Filter Design

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1 Introduction

We need to create both FIR and IIR filter designs for a specified filter type. Specifically, we're tasked with developing a bandpass filter, and we have the specifications for this filter.

2 Filter Specifications

2.1 The Digital Filter

1. Passband: The passband is from {4 + 0.6(j)}kHz to {4 + 0.6(j+2)}kHz.
   where

   $$j = (r - 11000) \mod \sigma \tag{1}$$

   where σ is sum of digits of roll number and r is roll number.

   $$r = 11025 \tag{2}$$
   $$\sigma = 9 \tag{3}$$
   $$j = 7 \tag{4}$$

   When substituting $j = 7$, the passband range for our bandpass filter spans from 8.2 kHz to 9.4 kHz.
   Therefore, the un-normalized discrete-time filter passband frequencies are $F_{p1} =$ 8.2 kHz and $F_{p2} = 9.4$ kHz.
   The corresponding normalized digital filter passband frequencies would be:

   $$\omega_{p1} = 2\pi \frac{F_{p1}}{F_s} = 0.34\pi \tag{5}$$
   $$\omega_{p2} = 2\pi \frac{F_{p2}}{F_s} = 0.39\pi \tag{6}$$

2. Tolerances: The passband (δ_1) and stopband (δ_2) tolerances are given to be equal, so we let $\delta_1 = \delta_2 = \delta = 0.15$.

3. Stopband: The transition band for bandpass filters is $\Delta F = 0.3$ kHz on either side of the passband.

$$F_{s1} = 8.2 - 0.3 = 7.9\text{KHz} \tag{7}$$
$$F_{s2} = 9.4 + 0.3 = 9.7\text{KHz} \tag{8}$$

$$\omega_{s1} = 2\pi\frac{F_{s1}}{F_s} = 0.329\pi \tag{9}$$
$$\omega_{s2} = 2\pi\frac{F_{s2}}{F_s} = 0.404\pi \tag{10}$$
$$\tag{11}$$

2.2 The Analog filter

In the bilinear transform, the relationship between the analog filter frequency (Ω) and the corresponding digital filter frequency (ω) is given by:

$$\Omega = \tan\frac{\omega}{2} \tag{12}$$

Using this relation, we obtain the analog passband and stopband frequencies as: $\Omega_{p1} = 0.5913$, $\Omega_{p2} = 0.702$ and $\Omega_{s1} = 0.5662$, $\Omega_{s2} = 0.7361$ respectively.

3 The IIR Filter Design

To ensure a monotonic stopband and an equiripple passband, we opt to design our bandpass IIR filter using the Chebyshev approximation method.

3.1 The Analog Filter

1. Low Pass Filter Specifications: If we denote $H_{a,BP}(j\Omega)$ as the desired analog bandpass filter with the specifications outlined in Section 2.2, and $H_{a,LP}(j\Omega_L)$ as the equivalent low-pass filter,

$$\Omega_L = \frac{\Omega^2 - \Omega_0^2}{B\Omega} \tag{13}$$

where $\Omega_0 = \sqrt{\Omega_{p1}\Omega_{p2}} = 0.644$ and $B = \Omega_{p2} - \Omega_{p1} = 0.1107$.

Substituting Ω_{s1} and Ω_{s2} in (13) we obtain the stopband edges of lowpass filter

$$\Omega_{Ls1} = \frac{\Omega_{s1}^2 - \Omega_0^2}{B\Omega_{s1}} = -1.502 \tag{14}$$
$$\Omega_{Ls2} = \frac{\Omega_{s2}^2 - \Omega_0^2}{B\Omega_{s2}} = 1.559 \tag{15}$$

And we choose the minimum of these two stopband edges

$$\Omega_{Ls} = \min(|\Omega_{Ls_1}|, |\Omega_{Ls_2}|) = 1.502. \tag{16}$$

2. The Low Pass Chebyschev Filter Paramters: The magnitude of frequency response of the low pass filter is given by

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1+\epsilon^2 c_N^2(\Omega_L/\Omega_{Lp})} \tag{17}$$

The passband edge of the low pass filter is chosen as $\Omega_{Lp} = 1$. Therfore ,

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1+\epsilon^2 c_N^2(\Omega_L)} \tag{18}$$

Here c_N denote the chebyshev polynomials for a particular order N of the filter.

$$c_N(x) = \cosh(N\cosh^{-1} x), x = \Omega_L \tag{19}$$
$$c_0(x) = 1 \tag{20}$$
$$c_1(x) = x \tag{21}$$

There exists a recurssive relation from which all the polynomials can be found out.

$$c_{N+2} = 2xc_{N+1} - c_N \tag{22}$$

Applying the band restrictions on (17)

$$|H_{a,LP}(j\Omega_L)|^2 < \delta_2 \text{ for } \Omega_L = \Omega_{Ls} \tag{23}$$
$$1-\delta_1 < |H_{a,LP}(j\Omega_L)|^2 < 1 \text{ for } \Omega_L = \Omega_{Lp} \tag{24}$$
$$\tag{25}$$

we obtain :

$$\frac{\sqrt{D_2}}{c_N(\Omega_{Ls})} \le \epsilon \le \sqrt{D_1},$$
$$N \ge \left\lceil \frac{\cosh^{-1}\sqrt{D_2/D_1}}{\cosh^{-1}\Omega_{Ls}} \right\rceil, \tag{26}$$

Where $D_1 = \frac{1}{(1-\delta)^2} - 1$ and $D_2 = \frac{1}{\delta^2} - 1$ and $\lceil . \rceil$ is known as the ceiling function .

Parameter	Value
D_1	0.384
D_2	43.44
N	4
$c_4(x)$	$8x^4 + 8x^2 + 1$

Table 1: Parameter Table

we get $N \geq 4$ and $0.278 \leq \epsilon \leq 0.61$

The below code plots (17) for different values of ϵ .

```
https://github.com/Gandubs/Signals−and−Systems/blob/master/filter%20design
      /codes/plot1.py
```

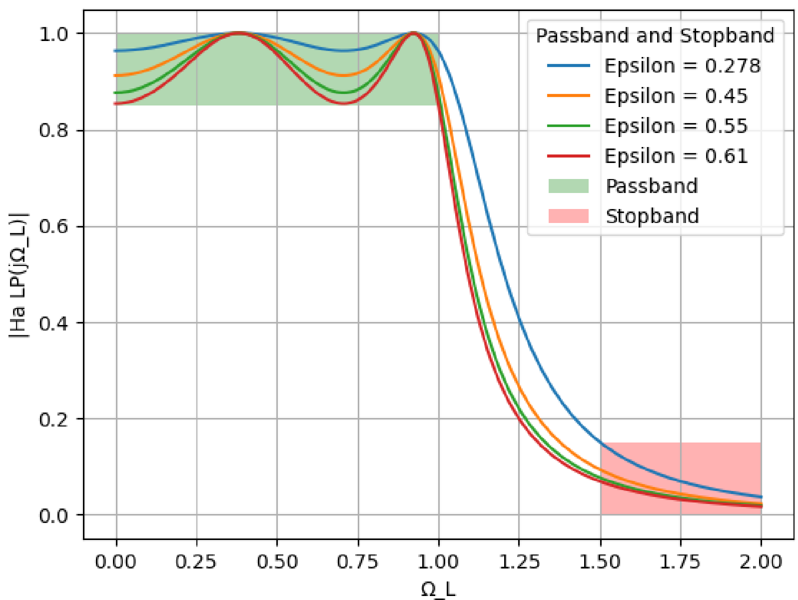

Figure 1: The Analog Low-Pass Frequency Response for $0.278 \leq \epsilon \leq 0.61$

In Figure 1 ($Fig.$ 1), we can observe the equiripple behavior in the passband and the monotonic behavior in the stopband. As the value of ϵ increases, the magnitude of $|H_{a,LP}(j\Omega_L)|$ decreases.

3. The Low Pass Chebyschev Filter: The subsequent step in the design process involves deriving an expression for the magnitude response in the s-domain.

   Using $s = j\Omega$ or in this case $s_L = j\Omega_L$ we obtain:

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\frac{s_L}{j})} \tag{27}$$

   To find poles equate the denominator to zero:

$$1 + \epsilon^2 c_N^2\left(\frac{s_L}{j}\right) = 0 \text{ where } \quad c_N(x) = cos\left(Ncos^{-1}(x)\right) \tag{28}$$

On solving (28) we obtain poles :

$$s_k = -\Omega_{Lp} \sin(A_k) \sinh(B_k) - j\Omega_{Lp} \cos(A_k) \cosh(B_k) \tag{29}$$

where k is the index of the pole and

$$A_k = (2k+1)\frac{\pi}{2N} \tag{30}$$

$$B_k = \frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right) \tag{31}$$

The below code computes the values of s_k and stores it in a text file.

```
https://github.com/Gandubs/Signals−and−Systems/blob/master/filter%20design
    /codes/sk_gen.c
```

The poles obtained are formulated in the table below.

Pole	*Value*
s_1	0.4604 + j0.4276
s_2	0.4604 - j0.4276
s_3	0.1907 - j1.0322
s_4	0.1907 + j1.0322
s_5	-0.1907 - j1.0322
s_6	-0.4604 - j0.4276
s_7	-0.4604 + j0.4276
s_8	-0.1907 + j1.0322

Table 2: Values of s_k

The below code plots the pole-zero plot.

```
https://github.com/Gandubs/Signals−and−Systems/blob/master/filter%20design
    /codes/pole_zero_plt.py
```

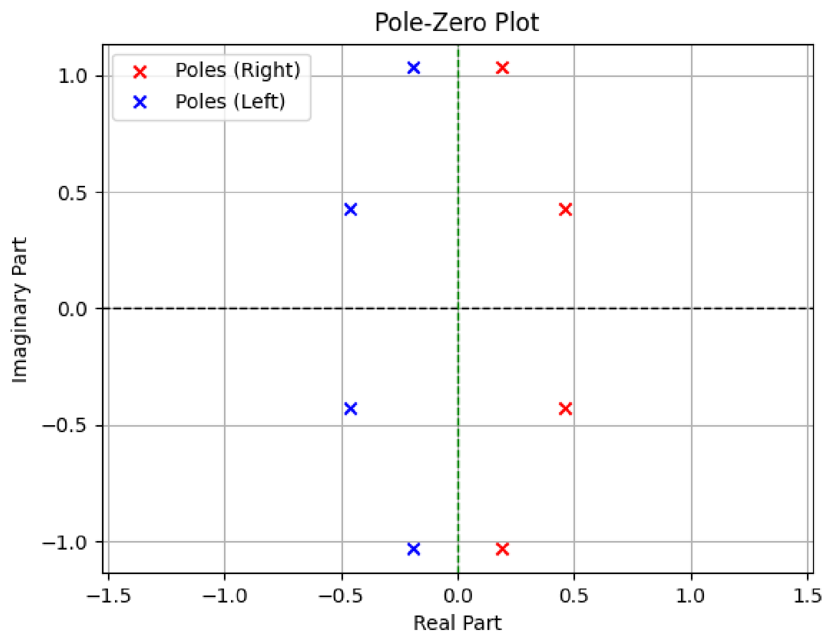

Figure 2: The Pole zero plot and all the poles lie on an ellipse. The left and right poles have been identified as shown.

The poles in the left half of the plane are considered in the design as we need to design a stable system.
Therefore the magnitude response is :-

$$H_{a,LP}(s_L) = \frac{G_{LP}}{(s_L - s_5)(s_L - s_6)(s_L - s_7)(s_L - s_8)} \tag{32}$$

where G_{LP} is the gain of the Low pass filter. Refer to Table 2 for s_k values.

We know that from (17):-

$$\left|H_{a,LP}(s_L)\right| = \frac{1}{\sqrt{1+\epsilon^2}} \text{at } \Omega_L = 1 \implies s_L = j \tag{33}$$

Substituting respective values in (33) we get $G_{LP} = 0.4166$

$$H_{a,LP}(s_L) = \frac{0.4166}{(s_L - s_5)(s_L - s_6)(s_L - s_7)(s_L - s_8)} \tag{34}$$

$$= \frac{0.4166}{s_L^4 + 1.3022 s_L^3 + 1.84781 s_L^2 + 1.16512 s_L + 0.435003} \tag{35}$$

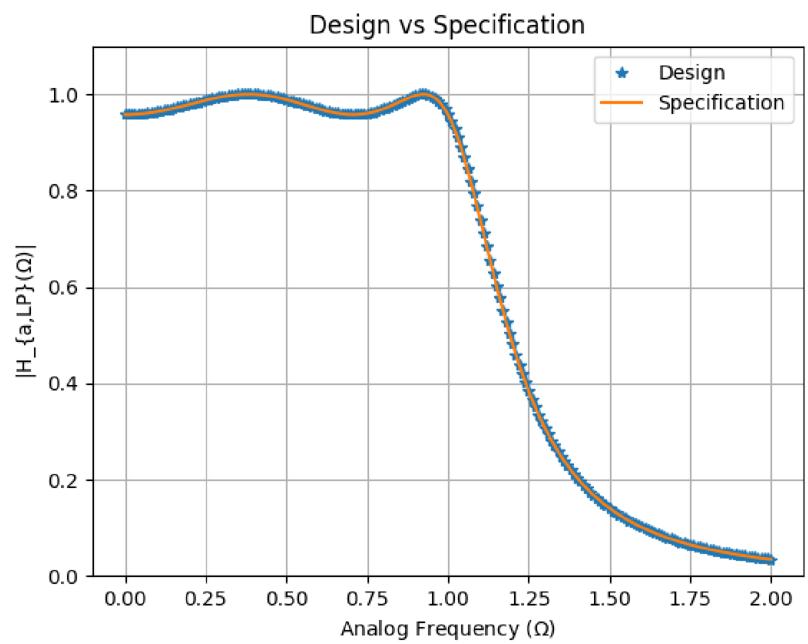

Figure 3: Design vs Specification corresponding to (35) and (18)

4. The Band Pass Chebyschev Filter: Once we've confirmed that the design meets the specified requirements, the next step is to transition to the desired type of filter using frequency transformation.

$$s_L = \frac{s^2 + \Omega_0^2}{Bs} \tag{36}$$

$$H_{a,BP}(s) = G_{BP} H_{a,LP}(s_L)\Big|_{s_L = \frac{s^2+\Omega_0^2}{Bs}}, \tag{37}$$

As there is one to one correspondence between the filters so $\Omega = \Omega_{p1}$ should correspond to Ω_{Lp}

$$s = j\Omega_{p1} \tag{38}$$

$$s_L = \frac{(j\Omega_{p1})^2 + \Omega_0^2}{B(j\Omega_{p1})} \tag{39}$$

$$\left|H_{a,BP}(j\Omega_{p1})\right| = 1 \tag{40}$$

$$G_{BP}\left|H_{a,LP}(s_L)\right| = 1 \tag{41}$$

Substituting (39) in (41) we obtain Gain of required bass pass filter:

$$G_{BP} = 1.0370 \tag{42}$$

Thus the response in s domain

$$H_{a,BP}\left(s\right) = \frac{6.49 \times 10^{-5} s^4}{s^8 + 0.144s^7 + 0.1682s^6 + 0.1810s^5 + 1.05s^4 + 0.750s^3 + 0.289s^2 + 0.0102s + 0.029} \tag{43}$$

The expressions in the s-domain and gain factors can be computed by writing a Python code. In Figure 3, we plot $|H_{a,BP}(j\Omega)|$ as a function of Ω for both positive and negative frequencies.
We observe that the passband and stopband frequencies in the figure closely match those obtained analytically through the bilinear transformation.

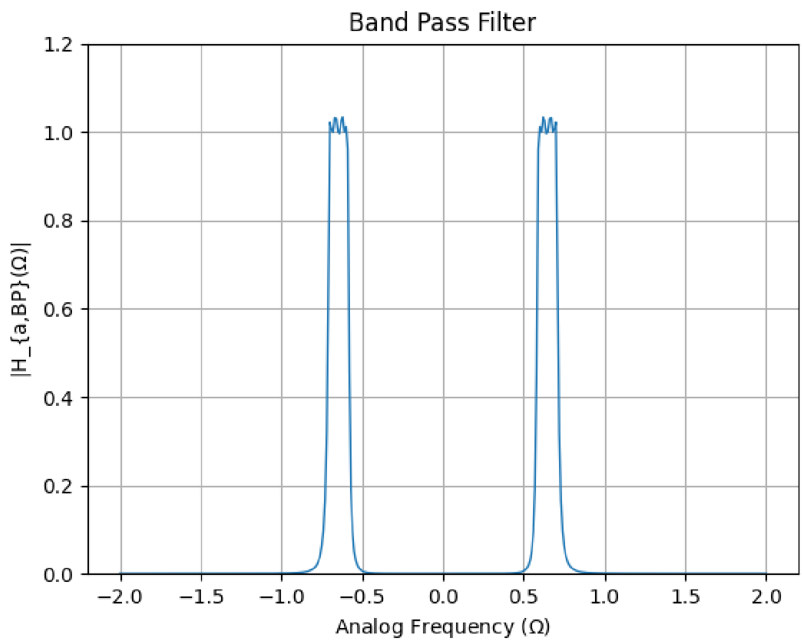

Figure 4: The Analog Bandpass Magnitude Response from (43).The filter design specifications are satisfied

3.2 The Digital Filter

From the bilinear transformation, we obtain the digital bandpass filter from the corresponding analog filter as

$$H_{d,BP}(z) = GH_{a,BP}(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \tag{44}$$

Substituting $s = \frac{1-z^{-1}}{1+z^{-1}}$ in (43) and calculating expression using a python code we get :

$$H_{d,BP}(z) = \frac{G\left(1 - 4z^{-2} + 6z^{-4} - 4z^{-6} + z^{-8}\right)}{3.6214 - 6.9496z^{-1} + 26.7376z^{-2} - 60.1464z^{-3} + 73.758z^{-4} - 49.572z^{-5} + 26.144z^{-6} - 7.62z^{-7} + 1.451z^{-8}} \tag{45}$$

where $G = 6.49 \times 10^{-5}$

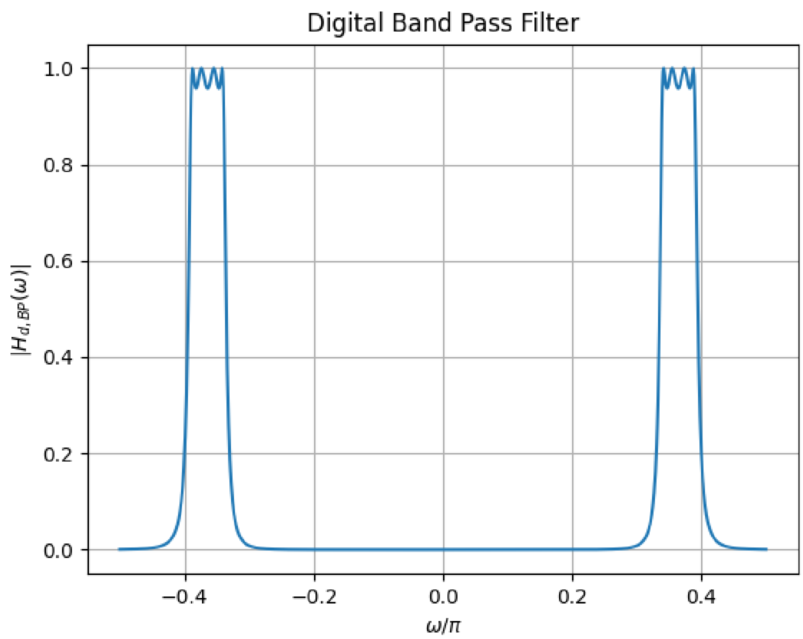

Figure 5: Digital Specifications are met. Passband and stopband frequencies are same

4 The FIR Filter

To design the FIR filter, we begin by obtaining the (non-causal) lowpass equivalent using the Kaiser window. Subsequently, we convert it to a causal bandpass filter.

4.1 The Equivalent Lowpass Filter

The lowpass filter has a passband frequency ω_l and transition band $\Delta\omega = 2\pi\frac{\Delta F}{F_s} = 0.0125\pi$. The stopband tolerance is $\delta = 0.15$.The cutoff-frequency is given by :

$$\omega_l = \frac{B}{2} \tag{46}$$

$$= 0.025\pi \tag{47}$$

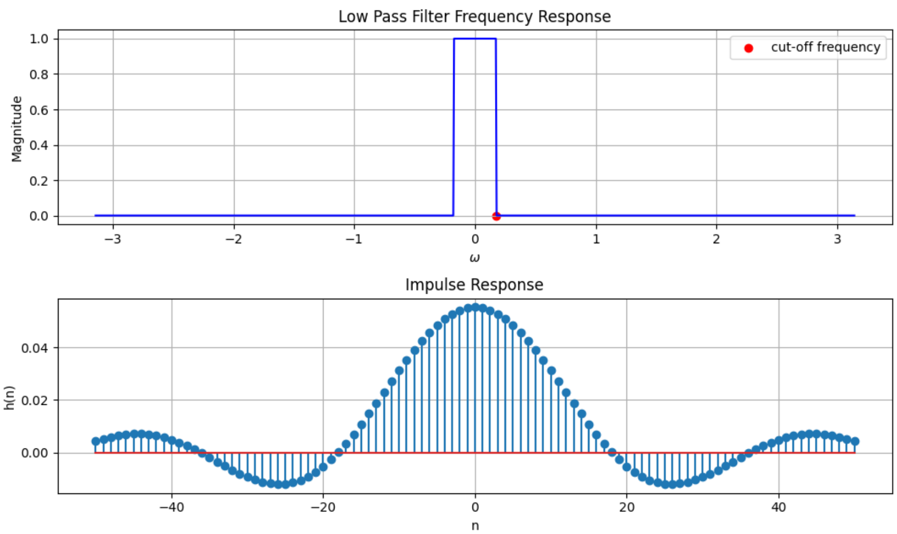

Figure 6: Frequency response and impulse response of an ideal Low Pass Filter

The impulse response of ideal Low Pass Filter is given by :

$$h(n) = \begin{cases} \frac{w_l}{\pi}, & \text{if } n = 0 \\ \frac{\sin(w_l n)}{n\pi}, & \text{if } n \neq 0 \end{cases} \tag{48}$$

From equation (48), we conclude that $h(n)$ for an ideal Low Pass Filter is not causal and cannot be made causal by introducing a finite delay. Additionally, $h(n)$ does not converge, resulting in an unstable system.

To address this, we move on to windowing the impulse response. A window function is chosen and multiplied to make the filter causal and stable. One such window function is the Kaiser window, defined as:

$$w(n) = \begin{cases} \frac{I_0\left[\beta N \sqrt{1-\left(\frac{n}{N}\right)^2}\right]}{I_0(\beta N)}, & -N \leq n \leq N, \quad \beta > 0 \\ 0 & \text{otherwise,} \end{cases}$$

1. N is chosen according to

$$N \geq \frac{A-8}{4.57\Delta\omega}, \tag{49}$$

   where $A = -20\log_{10}\delta$. Substituting the appropriate values from the design specifications, we obtain $A = 16.4782$ and $N \geq 48$.

2. β is chosen according to

$$\beta N = \begin{cases} 0.1102(A-8.7) & A > 50 \\ 0.5849(A-21)^{0.4} + 0.07886(A-21) & 21 \le A \le 50 \\ 0 & A < 21 \end{cases} \tag{50}$$

The window function is defined as :

$$w(n) = \begin{cases} 1, & \text{for } -48 \le n \le 48 \\ 0, & \text{otherwise} \end{cases} \tag{51}$$

Therefore the desired impulse response is :

$$h_{lp} = h_n w_n \tag{52}$$

$$h(n) = \begin{cases} \frac{\sin(w_l n)}{n\pi}, & \text{for } -48 \le n \le 48 \\ 0 & \text{otherwise} \end{cases} \tag{53}$$

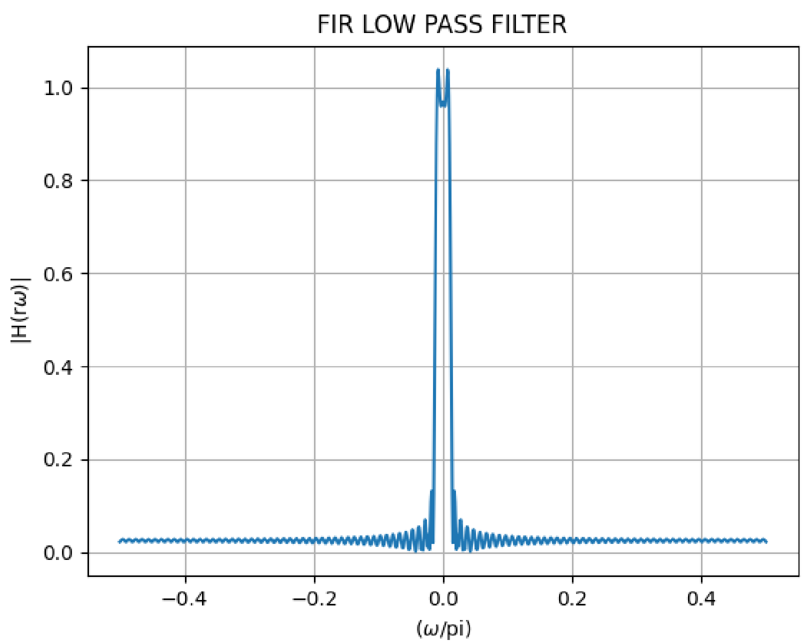

Figure 7: Magnitude Response of Low Pass Filter after using Kaiser Window

4.2 The Equivalent Band Pass Filter

A Band-Pass Filter (BPF) can be derived by subtracting the magnitude response of a Low-Pass Filter (LPF) with cutoff frequency ω_{p1} from another LPF magnitude response with cutoff frequency ω_{p2}.

$$h_{BP}(n) = \begin{cases} \frac{\sin(w_{p2}n)}{n\pi} - \frac{\sin\left(\omega_{p1}n\right)}{n\pi}, & \text{for } n \neq 0 \\ \frac{\omega_{p2}-\omega_{p1}}{\pi} & \text{for} n = 0 \end{cases} \tag{54}$$

$$\frac{\sin(\omega_{p2}n)}{n\pi} - \frac{\sin\left(\omega_{p1}n\right)}{n\pi} = 2\cos\left(\frac{\omega_{p2}n + \omega_{p1}n}{2}\right)\sin\left(\frac{\omega_{p2}n - \omega_{p1}n}{2}\right) \tag{55}$$

$$= \frac{2\cos(0.365n\pi)\sin(0.025n\pi)}{n\pi} \tag{56}$$

Multipying by window function we get :

$$h_{BP}(n) = \begin{cases} \frac{2\cos(0.365n\pi)\sin(0.025n\pi)}{n\pi}, & \text{for } -48 \leq n \leq 48 \\ 0 & \text{otherwise} \end{cases} \tag{57}$$

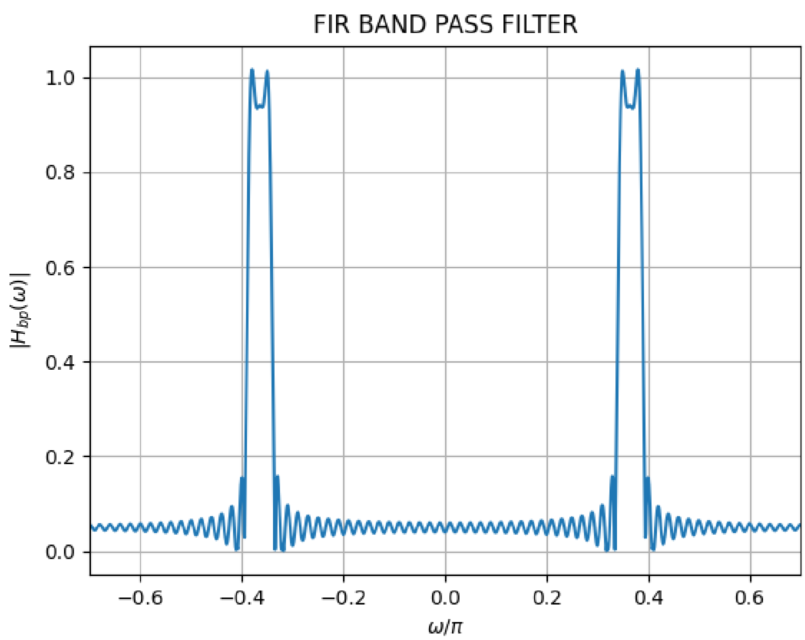

Figure 8: Magnitude Response of Band Pass Filter after using Kaiser Window