

NCERT-discrete : 10.5.3 - 2

EE23BTECH11025 - Anantha Krishnan

I. QUESTION

Find the sums given below:

- (i) $7 + 10\frac{1}{2} + 14 \dots + 84$
- (ii) $34 + 32 + 30 \dots + 10$
- (iii) $-5 + -8 + -11 \dots -230$

Solutions:

- (i) $7 + 10\frac{1}{2} + 14 \dots + 84$

Let $S_i(n)$ denote the sum of first $n+1$ terms in the i^{th} series, $x_i(0)$ denotes its first term, d_i denotes the common difference and $u_{(n)}$ denote the unit step function.

$$S_n = \frac{(n+1)u_{(n)}}{2}(2x_i(0) + (n)d_i) \quad (1)$$

For number of terms, we use

$$x_i(n) = (x_i(0) + nd_i)u_n \quad (2)$$

Where $x_i(n)$ is the $(n+1)^{th}$ term of the series. Putting the values

$$84 = 7 + \frac{7n}{2} \quad (3)$$

$$n = 22 \quad (4)$$

- 1) **Calculating $S_1(n)$ for $x_1(n)$** : We now use this result for calculating $S_1(23)$

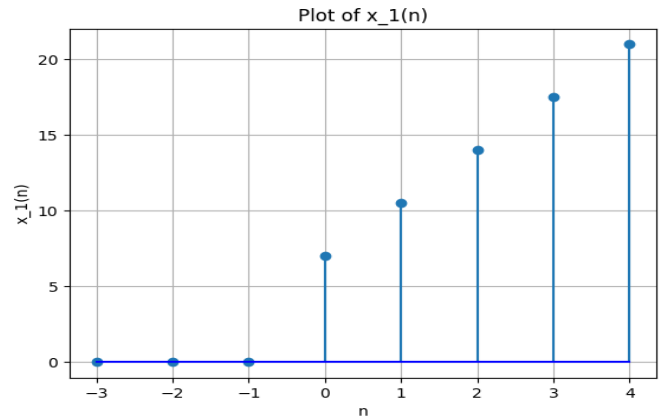
$$S_1(22) = \frac{23}{2}(14 + (22)\frac{7}{2}) \quad (5)$$

Solving this yields $S_1(22) = 1046.5$ We are now required to calculate $X_1(z)$ in terms of u_n and $x_1(n)$. Where $u_{(n)}$ is the unit step function.

$$x_1(n) = u_{(n)}(x_1(0) + \frac{7n}{2}) \quad (6)$$

- 2) **Z-Transform of $x_1(n)$** : By the Definition of Z-transform:

$$\sum_{n=-\infty}^{\infty} Z^{-n} x_i(n) = X_i(Z) \quad (7)$$



Graph:1 $x_1(n)$ vs n

Putting $x_1(n)$ in (7), we get

$$\sum_{n=-\infty}^{\infty} (x_1(0) + \frac{7n}{2})u_{(n)}Z^{-n} = X_1(z) \quad (8)$$

$$\sum_{n=-\infty}^{\infty} (7 + \frac{7n}{2})u_{(n)}Z^{-n} = X_1(z) \quad (9)$$

$$7z(z-1)^{-1} + 7z(2(z-1))^{-2} = X_1(z)$$

- 3) **Region of Convergence of $X_1(z)$**

$$|z| < 1 \quad (10)$$

- 4) **Expressing $S_1(n)$ as a convolution** : Using (2) and assuming

$$h[n] = u[n] \quad (11)$$

$$S_1(n) = x_1(n) * h[n] \quad (12)$$

Putting in the equations

$$S_1(n) = \sum_{k=0}^n (x_1(0) + kd_1)(u[k])(u[n-k])$$

$$S_1(n) = 7(n+1) + (1.75)(n+1)(n) \quad (13)$$

$$S_1(n) = 7 + 8.75n + 1.75n^2 \quad (14)$$

5) Using Z-Transform for $S_1(n)$:

$$S_1(z) = X_1(z) * H(z) \quad (15)$$

Where $X_1(z)$ comes from (10) For $H(z)$, it is Z-transform of unit-step function

$$H_1(z) = z(z-1)^{-1} \quad (16)$$

For $S_1(z)$:

$$S_1(z) = (7z(z-1)^{-1} + 7z(2(z-1))^{-2})z(z-1)^{-1} \quad (17)$$

ROC:

$$|z| > 1 \quad (18)$$

6) **Inversion of $S_1(z)$** : By using partial fractions :

$$S_1(z) = (7z^2(z-1)^{-2} + 7z^2(2(z-1))^{-3})$$

Using known results:

Inverse Z-transform of

$$z^2(z-1)^{-2} \leftrightarrow (n+1)u(n) \quad (19)$$

For $z^2(z-1)^{-3}$

we can differentiate (19) and get the inverse Z-transform as

$$z^2(z-1)^{-3} \leftrightarrow (n(n+1)/2)u(n) \quad (20)$$

Therefore:

$$S_1(n) = 7(n+1) + 1.75n(n+1) \quad (21)$$

$$(ii) \quad 34 + 32 + 30 \dots + 10$$

In this bit $x_2(0) = 34$, $d_2 = -2$.

Using equation (2)

$$10 = 34 - 2n \quad (22)$$

$$n = 12 \quad (23)$$

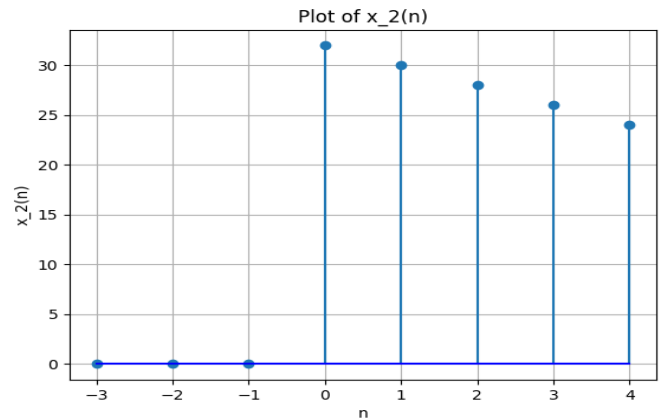
For $x_2(n)$

$$x_2(n) = x_2(0) + nd_2 \quad (24)$$

$$x_2(n) = x_2(0) - 2n \quad (25)$$

1) **Z-Transform of $x_2(n)$** : Using (7)

$$\sum_{n=-\infty}^{\infty} (x_2(0) - 2n)u(n)Z^{-n} = X_2(z) \quad (26)$$



Graph:2 $x_2(n)$ vs n

For $X_2(z)$

$$34z(z-1)^{-1} - 2z((z-1))^{-2} = X_2(z) \quad (27)$$

2) **Calculating $S_2(n)$ of $x_2(n)$** : For calculating the sum, we use (1)

$$S_2(12) = \frac{13}{2}(64 + 11(-2)) \quad (28)$$

$$S_2(12) = 286. \quad (29)$$

3) **Region of Convergence**

$$|z| < 1 \quad (30)$$

4) **Expressing $S_2(n)$ as a convolution** : Using (2) and assuming

$$h[n] = u[n] \quad (31)$$

$$S_2(n) = x_2(n) * h[n] \quad (32)$$

Putting in the equations

$$S_2(n) = \sum_{k=0}^n (x_2(0) + kd_2)(u[k])(u[n-k])$$

$$S_2(n) = 34(n+1) - (n+1)(n) \quad (33)$$

$$S_2(n) = 34 + 33n - n^2 \quad (34)$$

5) **Using Z-Transform for $S_2(n)$** :

$$S_2(z) = X_2(z) * H(z) \quad (35)$$

Where $X_2(z)$ comes from (27) and $H(z)$ from (16). For $S_2(z)$:

$$S_2(z) = (34z(z-1)^{-1} - 2z((z-1))^{-2})z(z-1)^{-1} \quad (36)$$

ROC:

$$|z| > 1 \quad (37)$$

6) **Inversion of $S_2(z)$** : By using partial fractions :

$$S_2(z) = (34z^2(z-1)^{-2} - 2z^2((z-1))^{-3})$$

Using results (19) and (20)

$$S_2(n) = (34(n+1) - n(n+1))u(n) \quad (38)$$

(iii) $-5 + -8 + -11 \dots -230$

Here $x_3(0) = -5$, $d_3 = -3$ From (2)

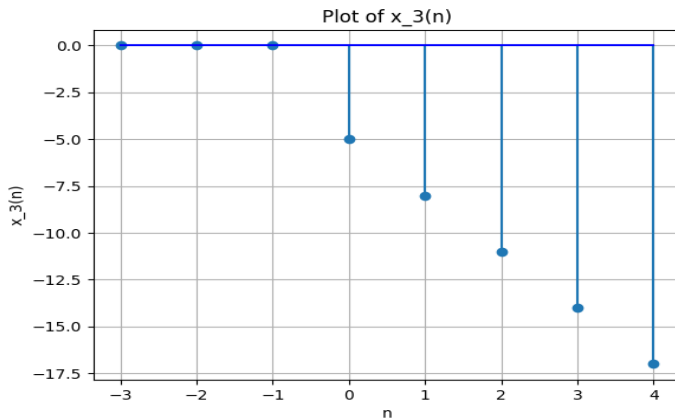
$$-230 = -5 - 3n \quad (39)$$

$$n = 75 \quad (40)$$

For $x_3(n)$

$$x_3(n) = x_3(0) + nd_3 \quad (41)$$

$$x_{3(n)} = x_3(0) - 3n \quad (42)$$



Graph:3 $x_3(n)$ vs n

1) **Z-Transform of $x_3(n)$** : Putting $x_3(n)$ in (7)

$$\sum_{n=-\infty}^{\infty} (x_3(0) - 3n)u(n)Z^{-n} = X_3(z) \quad (43)$$

For $X_3(z)$, we use the same process as in (i) bit

$$\begin{aligned} & -5z(z-1)^{-1} - \\ & 5z((z-1))^{-1} - \\ & 3z((z-1))^{-2} = X_3(z) \end{aligned} \quad (44)$$

2) **Calculating $S_3(n)$ of $x_3(n)$** : Using (1) :

$$S_3(75) = \frac{76}{2}(-10 + (76-1)(-3)) \quad (45)$$

$$S_3(75) = -8930 \quad (46)$$

3) **Region of Convergence**

$$|z| < 1 \quad (47)$$

4) **Expressing $S_3(n)$ as a convolution** : Using (2) and assuming

$$h[n] = u[n] \quad (48)$$

$$S_3(n) = x_3(n) * h[n] \quad (49)$$

Putting in the equations

$$S_3(n) = \sum_{k=0}^n (x_3(0) + kd_3)(u[k])(u[n-k])$$

$$S_3(n) = -5(n+1) - (1.5)(n+1)(n) \quad (50)$$

$$S_3(n) = -5 - 6.5n - (1.5)n^2 \quad (51)$$

5) **Using Z-Transform for $S_3(n)$** :

$$S_3(z) = X_3(z) * H(z) \quad (52)$$

Where $X_3(z)$ comes from (44) and $H(z)$ from (16). For $S_3(z)$:

$$S_3(z) = (-5z(z-1)^{-1} - (1.5)z((z-1))^{-2})z(z-1)^{-1} \quad (53)$$

ROC:

$$|z| > 1 \quad (54)$$

6) **Inversion of $S_2(z)$** : By using partial fractions :

$$S_3(z) = (-5z^2(z-1)^{-2} - (1.5)z^2((z-1))^{-3})$$

Using results (19) and (20)

$$S_3(n) = (-5(n+1) - (1.5)n(n+1))u(n) \quad (55)$$

Symbols	Description	Values
d_i	Common Difference	3.5, -2, -3
$x_i(n)$	Sequence	$(x_i(0) + nd_i)u(n)$
$X_i(z)$	Z-Transform of $x_i(n)$	$zx_i(0)(z-1)^{-1} + d_iz(z-1)^{-2}$
$S_i(n)$	Sum of (n+1) terms in i^{th} series	$\frac{(n+1)u(n)}{2}(2x_i(0) + kd_i)$
$h[n]$	Unit step function	$0 \forall n < 0, 1 \forall n \geq 0$
$H(z)$	Z-Transform of $H(z)$	$z(z-1)^{-1}$
$S_i(z)$	Z-Transform of $S_i(z)$	$X_i(z)*H(z)$

Table 1 : Parameters , Descriptions AND Values