Complex numbers

→ $i = \sqrt{-1}$, $i^2 = -1$

$i^1 = i^5 = i^9 = i^{13} = \cdots = i^{4n+1} = i$, $i^2 = i^6 = i^{10} = i^{14} = \cdots i^{4n+2} = -1$

$i^3 = i^7 = i^{11} = i^{15} = \cdots i^{4n+3} = -i$, $i^4 = i^8 = i^{12} = i^{16} = \cdots i^{4n+4} = 1$

NOTE: $i^2 = \sqrt{-1} \times \sqrt{-1} \neq 1$ it is equal to -1

→ Every complex number $z = a+ib$ can be represented by an ordered pair (a, b).

→ Square root of a complex number

(i) $\sqrt{a+ib} = \pm\left(\sqrt{\frac{c+a}{2}} + i\sqrt{\frac{c-a}{2}}\right)$ (ii) $\sqrt{a-ib} = \pm\left(\sqrt{\frac{c+a}{2}} - i\sqrt{\frac{c-a}{2}}\right)$

(iii) $\sqrt{a+ib} + \sqrt{a-ib} = \pm\sqrt{2}\sqrt{c+a}$ (iv) $\sqrt{a+ib} - \sqrt{a-ib} = \pm i\sqrt{2}\sqrt{c-a}$

Here $c = \sqrt{a^2+b^2}$

→ i) $z = x+iy$ its conjugate complex no. is $\bar{z} = x-iy$.

ii) If z is real then $\bar{z}$ is also real (itself).

Ex: $z = 2$, $\bar{z} = 2$

Properties of $\bar{z}$

→ $\overline{(\bar{z})} = z$

$\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$ $\overline{z_1-z_2} = \bar{z}_1 - \bar{z}_2$

$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

→ $\bar{z}_1 = \bar{z}_2 \Rightarrow z_1 = z_2$

$z_1 + \bar{z}_1 = 2Re(z)$, $z_1 - \bar{z}_1 = 2i\,Im(z)$

⇒ Modulus of a complex number:

If $z = x+iy$ then $|z| = r = \sqrt{x^2+y^2}$

NOTE: $|z| = |-z| = |\bar{z}| = |-\bar{z}| = \sqrt{x^2+y^2}$

→ Properties

$|z| = 0 \leftrightarrow z = 0$

$|z^n| = |z|^n$

If $|z| = 1$ then it is unimodular (z).

$z\bar{z} = |z|^2$

$|z_1 z_2| = |z_1||z_2|$

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

$|z_1+z_2|^2 = (z_1+z_2)(\bar{z}_1+\bar{z}_2)$

$= |z_1|^2 + |z_2|^2 + 2Re(z_1\bar{z}_2)$

$|z_1+z_2|^2 = |z_1|^2+|z_2|^2 \iff \frac{z_1}{z_2}$ is purely imaginary when $z_2 \neq 0$

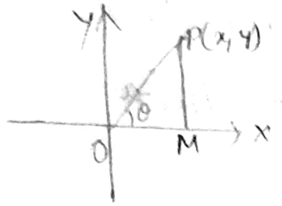
→ $|z_1+z_2| \le |z_1|+|z_2|$, $\big||z_1|-|z_2|\big| \le |z_1-z_2|$

→ If $|z - a/z| = K$ then (i) min value of $|z| = \frac{-K+\sqrt{K^2+4a}}{2}$

(ii) max. value of $|z| = \frac{K+\sqrt{K^2+4a}}{2}$

→ Least value of $|z-a| + |z-b|$ is $|a-b|$

→ Argument of a complex number (A.C.N):



$\theta = Arg(z) = \tan^{-1}(y/x)$

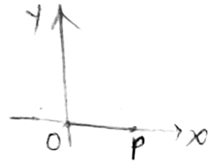
$-\pi < \theta \le \pi$

→ if $z = x+iy$ and $\theta = \tan^{-1}(y/x)$ then

$$arg(z) = \begin{cases} \theta & \text{if } x>0, y>0 \\ \pi-\theta & \text{if } x<0, y>0 \\ -(\pi-\theta) & \text{if } x<0, y<0 \\ -\theta & \text{if } x>0, y<0 \end{cases}$$

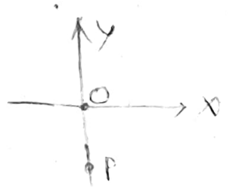
$\pi-\theta$	θ
$-(\pi-\theta)$	$-\theta$

→ $arg\,z = 0$ if $x>0, y=0$

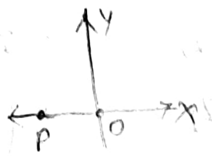


→ $arg\,z = \pi/2$ if $x=0, y>0$

→ $arg\,z = -\pi/2$ if $x=0, y<0$



→ $arg\,z = \pi$, if $x<0, y=0$



→ $arg\,\bar{z} = -arg\,z$

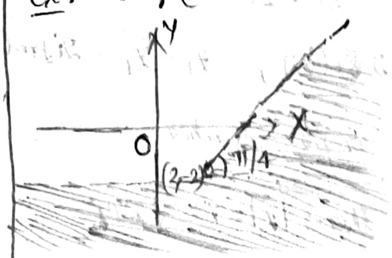
$arg(z^n) = n\,arg(z)$

$arg(z_1z_2) = arg\,z_1 + arg\,z_2$

$arg\left(\frac{z_1}{z_2}\right) = arg\,z_1 - arg\,z_2$

$arg(z_1\bar{z}_2) = arg\,z_1 - arg\,z_2$

Ex: $arg(z-2+2i) < \pi/4$



→ $|z_1+z_2| = |z_1-z_2| \iff arg\,z_1 - arg\,z_2 = \pi/2$

$|z_1+z_2| = |z_1|+|z_2| \iff arg\,z_1 - arg\,z_2 = 0$

→ Amplitude of a complex number (r) polar form:

$z = x+iy \Rightarrow z = r(\cos\theta + i\sin\theta)$ $\qquad r = \sqrt{a^2+b^2}$

$z = r\,cis\,\theta$

→ Euler's form: $Z = x+iy \Rightarrow Z = re^{i\theta}$

$$e^{i\theta} = cis\theta = \cos\theta + i\sin\theta, \quad e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \cosh(i\theta)$$

$$i\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2} = \sinh(i\theta)$$

→ $\frac{1}{cis\theta} = cis(-\theta)$

$cis\theta_1 \cdot cis\theta_2 = cis(\theta_1+\theta_2)$, $\frac{cis\theta_1}{cis\theta_2} = cis(\theta_1-\theta_2)$

NOTE: i) $1 = e^{i0}$, ii) $i = e^{i\pi/2}$, iii) $-i = e^{-i\pi/2}$, iv) $-1 = e^{i\pi}$

⇒ Logarithm of a complex number:

If $Z = x+iy$, then $\log(Z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$ ie,

$$\log Z = \log|Z| + i\arg Z$$

NOTE: i) $\log i = i\pi/2$

ii) $\log\left(\frac{a-ib}{a+ib}\right) = -2i\tan^{-1}(b/a)$

→ No. of soln. for $Z^m = c(\bar{Z})^n$ is $\boxed{m+n+1}$ where c is a non-zero complex number.

→ $|e^{-i\theta}| = |e^{i\theta}| = 1$

→ if Z = real no $= A+\bar{A}$ then $\arg(Z) = \pi$

→ Cube roots of unity:

$1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$, $\quad \omega = \frac{-1+i\sqrt{3}}{2}, \omega^2 = \frac{-1-i\sqrt{3}}{2}, \omega^3 = 1$

$1+\omega+\omega^2 = 0$ $\quad \omega^{3n} = 1, \omega^{3n+1} = \omega, \omega^{3n+2} = \omega^2$

De Moivre's theorem:

→ $(\cos\theta + i\sin\theta)^n = (\cos\theta - i\sin\theta)^{-n} = \cos n\theta + i\sin n\theta$

→ $(\cos\theta + i\sin\theta)^{-n} = (\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$

→ $\frac{cis\theta - 1}{cis\theta + 1} = i\tan\theta/2$, $cis\theta - 1 = 2i\sin\theta/2 \, cis\,\theta/2$, $cis\theta + 1 = 2\cos\theta/2 \, cis\,\theta/2$

→ $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$ then

i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$

ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

iii) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$

iv) $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

v) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 3/2$

vi) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 3/2$

for rational index

$(\cos\theta + i\sin\theta)^{p/q} = (\cos p\theta + i\sin p\theta)^{1/q}$

$= cis\left(\frac{2k\pi + p\theta}{q}\right)$

$k = 0, 1, 2, \ldots, q-1$

→ $\cos(2\beta-\gamma-\alpha)+\cos(2\alpha-\beta-\gamma)+\cos(2\gamma-\alpha-\beta)=3$

→ $\sin(2\alpha-\beta-\gamma)+\sin(2\beta-\gamma-\alpha)+\sin(2\gamma-\alpha-\beta)=0$

III) n^{th} roots of unity:

i) n^{th} root of unity are the soln $z^n-1=0$, they are given by $\text{cis}\left(\frac{2k\pi}{n}\right)$, $k=0,1,2,\dots,n-1$

ii) n^{th} roots of unity are $1,\alpha,\alpha^2,\dots,\alpha^{n-1}$ where $\alpha=\text{cis}\frac{2\pi}{n}$, $\alpha^n=1$

iii) sum of n^{th} roots of unity $=0$ if $n>2$ ⇒

① $1+\alpha+\alpha^2+\dots+\alpha^{n-1}=\left(\frac{1-\alpha^n}{1-\alpha}\right)=0$

② $\sum_{r=0}^{n-1}\cos\left(\frac{2k\pi}{n}\right)=0$, $\sum_{r=0}^{n-1}\sin\left(\frac{2k\pi}{n}\right)=0$

iv) product of n^{th} roots of unity $=(-1)^{n-1}=\begin{cases}-1, & \text{if } n \text{ is even}\\ 1, & \text{if } n \text{ is odd}\end{cases}$

v) If $1,\alpha_1,\alpha_2,\dots,\alpha_{n-1}$ are roots of $z^n-1=0$ ⇒ $z^n-1=(z-1)(z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_{n-1})$

→ $\left(\frac{1+\cos\theta+i\sin\theta}{1+\cos\theta-i\sin\theta}\right)^n=\text{cis}\,n\theta$, $\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n=\text{cis}\,n(\pi/2-\theta)$

Geometrical applications:

→ If Δ^{le} ABC is an equilateral Δ^{le} then

① $z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$

(ii) $(z_1-z_2)^2+(z_2-z_3)^2+(z_3-z_1)^2=0$

(iii) $\frac{1}{z_1-z_2}+\frac{1}{z_2-z_3}+\frac{1}{z_3-z_1}=0$

(iv) if z_0 is circumcentre of Δ^{le} ABC then $z_1^2+z_2^2+z_3^2=3z_0^2$

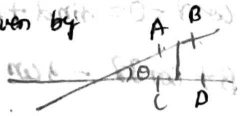
NOTE : i) if $z_1^2-z_1z_2+z_2^2=0$ then origin & z_1z_2 form an eq. Δle.

ii) if $z_1^2+z_1z_2+z_2^2=0$ then origin & z_1z_2 form an isosceles Δ^{le}

→ General eqn. of a st line is $\bar{a}z+a\bar{z}+b=0$

→ angle b/w 2 lines : if $A(z_1), B(z_2), C(z_3), D(z_4)$ are 4 points in Argand plane then angle b/w lines AB & CD is given by

$$\theta=\arg\left(\frac{z_1-z_2}{z_3-z_4}\right)$$



→ Length of ⊥ar from a point z_1 to the line $a\bar{z}+\bar{a}z+b=0$ is given by $\frac{|a\bar{z}_1+\bar{a}z_1+b|}{2|a|}$

→ $|z-z_1|=|z-z_2|$ is ⊥ar bisector of line (z_1z_2) formed

→ **Area of Δ^{le}:**

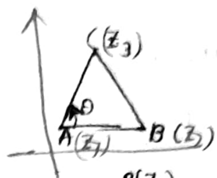
① Area of Δ^{le} whose vertices are $z_1, iz, z+iz$ is $\frac{1}{2}|z|^2$

② Area of Δ^{le} whose vertices are $-z, iz, z-iz$ is $\frac{3}{2}|z|^2$.

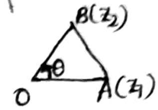
*③ Area of Δ^{le} whose vertices are $z, \omega z, z+\omega z$ is $\frac{\sqrt{3}}{4}|z|^2$

→ **Coni's theorem:** z_1, z_2, z_3 are vertices of Δ^{le} ABC & θ is in A.C.W sense then

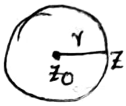
$$\frac{z_3 - z_1}{z_2 - z_1} = \left|\frac{z_3 - z_1}{z_2 - z_1}\right| e^{i\theta}$$



i) if one vertex is at origin then $\frac{z_2}{z_1} = \left|\frac{z_2}{z_1}\right| e^{i\theta}$



→ **Circle:** $|z - z_0| = r$



$z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + \beta = 0$ is eqn of cle.

$\arg\left(\frac{z-z_1}{z-z_2}\right) = \pi/2$ then locus of z represents circle as $A(z_1), B(z_2)$ as diameter.

locus of z satisfying $|z-z_1|^2 + |z-z_2|^2 = k$ if $k \geq \frac{|z_1 - z_2|}{2}$

→ **Ellipse:** Locus of $|z - z_1| + |z - z_2| = 2a$ is ellipse if $|z_1 - z_2| < 2a$ is ellipse.

$SP + S'P = 2a$

$$e = \frac{|z_1 - z_2|}{2a}$$

if $|z_1 - z_2| = 2a$ then it is line segment; $|z_1 - z_2| > 2a$ it is undefined.

→ **Hyperbola:** Locus of $||z - z_1| - |z - z_2|| = 2a$ is hyperbola if $|z_1 - z_2| > 2a$

$SP - S'P = 2a$

$$e = \frac{|z_1 - z_2|}{2a}$$

⇒ Sum of sin's in A.P

$$\sin a + \sin(a+d) + \sin(a+2d) + \cdots + \sin(a+(n-1)d) = \frac{\sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)} \sin\left(\frac{\text{1st angle} + \text{last angle}}{2}\right)$$

sum of cos's in AP

$$\cos a + \cos(a+d) + \cos(a+2d) + \cdots + \cos(a+(n-1)d) = \frac{\sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)} \cos\left(\frac{\text{first angle} + \text{last angle}}{2}\right)$$

product of cosine's in G.P

$$\cos\theta \cos 2\theta \cos 2^2\theta \cdots \cos 2^{n-1}\theta = \frac{\sin(2^n\theta)}{2^n \sin\theta}$$

→ if $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, $x^m y^n \Rightarrow \cos(m\alpha + n\beta) + i\sin(m\alpha + n\beta)$

$\frac{1}{x^m y^n} \Rightarrow \cos(m\alpha + n\beta) - i\sin(m\alpha + n\beta)$