ASSIGNMENT 5

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Download all python codes from

https://github.com/Ganesh−RB/AI1103prob−and−randomvariables/Assignment5/codes

and latex-tikz codes from

https://github.com/Ganesh−RB/AI1103prob−and−randomvariables/Assignment5

1 Problem

For $n \geq 1$, let X_n be a Poisson random variable with mean n^2. Which of the following's are equal to $\frac{1}{\sqrt{2\pi}} \int_2^\infty e^{-x^2/2}\,dx$

1) $\lim_{n\to\infty} \Pr\left(X_n > (n+1)^2\right)$
2) $\lim_{n\to\infty} \Pr\left(X_n \leqslant (n+1)^2\right)$
3) $\lim_{n\to\infty} \Pr\left(X_n < (n-1)^2\right)$
4) $\lim_{n\to\infty} \Pr\left(X_n < (n-2)^2\right)$

2 Solution

Let Y_i be a Poisson random variable with mean 1 for $i \in \left(1, n^2\right)$

By additive property of Poisson distribution

$$\sum_i^{n^2} Y_i = X_n \tag{2.0.1}$$

By central limit theorem

$$\lim_{n\to\infty} \frac{Y_1 + Y_2 + .. + Y_{n^2} - n^2}{n} = \mathcal{N}(0,1) \tag{2.0.2}$$

$$\implies \lim_{n\to\infty} \frac{X_n - n^2}{n} = \mathcal{N}(0,1) \tag{2.0.3}$$

Here, $\mathcal{N}(0,1)$ is normal distribution with unit mean and variance

Now

$$\Pr(X_n > k) = \Pr\left(\frac{X_n - n^2}{n} > \frac{k-n^2}{n}\right) \tag{2.0.4}$$

$$= \Pr\left(\mathcal{N}(0,1) > \frac{k-n^2}{n}\right) \tag{2.0.5}$$

$$= Q\left(\frac{k-n^2}{n}\right) \tag{2.0.6}$$

Here

$$Q(X) = 1 - Q(-x) \tag{2.0.7}$$

$$\because \frac{1}{\sqrt{2\pi}} \int_{-x}^{\infty} e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2}\,dx$$

$$\text{and} \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\,dx = 1$$

Also

$$\frac{1}{\sqrt{2\pi}} \int_2^\infty e^{-x^2/2}\,dx = Q(2) \tag{2.0.8}$$

1) $\lim_{n\to\infty} \Pr\left(X_n > (n+1)^2\right)$

$$\lim_{n\to\infty} \Pr\left(X_n > (n+1)^2\right) = \lim_{n\to\infty} Q\left(\frac{(n+1)^2 - n^2}{n}\right) \tag{2.0.9}$$

$$= Q(2) \tag{2.0.10}$$

$\therefore$ **Option 1 is correct**

2) $\lim_{n\to\infty} \Pr\left(X_n \leqslant (n+1)^2\right)$

$$\lim_{n\to\infty} \Pr\left(X_n \leqslant (n+1)^2\right) = 1 - \lim_{n\to\infty} \Pr\left(X_n > (n+1)^2\right) \tag{2.0.11}$$

$$= 1 - \lim_{n\to\infty} Q\left(\frac{(n+1)^2 - n^2}{n}\right) \tag{2.0.12}$$

$$= 1 - Q(2) \tag{2.0.13}$$

$$= Q(-2) > Q(2) \tag{2.0.14}$$

$\therefore$ **Option 2 is incorrect**

3) $\lim_{n\to\infty} \Pr\left(X_n < (n-1)^2\right)$

$$\lim_{n\to\infty} \Pr\left(X_n < (n-1)^2\right) = 1 - \lim_{n\to\infty} \Pr\left(X_n \geq (n-1)^2\right) \tag{2.0.15}$$

$$= 1 - \lim_{n\to\infty} Q\left(\frac{(n-1)^2 - n^2}{n}\right) \tag{2.0.16}$$

$$= 1 - Q(-2) \tag{2.0.17}$$

$$= Q(2) \tag{2.0.18}$$

$\therefore$ **Option 3 is also correct**

4) $\lim_{n\to\infty} \Pr\left(X_n < (n-2)^2\right)$

$$\lim_{n\to\infty} \Pr\left(X_n < (n-2)^2\right) = 1 - \lim_{n\to\infty} \Pr\left(X_n \geq (n-2)^2\right) \tag{2.0.19}$$

$$= 1 - \lim_{n\to\infty} Q\left(\frac{(n-2)^2 - n^2}{n}\right) \tag{2.0.20}$$

$$= 1 - Q(-4) \tag{2.0.21}$$

$$= Q(4) < Q(2) \tag{2.0.22}$$

$\therefore$ **Option 4 is incorrect**