Assignment 4

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Download all python codes from

```
https://github.com/Ganesh−RB/AI1103prob−and−
    randomvariables/Assignment4/codes
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and latex-tikz codes from

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https://github.com/Ganesh−RB/AI1103prob−and−
    randomvariables/Assignment4
```

1 Problem

CSIR UGC NET EXAM (Dec 2012) Q 51
Suppose X_1, X_2, X_3, X_4 are i.i.d random variables taking values 1 and −1 with probability 1/2 each. Then $E(X_1 + X_2 + X_3 + X_4)^4$ equals

1) 4
2) 76
3) 16
4) 12

2 Solution

Definition 1 (Taylor series for e^x)**.**

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \tag{2.0.1}$$

Definition 2 (Moment generating function)**.** *For a discrete random variable X*

$$M_X(t) \equiv E\left(e^{tX}\right) = \sum_{k=-\infty}^{\infty} e^{tk} \times p_X(k) \tag{2.0.2}$$

Theorem 2.1. *n^{th} Moment of X ($\equiv E(X^n)$) is coefficient of $\frac{t^n}{n!}$ in Taylor expansion of $M_X(t)$*

Proof.

$$e^{tX} = \sum_{k=0}^{\infty} \frac{(tX)^k}{k!} \tag{2.0.3}$$

$$= \sum_{k=0}^{\infty} X^k \frac{t^k}{k!} \tag{2.0.4}$$

$$E\left(e^{tX}\right) = \sum_{k=0}^{\infty} E\left(X^k\right) \frac{t^k}{k!} \tag{2.0.5}$$

$\therefore E(X^n)$ is coefficient of $\frac{t^n}{n!}$ in Taylor expansion of $M_X(t) \equiv E\left(e^{tX}\right)$ □

X_i are i.i.d random variable for $i \in \{1, 2, 3, 4\}$ with,

$$\Pr(X_i = +1) = \frac{1}{2} \tag{2.0.6}$$

$$\Pr(X_i = -1) = \frac{1}{2} \tag{2.0.7}$$

Lemma 2.1 (Moment generating function of X_i)**.**

$$M_{X_i}(t) = \frac{1}{2} \times e^{-t} + \frac{1}{2} \times e^{t} = \frac{e^t + e^{-t}}{2} \tag{2.0.8}$$

Let $Y = \sum_{i=0}^{4} X_i$, then by convolution

$$M_Y(t) = \prod_{i=1}^{4} M_{X_i}(t) \tag{2.0.9}$$

$$M_Y(t) = \left(\frac{e^t + e^{-t}}{2}\right)^4 \tag{2.0.10}$$

$$M_Y(t) = \frac{e^{4t} + 4e^{2t} + 6 + 4e^{-2t} + e^{-4t}}{16} \tag{2.0.11}$$

By using Taylor expansion of e^{4t}, e^{2t}, e^{-2t} and e^{-4t}

Corollary 2.2 (Taylor expansion of $M_Y(t)$)**.**

$$M_Y(t) = 1 + \frac{1}{16}\left(\sum_{k=1}^{\infty} 4^k + 4(2)^k + 4(-2)^k + (-4)^k \frac{t^k}{k!}\right) \tag{2.0.12}$$

By Theorem 2.1 and eqn: (2.0.12)

Theorem 2.3. *for $n \geq 1$*

$$E(Y^n) = \frac{4^n + 4(2)^n + 4(-2)^n + (-4)^n}{16} \tag{2.0.13}$$

Thus follows

$$E\left(Y^4\right) = \frac{4^4 + 4(2)^4 + 4(-2)^4 + (-4)^4}{16} \tag{2.0.14}$$

$$E\left(Y^4\right) = \frac{640}{16} = 40 \tag{2.0.15}$$

$$\therefore \mathbf{E(X_1 + X_2 + X_3 + X_4)^4 = 40}$$