Audio Filter Assignment

EE23BTECH11025 - ANANTHA KRISHNAN*

I. Digital Filter

I.1 Download the sound file used for this code from the below link

link here

I.2 Listed below is the python code for removal of out of band noise.

```
import soundfile as sf
import numpy as np
from scipy import signal
#read .wav file
input_signal,fs = sf.read('Ananth-singing.wav')

#sampling frequency of Input signal
sampl_freq=fs

#order of the filter
order=4

#cutoff frquency
cutoff_freq=1000.0

#digital frequency
Wn=2*cutoff_freq/sampl_freq

# b and a are numerator and denominator polynomials respectively
b, a = signal.butter(order, Wn, 'low')
#print(fs)

#filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a, input_signal,padlen=1)
#output_signal = signal.lfilter(b, a, input_signal)

#write the output signal into .wav file
sf.write('Sound_With_ReducedNoise.wav', output_signal, fs)
```

I.3 The audio file is now analyzed using a spectrogram from the website https://academo.org/demos/spectrum-analyzer.

Dark lines in a spectrogram often indicate areas of low intensity and bright lines represent areas of high intensity in the signal.
In the filtered image, there is a higher concentration of bright lines within the cutoff frequency($f_c = 1kHz$)

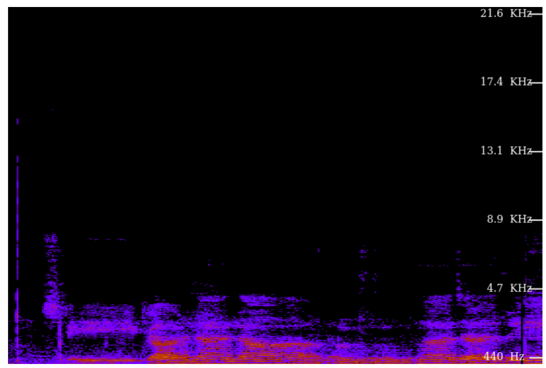

Fig. 1. Spectrogram of the audio file pre Filtering

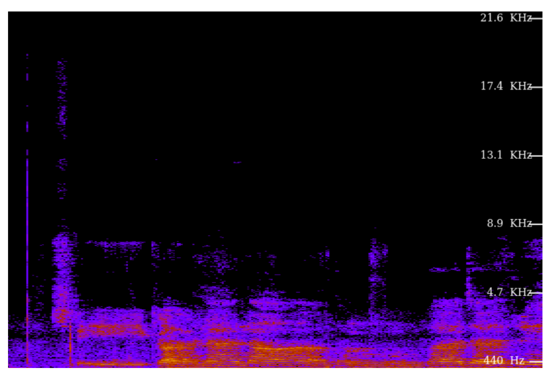

Fig. 2. Spectrogram of the audio file post Filtering

II. Difference Equation

II.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \quad (1)$$

Sketch $x(n)$.

II.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \quad (2)$$

Sketch $y(n)$.
Solve
Solution: The below C code computes the values of $y(n)$ till $n = 20$

link

The following code plots x[n] and $y[n]$

link

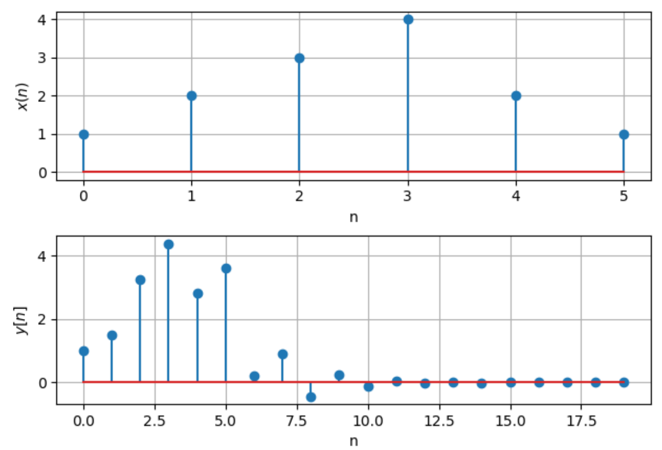

Fig. 3. Plot of $x(n)$ and $y(n)$

III. Z-Transform

III.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (3)$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \quad (4)$$

and find

$$\mathcal{Z}\{x(n-k)\} \quad (5)$$

Solution: From (3),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \quad (6)$$
$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (7)$$

resulting in (4). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \quad (8)$$

III.2 Find

$$H(z) = \frac{Y(z)}{X(z)} \quad (9)$$

from (2) assuming that the Z-transform is a linear operation.
Solution: Applying (8) in (2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \quad (10)$$
$$\implies \frac{Y(z)}{X(z)} = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \quad (11)$$

III.3 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \quad (12)$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (13)$$

is

$$U(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1 \quad (14)$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\longleftrightarrow} 1 \quad (15)$$

and from (13),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \quad (16)$$
$$= \frac{1}{1-z^{-1}}, \quad |z| > 1 \quad (17)$$

using the formula for the sum of an infinite geometric progression.

III.4 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}} \quad |z| > |a| \quad (18)$$

Solution: From (3)

$$a^n u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \sum_{n=0}^{\infty} \left(az^{-1}\right)^n \quad (19)$$
$$= \frac{1}{1-az^{-1}} \quad \forall\, |z| > |a| \quad (20)$$

III.5 Let
$$H\left(e^{j\omega}\right) = H\left(z = e^{j\omega}\right). \tag{21}$$
Plot $|H\left(e^{j\omega}\right)|$. Comment. $H(e^{j\omega})$ is known as the *Discrete Time Fourier Transform* (DTFT) of $x(n)$.
Solution: The below code plots $|H(e^{j\omega})|$.

link

Substituting $z = e^{j\omega}$ in (11), we get
$$|H\left(e^{j\omega}\right)| = \left|\frac{1+e^{-2j\omega}}{1+\frac{1}{2}e^{-j\omega}}\right| \tag{22}$$
$$= \sqrt{\frac{(1+\cos(2\omega))^2 + (\sin(2\omega))^2}{\left(1+\frac{1}{2}\cos(\omega)\right)^2 + \left(\frac{1}{2}\sin(\omega)\right)^2}} \tag{23}$$
$$= \frac{4|\cos(\omega)|}{\sqrt{5+4\cos(\omega)}} \tag{24}$$
$$\left|H\left(e^{j(\omega+2\pi)}\right)\right| = \frac{4|\cos(\omega+2\pi)|}{\sqrt{5+4\cos(\omega+2\pi)}} \tag{25}$$
$$= \frac{4|\cos(\omega)|}{\sqrt{5+4\cos(\omega)}} \tag{26}$$
$$= \left|H\left(e^{(j\omega)}\right)\right| \tag{27}$$
Its fundamental period is 2π , which verifies that the DTFT of a signal is always periodic.

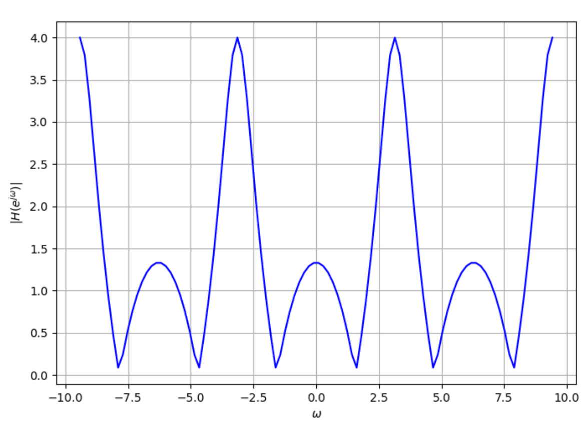

Fig. 4. $\left|H\left(e^{j\omega}\right)\right|$

IV. Impulse Response

IV.1 Find an expression for $h(n)$ using $H(z)$, given that
$$h(n) \overset{\mathcal{Z}}{\longleftrightarrow} H(z) \tag{28}$$
and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (2).
Solution: From (11),
$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2} \tag{29}$$
$$\implies h(n) = \left(-\frac{1}{2}\right)^n (u(n)) + \left(-\frac{1}{2}\right)^{n-2} (u(n-2)) \tag{30}$$
using (18) and (8).

IV.2 Sketch $h(n)$. Is it bounded? Convergent?
Solution: The following code plots $h(n)$

link

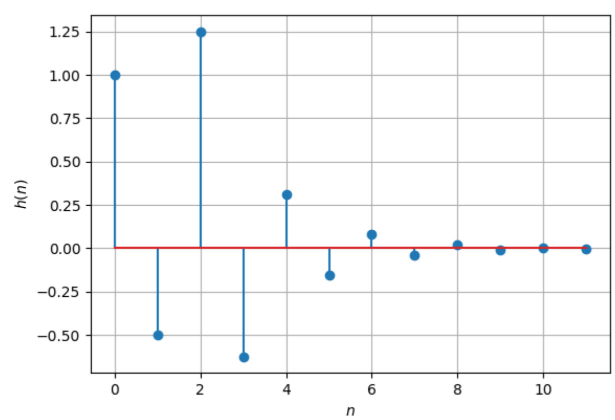

Fig. 5. $h(n)$

IV.3 The system with $h(n)$ is defined to be stable if
$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{31}$$
Is the system defined by (2) stable for the impulse response in (28)?
Solution: For a stable system, the R.O.C of $H(z)$ should contain the boundary of unit circle $|z| = 1$.From (29)
$$|z| = 1 \subset |z| > \frac{1}{2} \tag{32}$$
Therefore it converges and hence stable.

IV.4 Compute and sketch $h(n)$ using
$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{33}$$

This is the definition of $h(n)$.
Solution:
Below is the plot of $h(n)$ using the difference equations. 5.

link 4.2.py

Fig. 6. $h(n)$ from definition is same as Fig. 5

IV.5 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \tag{34}$$

Comment. The operation in (34) is known as *convolution*.
Solution: The following code plots Fig. 7 using convolution. 3.

link

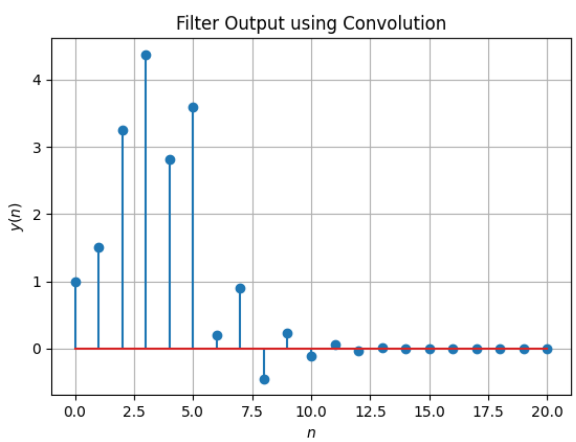

Fig. 7. $y(n)$ from the definition of convolution

IV.6 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{35}$$

Solution: In (34),substituting $k \to n-k$, we get

$$y(n) = \sum_{n-k=-\infty}^{\infty} x(n-k)\, h(k) \tag{36}$$

$$= \sum_{k=-\infty}^{\infty} x(n-k)\, h(k) \tag{37}$$

V. DFT and FFT

V.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{38}$$

and $H(k)$ using $h(n)$.

V.2 Compute

$$Y(k) = X(k)H(k) \tag{39}$$

V.3 Compute

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{\jmath 2\pi kn/N}, \quad n = 0, 1, \ldots, N-1 \tag{40}$$

Solution: The above three questions are solved using the following code :

link 5_sol.py

V.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
Solution: The solution of this question can be found in the code below.

link 5.4.py

This code verifies the result by plotting the obtained result with the result obtained previously.

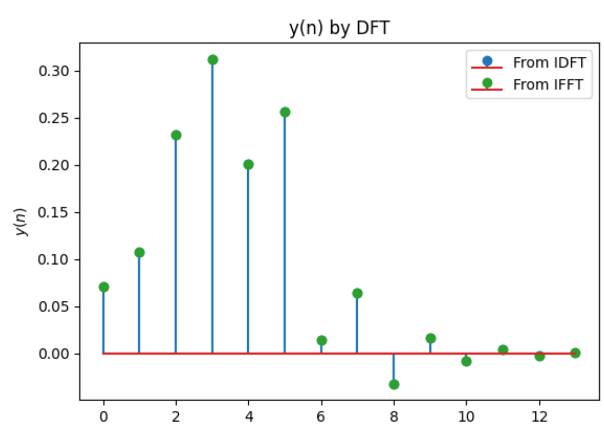

Fig. 8. $y(n)$ obtained from IDFT and IFFT is plotted and verified

V.5 Wherever possible, express all the above equations as matrix equations.
Solution: The DFT matrix is defined as :

$$\mathbf{W} = \begin{pmatrix} \omega^0 & \omega^0 & \dots & \omega^0 \\ \omega^0 & \omega^1 & \dots & \omega^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^0 & \omega^{N-1} & \dots & \omega^{(N-1)(N-1)} \end{pmatrix} \tag{41}$$

where $\omega = e^{-\frac{j2\pi}{N}}$. Now any DFT equation can be written as

$$\mathbf{X} = \mathbf{W}\mathbf{x} \tag{42}$$

where

$$\mathbf{x} = \begin{pmatrix} x(0) \\ x(1) \\ \vdots \\ x(n-1) \end{pmatrix} \tag{43}$$

$$\mathbf{X} = \begin{pmatrix} X(0) \\ X(1) \\ \vdots \\ X(n-1) \end{pmatrix} \tag{44}$$

Thus we can rewrite (39) as:

$$\mathbf{Y} = \mathbf{X} \odot \mathbf{H} = (\mathbf{W}\mathbf{x}) \odot (\mathbf{W}\mathbf{h}) \tag{45}$$

where the $\odot$ represents the Hadamard product which performs element-wise multiplication.

The below code computes $y(n)$ by DFT Matrix and then plots it.

link 5.5.py

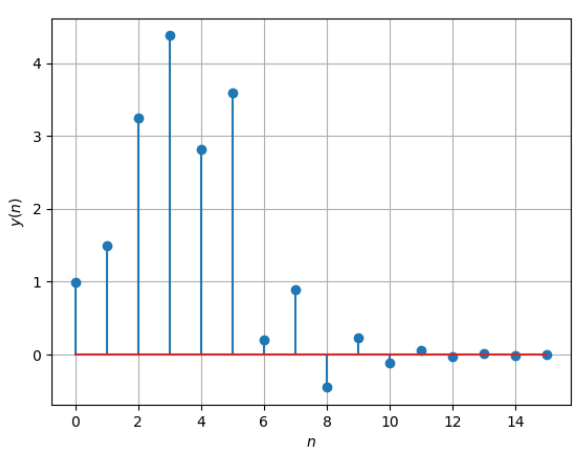

Fig. 9. $y(n)$ obtained from DFT through matrix computations

VI. EXERCISES

Answer the following questions by looking at the python code in Problem I.2.

VI.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem I.2 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \tag{46}$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal. filtfilt** with your own routine and verify.

Solution: The below code gives the output of an Audio Filter without using the built in function signal.lfilter(Slightly modified audio file here).

link 6.1.py

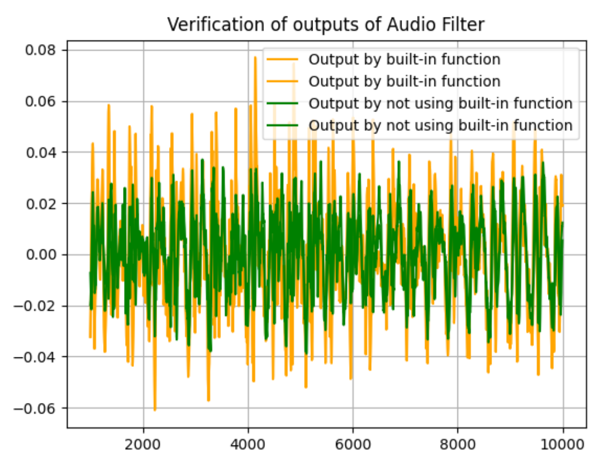

Fig. 10. Both the outputs using and without using function overlap

VI.2 Repeat all the exercises in the previous sections for the above a and b.
Solution: The code in I.2 generates the values of a and b which can be used to generate a difference equation.
And,

$$M = 5 \tag{47}$$

$$N = 5 \tag{48}$$

From 46

$$a(0)\,y(n) + a(1)\,y(n-1) + a(2)\,y(n-2) + a(3) \tag{49}$$

$$y(n-3) + a(4)\,y(n-4) = b(0)\,x(n) + b(1)\,x(n-1) + b(2)\,x(n-2) + b(3)\,x(n-3) + b(4)\,x(n-4)$$

Difference Equation is given by :

$$\begin{aligned} &y(n) - (3.658)\,y(n-1) + (5.0314)\,y(n-2) \\ &- (3.083)\,y(n-3) + (0.710)\,y(n-4) \\ &= \left(1.555 \times 10^{-5}\right) x(n) + \left(6.220 \times 10^{-5}\right) x(n-1) \\ &+ \left(9.331 \times 10^{-5}\right) x(n-2) + \left(6.220 \times 10^{-5}\right) x(n-3) \\ &+ \left(1.555 \times 10^{-5}\right) x(n-4) \end{aligned} \tag{50}$$

From (46)

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-N}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-M}} \tag{51}$$

$$H(z) = \frac{\sum_{k=0}^{N} b(k) z^{-k}}{\sum_{k=0}^{M} a(k) z^{-k}} \tag{52}$$

Partial fraction on (52) can be generalised as:

$$H(z) = \sum_i \frac{r(i)}{1 - p(i) z^{-1}} + \sum_j k(j) z^{-j} \tag{53}$$

Now,

$$a^n u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - a z^{-1}} \tag{54}$$

$$\delta(n-k) \overset{\mathcal{Z}}{\longleftrightarrow} z^{-k} \tag{55}$$

Taking inverse z transform of (53) by using (54) and (55)

$$h(n) = \sum_i r(i)[p(i)]^n u(n) + \sum_j k(j)\delta(n-j) \tag{56}$$

The below code computes the values of $r(i), p(i), k(i)$ and plots $h(n)$

link 6.2.py

$r(i)$	$p(i)$	$k(i)$
$0.06029142 - 0.14682007j$	0.88475217+0.0445749j	$2.19006287e-05$
$0.06029142 + 0.14682007j$	0.88475217-0.0445749j	–
$-0.06029459 + 0.02518904j$	0.94427798+0.11485352j	–
$-0.06029459 - 0.02518904j$	0.94427798-0.11485352j	–

TABLE I
VALUES OF $r(i), p(i), k(i)$

Stability of h(n):
According to (31)

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} \tag{57}$$

$$H(1) = \sum_{n=0}^{\infty} h(n) = \frac{\sum_{k=0}^{N} b(k)}{\sum_{k=0}^{M} a(k)} < \infty \tag{58}$$

As both $a(k)$ and $b(k)$ are finite length sequences they converge.
The below code plots Filter frequency response

link 6_filter_response.py

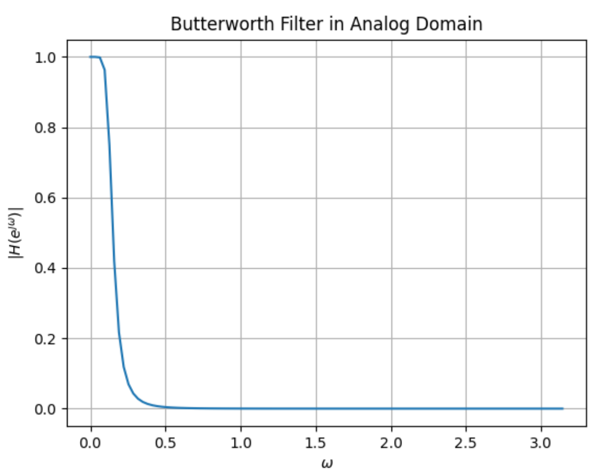

Fig. 11. Frequency Response of Audio Filter

The below code plots the Butterworth Filter in analog domain by using bilinear transform.

$$z = \frac{1 + sT/2}{1 - sT/2} \tag{59}$$

link analog_filt.py

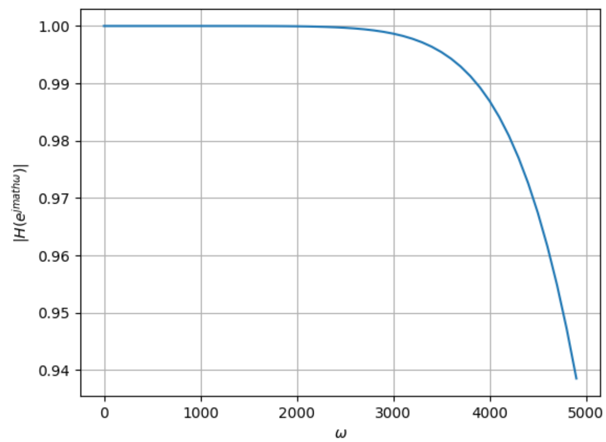

Fig. 12. Butterworth Filter Frequency response in analog domain

The below code plots the Pole-Zero Plot of the frequency response.

link pole zero

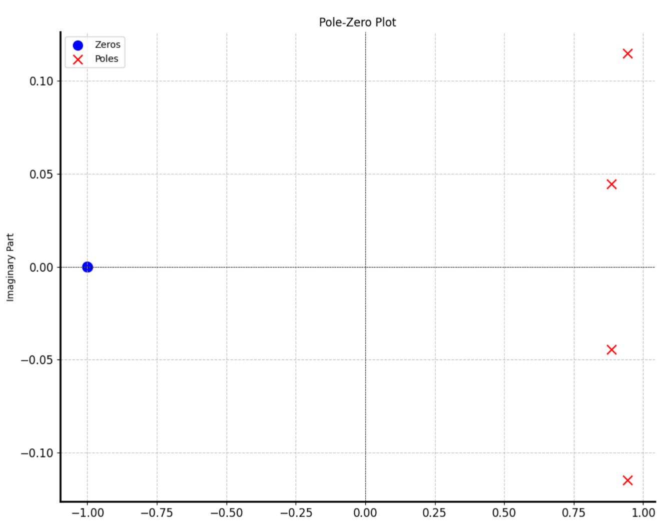

Fig. 13. There are complex poles. So $h(n)$ should be damped sinusoid.

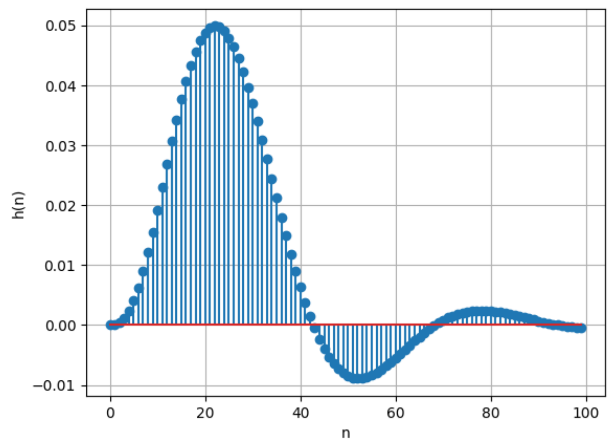

Fig. 14. h(n) of Audio Filter.It is a damped sinusoid.

VI.3 What is the sampling frequency of the input signal?
Solution: The Sampling Frequency is 48KHz

VI.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low-pass with order=4 and cutoff-frequency=1kHz.

VI.5 Modify the code with different input parameters and get the best possible output.
Solution: A better filtering was found on setting the order of the filter to be 5.