ASSIGNMENT 5

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1 Problem

For $n \geq 1$, let X_n be a Poisson random variable with mean n^2. Which of the following's are equal to $\dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{2}^{\infty} e^{-x^2/2} \, dx$

1) $\lim\limits_{n \to \infty} \Pr\left(X_n > (n+1)^2\right)$
2) $\lim\limits_{n \to \infty} \Pr\left(X_n \leqslant (n+1)^2\right)$
3) $\lim\limits_{n \to \infty} \Pr\left(X_n < (n-1)^2\right)$
4) $\lim\limits_{n \to \infty} \Pr\left(X_n < (n-2)^2\right)$

2 Solution

Let Y_i be a Poisson random variable with mean 1 for $i \in \left(1, n^2\right)$

By additive property of Poisson distribution

$$\sum_{i}^{n^2} Y_i = X_n \qquad (2.0.1)$$

By central limit theorem

$$\lim_{n \to \infty} \frac{Y_1 + Y_2 + .. + Y_{n^2} - n^2}{n} = \mathcal{N}(0,1) \qquad (2.0.2)$$

$$\implies \lim_{n \to \infty} \frac{X_n - n^2}{n} = \mathcal{N}(0,1) \qquad (2.0.3)$$

Here, $\mathcal{N}(0,1)$ is normal distribution with unit mean and variance

Now

$$\Pr\left(X_n < k\right) = \Pr\left(\frac{X_n - n^2}{n} < \frac{k - n^2}{n}\right) \qquad (2.0.4)$$

$$= \Pr\left(\mathcal{N}(0,1) < \frac{k - n^2}{n}\right) \qquad (2.0.5)$$

$$= \Phi\left(\frac{k - n^2}{n}\right) \qquad (2.0.6)$$

1) $\lim\limits_{n \to \infty} \Pr\left(X_n > (n+1)^2\right)$

$$\lim_{n \to \infty} \Pr\left(X_n > (n+1)^2\right) = 1 - \lim_{n \to \infty} \Pr\left(X_n < (n+1)^2\right) \qquad (2.0.7)$$

$$= 1 - \lim_{n \to \infty} \Phi\left(\frac{(n+1)^2 - n^2}{n}\right) \qquad (2.0.8)$$

$$= 1 - \Phi(2) \qquad (2.0.9)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{2}^{\infty} e^{-x^2/2} \, dx \qquad (2.0.10)$$

∴ **Option 1 is correct**

2) $\lim\limits_{n \to \infty} \Pr\left(X_n \leqslant (n+1)^2\right)$

$$\lim_{n \to \infty} Pr\left(X_n \leqslant (n+1)^2\right) = \lim_{n \to \infty} \Phi\left(\frac{(n+1)^2 - n^2}{n}\right) \qquad (2.0.11)$$

$$= \Phi(2) \qquad (2.0.12)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{2} e^{-x^2/2} \, dx \qquad (2.0.13)$$

∴ **Option 2 is incorrect**

3) $\lim\limits_{n \to \infty} \Pr\left(X_n < (n-1)^2\right)$

$$\lim_{n \to \infty} Pr\left(X_n < (n-1)^2\right) = \lim_{n \to \infty} \Phi\left(\frac{(n-1)^2 - n^2}{n}\right) \qquad (2.0.14)$$

$$= \Phi(-2) \qquad (2.0.15)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-2} e^{-x^2/2} \, dx \qquad (2.0.16)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{2} e^{-x^2/2} \, dx \qquad (2.0.17)$$

∴ **Option 3 is also correct**

4) $\lim\limits_{n\to\infty} \Pr\left(X_n < (n-2)^2\right)$

$$\lim_{n\to\infty} Pr\left(X_n < (n-2)^2\right) = \lim_{n\to\infty} \Phi\left(\frac{(n-2)^2 - n^2}{n}\right) \qquad (2.0.18)$$

$$= \Phi(-4) \qquad (2.0.19)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-4} e^{-x^2/2}\, dx \qquad (2.0.20)$$

∴ **Option 4 is incorrect**