Rotational Dynamics

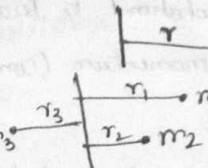
⇒ (1) Moment of inertia – It is measurement of the resistance of a body to a change in its rotational motion.

(2)

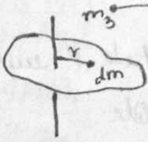
Translatory motion	Rotatory motion
mass (m)	(I) moment of inertia
displacement (s)	angular displacement (θ)
velocity (v)	angular velocity (ω)
acceleration (a)	angular acceleration (α)
momentum (P)	angular moment (L) $L = I\omega$
Force (F)	Torque (τ)
work (W) = F·S	work (W) = $\tau\theta$

→ for single particle $I = mr^2$

system of particles $I = \sum m_i r_i^2$



for rigid bodies $I = \int r^2 dm$



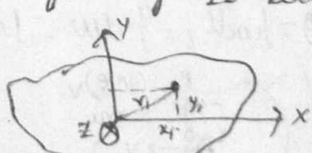
(3) Parallel axes theorem – Applicable for 2D & 3D bodies

$$\boxed{I = I_{COM} + Mr^2}$$

r = ⊥ar distance b/w axis to point.

Perpendicular axes theorem – applicable for only 2D bodies.

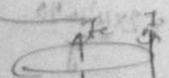
$$I_z = I_x + I_y$$



Radius of gyration (k) – a distance from the axis where the whole mass can be assumed to be concentrated $\boxed{K = \sqrt{\frac{I}{M}}}$

(4) M.O.I of standard bodies

i) Ring

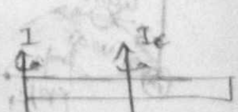


$I_C = MR^2$ (axis ⊥ar to plane), $I = 2MR^2$ (tgt ⊥ar to plane),

diameter

$I = MR^2/2$ (axis lie in the plane)

iii)



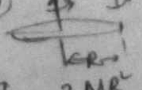
$I_C = \frac{ml^2}{12}$

$I = \frac{ml^2}{3}$

ii) Disc

(a) $I_C = MR^2/2$,

(b)



$I = \frac{3MR^2}{2}$

(c)

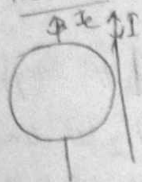
$I = \frac{MR^2}{4}$

(iv) solid

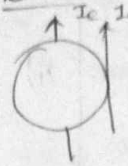
$I_C = \frac{MR^2}{2}$ $\quad I = \frac{3}{2}MR^2$

(v) hollow

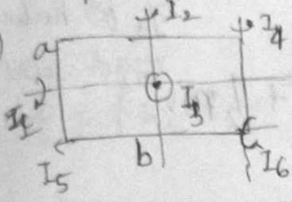
$I_c = MR^2$

(vi) Hollow sphere

$I_c = \frac{2}{3}MR^2$

$I = \frac{5}{3}MR^2$

(vii) solid

$I_C = \frac{2}{5}MR^2$

$I = \frac{7}{5}MR^2$

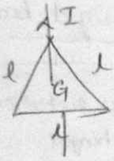
(viii)

$I_1 = I_1 + I_2 =$ $\quad I_4 = Mb^2/3$

$I_1 = Ma^2/12$ $\quad I_5 = Ma^2/3$

$I_2 = Mb^2/12$ $\quad I_6 = I_4 + I_3$

(ix) Hollow cone

$I_C = \frac{MR^2}{2}$

(x)

$I = \frac{ml^2}{12}$

(xi) Solid cone

$I = \frac{3MR^2}{10}$

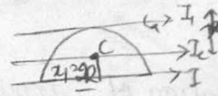
(xii) Semicircular ring

ⓐ

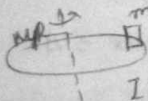
$I = MR^2$

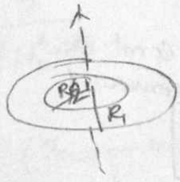
ⓑ

$I = \frac{MR^2}{2}$

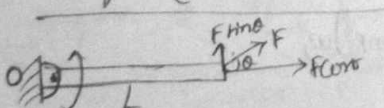
ⓒ

$I_1 = I + M(x_1^2 - x_2^2)$

⑤ Addition (or) subtraction of MOI

ⓐ

$I = \frac{MR^2}{2} + mR^2$

ⓑ

$I = \frac{M(R_1^2 + R_2^2)}{2}$

⑥ Torque $(\vec{T} = \vec{r} \times \vec{F}) = N\cdot m$



i) $T = LF\sin\theta$, ACW

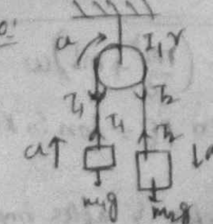
ii) $F\cos\theta$ passes through point O, can't have rotational effect

$\sum \vec{T} = I\vec{\alpha}$ — angular accl.

static equilibrium problems

$\sum \vec{T} = 0$ & $\sum \vec{F} = 0$ both

ω & $V_{cm} = 0$

Ex:


$m_2 g - T_2 = m_2 a \rightarrow ①$

$T_1 - m_1 g = m_1 a \rightarrow ②$

$T_2 r - T_1 r = I\alpha$

$(T_2 - T_1)r = I \cdot a/r$

$T_2 - T_1 = \frac{Ia}{r^2} \rightarrow ③$

from ① + ② + ③ ⇒

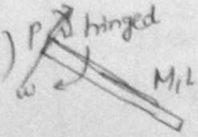
$a = \frac{(m_2 - m_1)g}{\left(m_2 + m_1 + \frac{I}{r^2}\right)}$

$a = \frac{g\sin\theta}{1 + k^2/R^2}$

$t = \sqrt{\frac{2s}{g\sin\theta}\left(1 + k^2/R^2\right)}$

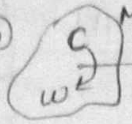
7) (a) $KE \Rightarrow \int d(KE) = \frac{1}{2}\int dm\, v^2$

$KE = \frac{1}{2} I\omega^2$

i) 

$KE = \frac{1}{2} I_P \omega^2$

$= \frac{1}{2}\frac{ML^2}{3}\omega^2$

(b) 

here Axis of rotation is COM

$KE = KE_R + KE_T$

$$\boxed{KE = \frac{1}{2} I\omega^2 + \frac{1}{2} m v_{cm}^2}$$

ii) 

$I_C = K m r^2$

K = 1 for ring / H. cyl.

1/2 for disc / solid cyl.

2/3 for hollow sphere

2/5 for solid sphere

$$\boxed{KE = \frac{1}{2} I_C \omega^2 + \frac{1}{2} m v_{cm}^2}$$

8) Angular momentum (L)

i) (a) for point mass

$L_P = mvd$

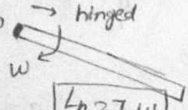
* $\vec{L} = \vec{r} \times \vec{p}$

$= mv\underbrace{r\sin\theta}_{d}$

(b) rigid body

→ pure rotation

→ rolling (rot + trans.)



$\boxed{L_P = I_P \omega}$

$L_P = I_{CM}\vec{\omega} + \vec{r} \times \vec{P}_{CM}$

ii) Conservation of angular momentum

(a) $\tau = \frac{dL}{dt}$

if $\tau_{ext} = 0 \Rightarrow L$ is const. (or) conserved

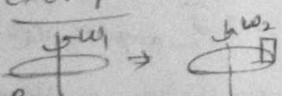
[IIly to $F = \frac{dp}{dt}$]

(b) If $\tau_{ext} \neq 0$

$\tau dt = dL \Rightarrow \boxed{\int \tau dt = \Delta L}$

angular impulse $= \Delta L$

iii) Examples

(a) 

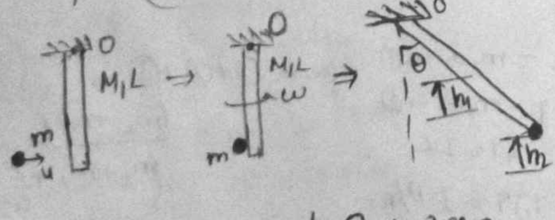
$\Rightarrow L_i = L_f$

$\frac{MR^2}{2}\omega_1 = \left(\frac{MR^2}{2} + mR^2\right)\omega_2$

(b) 

(i) $I_1\omega = (I_1 + I_2)\omega$

(ii) Loss of $KE = \frac{1}{2} I_1\omega_1^2 - \frac{1}{2}(I_1 + I_2)\omega_2^2$

(c) 

$$\boxed{mul = (I_{rod} + I_{mass})\omega}$$

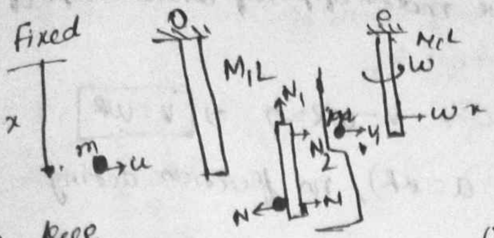
$mul = \left(\frac{mL^2}{3} + mL^2\right)\omega$

gain in PE = loss in KE

τ_{net} about O is zero during impact, L is same just before & after impact

(d) fixed & free collision (e = coeff of restitution)

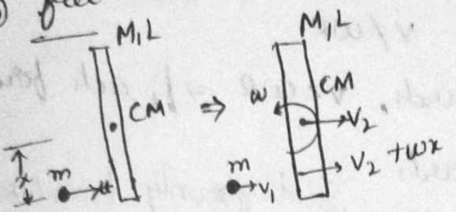
(a) Fixed



$L_i = L_f$
$mux = mv_1 x + \frac{mL^2}{3}\omega \rightarrow$ (i)
$e = \frac{\omega x - v_1}{u} \rightarrow$ (ii)

hence solve 1 & 2 we get v_1 & ω

(b) free



(i) L is const about any point
about CM $\Rightarrow L_i = L_f$

$mux = mv_1 x + \frac{mL^2}{12}\omega$ — (1)

(ii) P is const (As $F_{net} = 0$)

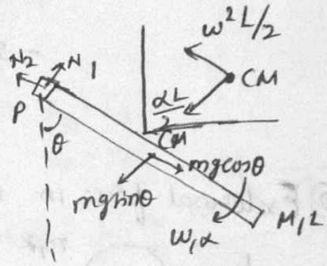
$P_i = P_f \Rightarrow mu = mv_1 + mv_2 \rightarrow$ (2)

(iii) $e = \frac{v_2 + \omega x - v_1}{u} \rightarrow$ (3)

by solving (1) & (2), (3) we get v_1, v_2 & ω

(e) hinge reaction

(i) Always take 2 ⊥ar components of this model



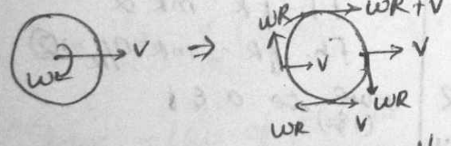
$mg\sin\theta - N_1 = \frac{m\alpha L}{2}$

$N_2 - mg\cos\theta = m\omega^2 L/2$

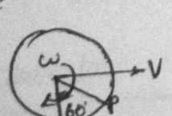
$\therefore rxn = \sqrt{N_1^2 + N_2^2}$

(Tran + rotational) = rolling

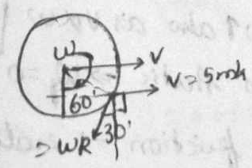
(9) Velocity of point on circumference



Ex1- $V = 5$ m/s, $\omega = 2.5$ rad/s, $R = 2$m, find V_p ??

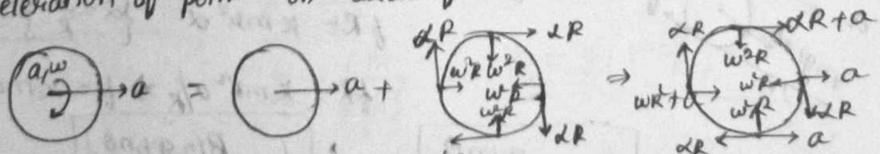


sol:



$V_P = \sqrt{5^2 + 5^2 + 2(5)^2\cos 120°}$
$= 5$ m/s

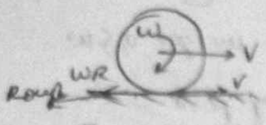
(10) Acceleration of point on circumference



(10) Rolling motion

(a) <u>Pure rolling</u>: No relative motion of pt. of contact (&) pt. of contacts have same velocity

$\because V_{ground} = 0 \Rightarrow V - \omega R = 0 \Rightarrow \boxed{V = \omega R}$



(i) $V = \omega R$ ($\because a = \alpha R$), no friction acting

(b) <u>rolling without slipping</u>: $V \neq \omega R$

→ If $V > \omega R \Rightarrow f_k$ acts backwards, $V < \omega R \Rightarrow f_k$ acts forward

f_k until pure rolling starts

Sliding → only translational

(11) <u>KE in pure rolling</u>

$\beta = K^2/R^2 \rightarrow \beta = 1$ if ring / H. cyli

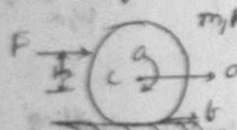
$= 1/2$ if disc / sol. cylin

$= 2/3$ for H. sphere

$= 2/5$ for sol. sphere

$\boxed{KE_T = 1/2\, mv^2 (1+\beta)}$

(12) External force in rolling



→ Enough to ensure pure rolling

Note: (i) pure rolling → f is static

(ii) pt of contact is at instantaneous rest $\rightarrow W_f = 0$

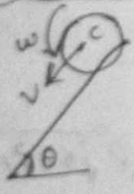
$F + f = ma \rightarrow ①$ $\left[\alpha = \frac{a}{R}\right]$

$Fh - fR = mR^2\alpha$

$Fh - fR = mR^2\, a/R \rightarrow ②$

solve (1 & 2) to a & f

(13) <u>Energy conservation</u> (pure rolling)



↓

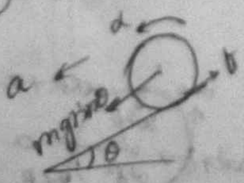
$V = \omega r$ & $a = \alpha R$

i) V ↑ due to gravity

ii) ω ↑ also as $V \propto \omega$

f acts static $\Rightarrow W_f = 0$

(14) acceleration & friction → rolling on inclined



$mg\sin\theta - f = ma \rightarrow ①$

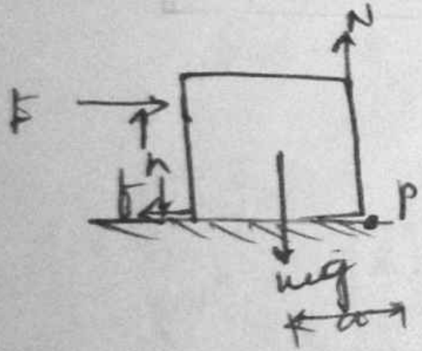
$fR = K m R^2 \alpha$ $\left\{\alpha = \frac{a}{R}\right\}$

$fR = K m R^2\, a/R \Rightarrow f = K m a \rightarrow ②$

$\boxed{a = \frac{g\sin\theta}{1+\beta}}$ & $\boxed{f = \frac{\beta(mg\sin\theta)}{1+\beta}}$

$a_{solid\text{-}sph} > a_{disc} > a_{H\cdot s} > a_{ring}$

(15) Toppling



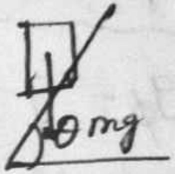
N should pass through P
for toppling about P

$\tau_F > \tau_{mg}$

$Fh > mga$

$F > mg\,a/h$

$$\boxed{F_{min} = \frac{mga}{h}}$$



$\theta \uparrow$ in inclined plane

toppling also $\uparrow$