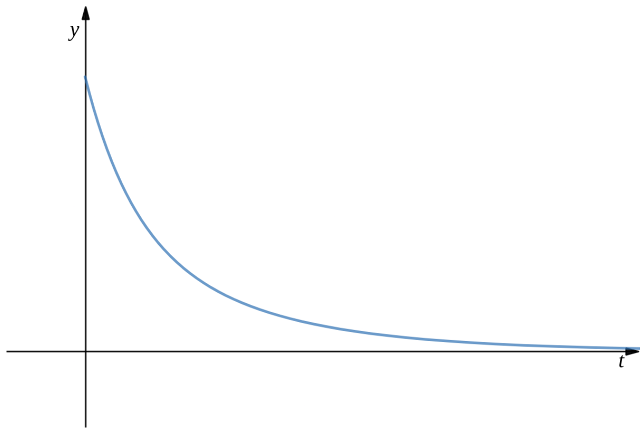

Figure 2: Possible graph of x vs t.