Assignment 4

Ganesh Bombatkar - CS20BTECH11016

Download all python codes from

https://github.com/Ganesh−RB/AI1103prob−and−randomvariables/Assignment4/codes

and latex-tikz codes from

https://github.com/Ganesh−RB/AI1103prob−and−randomvariables/Assignment4

1 Problem

CSIR UGC NET EXAM (Dec 2012) Q 51
Suppose X_1, X_2, X_3, X_4 are i.i.d random variables taking values 1 and -1 with probability $1/2$ each. Then $E\left(X_1 + X_2 + X_3 + X_4\right)^4$ equals

1) 4 2) 76 3) 16 4) 12

2 Solution

Definition 1 (Taylor series for e^x)**.**

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \tag{2.0.1}$$

Definition 2 (Moment generating function)**.** *For a discrete random variable X*

$$M_X(t) \equiv E\left(e^{tX}\right) = \sum_{k=-\infty}^{\infty} e^{tk} \times p_X(k) \tag{2.0.2}$$

Theorem 2.1. *n^{th} Moment of X ($\equiv E(X^n)$) is coefficient of $\frac{t^n}{n!}$ in Taylor expansion of $M_X(t)$*

Proof.

$$e^{tX} = \sum_{k=0}^{\infty} \frac{(tX)^k}{k!} \tag{2.0.3}$$

$$= \sum_{k=0}^{\infty} X^k \frac{t^k}{k!} \tag{2.0.4}$$

$$E\left(e^{tX}\right) = \sum_{k=0}^{\infty} E\left(X^k\right) \frac{t^k}{k!}. \tag{2.0.5}$$

$\therefore E(X^n)$ is coefficient of $\frac{t^n}{n!}$ in Taylor expansion of $M_X(t) \equiv E\left(e^{tX}\right)$ □

X_i are i.i.d random variable for $i \in \{1, 2, 3, 4\}$ with,

$$\Pr(X_i = +1) = \frac{1}{2} \tag{2.0.6}$$

$$\Pr(X_i = -1) = \frac{1}{2} \tag{2.0.7}$$

Lemma 2.1. *Moment Generating Function of random variable X_i is given by*

$$M_{X_i}(t) = \frac{1}{2} \times e^{-t} + \frac{1}{2} \times e^{t} = \frac{e^t + e^{-t}}{2} \tag{2.0.8}$$

Above result follows from definition 1

Let $Y = \sum_{i=0}^{4} X_i$, then by convolution

$$M_Y(t) = \prod_{i=1}^{4} M_{X_i}(t) \tag{2.0.9}$$

$$M_Y(t) = \left(\frac{e^t + e^{-t}}{2}\right)^4 \tag{2.0.10}$$

$$M_Y(t) = \frac{e^{4t} + 4e^{2t} + 6 + 4e^{-2t} + e^{-4t}}{16} \tag{2.0.11}$$

By using Taylor expansion of e^{4t}, e^{2t}, e^{-2t} and e^{-4t}

Corollary 2.2 (Taylor expansion of $M_Y(t)$)**.**

$$M_Y(t) = 1 + \frac{1}{16}\left(\sum_{k=1}^{\infty} 4^k + 4(2)^k + 4(-2)^k + (-4)^k \frac{t^k}{k!}\right) \tag{2.0.12}$$

Theorem 2.3. *for $n \geq 1$, $E(Y^n)$ is given by*

$$E(Y^n) = \frac{4^n + 4(2)^n + 4(-2)^n + (-4)^n}{16} \tag{2.0.13}$$

It follows from Theorem 2.1 and eqn: (2.0.12)

$$\therefore E\left(Y^4\right) = \frac{4^4 + 4(2)^4 + 4(-2)^4 + (-4)^4}{16} \tag{2.0.14}$$

$$E\left(Y^4\right) = \frac{640}{16} = 40 \tag{2.0.15}$$

$$\therefore \mathbf{E\left(X_1 + X_2 + X_3 + X_4\right)^4 = 40}$$

3 Verification

Theorem 3.1. *If $X_1, \ldots, X_n$ are i.i.d. random variables, all Bernoulli trials with success probability p, then their sum is distributed according to a binomial distribution with parameters n and p*

$$\sum_{k=1}^{n} X_k \sim B(n, p)$$

For a binomial random variable X with parameters n and p and $q = 1 - p$

$$E(X) = np \tag{3.0.1}$$

$$E\left(X^2\right) = np\,(np + q) \tag{3.0.2}$$

$$E\left(X^3\right) = np\left(n^2p^2 + 3npq - 2pq + q\right) \tag{3.0.3}$$

$$E\left(X^4\right) = np\Big(n^3p^3 + 6n^2p^2q - 11np^2q + 7npq - 6pq^2 + q\Big) \tag{3.0.4}$$

Let random variables $X_i' = \frac{X_i+1}{2}$ then,

$$\Pr(X_i' = 1) = \frac{1}{2} \tag{3.0.5}$$

$$\Pr(X_i' = 0) = \frac{1}{2} \tag{3.0.6}$$

So X_i' are Bernoulli random variables with $p = 0.5$
As

$$Y = \sum_{i=1}^{4} X_i \tag{3.0.7}$$

similarly let

$$Z = \sum_{i=1}^{4} X_i' \tag{3.0.8}$$

$$Z = \sum_{i=1}^{4} \frac{X_i + 1}{2} \tag{3.0.9}$$

$$Z = \frac{Y}{2} + 2 \tag{3.0.10}$$

Here Z will be binomial random variable with parameter $n = 4$, $p = 0.5$ and $q = 0.5$
Thus follows

$$E(Z) = 2 \tag{3.0.11}$$

$$E\left(Z^2\right) = 5 \tag{3.0.12}$$

$$E\left(Z^3\right) = 14 \tag{3.0.13}$$

$$E\left(Z^4\right) = 42.5 \tag{3.0.14}$$

Now

$$Y^4 = 16(Z - 2)^4 \tag{3.0.15}$$

$$Y^4 = 16\left(Z^4 - 8z^3 + 24Z^2 - 32Z + 16\right) \tag{3.0.16}$$

$$E\left(Y^4\right) = 16\Big(E\left(Z^4\right) - 8E\left(z^3\right) + 24E\left(Z^2\right) - 32E(Z) + 16\Big) \tag{3.0.17}$$

$$= 16\,(42.5 - 8 \times 14 + 24 \times 5 - 32 \times 2 + 16) \tag{3.0.18}$$

$$= 16 \times 2.5 \tag{3.0.19}$$

$$= 40 \tag{3.0.20}$$

$$\therefore \mathbf{E\left(X_1 + X_2 + X_3 + X_4\right)^4 = 40}$$