Proposition 0.1. *Soient* $F(t) = \sum_{n\geq 0} F_n t^n$ *la fonction génératrice ordinaire des nombres de Fine. On convient que* $F_0 = 1$*. On a :*

$$F(t) = \cfrac{1}{1 - \cfrac{t^2}{1 - 2t - \cfrac{t^2}{\ddots}}}$$

<u>Démonstration</u> : Posons $\mathcal{F} = \{c \in \Gamma_n : \text{c ne contient ni palier rouge ni bleu de niveau } 0\}$ et $\widetilde{HL}(n) = \{(c,p) \in HL(n) : c \in \mathcal{F}\}$. $\exists! \widetilde{S}_n, \widetilde{S}_n \subset S_n$, tel que $\psi_{F.V}^{-1}(\widetilde{HL}(n)) = \widetilde{S}_n$ car $\psi_{F.V}$ est une bijection.
Posons maintenant $\widetilde{S}_n(c) = \{\sigma \in \widetilde{S}_n : \psi_{F.V}(\sigma) = (c,p) \in \widetilde{HL}(n)\}$. On a :

$$|\widetilde{S}_n(c)| = \prod_{c_i=m} m_{\gamma_{i-1}} \prod_{c_i=d} d_{\gamma_{i-1}} \prod_{c_i=b} b_{\gamma_{i-1}} \prod_{c_i=r} r_{\gamma_{i-1}} = w(c) \text{ où } w(c) = 0 \text{ si } c \notin \mathcal{F}$$

Alors,

$$\sum_{c\in\mathcal{F}} |\widetilde{S}_n(c)| = \sum_{c\in\mathcal{F}} w(c) = \sum_{c\in\Gamma_n} w(c) = |\widetilde{S}_n|$$

De plus, $F_n = |\mathcal{F}| = \sum_{c\in\mathcal{F}} 1$. On a l'équivalence suivante :

$$\begin{aligned} \sum_{c\in\mathcal{F}} 1 = \sum_{c\in\mathcal{F}} |\widetilde{S}_n(c)| \quad &\Longleftrightarrow \quad |\widetilde{S}_n(c)| = 1 \\ &\Longleftrightarrow \quad \left\{ \begin{array}{lcl} m_{\gamma_{i-1}} & = & d_{\gamma_{i-1}} = 1 \\ & et & \\ b_{\gamma_{i-1}} & = & r_{\gamma_{i-1}} = \left\{ \begin{array}{l} 1 \text{ si } \gamma_{i-1} \neq 0 \\ 0 \text{ sinon} \end{array} \right. \end{array} \right. \end{aligned}$$

Ainsi, $1 + \sum_{n\geq 1} F_n t^n = 1 + \sum_{n\geq 1} |\widetilde{S}_n| t^n = \cfrac{1}{1 - \cfrac{t^2}{1 - 2t - \cfrac{t^2}{\ddots}}}$