Assignment 2

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Download all python codes from

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https://github.com/Ganesh−RB/AI1103prob−and−
    randomvariables/Assignment3/codes
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and latex-tikz codes from

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https://github.com/Ganesh−RB/AI1103prob−and−
    randomvariables/Assignment3
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1 Problem

(GATE EC Q44)Consider a communication scheme where the binary valued signal X satisfies $\Pr(X = +1) = 0.75$ and $\Pr(X = -1) = 0.25$. The received signal $Y = X + Z$, where Z is a Gaussian random variable with zero mean and variance σ^2. The received signal Y is fed to the threshold detector. The output of the threshold detector $\hat{X}$ is:

$$\hat{X} = \begin{cases} +1 & Y > \tau \\ -1 & Y \leqslant \tau \end{cases} \tag{1.0.1}$$

To achieve minimum probability of error $P\{\hat{X} \neq X\}$, the thresholds τ should be

1) strictly positive
2) zero
3) strictly negative
4) strictly positive, zero or strictly negative depending on the nonzero value of σ^2

2 Solution

It is given that

$$\Pr(X = +1) = \frac{3}{4} \tag{2.0.1}$$

$$\Pr(X = -1) = \frac{1}{4} \tag{2.0.2}$$

Z is a Gaussian random variable with mean $\mu = 0$ and variance $= \sigma^2$

$$F_Z(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dz \tag{2.0.3}$$

$$F'_Z(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \tag{2.0.4}$$

As $Y = X + Z$

$$\Pr(Y \leqslant \tau | X = +1) = \Pr(1 + Z \leqslant \tau) \tag{2.0.5}$$

$$= F_Z(\tau - 1) \tag{2.0.6}$$

$$\Pr(Y > \tau | X = -1) = \Pr(-1 + Z > \tau) \tag{2.0.7}$$

$$= 1 - \Pr(Z \leqslant \tau + 1) \tag{2.0.8}$$

$$= 1 - F_Z(\tau + 1) \tag{2.0.9}$$

It follows from eqn (1.0.1) that

$$\Pr\left(\hat{X} = +1\right) = \Pr(Y > \tau) \tag{2.0.10}$$

$$\Pr\left(\hat{X} = -1\right) = \Pr(Y \leqslant \tau) \tag{2.0.11}$$

$$\Pr\left(\hat{X} = +1 | X = -1\right) = \Pr(Y > \tau | X = -1) \tag{2.0.12}$$

Now

$$\Pr\left(\hat{X} \neq X\right) = \Pr\left(\hat{X} = 1, X = -1\right) + \Pr\left(\hat{X} = -1, X = 1\right) \tag{2.0.13}$$

$$= \Pr(X = +1) \times \Pr\left(\hat{X} = -1 | X = +1\right) + \Pr(X = -1) \times \Pr\left(\hat{X} = +1 | X = -1\right) \tag{2.0.14}$$

By substitution from (2.0.6) and (2.0.9)

$$= \frac{3}{4} \times F_Z(\tau - 1) + \frac{1}{4} \times (1 - F_Z(\tau + 1)) \tag{2.0.15}$$

We have to Minimize $\Pr\left(\hat{X} \neq X\right)$

i.e. $\Pr'\left(\hat{X} \neq X\right) = 0$

$$\implies \frac{3}{4} \times (F_Z(\tau - 1))' - \frac{1}{4} \times (1 - F_Z(\tau + 1))' = 0 \tag{2.0.16}$$

$$\implies 3 \times F'_Z(\tau - 1) - F'_Z(\tau + 1) = 0 \tag{2.0.17}$$

$$\implies 3 \times \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\tau-1)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\tau+1)^2}{2\sigma^2}} \tag{2.0.18}$$

$$\implies e^{\frac{(\tau-1)^2 - (\tau+1)^2}{2\sigma^2}} = 3 \tag{2.0.19}$$

$$\implies e^{-\frac{2\tau}{\sigma^2}} = 3 \tag{2.0.20}$$

$$\implies \tau = \frac{-\sigma^2 \ln 3}{2} \tag{2.0.21}$$

$$\implies \tau < 0 \text{ for all nonzero values of } \sigma^2 \tag{2.0.22}$$

$\therefore$ option 3 is correct

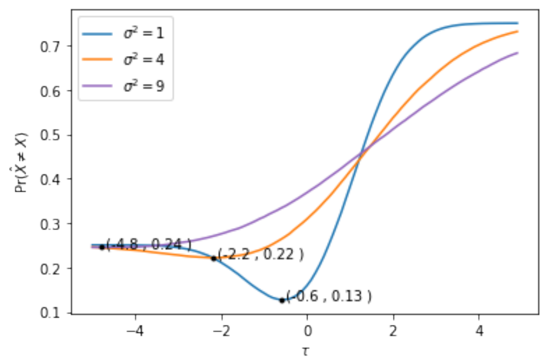

Fig. 1: Plot to show $\Pr\left(\hat{X} \neq X\right)$ is minimum at negative value of τ for all nonzero values of σ^2

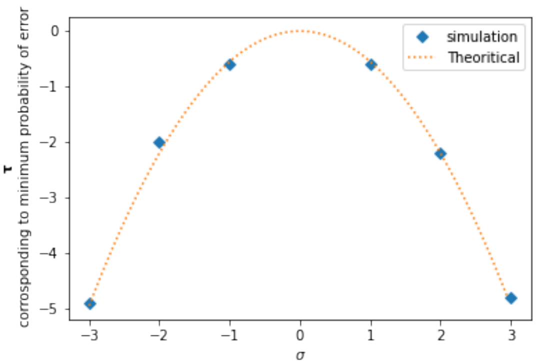

Fig. 2: Plot to show $\tau = -\frac{\sigma^2 \ln 3}{2}$ corresponds to minimum probability of error