

# Assignment 4

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Download all python codes from

<https://github.com/Ganesh-RB/AI1103prob-and-randomvariables/Assignment4/codes>

and latex-tikz codes from

<https://github.com/Ganesh-RB/AI1103prob-and-randomvariables/Assignment4>

## 1 PROBLEM

CSIR UGC NET EXAM (Dec 2012) Q 51

Suppose  $X_1, X_2, X_3, X_4$  are i.i.d random variables taking values 1 and  $-1$  with probability  $1/2$  each. Then  $E(X_1 + X_2 + X_3 + X_4)^4$  equals

- 1) 4                      2) 76                      3) 16                      4) 12

## 2 SOLUTION

**Definition 1** (Taylor series for  $e^x$ ).

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \quad (2.0.1)$$

**Definition 2** (Moment generating function). For a discrete random variable  $X$

$$M_X(t) \equiv E(e^{tX}) = \sum_{k=-\infty}^{\infty} e^{tk} \times p_X(k) \quad (2.0.2)$$

**Theorem 2.1.**  $n^{\text{th}}$  Moment of  $X$  ( $\equiv E(X^n)$ ) is coefficient of  $\frac{t^n}{n!}$  in Taylor expansion of  $M_X(t)$

*Proof.*

$$e^{tX} = \sum_{k=0}^{\infty} \frac{(tX)^k}{k!} \quad (2.0.3)$$

$$= \sum_{k=0}^{\infty} X^k \frac{t^k}{k!} \quad (2.0.4)$$

$$E(e^{tX}) = \sum_{k=0}^{\infty} E(X^k) \frac{t^k}{k!} \quad (2.0.5)$$

$\therefore E(X^n)$  is coefficient of  $\frac{t^n}{n!}$  in Taylor expansion of  $M_X(t) \equiv E(e^{tX})$   $\square$

$X_i$  are i.i.d random variable for  $i \in \{1, 2, 3, 4\}$  with,

$$\Pr(X_i = +1) = \frac{1}{2} \quad (2.0.6)$$

$$\Pr(X_i = -1) = \frac{1}{2} \quad (2.0.7)$$

**Lemma 2.1.** Moment Generating Function of random variable  $X_i$  is given by

$$M_{X_i}(t) = \frac{1}{2} \times e^{-t} + \frac{1}{2} \times e^t = \frac{e^t + e^{-t}}{2} \quad (2.0.8)$$

Above result follows from definition 1

Let  $Y = \sum_{i=1}^4 X_i$ , then by convolution

$$M_Y(t) = \prod_{i=1}^4 M_{X_i}(t) \quad (2.0.9)$$

$$M_Y(t) = \left( \frac{e^t + e^{-t}}{2} \right)^4 \quad (2.0.10)$$

$$M_Y(t) = \frac{e^{4t} + 4e^{2t} + 6 + 4e^{-2t} + e^{-4t}}{16} \quad (2.0.11)$$

By using Taylor expansion of  $e^{4t}$ ,  $e^{2t}$ ,  $e^{-2t}$  and  $e^{-4t}$

**Corollary 2.2** (Taylor expansion of  $M_Y(t)$ ).

$$M_Y(t) = 1 + \frac{1}{16} \left( \sum_{k=1}^{\infty} 4^k + 4(2)^k + 4(-2)^k + (-4)^k \frac{t^k}{k!} \right) \quad (2.0.12)$$

**Theorem 2.3.** for  $n \geq 1$ ,  $E(Y^n)$  is given by

$$E(Y^n) = \frac{4^n + 4(2)^n + 4(-2)^n + (-4)^n}{16} \quad (2.0.13)$$

It follows from Theorem 2.1 and eqn: (2.0.12)

$$\therefore E(Y^4) = \frac{4^4 + 4(2)^4 + 4(-2)^4 + (-4)^4}{16} \quad (2.0.14)$$

$$E(Y^4) = \frac{640}{16} = 40 \quad (2.0.15)$$

$$\therefore E(X_1 + X_2 + X_3 + X_4)^4 = 40$$

## 3 VERIFICATION

Now

**Theorem 3.1.** *If  $X_1, \dots, X_n$  are i.i.d. random variables, all Bernoulli trials with success probability  $p$ , then their sum is distributed according to a binomial distribution with parameters  $n$  and  $p$*

$$\sum_{k=1}^n X_k \sim B(n, p)$$

For a binomial random variable  $X$  with parameters  $n$  and  $p$

$$E(X) = np \quad (3.0.1)$$

$$E(X^2) = p^2 n(n-1) + pn \quad (3.0.2)$$

$$E(X^3) = p^3 n(n-1)(n-2) + 3p^2 n(n-1) + pn \quad (3.0.3)$$

$$\begin{aligned} E(X^4) &= p^4 n(n-1)(n-2)(n-3) \\ &\quad + 6p^3 n(n-1)(n-2) \\ &\quad + 7p^2 n(n-1) + np \end{aligned} \quad (3.0.4)$$

Let random variables  $X'_i = \frac{X_i+1}{2}$  then,

$$\Pr(X'_i = 1) = \frac{1}{2} \quad (3.0.5)$$

$$\Pr(X'_i = 0) = \frac{1}{2} \quad (3.0.6)$$

So  $X'_i$  are Bernoulli random variables with  $p = 0.5$   
As

$$Y = \sum_{i=1}^4 X_i \quad (3.0.7)$$

similarly let

$$Z = \sum_{i=1}^4 X'_i \quad (3.0.8)$$

$$Z = \sum_{i=1}^4 \frac{X_i + 1}{2} \quad (3.0.9)$$

$$Z = \frac{Y}{2} + 2 \quad (3.0.10)$$

Here  $Z$  will be binomial random variable with parameter  $n = 4$  and  $p = 0.5$

Thus follows

$$E(Z) = 2 \quad (3.0.11)$$

$$E(Z^2) = 5 \quad (3.0.12)$$

$$E(Z^3) = 14 \quad (3.0.13)$$

$$E(Z^4) = 42.5 \quad (3.0.14)$$

$$Y^4 = 16(Z-2)^4 \quad (3.0.15)$$

$$Y^4 = 16(Z^4 - 8Z^3 + 24Z^2 - 32Z + 16) \quad (3.0.16)$$

$$\begin{aligned} E(Y^4) &= 16(E(Z^4) - 8E(Z^3) \\ &\quad + 24E(Z^2) - 32E(Z) + 16) \end{aligned} \quad (3.0.17)$$

$$= 16(42.5 - 8 \times 14 + 24 \times 5 - 32 \times 2 + 16) \quad (3.0.18)$$

$$= 16 \times 2.5 \quad (3.0.19)$$

$$= 40 \quad (3.0.20)$$