3. Indefinite Integrals

We now want to ask what function we differentiated to get the function $f'(x)$.
We show the notation with the following example:

$$\int x^4 + 3x - 9 \; dx$$

Where:

- The function to integrate is the one between $\int$ and dx.
- dx indicates the variable selected for integration (here it's x).

Properties

Factoring out multiplicative constants

$$\int kf(x) \; dx \quad = \quad k \int f(x) \; dx$$

Also works if $k = -1$

$$\int -f(x) \; dx \quad = \quad -\int f(x) \; dx$$

Sum/Difference property

$$\int f(x) \pm g(x) \; dx \quad = \quad \int f(x) \; dx \pm \int g(x) \; dx$$

1. $\int k\, f(x) \; dx = k \int f(x) \; dx$ where k is any number. So, we can factor multiplicative constants out of indefinite integrals.

   See the **Proof of Various Integral Formulas** section of the Extras chapter to see the proof of this property.

2. $\int -f(x) \; dx = -\int f(x) \; dx$. This is really the first property with $k = -1$ and so no proof of this property will be given.

3. $\int f(x) \pm g(x) \; dx = \int f(x) \; dx \pm \int g(x) \; dx$. In other words, the integral of a sum or difference of functions is the sum or difference of the individual integrals. This rule can be extended to as many functions as we need.

   See the **Proof of Various Integral Formulas** section of the Extras chapter to see the proof of this property.

How to solve (simple integrals)

If the integral is simple, we just guess what the function originated from, by using logic and by knowing how derivatives work. (and adding c at the end)

Constant term

We always add a constant term c in the final result, because any constant will be eliminated by differentiating, and we can't know what it was originally.

Example

$$\int x^4 + 3x - 9$$

We know that:

- $\frac{d}{dx}(\frac{1}{5}x^5) = x^4$
- $\frac{d}{dx}(\frac{3}{2}x^2) = 3x$
- $\frac{d}{dx}(-9x) = -9$

So the original function was:

$$\frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + c$$

Substitution method

Whenever we have an integral of the form:

$$\int f(g(x)) \cdot g'(x)\ dx$$

We can use the following rule to solve it:

$$\int f(g(x)) \cdot g'(x)\ dx \quad = \quad \int f(u)\ du$$

knowing that:

$$du = g'(x) \cdot dx$$

Then you can solve the simplified integral and substitute u with it's actual value, $g(x)$.

Hint

$$\int f(\underbrace{g(x)}_{u}) \cdot \underbrace{g'(x)\ dx}_{du} \quad = \quad \int f(u)\ du$$

There are two key points to this:

- Turning the inner function to u
- Converting dx to du (swapping the differential).

Hint

In order to swap the differential, you need to solve for dx and substitute the result in the integral.

Example

If $du = -8x\ dx$, and the integral is $\int x \cdot u^{-\frac{1}{2}}\ dx$, then it becomes $\int u^{-\frac{1}{2}} \cdot (-\frac{1}{8})\ du$

Example

(a) $\displaystyle\int \left(1 - \frac{1}{w}\right) \cos(w - \ln w)\, dw$ ***Hide Solution*** ▾

In this case it looks like we have a cosine with an inside function and so let's use that as the substitution.

$$u = w - \ln w \qquad du = \left(1 - \frac{1}{w}\right) dw$$

So, as with the first example we worked the stuff in front of the cosine appears exactly in the differential. The integral is then,

$$\begin{aligned}\int \left(1 - \frac{1}{w}\right) \cos(w - \ln w)\, dw &= \int \cos(u)\, du \\ &= \sin(u) + c \\ &= \sin(w - \ln w) + c\end{aligned}$$

Don't forget to go back to the original variable in the problem.

Example

(b) $\displaystyle\int 3\,(8y - 1)\, \mathbf{e}^{4y^2 - y}\, dy$ ***Hide Solution*** ▾

Again, it looks like we have an exponential function with an inside function (*i.e.* the exponent) and it looks like the substitution should be,

$$u = 4y^2 - y \qquad du = (8y - 1)\, dy$$

Now, with the exception of the 3 the stuff in front of the exponential appears exactly in the differential. Recall however that we can factor the 3 out of the integral and so it won't cause any problems. The integral is then,

$$\begin{aligned}\int 3\,(8y - 1)\, \mathbf{e}^{4y^2 - y}\, dy &= 3\int \mathbf{e}^{u}\, du \\ &= 3\mathbf{e}^{u} + c \\ &= 3\mathbf{e}^{4y^2 - y} + c\end{aligned}$$

Example

(c) $\int x^2 \left(3-10x^3\right)^4 dx$ *Hide Solution* ▾

In this case it looks like the following should be the substitution.

$$u = 3 - 10x^3 \qquad du = -30x^2\,dx$$

Okay, now we have a small problem. We've got an x^2 out in front of the parenthesis but we don't have a "-30". This is not really the problem it might appear to be at first. We will simply rewrite the differential as follows.

$$x^2\,dx = -\frac{1}{30}\,du$$

With this we can now substitute the $x^2\,dx$ away. In the process we will pick up a constant, but that isn't a problem since it can always be factored out of the integral.

We can now do the integral.

$$\begin{aligned}\int x^2 \left(3-10x^3\right)^4 dx &= \int \left(3-10x^3\right)^4 x^2\,dx \\ &= \int u^4 \left(-\frac{1}{30}\right) du \\ &= -\frac{1}{30}\left(\frac{1}{5}\right) u^5 + c \\ &= -\frac{1}{150}\left(3-10x^3\right)^5 + c\end{aligned}$$

Note that in most problems when we pick up a constant as we did in this example we will generally factor it out of the integral in the same step that we substitute it in.

Example

(c) $\int \frac{x}{\sqrt{1-4x^2}}\,dx$ *Hide Solution* ▾

Here, if we ignore the root we can again see that the derivative of the stuff under the radical differs from the numerator by only a multiplicative constant and so we'll use that as the substitution.

$$u = 1 - 4x^2 \qquad du = -8x\,dx \qquad \Rightarrow \qquad x\,dx = -\frac{1}{8}du$$

The integral is then,

$$\begin{aligned}\int \frac{x}{\sqrt{1-4x^2}}\,dx &= -\frac{1}{8}\int u^{-\frac{1}{2}}\,du \\ &= -\frac{1}{8}(2)\,u^{\frac{1}{2}} + c \\ &= -\frac{1}{4}\sqrt{1-4x^2} + c\end{aligned}$$

Pattern recognition in substitution method

This shit is all about recognizing patterns.
For example when you have a fraction, you can **try to make the numerator 1** and make it **match the derivative of something**, for example a $\tan$.
Or try to make the denominator 1 so it's **not a fraction anymore** (easier to solve).

Hint

Usually, things can get eliminated or simplified when you convert dx to du.

When you select u it's usually to kill it's derivative in the numerator, because it will be of the form $dx = \frac{1}{\text{thing you want to get rid of}} \cdot du$.

⚠ **Warning**

Be sure to know and remember a lot of the derivatives of the trigonometric functions.

Turning fraction to $\tan$

$$\begin{aligned}\int \frac{\mathrm{e}^{2t}}{1+\mathrm{e}^{4t}}\,dt &= \frac{1}{2}\int \frac{1}{1+u^2}\,du\\ &= \frac{1}{2}\tan^{-1}(u)+c\\ &= \frac{1}{2}\tan^{-1}\left(\mathrm{e}^{2t}\right)+c\end{aligned}$$

Turning fraction to natural logarithm

$$\begin{aligned}\int \frac{1}{x\,\ln x}\,dx &= \int \frac{1}{u}\,du\\ &= \ln|u|+c\\ &= \ln|\ln x|+c\end{aligned}$$

Killing denominator from fraction

$$\begin{aligned}\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx &= \int u\,du\\ &= \frac{1}{2}u^2+c\\ &= \frac{1}{2}\left(\sin^{-1}x\right)^2+c\end{aligned}$$

$$u = \sin^{-1}(x) \qquad du = \frac{1}{\sqrt{1-x^2}}dx$$

Integration by parts

We remember the product rule:

$$\frac{d}{dx} f \cdot g = f'g + gf'$$

If we wanted to find the antiderivative, that would be:

$$f \cdot g = \int f'g \, dx + \int fg' \, dx$$

Now we introduce the actual technique, by rearranging the formula above:

$$\int fg' \, dx = f \cdot g - \int f'g \, dx$$

Hint

Looking at this, you might think:

"Isn't this useless? we are substituting the integral of a product with **ANOTHER integral of a product**".

But the usefulness is **what function you choose** to be f.

Example

$$\int \underbrace{x}_{d(x)=1} \cos(x)$$

Basically it's useful when f' **becomes a constant**.