NCERT-discrete : 10.5.3 - 2

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I. Question

Find the sums given below:

(i) 7 + $10\frac{1}{2}$ + 14 ... + 84
(ii) 34 + 32 + 30 ... + 10
(iii) -5 + -8 + -11 ... -230

Solutions:

(i) By observing the consecutive common differences in the given series , we observe that it is a constant value , which is $\frac{7}{2}$.
Since this an arithmetic progression , we can use the formula which dictates the sum of "n" terms of such a series
Let "S_n" denote the sum of n terms in a series ,"a" denotes its first term and "d" denotes the common difference.It is known that

$$S_n = \frac{n}{2}(2a + (n - 1)d) \tag{1}$$

In the question, a=7 and d=$\frac{7}{2}$, and "n" is unknown
For calculating the number of terms, we use the formula

$$T_n = a + (n - 1)d \tag{2}$$

Where T_n is the nth term of the series
Given that T_n is 84, we solve for "n"

$$84 = 7 + (n - 1)\frac{7}{2} \tag{3}$$

Solving this yields n=23.
We now use this result for calculating S_{23}

$$S_{23} = \frac{23}{2}(14 + (22)\frac{7}{2}) \tag{4}$$

Again, solving this yields S_{23} as 1046.5

(ii) Based on the analysis of the previous bit, we observe that in this bit
a=34 , d=-2 For calculating the number of terms, we use the formula ((2)
Substituting the values, we get

$$10 = 34 + (n - 1)(-2) \tag{5}$$

Solving this yields n=13
For calculating the sum , we use (1)

$$S_{13} = \frac{13}{2}(64 + 11(-2)) \tag{6}$$

Solving this, we get S_n= 286.

(iii) By using the previous analysis, we can conclude that a=-5, d=-3
Again, for n , we use the formula (2)

$$-230 = -5 + (n - 1)(-3) \tag{7}$$

Solving this yields n=76
Now, for the sum we use equation (1) :

$$S_{76} = \frac{76}{2}(-10 + (76 - 1)(-3)) \tag{8}$$

Solving this we obtain S_{76}=-8930.

Extension - 1: It is told to replace S_n with x(n) and therefore calculate X(Z). The equation (1) and calculate the general form of X(Z) . It is known that

$$\sum_{n=-\infty}^{\infty} Z^{-n}x(n) = X(Z) \tag{9}$$

For nϵ[$-\infty, 0$]
The summation is 0 in the above question. So we can start the sum from n=1 and so on... Putting the equation (1) in (9)

$$\sum_{n=1}^{\infty} Z^{-n}\frac{n}{2}(2a + (n - 1)d) = X(Z) \tag{10}$$

Writing another equation by multiplying (10) with Z, we get

$$\sum_{n=1}^{\infty} Z^{-n+1}\frac{n}{2}(2a+(n-1)d) = ZX(Z) \qquad (11)$$

Now we subtract (10) from (11) by displacing it with one term , i.e we subtract the first term of (10) from the second term of (11) and so on.
By simplifying this, we get

$$\sum_{n=0}^{\infty}(a-\frac{d}{2})Z^{-n} + \frac{n}{2}dZ^{-n} = ZX(Z) - X(Z) \qquad (12)$$

The first part is a GP and the second part is an AGP. Since the nature of Z is not known, we calculate the general sum for finite terms(Assuming k) and then use it for each bit.
Calculating the individual sums, we get

$$\frac{(a-\frac{d}{2})Z^{1-k}\frac{(Z^k-1)}{Z-1} + \frac{d}{2Z^{k-2}}\frac{Z^{k-1}-1}{2(Z-1)^2} - \frac{d(k-1)}{2(Z-1)Z^{k-1}}}{Z-1} \qquad (13)$$
$$= X(Z)$$

Now if we put the values of (i),(ii) and (iii) , we get the respective X(Z) as :

1) $$\frac{(\frac{21}{4})Z^{-22}\frac{(Z^{23}-1)}{Z-1} + \frac{7}{4Z^{21}}\frac{Z^{22}-1}{2(Z-1)^2} - \frac{7(22)}{4(Z-1)Z^{22}}}{Z-1}$$

2) $$\frac{35Z^{-12}\frac{(Z^{13}-1)}{Z-1} - \frac{1}{Z^{11}}\frac{Z^{12}-1}{2(Z-1)^2} + \frac{2(12)}{(Z-1)Z^{12}}}{Z-1}$$

3) $$\frac{(\frac{7}{2})Z^{-75}\frac{(Z^{76}-1)}{Z-1} - \frac{-3}{2Z^{74}}\frac{Z^{75}-1}{2(Z-1)^2} + \frac{-3(75)}{2(Z-1)Z^{75}}}{Z-1}$$

Extension - 2 :

For the n^{th} term of (i) bit , we are now required to calculate X(z) given that n^{th} term is x(n)
x(n) = 7 + (n-1) $\frac{7}{2}$

Now , putting this x(n) in (9) , we obtain (Starting n from 0 as it cannot be negative for a series)

$$\sum_{n=1}^{\infty}(7+(n-1)\frac{7}{2})Z^{-n} = X(z) \qquad (14)$$

$$\sum_{n=1}^{\infty}(\frac{7}{2}+\frac{7n}{2})Z^{-n} = X(z) \qquad (15)$$

For the **region of convergence**
X(z) = convergent $\forall$ z ϵ(-∞, $-1)U(1,\infty)$

This can be proved from ratio test .
We now calculate the sum using the formula for gp and agp(Assuming k terms), we obtain.

$$\frac{7}{2}\frac{1-z^k}{z^k(1-z)} + \frac{7(z^k-1)z}{2z^k(z-1)^2} - \frac{7kz}{2z^{k+1}(z-1)} = X(z) \qquad (16)$$