GATE:2022 - CE 48

EE23BTECH11025 - Anantha Krishnan

I. QUESTION

Consider the differential equation

$$\frac{dy}{dx} = 4(x+2) - y$$

For the initial condition $y = 3$ at $x = 1$, the value of y at $x = 1.4$ obtained using Euler's method with a step-size of 0.2 is ? (*round off to one decimal place*) (GATE CE 2022)

Solutions :

Symbols	Description	Values
R	Residue Formula	$\frac{1}{(m-1)!} \lim_{s \to a} \frac{d^{m-1}}{ds^{m-1}} \left((s-a)^m f(s) e^{st}\right)$
$\phi(x)$	Transformation of $y(x)$	$y(x-1)$
$g(x)$	Euler's Approximated function of f(x)	$g_{(n-1)}(x) + hf'(x_{n-1}, y_{n-1})$
h	Step-size	0.2

TABLE I
PARAMETERS, DESCRIPTIONS, AND VALUES

1) Solution of the differential:
   Applying the transformation from table I and laplace transform

$$s\mathcal{L}(\phi(x)) - \phi(0) = 4\left(\frac{1}{s^2} + \frac{2}{s}\right) - \mathcal{L}(\phi(x)) \tag{1}$$

$$\mathcal{L}(\phi(x)) = \frac{3}{s+1} + 4\left(\frac{1}{s^2(s+1)} + \frac{2}{s(s+1)}\right) \tag{2}$$

$$= \frac{-1}{s+1} + \frac{4}{s} + \frac{4}{s^2} \tag{3}$$

2) Inverse Laplace Transform:
   Using Bromwich integrals and extension of Jordans lemma :

$$\phi(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \mathcal{L}(\phi(x)) e^{st}\, dt, c > 0 \tag{4}$$

$$= \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \left(\frac{-1}{s+1} + \frac{4}{s} + \frac{4}{s^2}\right) e^{st}\, dt \tag{5}$$

Here, the poles $s = -1$ (non repeated, $m = 1$) and $s = 0$ (repeated, $m = 2$) lie inside a semicircle for some $c > 0$. Using method of residues from I:

$$\begin{aligned}
R_1 &= \lim_{s\to -1}\left((s+1)\left(\frac{-1}{s+1}\right)e^{st}\right) && (6)\\
&= -e^{-t} && (7)\\
R_2 &= \lim_{s\to 0}\left((s)\left(\frac{4}{s}\right)e^{st}\right) && (8)\\
&= 4 && (9)\\
R_3 &= \frac{1}{(1)!}\lim_{s\to 0}\frac{d}{dz}\left((s)^2\left(\frac{4}{s^2}\right)e^{st}\right) && (10)\\
&= 4t && (11)\\
\phi(t) &= R_1 + R_2 + R_3 && (12)\\
&= -e^{-t} + 4 + 4t && (13)
\end{aligned}$$

Reverting to $y(x)$, we get :

$$y(x) = -e^{-t+1} + 4 + 4\,(t-1) \tag{14}$$

Now, approaching to $y(1.4)$ using euler's approximation. Let the approximated function be $g(x)$

$$\begin{aligned}
g_{(1.2)} &= 3 + (0.2)\, f'\,(1,3) && (15)\\
&= 4.8 && (16)\\
g_{(1.4)} &= 4.8 + (0.2)\, f'\,(1.2, 4.8) && (17)\\
&= 6.4 && (18)\\
&\implies y(1.4) \approx 6.4 && (19)
\end{aligned}$$

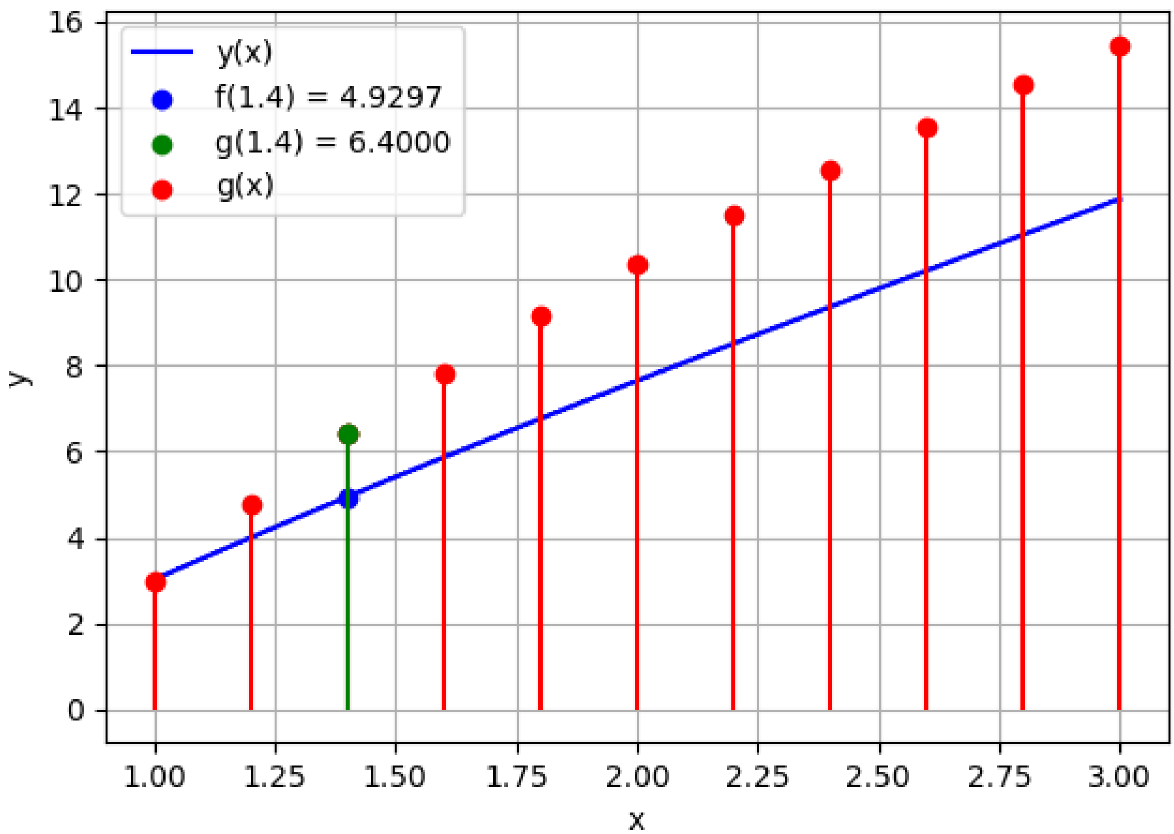

Fig. 1. Euler's approximated function vs Original function