NCERT-discrete : 10.5.3 - 2

EE23BTECH11025 - Anantha Krishnan

I. Question

Find the sums given below:

(i) $7 + 10\frac{1}{2} + 14 \dots + 84$
(ii) $34 + 32 + 30 \dots + 10$
(iii) $-5 + -8 + -11 \dots -230$

Solutions:

(i) By observing the consecutive common differences in the given series , we observe that it is a constant value , which is $\frac{7}{2}$.
Since this an arithmetic progression , we can use the formula which dictates the sum of "n" terms of such a series
Let "S_n" denote the sum of n terms in a series ,"a" denotes its first term and "d" denotes the common difference.It is known that

$$S_n = \frac{n}{2}(2a + (n-1)d) \tag{1}$$

In the question, a=7 and d=$\frac{7}{2}$, and "n" is unknown
For calculating the number of terms, we use the formula

$$T_n = a + (n-1)d \tag{2}$$

Where T_n is the nth term of the series
Given that T_n is 84, we solve for "n"

$$84 = 7 + (n-1)\frac{7}{2} \tag{3}$$

Solving this yields n=23.
We now use this result for calculating S_{23}

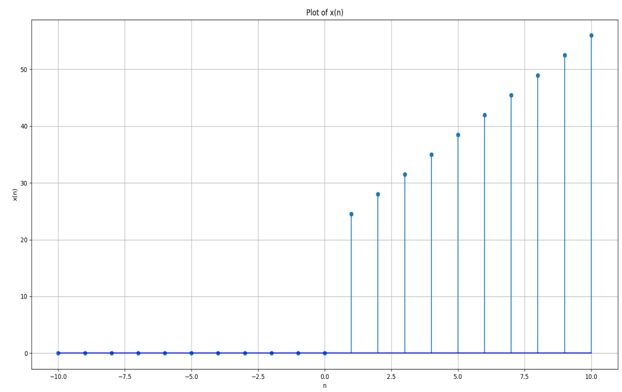
$$S_{23} = \frac{23}{2}(14 + (22)\frac{7}{2}) \tag{4}$$

Again, solving this yields S_{23} as 1046.5

For the n^{th} term of (i) bit , we are now required to calculate X(z) given that (T^{th})*$u_{(n)}$ is x(n) $\forall n \neq 0, where u_{(n)}$ is the unit step function. We take x(0) to be 0.

x(n) = $u_{(n)}(7 + (n-1)\,\frac{7}{2})$

The graph of x(n) vs n is shown below.



Now , putting this x(n) in (8) , we obtain

$$\sum_{n=-\infty}^{\infty}(7 + (n-1)\frac{7}{2})u_{(n)}Z^{-n} = X(z) \tag{5}$$

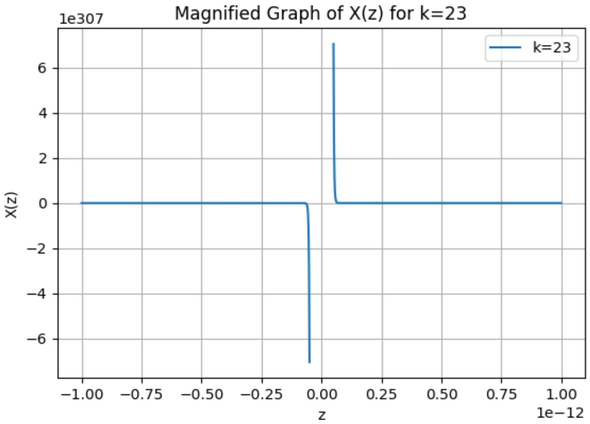
$$\sum_{n=-\infty}^{\infty}(\frac{7}{2} + \frac{7n}{2})u_{(n)}Z^{-n} = X(z) \tag{6}$$

For the **region of convergence**
X(z) = convergent $\forall$ z $\epsilon(-\infty, -1)U(1, \infty)$

This can be proved from ratio test .
We now calculate the sum using the formula for gp and agp(Assuming k terms), we obtain.

$$\frac{7}{2}(1 - z^k)(z^k(1 - z))^{-1} + (7(z^k - 1)z)(2z^k(z - 1)^2)^{-1} - (7kz)(2z^{k+1}(z - 1))^{-1} = X(z) \tag{7}$$

Below is the graph of X(z) vs z



It is told to replace S_n*$u_{(n-1)}$ with x(n) and therefore calculate X(Z). The equation (1) and calculate the general form of X(Z) . It is known that

$$\sum_{n=-\infty}^{\infty} Z^{-n}x(n) = X(Z) \tag{8}$$

Putting the equation (1) in (8)

$$\sum_{n=-\infty}^{\infty} u_{(n)}Z^{-n}\frac{n}{2}(2a+(n-1)d) = X(Z) \tag{9}$$

Writing another equation by multiplying (9) with Z, we get

2)

$$\sum_{n=-\infty}^{\infty} u_{(n)}Z^{-n+1}\frac{n}{2}(2a+(n-1)d) = ZX(Z) \tag{10}$$

Now we subtract (9) from (10) by displacing it with one term , i.e we subtract the first term of (9) from the second term of (10) and so on.
By simplifying this, we get

$$\sum_{n=-\infty}^{\infty}(a-\frac{d}{2})u_{(n)}Z^{-n}+\frac{n}{2}dZ^{-n} = ZX(Z)-X(Z) \tag{11}$$

The first part is a GP and the second part is an AGP. Since the nature of Z is not known, we calculate the general sum for finite terms(Assuming k) and then use it for each bit.
Calculating the individual sums, we get

$$\begin{aligned}((a-\frac{d}{2})Z^{1-k}((Z^{k}-1))(Z-1)^{-1}+\\ (d)(2Z^{k-2})^{-1}(Z^{k-1}-1)^{(}2(Z-1)^{2})^{-1}-\\ (d(k-1))(2(Z-1)Z^{k-1})^{-1})(Z-1)^{-1}\\ =X(Z)\end{aligned} \tag{12}$$

By substituting the respective values of k,a,d we obtain the following:

1)

$$\begin{aligned}X(Z) = ((\frac{21}{4})Z^{-22}((Z^{23}-1))(Z-1)^{-1}+\\ 7(4Z^{21})^{-1}(Z^{22}-1)(2(Z-1)^{2})^{-1}-\\ (7(22))(4(Z-1)Z^{22})^{-1})(Z-1)^{-1}\end{aligned}$$

(ii) Based on the analysis of the previous bit, we observe that in this bit
a=34 , d=-2 For calculating the number of terms, we use the formula ((2)
Substituting the values, we get

$$10 = 34+(n-1)(-2) \tag{13}$$

Solving this yields n=13
For calculating the sum , we use (1)

$$S_{13} = \frac{13}{2}(64+11(-2)) \tag{14}$$

Solving this, we get S_n= 286.

Using the equation(12) , we obtain X(Z) as:

$$\begin{aligned}X(Z) = (35Z^{-12}((Z^{13}-1))(Z-1)^{-1}-\\ (Z^{11})^{-1}(Z^{12}-1)(2(Z-1)^{2})^{-1}+\\ (24)((Z-1)Z^{12})^{-1})(Z-1)^{-1}\end{aligned}$$

(iii) By using the previous analysis, we can conclude that a=-5, d=-3
Again, for n , we use the formula (2)

$$-230 = -5+(n-1)(-3) \tag{15}$$

Solving this yields n=76
Now, for the sum we use equation (1) :

$$S_{76} = \frac{76}{2}(-10+(76-1)(-3)) \tag{16}$$

Solving this we obtain S_{76}=-8930.

Using the equation(12) , we obtain X(Z) as:

3)

$$X(Z) = ((\frac{7}{2})Z^{-75}((Z^{76} - 1))(Z - 1)^{-1} -$$
$$(-3)(2Z^{74})^{-1}(Z^{75} - 1)(2(Z - 1)^{2})^{-1} +$$
$$(-3(75))(2(Z - 1)Z^{75})^{-1})(Z - 1)^{-1}$$