# NCERT-discrete : 10.5.3 - 2

EE23BTECH11025 - Anantha Krishnan

## I. Question

Find the sums given below:

(i) $7 + 10\frac{1}{2} + 14 ... + 84$
(ii) $34 + 32 + 30 ... + 10$
(iii) -5 + -8 + -11 ... -230

**Solutions**:

(i) By observing the consecutive common differences in the given series , we observe that it is a constant value , which is $\frac{7}{2}$ .
Since this an arithmetic progression , we can use the formula which dictates the sum of "n" terms of such a series
Let "$S_n$" denote the sum of n terms in a series ,"a" denotes its first term and "d" denotes the common difference.It is known that

$$S_n = \frac{n}{2}(2a + (n - 1)d) \tag{1}$$

In the question, a=7 and d=$\frac{7}{2}$ , and "n" is unknown
For calculating the number of terms, we use the formula

$$x(n) = x(0) + nd \tag{2}$$

Where x(n) is the nth term of the series
Given that x(n) is 84, we solve for "n"

$$84 = \frac{7}{2} + (n)\frac{7}{2} \tag{3}$$

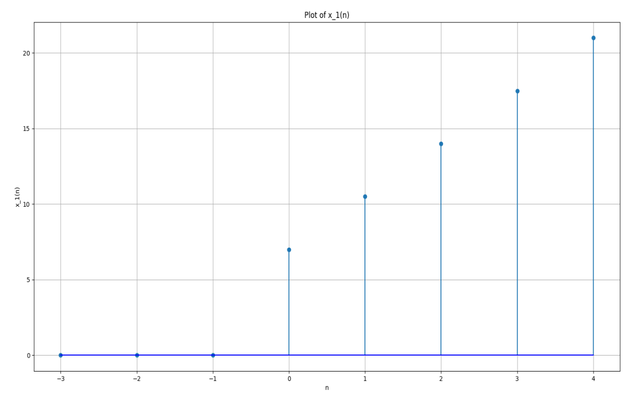
Solving this yields n=23.
We now use this result for calculating $S_{23}$

$$S_{23} = \frac{23}{2}(14 + (22)\frac{7}{2}) \tag{4}$$

Again, solving this yields $S_{23}$ as 1046.5

For the $n^{th}$ term of (i) bit , we are now required to calculate $X_{1(z)}$ in terms of $u_n$ and $x_{1(n)}$.Where u$_{(n)}$ is the unit step function.Here, $x_{1(0)}$ equals 7.

$$x_{1(n)} = u_{(n)}(x_{1(0)} + \frac{7n}{2}) \tag{5}$$




The graph of $x_{1(n)}$ vs n is shown above.

Now , putting this $x_{1(n)}$ in (9) , we obtain

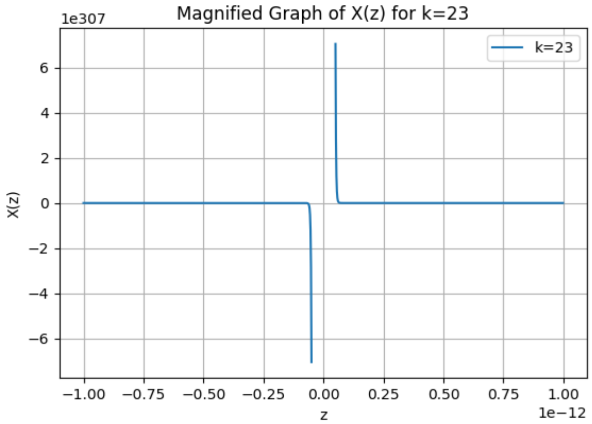
$$\sum_{n=-\infty}^{\infty}(x_{1(0)} + \frac{7n}{2})u_{(n)}Z^{-n} = X_{1(z)} \tag{6}$$

$$\sum_{n=-\infty}^{\infty}(7 + \frac{7n}{2})u_{(n)}Z^{-n} = X_{1(z)} \tag{7}$$

For the **region of convergence**
$X_{1(z)}$ = convergent ∀ z $\epsilon(-\infty, -1)U(1, \infty)$

This can be proved from ratio test .
We now calculate the sum using the formula for gp and agp(Assuming $(k-1)^{th}$ term is the last term and also where $x_{1(0)}$ is the first term), we obtain.

$$\begin{aligned} &7(1 - z^k)(z^k(1 - z))^{-1} + \\ &(7(z^k - 1)z)(2z^k(z - 1)^2)^{-1} - \\ &(7kz)(2z^{k+1}(z - 1))^{-1} = X_{1(z)} \end{aligned} \tag{8}$$




Above is the graph of $X_{1(z)}$ vs z

It is told to replace $S_n$*$u_{(n)}$ with x(n) and therefore calculate X(Z). The equation (1) and calculate the general form of X(Z) . It is known that

$$\sum_{n=-\infty}^{\infty} Z^{-n}x(n) = X(Z) \tag{9}$$

Putting the equation (1) in (9)

$$\sum_{n=-\infty}^{\infty} u_{(n)}Z^{-n}\frac{n}{2}(2a+(n-1)d) = X(Z) \tag{10}$$

Writing another equation by multiplying (10) with Z, we get

$$\sum_{n=-\infty}^{\infty} u_{(n)}Z^{-n+1}\frac{n}{2}(2a+(n-1)d) = ZX(Z) \tag{11}$$

Now we subtract (10) from (11) by displacing it with one term , i.e we subtract the first term of (10) from the second term of (11) and so on.
By simplifying this, we get

$$\sum_{n=-\infty}^{\infty} (a-\frac{d}{2})u_{(n)}Z^{-n} + \frac{n}{2}dZ^{-n} = ZX(Z) - X(Z) \tag{12}$$

The first part is a GP and the second part is an AGP. Since the nature of Z is not known, we calculate the general sum for finite terms(Assuming k) and then use it for each bit.
Calculating the individual sums, we get

$$\begin{aligned}((a-\frac{d}{2})Z^{1-k}((Z^k-1))(Z-1)^{-1}+\\(d)(2Z^{k-2})^{-1}(Z^{k-1}-1)^{(}2(Z-1)^2)^{-1}-\\(d(k-1))(2(Z-1)Z^{k-1})^{-1})(Z-1)^{-1}\\=X(Z)\end{aligned} \tag{13}$$

By substituting the respective values of k,a,d we obtain the following:

1)

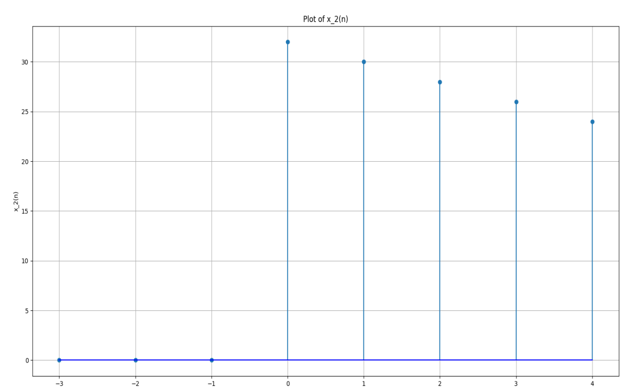
$$\begin{aligned}X(Z) = ((\frac{21}{4})Z^{-22}((Z^{23}-1))(Z-1)^{-1}+\\7(4Z^{21})^{-1}(Z^{22}-1)(2(Z-1)^2)^{-1}-\\(7(22))(4(Z-1)Z^{22})^{-1})(Z-1)^{-1}\end{aligned}$$

(ii) Based on the analysis of the previous bit, we observe that in this bit
a=34 , d=-2 For calculating the number of terms, we use the formula ((2)
Substituting the values, we get

$$10 = 34 + (n-1)(-2) \tag{14}$$

Solving this yields n=13
For calculating $X_{2(z)}$, we first write $x_{2(n)}$ ( here x(0) is 34)

$$x_{2(n)} = x_{2(0)} + nd \tag{15}$$
$$x_{2(n)} = x_{2(0)} - 2n \tag{16}$$




The graph of $x_{2(n)}$ vs n is shown above

Now , putting this value in (9), we obtain

$$\sum_{n=-\infty}^{\infty} (x_{2(0)} - 2n)u_{(n)}Z^{-n} = X_{2(z)} \tag{17}$$

It can be observed that the region of convergence does not change as the nature of the function remains the same as (i)bit
For $X_{2(z)}$ , we use the same analogy as in (i)bit to

obtain X(z)

$$32(1-z^k)(z^k(1-z))^{-1}-$$
$$(2(z^k-1)z)(z^k(z-1)^2)^{-1}+$$
$$(2kz)(z^{k+1}(z-1))^{-1} = X_{2(z)} \tag{18}$$

As For calculating the sum , we use (1)

$$S_{13} = \frac{13}{2}(64 + 11(-2)) \tag{19}$$

Solving this, we get $S_n$= 286.

Using the equation(12) , we obtain X(Z) as:

2)

$$X(Z) = (35Z^{-12}((Z^{13}-1))(Z-1)^{-1}-$$
$$(Z^{11})^{-1}(Z^{12}-1)(2(Z-1)^2)^{-1}+$$
$$(24)((Z-1)Z^{12})^{-1})(Z-1)^{-1}$$

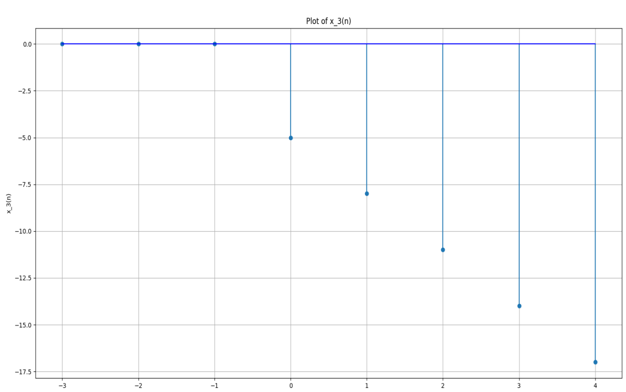
3)

(iii) By using the previous analysis, we can conclude that a=-5, d=-3
Again, for n , we use the formula (2)

$$-230 = -5 + (n-1)(-3) \tag{20}$$

Solving this yields n=76
Now, we write the equation for $x_{3(n)}$ ( given x(0) is -5)

$$x_{3(n)} = x_3(0) + nd \tag{21}$$
$$x_{3(n)} = x_3(0) - 3n \tag{22}$$




The graph of $x_{3(n)}$ vs n is shown above.
Now , putting this value in (9), we obtain

$$\sum_{n=-\infty}^{\infty} (x_{3(0)} - 3n)u_{(n)}Z^{-n} = X_{3(z)} \tag{23}$$

Again , it can be observed that the region of convergence does not change as the nature of the function remains the same as (i)bit
For $X_{3(z)}$ , we use the same analogy as in (i)bit to obtain

$$-5(1-z^k)(z^k(1-z))^{-1}-$$
$$(3(z^k-1)z)(z^k(z-1)^2)^{-1}+$$
$$(3kz)(z^{k+1}(z-1))^{-1} = X_{3(z)} \tag{24}$$

Now, for the sum we use equation (1) :

$$S_{76} = \frac{76}{2}(-10 + (76-1)(-3)) \tag{25}$$

Solving this we obtain $S_{76}$=-8930.

Using the equation(12) , we obtain X(Z) as:

$$X(Z) = ((\frac{7}{2})Z^{-75}((Z^{76}-1))(Z-1)^{-1}-$$
$$(-3)(2Z^{74})^{-1}(Z^{75}-1)(2(Z-1)^2)^{-1}+$$
$$(-3(75))(2(Z-1)Z^{75})^{-1})(Z-1)^{-1}$$

| Parameters | Equations |
|---|---|
| $x_{1(n)}$ | $x_{1(0)}$ +3.5n |
| $x_{2(n)}$ | $x_{2(0)}$ -2n |
| $x_{3(n)}$ | $x_{3(0)}$ -3n |
| $X_{1(z)}$ | $7(1-z^k)(z^k(1-z))^{-1}+(7(z^k-1)z)(2z^k(z-1)^2)^{-1}-(7kz)(2z^{k+1}(z-1))^{-1}$ |
| $X_{2(z)}$ | $32(1-z^k)(z^k(1-z))^{-1}-(2(z^k-1)z)(z^k(z-1)^2)^{-1}+(2kz)(z^{k+1}(z-1))^{-1}$ |
| $Z_{3(z)}$ | $-5(1-z^k)(z^k(1-z))^{-1}-(3(z^k-1)z)(z^k(z-1)^2)^{-1}+(3kz)(z^{k+1}(z-1))^{-1}$ |

Given above is the list of all parameters used in the analysis so far.