

GATE:2022 - CE 48

EE23BTECH11025 - Anantha Krishnan

I. QUESTION

Consider the differential equation

$$\frac{dy}{dx} = 4(x + 2) - y$$

For the initial condition $y = 3$ at $x = 1$, the value of y at $x = 1.4$ obtained using Euler's method with a step-size of 0.2 is ? (round off to one decimal place) (GATE CE 2022)

Solutions :

| Symbols | Description | Values |
|-----------|---|--|
| R | Residue Formula | $\frac{1}{(m-1)!} \lim_{s \rightarrow a} \frac{d^{m-1}}{ds^{m-1}} ((s-a)^m f(s) e^{st})$ |
| $\phi(x)$ | Transformation of $y(x)$ | $y(x-1)$ |
| $g(x)$ | Euler's Approximated function of $f(x)$ | $g_{(n-1)}(x) + hf'(x_{n-1}, y_{n-1})$ |
| h | Step-size | 0.2 |

TABLE I
PARAMETERS, DESCRIPTIONS, AND VALUES

1) Solution of the differential: Applying the transformation from table I and laplace transform

$$s\mathcal{L}(\phi(x)) - \phi(0) = 4\left(\frac{1}{s^2} + \frac{2}{s}\right) - \mathcal{L}(\phi(x)) \quad (1)$$

$$\mathcal{L}(\phi(x)) = \frac{3}{s+1} + 4\left(\frac{1}{s^2(s+1)} + \frac{2}{s(s+1)}\right) \quad (2)$$

2) Inverse Laplace Transform: Using Bromwich integrals and extension of Jordans lemma :

$$\phi(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \mathcal{L}(\phi(x)) e^{st} dt, c > 0 \quad (3)$$

$$= \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \left(\frac{3}{s+1} + 4\left(\frac{1}{s^2(s+1)} + \frac{2}{s(s+1)}\right) \right) e^{st} dt \quad (4)$$

$$= \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \left(\frac{-1}{s+1} + \frac{4}{s} + \frac{4}{s^2} \right) e^{st} dt \quad (5)$$

Here, the poles $s = -1$ (non repeated, $m = 1$) and $s = 0$ (repeated, $m = 2$) lie inside a semicircle for some $c > 0$. Using method of residues from I:

$$R_1 = \lim_{s \rightarrow -1} (s+1) \left(\frac{-1}{s+1} \right) e^{st} \quad (6)$$

$$= -e^{-t} \quad (7)$$

$$R_2 = \lim_{s \rightarrow 0} \left(s \left(\frac{4}{s} \right) e^{st} \right) \quad (8)$$

$$= 4 \quad (9)$$

$$R_3 = \frac{1}{(1)!} \lim_{s \rightarrow 0} \frac{d}{dz} \left((s)^2 \left(\frac{4}{s^2} \right) e^{st} \right) \quad (10)$$

$$= 4t \quad (11)$$

$$\phi(t) = R_1 + R_2 + R_3 \quad (12)$$

$$= -e^{-t} + 4 + 4t \quad (13)$$

Reverting to $y(x)$, we get :

$$y(x) = -e^{-t+1} + 4 + 4(t-1) \quad (14)$$

Now, approaching to $y(1.4)$ using euler's approximation. Let the approximated function be $g(x)$

$$g_{(1.2)} = 3 + (0.2) f'(1, 3) \quad (15)$$

$$= 4.8 \quad (16)$$

$$g_{(1.4)} = 4.8 + (0.2) f'(1.2, 4.8) \quad (17)$$

$$= 6.4 \quad (18)$$

$$\Rightarrow y(1.4) \approx 6.4 \quad (19)$$

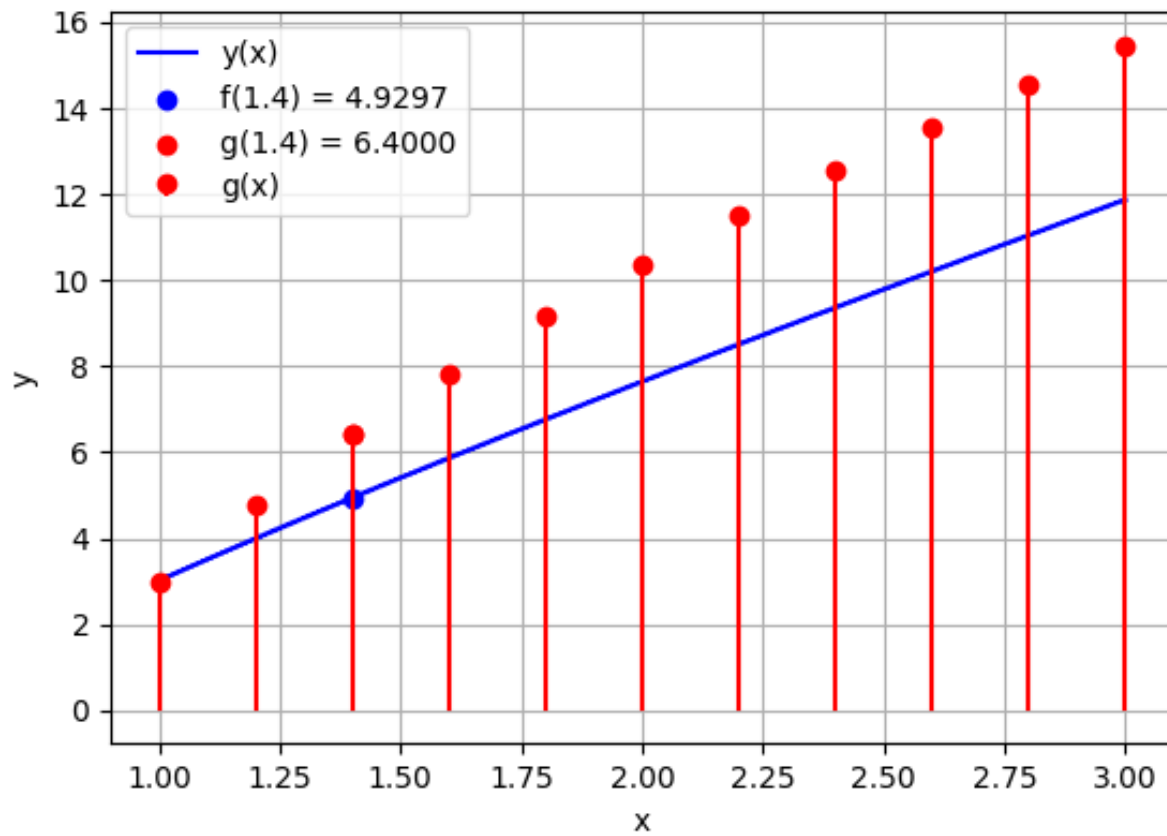


Fig. 1. Euler's approximated function vs Original function