3D PLANES & LINES

(I) PLANES

→ dir's = (a,b,c)

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ (for llel also same)

(x_1, y_1, z_1)

→ $ax+by+cz+d=0$ llel then $ax+by+cz+k=0$

→ Eqn. of plane which is llel to lines:

i) Point (x_1, y_1, z_1) & llel to lines whose dir's are $(a_1, b_1, c_1)(a_2, b_2, c_2)$ is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$

ii) if passing through 2 points also & llel to line of dir's (a,b,c) is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a & b & c \end{vmatrix}$

iii) through 3 non-collinear points is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$

iv) if 4 points then $\begin{vmatrix} x_4-x_1 & y_4-y_1 & z_4-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$

→ Eqn. of plane (π) with diff. conditions:

i) with dir's of normal as (a,b,c) is $ax+by+cz+d=0$

ii) $a=0, b\neq 0, c\neq 0$ then eqn. of π is $by+cz+d=0$ which is llel to x-axis & ⊥r to YZ-plane. Illy for $b=0$, $c=0$.

iii) eqn. of π passing through (x_1, y_1, z_1) & llel to

① YZ π & ⊥ar X-axis is $x=x_1$

② ZX π & ⊥ar Y-axis is $y=y_1$

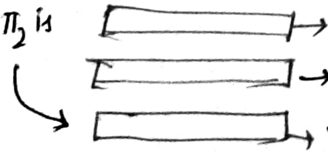
③ XY π & ⊥ar to Z-axis is $z=z_1$

iv) distance b/w 2 llel π's $= \dfrac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$ $\left[\begin{array}{l}\pi_1 \equiv ax+by+cz+d_1 \\ \pi_2 \equiv ax+by+cz+d_2\end{array}\right]$

v) Eqn. of π in midway of 2 π's $ax+by+cz+d_1 = \pi_1$ & π_2 is

$$ax+by+cz+\left(\frac{d_1+d_2}{2}\right)=0$$

vi) Reflection of π_1 in π_2 is



$a'x+b'y+c'z+d'=0 \rightarrow \pi_1$

$ax+by+cz+d=0 \rightarrow \pi_2$

$$\frac{\pi_1}{\pi_2} = \frac{2(aa'+bb'+cc')}{a^2+b^2+c^2}$$

→ **Ratio:** $\pi \equiv ax+by+cz+d=0$ $\quad A=(x_1,y_1,z_1)\ \ B=(x_2,y_2,z_2)$

Ratio $= -\pi_{111} : \pi_{222}$ $\quad \pi_{111} = ax_1+by_1+cz_1+d$

$\pi_{222} = ax_2+by_2+cz_2+d$

i)  $\pi_{111}\cdot\pi_{222}>0 \Rightarrow \pi_{111}>0,\ \pi_{222}>0$ (or) $\pi_{111}<0,\ \pi_{222}<0$

ii)  $\pi_{111}\cdot\pi_{222}<0 \Rightarrow \pi_{111}>0,\ \pi_{222}<0$ (or) $\pi_{111}<0,\ \pi_{222}>0$

→ Normal form: $d<0$ $\quad \dfrac{a}{\sqrt{a^2+b^2+c^2}}x + \dfrac{b}{\sqrt{a^2+b^2+c^2}}y + \dfrac{c}{\sqrt{a^2+b^2+c^2}}z = \dfrac{-d}{\sqrt{a^2+b^2+c^2}}$, $d>0$ we ~~have~~ multiply with (−ve)

→ Lar distance from point to π:

i) (x_1,y_1,z_1) to π is $\dfrac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

→ Intercept form:

$x/a + y/b + z/c = 1$

a,b,c are x,y,z-intercepts

→ **Areas:**

i) Area of Δ^{le} formed by $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ with

(a) x-axis, y-axis is $\frac{1}{2}|ab|$ sq.units (b) y, z-axes then $\frac{1}{2}|bc|$

ii) π $x/a+y/b+z/c=1$ meets co-ordinate axes in the points A, B, C then the area of Δ^{le} ABC is $\frac{1}{2}\sqrt{(ab)^2+(bc)^2+(ca)^2}$

→ Angle b/w 2 π's:

i) $\cos\theta = \dfrac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\ \sqrt{a_2^2+b_2^2+c_2^2}}$ for π_1, π_2

ii) if π_1, π_2 are ||el then $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$, if π_1,π_2 are lar then $a_1a_2+b_1b_2+c_1c_2=0$

→ foot & image:

foot $\Rightarrow \dfrac{h-x_1}{a} = \dfrac{k-y_1}{b} = \dfrac{l-z_1}{c} = \dfrac{-(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$

image $\Rightarrow \dfrac{h-x_1}{a} = \dfrac{k-y_1}{b} = \dfrac{l-z_1}{c} = \dfrac{-2(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$

→ Eqn. of π's bisecting the angle b/w given π's:

i) π_1, π_2 are $\dfrac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm \dfrac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$

ii) $d_1, d_2>0$

condition	Acute	obtuse
$a_1a_2+b_1b_2+c_1c_2>0$	−	+
$a_1a_2+b_1b_2+c_1c_2<0$	+	−

* (here bisectors are lar & +ve sign indicates bisector contains origin) *

(II) Lines :

→ Unsymmetrical form –

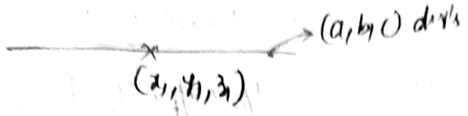
$a_1x+b_1y+c_1z+d_1=0,\quad a_2x+b_2y+c_2z+d_2=0$ is a line

→ symmetrical form

if point (x_1,y_1,z_1) & d.c's (l,m,n) is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

→ vector form :



$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

→ Conversion of non-symmetrical form to symmetrical form :

unsym– $\quad a_1x+b_1y+c_1z+d_1=0$

$\qquad\qquad a_2x+b_2y+c_2z+d_2=0$

we must know (i) dr's of it

(ii) coordinates of any point on it.

Step-1: Let (l,m,n) are dr's of line then

$a_1l+b_1m+c_1n=0,\quad a_2l+b_2m+c_2n=0$ by m.e.f.m method we find l,m,n

step 2: point on line.

atleast one of the dr's is non-zero & $a_1b_2-a_2b_1\neq0 \Rightarrow$ the line not ∥ to xy plane

→ Let it intersect xy plane in $(x_1,y_1,0)$ then $a_1x_1+b_1y_1+d_1=0$ &

$a_2x_1+b_2y_1+d_2=0$ by solving we get $(x_1,y_1,0)$ on line

Hence, eqn. of line is $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-0}{n}$

NOTE : If $l\neq0$, take a point on yz-plane as $(0,y_1,z_1)$ &

if $m\neq0$ take a point on xz-plane as $(x_1,0,z_1)$

→ Parametric form : $P(x_1,y_1,z_1)$ & d.c's (l,m,n)

$x=x_1+lr,\quad y=y_1+mr,\quad z=z_1+nr$ where $r=OP$ distance

$\boxed{\frac{x-x_1}{l}=r}$

d.c's $=(l,m,n)$ the point (x_1,y_1,z_1) at a distance r is

(lr,mr,nr)

Angle b/w 2 lines:

$L_1 \equiv \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ & $L_2 \equiv \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ then

$$\cos\theta = \left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{\Sigma a_1^2}\sqrt{\Sigma a_2^2}}\right|$$

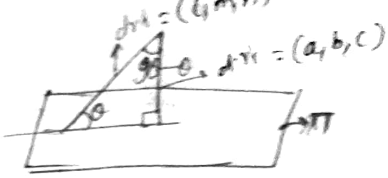
i) If lines are ||el then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

ii) If lines are lar then $a_1a_2+b_1b_2+c_1c_2=0$

Lines & Plane:

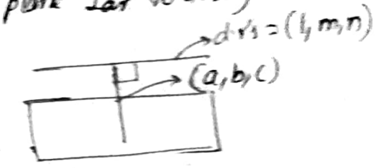
① If 'θ' is acute angle b/w line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ & $\pi = ax+by+cz+d=0$ is.

$$\cos(90°-\theta) = \sin\theta = \frac{|la+mb+nc|}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$$

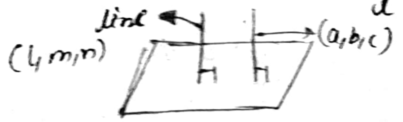


② If line & plane (π) are ||el. (Normal to plane lar to line)

$$\boxed{al+bm+nc=0}$$



③ If line & plane (π) are lar then $\frac{a}{l} = \frac{b}{m} = \frac{c}{n}$



④ d.c's of line make equal angles with co-ordinate axes are $\pm\left(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\right)$

& d.r's of line are (1,1,1)

→ Coplanar lines — 2 lines are coplanar if they either intersect (or) ||el.

Non-coplanar — 2 lines are non-coplanar or skew lines if neither intersect or ||el.

→ condition for coplanarity of line

① Line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ lies in plane $ax+by+cz+d=0$ if

$ax_1+by_1+cz_1+d=0$, $al+bm+cn=0$

(ii) Lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ & $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ are coplanar

$$\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$

→ Eqn. of plane containing lines –

i) $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ & $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0 \quad (or) \quad \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$

ii) If lines are coplanar then $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$, $a_1x+b_1y+c_1z+d_1=0$, $a_2x+b_2y+c_2z+d_2=0$ are coplanar then

$$\frac{a_1x_1+b_1y_1+c_1z_1+d_1}{a_1l+b_1m+c_1n}=\frac{a_2x_1+b_2y_1+c_2z_1+d_2}{a_2l+b_2m+c_2n}$$

⇒ Shortest distance b/w skew lines:

$\bar{r}_1=\bar{a}_1+\lambda\bar{b}_1$; $\bar{r}_2=\bar{a}_2+\mu\bar{b}_2$ is $\frac{|(\bar{a}_1-\bar{a}_2)(\bar{b}_1\times\bar{b}_2)|}{|\bar{b}_1\times\bar{b}_2|}$

i) if intersecting then $[\bar{a}_1-\bar{a}_2\ \bar{b}_1\ \bar{b}_2]=0$

ii) shortest distance b/w lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ & $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$

$$\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{\Sigma(b_1c_2-b_2c_1)^2}}$$

iii) distance b/w ||el lines $\bar{r}=\bar{a}_1+\lambda\bar{b}$, $\bar{r}=\bar{a}_2+\mu\bar{b}$ is

$$|d|=\frac{|\bar{b}\times(\bar{a}_1-\bar{a}_2)|}{|\bar{b}|}$$