

Assignment 2

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Download all python codes from

<https://github.com/Ganesh-RB/AI1103prob-and-randomvariables/Assignment3/codes>

and latex-tikz codes from

<https://github.com/Ganesh-RB/AI1103prob-and-randomvariables/Assignment3>

1 PROBLEM

(GATE EC Q44) Consider a communication scheme where the binary valued signal X satisfies $P\{X = +1\} = 0.75$ and $P\{X = -1\} = 0.25$. The received signal $Y = X + Z$, where Z is a Gaussian random variable with zero mean and variance σ^2 . The received signal Y is fed to the threshold detector. The output of the threshold detector \hat{X} is:

$$\hat{X} = \begin{cases} +1 & Y > \tau \\ -1 & Y \leq \tau \end{cases} \quad (1.0.1)$$

To achieve minimum probability of error $P\{\hat{X} \neq X\}$, the thresholds τ should be

- | | |
|----------------------|------------------------------------|
| 1) strictly positive | 4) strictly positive, zero or |
| 2) zero | strictly negative depending |
| 3) strictly negative | on the nonzero value of σ^2 |

2 SOLUTION

It is given that

$$\Pr(X = +1) = \frac{3}{4} \quad (2.0.1)$$

$$\Pr(X = -1) = \frac{1}{4} \quad (2.0.2)$$

Z is a Gaussian random variable with mean $\mu = 0$ and variance $= \sigma^2$

$$F_Z(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}} dz \quad (2.0.3)$$

$$F'_Z(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}} \quad (2.0.4)$$

As $Y = X + Z$

$$\Pr(Y \leq \tau | X = +1) = \Pr(1 + Z \leq \tau) \quad (2.0.5)$$

$$= F_Z(\tau - 1) \quad (2.0.6)$$

$$\Pr(Y > \tau | X = -1) = \Pr(-1 + Z > \tau) \quad (2.0.7)$$

$$= 1 - \Pr(Z \leq \tau + 1) \quad (2.0.8)$$

$$= 1 - F_Z(\tau + 1) \quad (2.0.9)$$

It follows from eqn (1.0.1) that

$$\Pr(\hat{X} = +1) = \Pr(Y > \tau) \quad (2.0.10)$$

$$\Pr(\hat{X} = -1) = \Pr(Y \leq \tau) \quad (2.0.11)$$

$$\Pr(\hat{X} = +1 | X = -1) = \Pr(Y > \tau | X = -1) \quad (2.0.12)$$

Now

$$\Pr(\hat{X} \neq X) = \Pr(\hat{X} = 1, X = -1) \quad (2.0.13)$$

$$+ \Pr(\hat{X} = -1, X = 1)$$

$$= \Pr(X = +1) \times \Pr(\hat{X} = -1 | X = +1)$$

$$+ \Pr(X = -1) \times \Pr(\hat{X} = +1 | X = -1) \quad (2.0.14)$$

By substitution from (2.0.6) and (2.0.9)

$$= \frac{3}{4} \times F_Z(\tau - 1) + \frac{1}{4} \times (1 - F_Z(\tau + 1)) \quad (2.0.15)$$

We have to Minimize $\Pr(\hat{X} \neq X)$

$$i.e. \quad \Pr'(\hat{X} \neq X) = 0$$

$$\Rightarrow \frac{3}{4} \times (F_Z(\tau - 1))' - \frac{1}{4} \times (1 - F_Z(\tau + 1))' = 0 \quad (2.0.16)$$

$$\Rightarrow 3 \times F'_Z(\tau - 1) - F'_Z(\tau + 1) = 0 \quad (2.0.17)$$

$$\Rightarrow 3 \times \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\tau-1)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\tau+1)^2}{2\sigma^2}} \quad (2.0.18)$$

$$\Rightarrow e^{\frac{(\tau-1)^2 - (\tau+1)^2}{2\sigma^2}} = 3 \quad (2.0.19)$$

$$\Rightarrow e^{-\frac{2\tau}{\sigma^2}} = 3 \quad (2.0.20)$$

$$\Rightarrow \tau = \frac{-\sigma^2 \ln 3}{2} \quad (2.0.21)$$

$$\Rightarrow \tau < 0 \text{ for all nonzero values of } \sigma^2 \quad (2.0.22)$$

\therefore option 3 is correct

Plot to show $\Pr(\hat{X} \neq X)$ is minimum at negative value of τ
for all nonzero values of σ^2

