NCERT-discrete : 10.5.3 - 2

EE23BTECH11025 - Anantha Krishnan

I. Question

Find the sums given below:

(i) $7 + 10\frac{1}{2} + 14 ... + 84$
(ii) 34 + 32 + 30 ... + 10
(iii) -5 + -8 + -11 ... -230

Solutions:

(i) $7 + 10\frac{1}{2} + 14 ... + 84$

Let $S_i(n)$ denote the sum of first n+1 terms in the i^{th} series , $x_i(0)$ denotes its first term , d_i denotes the common difference and $u_{(n)}$ denote the unit step function.

$$S_n = \frac{(n+1)u_{(n)}}{2}(2x_i(0) + (n)d_i) \tag{1}$$

For number of terms , we use

$$x_i(n) = (x_i(0) + nd_i)u_n \tag{2}$$

Where $x_i(n)$ is the $(n+1)^{th}$ term of the series. Putting the values

$$84 = 7 + \frac{7n}{2} \tag{3}$$
$$n = 22 \tag{4}$$

1) **Calculating $S_1(n)$ for $x_1(n)$** : We now use this result for calculating $S_1(23)$

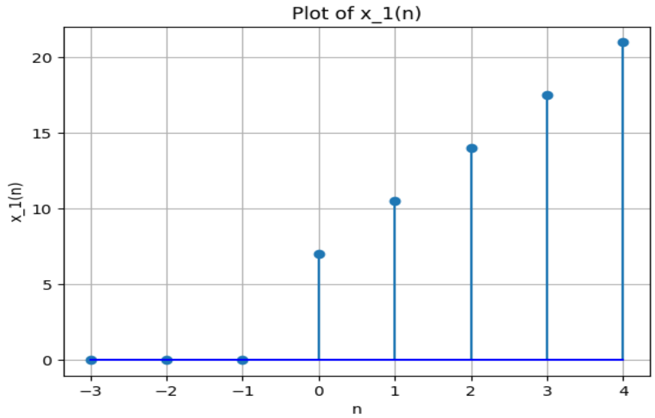
$$S_1(22) = \frac{23}{2}(14 + (22)\frac{7}{2}) \tag{5}$$

Solving this yields $S_1(22) = 1046.5$ We are now required to calculate $X_1(z)$ in terms of u_n and $x_1(n)$.Where $u_{(n)}$ is the unit step function.

$$x_1(n) = u_{(n)}(x_1(0) + \frac{7n}{2}) \tag{6}$$

2) **Z-Transform of $x_1(n)$** : By the Definition of Z-transform:

$$\sum_{n=-\infty}^{\infty} Z^{-n} x_i(n) = X_i(Z) \tag{7}$$



Graph:1 $x_1(n)$ vs n

Putting $x_1(n)$ in (7) , we get

$$\sum_{n=-\infty}^{\infty} (x_1(0) + \frac{7n}{2})u_{(n)}Z^{-n} = X_1(z) \tag{8}$$

$$\sum_{n=-\infty}^{\infty} (7 + \frac{7n}{2})u_{(n)}Z^{-n} = X_1(z) \tag{9}$$

$$7z(z-1)^{-1} + 7z(2(z-1))^{-2} = X_1(z)$$

3) **Region of Convergence of $X_1(z)$**

$$|z| < 1 \tag{10}$$

4) **Expressing $S_1(n)$ as a convolution** : Using (2) and assuming

$$h[n] = u[n] \tag{11}$$
$$S_1(n) = x_1(n) * h[n] \tag{12}$$

Putting in the equations

$$S_1(n) = \sum_{k=0}^{n} (x_1(0) + kd_1)(u[k])(u[n-k])$$
$$S_1(n) = 7(n+1) + (1.75)(n+1)(n) \tag{13}$$
$$S_1(n) = 7 + 8.75n + 1.75n^2 \tag{14}$$

5) **Using Z-Transform for** $S_1(n)$:

$$S_1(z) = X_1(z) * H_(z) \tag{15}$$

Where $X_1(z)$ comes from (10) For $H_(z)$, it is Z-transform of unit-step function

$$H_1(z) = z(z-1)^{-1} \tag{16}$$

For $S_1(z)$:

$$S_1(z) = (7z(z-1)^{-1} + 7z(2(z-1))^{-2})z(z-1)^{-1} \tag{17}$$

ROC:

$$|z| > 1 \tag{18}$$

6) **Inversion of** $S_1(z)$: By using partial fractions :

$$S_1(z) = (7z^2(z-1)^{-2} + 7z^2(2(z-1))^{-3})$$

Using known results:
Inverse Z-transform of

$$z^2(z-1)^{-2} \leftrightarrow (n+1)u(n) \tag{19}$$

For $z^2(z-1)^{-3}$

we can differentiate (19) and get the inverse Z-transform as

$$z^2(z-1)^{-3} \leftrightarrow (n(n+1)/2)u(n) \tag{20}$$

Therefore:

$$S_1(n) = 7(n+1) + 1.75n(n+1) \tag{21}$$

(ii) 34 + 32 + 30 ... + 10

In this bit $x_2(0)$ = 34 , d_2 = -2.

Using equation (2)

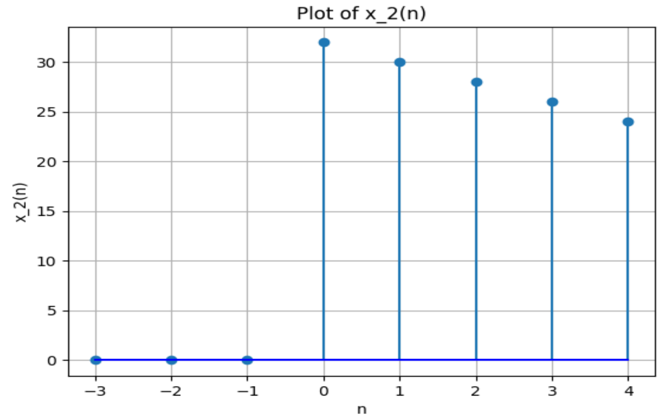
$$10 = 34 - 2n \tag{22}$$
$$n = 12 \tag{23}$$

For $x_2(n)$

$$x_2(n) = x_2(0) + nd_2 \tag{24}$$
$$x_2(n) = x_2(0) - 2n \tag{25}$$

1) **Z-Transform of** $x_2(n)$: Using (7)

$$\sum_{n=-\infty}^{\infty} (x_2(0) - 2n)u_{(n)}Z^{-n} = X_2(z) \tag{26}$$



Graph:2 $x_2(n)$ vs n

For $X_2(z)$

$$34z(z-1)^{-1} - 2z((z-1))^{-2} = X_2(z) \tag{27}$$

2) **Calculating** $S_2(n)$ **of** $x_2(n)$: For calculating the sum , we use (1)

$$S_2(12) = \frac{13}{2}(64 + 11(-2)) \tag{28}$$
$$S_2(12) = 286. \tag{29}$$

3) **Region of Convergence**

$$|z| < 1 \tag{30}$$

4) **Expressing** $S_2(n)$ **as a convolution** : Using (2) and assuming

$$h[n] = u[n] \tag{31}$$
$$S_2(n) = x_2(n) * h[n] \tag{32}$$

Putting in the equations

$$S_2(n) = \sum_{k=0}^{n} (x_2(0) + kd_2)(u[k])(u[n-k])$$
$$S_2(n) = 34(n+1) - (n+1)(n) \tag{33}$$
$$S_2(n) = 34 + 33n - n^2 \tag{34}$$

5) **Using Z-Transform for** $S_2(n)$:

$$S_2(z) = X_2(z) * H_(z) \tag{35}$$

Where $X_2(z)$ comes from (27) and $H_(z)$ from (16). For $S_2(z)$:

$$S_2(z) = (34z(z-1)^{-1} - 2z((z-1))^{-2})z(z-1)^{-1} \tag{36}$$

ROC:

$$|z| > 1 \tag{37}$$

6) **Inversion of $S_2(z)$** : By using partial fractions :

$$S_2(z) = (34z^2(z-1)^{-2} - 2z^2((z-1))^{-3})$$

Using results (19) and (20)

$$S_2(n) = (34(n+1) - n(n+1))u(n) \tag{38}$$

(iii) -5 + -8 + -11 ... -230

Here $x_3(0)$ = -5, d_3 = -3 From (2)

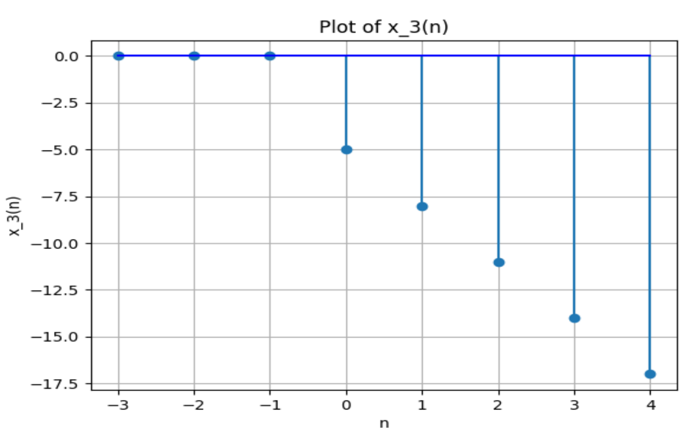
$$-230 = -5 - 3n \tag{39}$$

$$n = 75 \tag{40}$$

For $x_3(n)$

$$x_3(n) = x_3(0) + nd_3 \tag{41}$$

$$x_{3(n)} = x_3(0) - 3n \tag{42}$$



Graph:3 $x_3(n)$ vs n

1) **Z-Transform of $x_3(n)$** : Putting $x_3(n)$ in (7)

$$\sum_{n=-\infty}^{\infty} (x_3(0) - 3n)u_{(n)}Z^{-n} = X_3(z) \tag{43}$$

For $X_3(z)$, we use the same process as in (i) bit

$$-5z(z-1)^{-1} - 5z((z-1))^{-1} - 3z((z-1))^{-2} = X_3(z) \tag{44}$$

2) **Calculating $S_3(n)$ of $x_3(n)$** : Using (1) :

$$S_3(75) = \frac{76}{2}(-10 + (76-1)(-3)) \tag{45}$$

$$S_3(75) = -8930 \tag{46}$$

3) **Region of Convergence**

$$|z| < 1 \tag{47}$$

4) **Expressing $S_3(n)$ as a convolution** : Using (2) and assuming

$$h[n] = u[n] \tag{48}$$

$$S_3(n) = x_3(n) * h[n] \tag{49}$$

Putting in the equations

$$S_3(n) = \sum_{k=0}^{n} (x_3(0) + kd_3)(u[k])(u[n-k])$$

$$S_3(n) = -5(n+1) - (1.5)(n+1)(n) \tag{50}$$

$$S_3(n) = -5 - 6.5n - (1.5)n^2 \tag{51}$$

5) **Using Z-Transform for $S_3(n)$** :

$$S_3(z) = X_3(z) * H_(z) \tag{52}$$

Where $X_3(z)$ comes from (44) and $H_(z)$ from (16). For $S_3(z)$:

$$S_3(z) = (-5z(z-1)^{-1} - (1.5)z((z-1))^{-2})z(z-1)^{-1} \tag{53}$$

ROC:

$$|z| > 1 \tag{54}$$

6) **Inversion of $S_2(z)$** : By using partial fractions :

$$S_3(z) = (-5z^2(z-1)^{-2} - (1.5)z^2((z-1))^{-3})$$

Using results (19) and (20)

$$S_3(n) = (-5(n+1) - (1.5)n(n+1))u(n) \tag{55}$$

Symbols	Description	Values
d_i	Common Difference	3.5, -2, -3
$x_i(n)$	Sequence	$(x_i(0) + nd_i)u_{(k)}$
$X_i(z)$	Z-Transform of $x_i(n)$	$zx_i(0)(z-1)^{-1} + d_iz(z-1)^{-2}$
$S_i(n)$	Sum of (n+1)terms in i^{th} series	$\frac{(n+1)u_{(u)}}{2}(2x_i(0) + kd_i)$
$h[n]$	Unit step function	$0\ \forall n < 0, 1 \forall n \geq 0$
$H(z)$	Z-Transform of $h[n]$	$z(z-1)^{-1}$
$S_i(z)$	Z-Transform of $S_i(z)$	$X_i(z)*H(z)$

Table 1 : Parameters , Descriptions AND Values