Central limit theorem and Application

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PREREQUISITES

1. Normal distribution
2. Central limit theorem
3. Poisson distribution and some properties

Normal distribution

Normal (Gaussian) distribution it's probability density given by

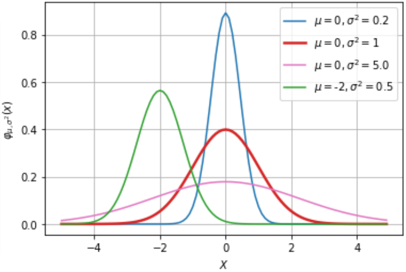
$$F(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{1}$$

The parameter μ and σ are mean and standard deviation respectively

Standard normal distribution

Normal distribution with $\mu = 0$ and $\sigma^2 = 1$ is Standard normal distribution . It's PDF denoted by φ

$$\varphi = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{2}$$



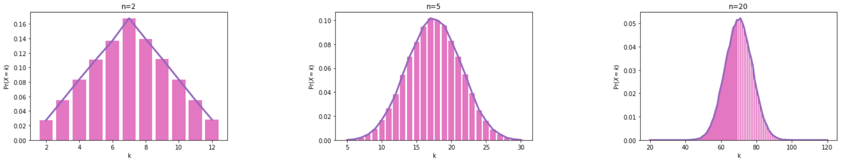
Central limit theorem

When independent random variables are added, their properly normalized sum tends toward a normal distribution even if the original variables themselves are not normally distributed

Let $X_1, X_2, .., X_n$ are i.i.d and $\hat{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ has mean μ and variance σ^2

$$Z = \lim_{n\to\infty} \frac{\hat{X} - \mu}{\sigma} \tag{3}$$

then Z approaches standard normal distribution with $\mu = 0$ and $\sigma^2 = 1$



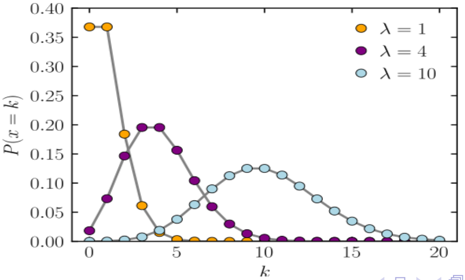
plots to show Central limit theorem on event of rolling n dices

Poisson distribution

Poisson distribution is used to model the number of events occurring within a given time interval

$$\Pr(X = k) = \frac{e^{-\lambda}\lambda^k}{k!} \quad \text{for } k = 0, 1, 2, ... \tag{4}$$

Here λ is shape parameter indicating average number of events in given time interval



Summation of Poisson random variable

Let X and Y are independent Poisson random variables with parameter α and β respectively then $Z = X + Y$ is a Poisson random variable with parameter $\alpha + \beta$

proof:

$$\Pr(Z = z) = \sum_{k=0}^{z} \Pr(X = k) \times \Pr(Y = z - k) \tag{5}$$

$$= \sum_{k=0}^{z} \frac{e^{-\alpha}\alpha^{k}}{k!} \times \frac{e^{-\beta}\beta^{z-k}}{(z-k)!} \tag{6}$$

$$= \frac{e^{-(\alpha+\beta)}}{z!} \sum_{k=0}^{z} \binom{z}{k} \alpha^{k}\beta^{z-k} \tag{7}$$

$$= \frac{e^{-\alpha+\beta}(\alpha+\beta)^{z}}{z!} \tag{8}$$

Question:

UGC/MATH (math Dec 2017), Q.107

For $n \geq 1$, let X_n be a Poisson random variable with mean n^2. Which of the following's are equal to $\frac{1}{\sqrt{2\pi}} \int_{2}^{\infty} e^{-x^2/2}\, dx$

1. $\lim_{n\to\infty} \Pr\left(X_n > (n+1)^2\right)$
2. $\lim_{n\to\infty} \Pr\left(X_n \leqslant (n+1)^2\right)$
3. $\lim_{n\to\infty} \Pr\left(X_n < (n-1)^2\right)$
4. $\lim_{n\to\infty} \Pr\left(X_n < (n-2)^2\right)$

Solution:

X_n is a Poisson random variable with parameter n^2 and Let Y_i be a Poisson random variable with parameter 1, for $i \in \left(1, n^2\right)$

By additive property of Poisson distribution

$$\sum_{i=1}^{n^2} Y_i = X_n \tag{9}$$

By central limit theorem

$$\lim_{n\to\infty} \frac{Y_1 + Y_2 + .. + Y_{n^2} - n^2}{n} = \mathcal{N}(0,1) \tag{10}$$

$$\implies \lim_{n\to\infty} \frac{X_n - n^2}{n} = \mathcal{N}\left(0,1\right) \tag{11}$$

Here, $\mathcal{N}\left(0,1\right)$ is normal distribution with zero mean and unit variance

$$\Pr\left(X_n > k\right) = \Pr\left(\frac{X_n - n^2}{n} > \frac{k - n^2}{n}\right) \tag{12}$$

$$= \Pr\left(\mathcal{N}\left(0,1\right) > \frac{k - n^2}{n}\right) \tag{13}$$

$$= Q\left(\frac{k - n^2}{n}\right) \tag{14}$$

$$Q(X) = 1 - Q(-x) \tag{15}$$

$$\because \frac{1}{\sqrt{2\pi}} \int_{-x}^{\infty} e^{-x^2/2}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2}\, dx \quad \text{and} \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = 1$$

$$\frac{1}{\sqrt{2\pi}} \int_{2}^{\infty} e^{-x^2/2}\, dx = Q(2) \tag{16}$$

$$\frac{1}{\sqrt{2\pi}} \int_{2}^{\infty} e^{-x^2/2}\, dx = 1 - Q(-2) \tag{17}$$

1. $\lim_{n\to\infty} \Pr\left(X_n > (n+1)^2\right)$

$$\lim_{n\to\infty} \Pr\left(X_n > (n+1)^2\right) = \lim_{n\to\infty} Q\left(\frac{(n+1)^2 - n^2}{n}\right) \tag{18}$$

$$= Q\left(2\right) \tag{19}$$

$\therefore$ **Option 1 is correct**

2 $\lim_{n\to\infty} \Pr\left(X_n \leqslant (n+1)^2\right)$

$$\begin{aligned}
\lim_{n\to\infty} \Pr\left(X_n \leqslant (n+1)^2\right) &= 1 - \lim_{n\to\infty} \Pr\left(X_n > (n+1)^2\right) && (20)\\
&= 1 - \lim_{n\to\infty} Q\left(\frac{(n+1)^2 - n^2}{n}\right) && (21)\\
&= 1 - Q(2) && (22)\\
&= Q(-2) > Q(2) && (23)
\end{aligned}$$

$\therefore$ **Option 2 is incorrect**

3 $\lim_{n\to\infty} \Pr\left(X_n < (n-1)^2\right)$

$$\begin{aligned}
\lim_{n\to\infty} \Pr\left(X_n < (n-1)^2\right) &= 1 - \lim_{n\to\infty} \Pr\left(X_n \geq (n-1)^2\right) && (24)\\
&= 1 - \lim_{n\to\infty} Q\left(\frac{(n-1)^2 - n^2}{n}\right) && (25)\\
&= 1 - Q(-2) && (26)\\
&= Q(2) && (27)
\end{aligned}$$

$\therefore$ **Option 3 is also correct**

4. $\lim_{n\to\infty} \Pr\left(X_n < (n-2)^2\right)$

$$\lim_{n\to\infty} \Pr\left(X_n < (n-2)^2\right) = 1 - \lim_{n\to\infty} \Pr\left(X_n \geq (n-2)^2\right) \tag{28}$$

$$= 1 - \lim_{n\to\infty} Q\left(\frac{(n-2)^2 - n^2}{n}\right) \tag{29}$$

$$= 1 - Q(-4) \tag{30}$$

$$= Q(4) < Q(2) \tag{31}$$

$\therefore$ **Option 4 is incorrect**