GATE:2021 - EC 48

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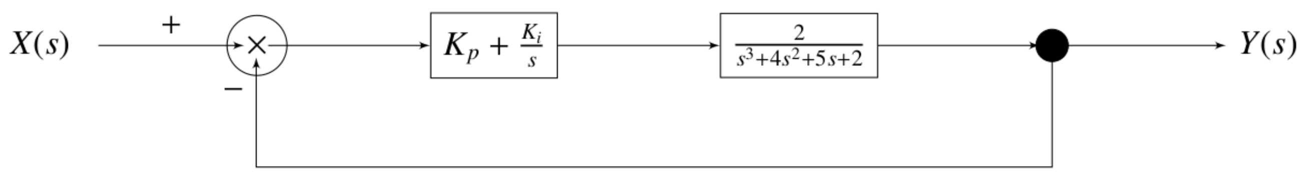
I. Question

A unity feedback system that uses proportional-integral (PI) control is shown in the figure. The stability



of the overall system is controlled by tuning the PI control parameters K_p and K_i. The maximum value of K_i that can be chosen so as to keep the overall system stable or, in the worst case, marginally stable (*rounded off to three decimal places*) is? (GATE EC 2021)

Solutions :

Symbols	Description	Values
$P(s)$	Plant transfer function	$\frac{2}{s^3+4s^2+5s+2}$
$C(s)$	PI controller transfer function	$K_p + \frac{K_i}{s}$
$G(s)$	Closed loop transfer function	$\frac{P(s)C(s)}{1+P(s)C(s)}$
$x_{(m,n)}$	Element in m^{th} row and n^{th} column in Routh array $(m > 2)$	$\frac{x_{(m-1,n)}x_{(m-2,n+1)}-x_{(m-1,n+1)}x_{(m-2,n)}}{x_{(m-1,n)}}$

TABLE I
PARAMETERS, DESCRIPTIONS, AND VALUES

From table I, the characteristic equation is given as:

$$1 + C(s)P(s) = 0 \tag{1}$$

$$1 + \left(K_p + \frac{K_i}{s}\right)\left(\frac{2}{s^3 + 4s^2 + 5s + 2}\right) = 0 \tag{2}$$

Rearranging the terms

$$s^4 + 4s^3 + 5s^2 + (2 + 2K_p)s + 2K_i = 0 \tag{3}$$

For the system to be stable, there must be no sign changes in the first coloumn of the routh array for the above equation. From I

$$\begin{array}{c|ccc} s^4 & 1 & 5 & 2K_i \\ s^3 & 4 & (2+2K_p) & 0 \\ s^2 & \frac{18-2K_p}{4} & 2K_i & 0 \\ s^1 & \frac{\left(\frac{18-2K_p}{4}\right)(2+2K_p)-8K_i}{\frac{18-2K_p}{4}} & 0 & 0 \\ s^0 & 2K_i & 0 & 0 \end{array} \tag{4}$$

(5)

$$\frac{18 - 2K_p}{4} > 0 \tag{6}$$

$$\implies K_p < 9 \tag{7}$$

$$\frac{\left(\frac{18-2K_p}{4}\right)\left(2 + 2K_p\right) - 8K_i}{\frac{18-2K_p}{4}} > 0 \tag{8}$$

$$K_i > 0 \tag{9}$$

For marginal stability, one of the above constraints must assume equality, so we assume 3 cases while simultaneously maximising K_i if necessary.

1) $K_p = 9$
   Checking if (8) and (9) hold. Limits are introduced to deal with (8)

$$\left(\lim_{K_p \to 9^-} \frac{\left(\frac{18-2K_p}{4}\right)\left(2 + 2K_p\right) - 8K_i}{\frac{18-2K_p}{4}} > 0\right) \cap (K_i > 0) \tag{10}$$

$$\left(\lim_{K_p \to 9^-} -8K_i > 0\right) \cap (K_i > 0) \tag{11}$$

$$\implies K_p = 9 \forall K_i \epsilon (\phi) \tag{12}$$

2) $K_i = 0$
   Checking if (7) and (8) hold

$$\left(\left(\frac{18 - 2K_p}{4}\right)\left(2 + 2K_p\right) > 0\right) \cap \left(K_p < 9\right) \tag{13}$$

$$\implies K_i = 0 \forall K_p \epsilon (-1, 9) \tag{14}$$

3) $\frac{\left(\frac{18-2K_p}{4}\right)\left(2+2K_p\right)-8K_i}{\frac{18-2K_p}{4}} = 0$
   Checking if (7) and (9) hold while maximising K_i.

$$\left(\frac{18 - 2K_p}{4}\right)\left(2 + 2K_p\right) = 8K_i \tag{15}$$

$$-K_p^2 + 8K_p + 9 = 8K_i \tag{16}$$

   Since the L.H.S is a downward parabola, and it's vertex ($K_p = 4$) satisfies (7):

$$K_i = 3.125 \forall K_p < 9 \tag{17}$$

Based on the all the three cases, it is concluded that the maximum value of K_1 is 3.125, $\forall K_p < 9$.