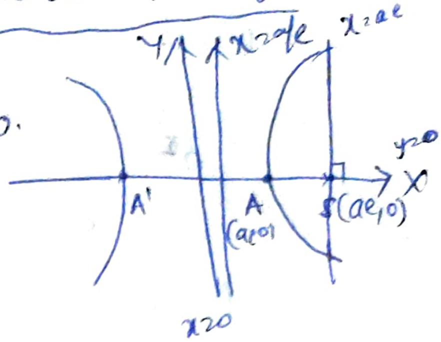
HYPERBOLA

→ locus of the centre of circle which touches two given circles externally is a hyperbola.



STD form:

$\rightarrow h^2 > ab,\ \Delta \neq 0.$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad e > 1$$

1. $C = (0,0)$
2. $e = \sqrt{\frac{a^2+b^2}{a^2}}$, $b^2 = a^2(e^2-1)$
3. $S = (\pm ae, 0)$
4. directrices $\Rightarrow x = \pm a/e$
5. L.L.R $= \frac{2b^2}{a}$
6. Ends of L.R $= (\pm ae, \pm b^2/a)$
7. Eqn of Transverse axes $\Rightarrow y = 0$
   conjugate axes $\Rightarrow x = 0$
8. Eqn of L.R $\Rightarrow x = \pm ae$
9. $|SP - S'P| = 2a$

$SP = (ex_1 - a)$
$S'P = (ex_1 + a)$

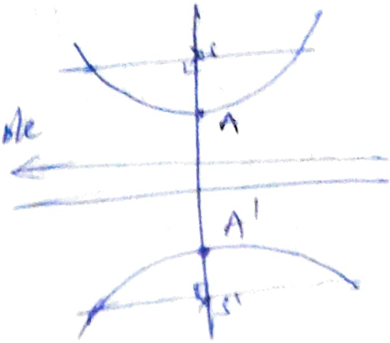
Length of T.A $= 2a$
C.A $= 2b$

$P(\theta) = (a\sec\theta, b\tan\theta)$
(or)
$(a\cosh\theta, b\sinh\theta)$

$$\Rightarrow \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

1. $C = (h, k)$
2. $e = \sqrt{\frac{a^2+b^2}{a^2}}$ $\qquad b^2 = a^2(e^2-1)$
3. $S = (h \pm ae, k)$
4. directrices $\Rightarrow x = h \pm a/e$
5. L.L.R $= 2b^2/a$
6. Ends of L.R $= (h \pm ae, k \pm b^2/a)$
7. Eqn of T.A $\Rightarrow y = k$ $\qquad$ length of T.A $= 2a$
   C.A $\Rightarrow x = h$ $\qquad$ length of C.A $= 2b$
8. Eqn of L.R $\Rightarrow x = h \pm ae$ $\qquad P(\theta) = (h + a\sec\theta, k + b\tan\theta)$
   $|SP - S'P| = 2a$

$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ (conjugate hyperbola)



$C = (0,0)$

$e = \sqrt{\frac{a^2+b^2}{b^2}}$ where $a^2 = b^2(e^2-1)$

$S = (0, \pm be)$

directrices $\Rightarrow y = \pm b/e$

$L.L.R \Rightarrow 2a^2/b$

ends of $L.R$ = ~~$(\pm be, \pm a^2/b)$~~ $(\pm a^2/b, \pm be)$

eqn of $T.A \Rightarrow x=0$ — length of $T.A = 2b$

eqn of $C.A \Rightarrow y=0$ — length of $C.A = 2a$

Eqn of $L.R \Rightarrow y = \pm be$

$SP = (ey_1 - b)$

$S'P = (ey_1 + b)$

$|SP - S'P| = 2b$

$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = -1$

① $C = (h,k)$

$e = \sqrt{\frac{a^2+b^2}{b^2}}$

$S = (h, \pm be + k)$

directrices $\Rightarrow y = k \pm b/e$

Eqn of $L.R \Rightarrow y = k \pm be$

$L.L.R = 2a^2/b$

Ends of $L.R = (h \pm a^2/b, k \pm be)$

Eqn of $T.A \Rightarrow x = h$ — length of $T.A = 2b$

of $C.A \Rightarrow y = k$ — Length of $C.A = 2a$

$\Rightarrow$ ** Curve does not exist b/w the line $x = a$ & $x = -a$

$\rightarrow$ "e" is eccentricity of H

"e'" is eccentricity of C.H. then

$\frac{1}{e^2} + \frac{1}{(e')^2} = 1$

$\rightarrow$ Position of point

$S_{11} < 0$ then $P(x_1, y_1)$ is an external point

$S_{11} = 0$ then $P(x_1, y_1)$ is on hyperbola

$S_{11} > 0$ then $P(x_1, y_1)$ is internal point

Tgts :

→ 2 tgts can be drawn to a hyperbola from an external point

→ Eqn of tgt point form ⇒ $S_1 = 0$

slope form ⇒ $y = mx \pm \sqrt{a^2m^2 - b^2}$

condition for tangency $c^2 = a^2m^2 - b^2$

Condition for c of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is $c^2 = b^2 - a^2m^2$

Parametric form ⇒ $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$

Angle b/w tgts $\tan\theta = \frac{2ab\sqrt{-S_{11}}}{x_1^2 + y_1^2 - a^2 + b^2}$

$m_1 m_2 = \frac{y_1^2 + b^2}{x_1^2 - a^2}$, $m_1 + m_2 = \frac{2x_1y_1}{x_1^2 - a^2}$

$P.O.C = \left(\frac{-a^2m}{c}, \frac{-b^2}{c}\right)$

for the line $lx + my + n = 0$ be a tgt

$a^2l^2 - b^2m^2 = n^2$ $P.O.C = \left(\frac{-a^2l}{n}, \frac{b^2m}{n}\right)$

→ Auxiliary circle: locus of foot of $\perp$ars for focii upon any tgt

$x^2 + y^2 = a^2$

Director circle: locus of point of intersection of $\perp$ar tangents to a hyperbola.

$x^2 + y^2 = a^2 - b^2$

→ ① Eqn of chord joining the two points $A(x_1, y_1)$, $B(x_2, y_2)$ is $S_1 + S_2 = S_{12}$

② chord of contact is $S_1 = 0$

③ eqn of chord having $P(x_1, y_1)$ as midpoint is $S_1 = S_{11}$

④. condition that the line $lx + my + n = 0$ to be a normal to the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}$

Normal :

condition for normality $c = \pm\frac{m(a^2 - b^2)}{\sqrt{a^2 - b^2m^2}}$

i) slope $y = mx \pm \frac{m(a^2 - b^2)}{\sqrt{a^2 - b^2m^2}}$

ii) $P(\theta)$ $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$

→ at most 4 normals can be drawn to a hyperbola.

Eqn of chord joining two points $P(\alpha)$, $Q(\beta)$ is

$$\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) - \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$$

→ $\alpha, \beta, \gamma, \delta$ are eccentric angles of co-normal points then $\alpha+\beta+\gamma+\delta = (2n+1)\pi$, $n \in W$

→ diameter: $y = \frac{b^2 x}{a^2 m}$

Some properties:

→ Tgt at the extremities of L.R intersects at corresponding directrix.

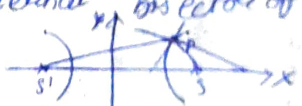
→ Product of $\perp$ar drawn from foci upon any tgt to auxiliary circle is b^2.

→ Normal at $P(\theta)$ meets T.A at G then $(A\cdot G)(A'G) = a^2(e^4\sec^2\theta - 1)$

→ Normal at $P(x_1, y_1)$ meets T.A at N then $eSP = SN$, $S'N = eS'P$

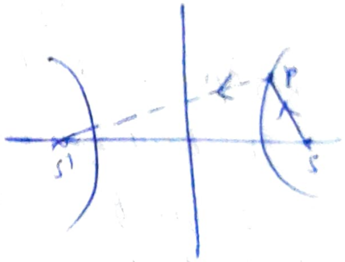
→ Area formed by foci of CH & H $= 2abe_1e_2$

→ tgt & normal are internal & external bisector of $\angle$le SPS'.



→ Four concyclic points lie on co-ordinate axes $m_1 m_2 = 1$

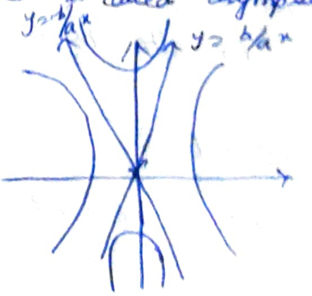
→ Reflection: Incident ray passes through SP then reflected ray passes through S'P.



→ Asymptotes: A tgt at infinity to the hyperbola is called asymptote.

for hyperbola $y = \pm \frac{b}{a}x$

pair of asymptotes $= \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$



Properties of asymptotes:

① Asymptotes are passing through centre of hyperbola ((0,0)).

② Asymptotes are equally inclined with co-ordinate axes $(m = \pm 1)$ $\tan \pi/4$

③ Asymptotes are bisectors of T.A & C.A

④ Asymptotes are same for hyperbola & conjugate hyperbola.

$S + 0K = 0$ (hyperbola)
$S + K = 0$ (asymptotes)
$S + 2K = 0$ (hyperbola)
$S + 3K = 0$ (asymptotes)

⑤ $H: \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$

$C: \frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 = 0$

$A: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ ie, $\boxed{C + H = 2A}$ C, H, A are in AP

⑥ Hyperbola & asymptotes are differ by constant.

Angle b/w asymptotes: $\theta = 2\tan^{-1}(b/a)$

$\theta = 2\sec^{-1}(e)$

⇒ Rectangular hyperbola (or) equilateral hyperbola:

→ If C.A = T.A then hyperbola is called rectangular hyperbola.

$\boxed{a = b}$

$x^2 - y^2 = a^2$

$e = \sqrt{2}$

asymptotes ⇒ $y = \pm x$

$x = y,\ y = -x$

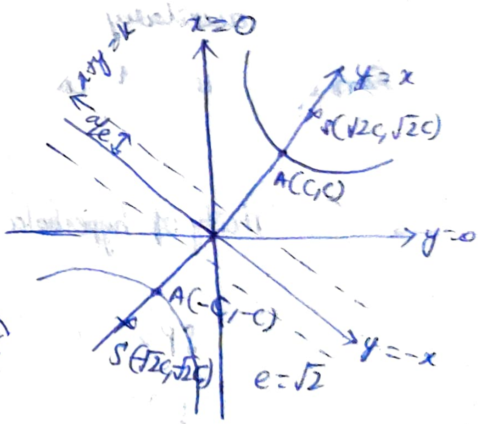
pair of asymptotes ⇒ $x^2 - y^2 = c$

$\theta = \pi/2$ in R.H asymptotes are ⊥ar

⇒ Asymptotes are rotated by 45° in clockwise direction then

$x^2 - y^2 = a^2$ reduces into $xy = c^2$

$xy = c^2 = a^2/2$



C = (0,0)

$e = \sqrt{2}$

vertices = $(\pm c, \pm c)$, $S = (\pm\sqrt{2}c, \pm\sqrt{2}c)$

T.A ⇒ $y = x$

C.A ⇒ $y = -x$

asymptotes $x = 0, y = 0$ $(a^2 = 2c^2)$

A.C ⇒ $x^2 + y^2 = 2c^2$

D.C ⇒ $x^2 + y^2 = 0$

L.L.R = $2a = 2\sqrt{2}c$

Equ of directrices : $x+y=\pm\sqrt{2}c$

Parametric co-ordinates = $(ct, c/t) \to P(t)$

Equ of tgt at $P(x_1, y_1)$ to $xy=c^2$ is $S_1=0$

point form → $\frac{1}{2}(xy_1+yx_1)=c^2 \Rightarrow S_1$

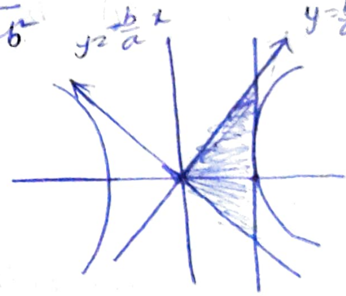
(or)

$$\frac{x}{x_1}+\frac{y}{y_1}=2$$

Parametric form ~~slope form~~ → $x+yt^2-2ct=0$ slope : $\frac{-1}{t^2}$

Equ of normal is $t^3x-yt=ct^4-c$

⇒ product of ⊥ distances drawn from any point to the hyperbola to its asymptotes is $\frac{a^2b}{a^2+b^2}$



⇒ Area of Δle formed by asymptotes & any one of the tgt is equal to ab

→ Equ of double ordinate is $x=a\sec\theta$

→ $$\frac{\left[\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}\right]^2}{a^2}-\frac{\left[\frac{b_2x-a_2y+c_2}{\sqrt{a_2^2+b_2^2}}\right]^2}{b^2}=1$$

→ Eqn of auxiliary circle $x^2+y^2=2c^2$

Eqn of director circle $x^2+y^2=0$

→ Eccentricity of hyperbola is $ax^2-by^2+cx+dy+e=0$ is $e=\sqrt{1+\frac{\text{coeff of } x^2}{|\text{coeff of } y^2|}}$

SP = ey_1-b, ex_1-a small focal distance

S'P = ey_1+b, ex_1+a large focal distance