Moment Generating Function

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Problem

Suppose X_1, X_2, X_3, X_4 are i.i.d random variables taking values 1 and -1 with probability $1/2$ each. Then $E\left(X_1 + X_2 + X_3 + X_4\right)^4$ equals

1. 40
2. 76
3. 16
4. 12

Solution

X_i, for $i \in \{1..4\}$ are i.i.d random variables with

$$\Pr(X_i = +1) = \frac{1}{2} \tag{1}$$

$$\Pr(X_i = -1) = \frac{1}{2} \tag{2}$$

Let $Y = X_1 + X_2 + X_3 + X_4$

$Y = k$	-4	-2	0	2	4
$p_Y(k)$	$1/16$	$1/4$	$3/8$	$1/4$	$1/16$

Table: pmf Y

Expectation

Expectation of discrete random variable X denoted with $E(X)$

$$E(x) = \sum_k k \times p_X(k) \tag{3}$$

$$E\left(X^n\right) = \sum_k k^n \times p_X(k) \tag{4}$$

$$E\left(Y^4\right) = \sum_k k^4 \times p_X(k) \tag{5}$$

$$= 256 \times \frac{1}{16} + 16 \times \frac{1}{4} + 0 \times \frac{3}{8} + 16 \times \frac{1}{4} + 256 \times \frac{1}{16} \tag{6}$$

$$E\left(Y^4\right) = 40 \tag{7}$$

Moment Generating Function

Definition (Moment)

n^{th} Moment of a random variable X is $E\left(X^n\right)$

Moments are useful in describing shape of any distribution

- First moment represent mean.
- Second represents variance.
- Third represent skewness, indicates any asymmetric 'leaning' to either left or right.

Moment Generating Function

Definition (Moment generating function)

Moment Generating Function (MGF) of a random variable X is a function $M_X(t)$ defined as

$$M_X(t) = E\left(e^{tX}\right) \tag{8}$$

$M_X(t)$ exist, if there exists *a* positive constant a such that $M_X(t)$ is finite for all $t \in [-a, a]$

If X is a discrete random variable

$$M_X(t) = \sum_{k=-\infty}^{\infty} e^{tk} \times p_X(k) \tag{9}$$

Taylor series

Taylor series for e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \tag{10}$$

Theorem

n^{th} Moment of X $(E(X^n))$ is coefficient of $\frac{t^n}{n!}$ in Taylor expansion of $M_X(t)$

Proof.

Taylor series of e^{tX}

$$e^{tX} = \sum_{k=0}^{\infty} \frac{X^k t^k}{k!} \tag{11}$$

Taking expectation on both side

$$E\left(e^{tX}\right) = \sum_{k=0}^{\infty} E\left(X^k\right) \frac{t^k}{k!} \tag{12}$$

$\therefore E\left(X^n\right)$ is coefficient of $\frac{t^n}{n!}$ in Taylor expansion of $M_X(t) \equiv E\left(e^{tX}\right)$ $\square$

Solution using MGF

X_i are i.i.d random variable for $i \in \{1, 2, 3, 4\}$ with,

$$\Pr(X_i = +1) = \frac{1}{2} \tag{13}$$

$$\Pr(X_i = -1) = \frac{1}{2} \tag{14}$$

By defining Moment generating function of X_i

$$M_{X_i}(t) = \frac{1}{2} \times e^{-t} + \frac{1}{2} \times e^{t} = \frac{e^{t} + e^{-t}}{2} \tag{15}$$

Let $Y = \sum_{i=0}^{4} X_i$, then by convolution

$$M_Y(t) = \prod_{i=1}^{4} M_{X_i}(t) \tag{16}$$

$$M_Y(t) = \left(\frac{e^t + e^{-t}}{2}\right)^4 \tag{17}$$

$$M_Y(t) = \frac{e^{4t} + 4e^{2t} + 6 + 4e^{-2t} + e^{-4t}}{16} \tag{18}$$

By using Taylor expansion of e^{4t}, e^{2t}, e^{-2t} and e^{-4t}

$$M_Y(t) = 1 + \frac{1}{16}\left(\sum_{k=1}^{\infty}\left(4^k + 4(2)^k + 4(-2)^k + (-4)^k\right)\frac{t^k}{k!}\right) \tag{19}$$

By Theorem 3 and eqn: (19)

$$E\left(Y^n\right) = \frac{4^n + 4(2)^n + 4(-2)^n + (-4)^n}{16} \tag{20}$$

$$E\left(Y^4\right) = \frac{4^4 + 4(2)^4 + 4(-2)^4 + (-4)^4}{16} \tag{21}$$

$$E\left(Y^4\right) = \frac{640}{16} = 40 \tag{22}$$

$$\therefore \mathrm{E}\left(\mathrm{X}_1 + \mathrm{X}_2 + \mathrm{X}_3 + \mathrm{X}_4\right)^4 = 40$$