Permutations

→ Addition principle : One can be performed m ways or other can be done in n ways, either of both can be done in m+n ways.

→ Multiplication principle : One can be done in m ways & other can be done in n ways, these two can be done one after other in mn ways.

→ Factorial : continuous product of 'n' natural numbers

$\lfloor n$ (or) $n!$.

$n! = n(n-1)!$

→ $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$, $6! = 720$, $7! = 5040$, $8! = 40320$, $9! = 362880$, $10! = 3628800$

→ Permutation - arrangement of some or all of finite set of things.

→ Linear permutation : arranged in line.

→ Permutations of 'n' dissimilar things $= {}^nP_r$ (or) $P(n,r)$ (or) $P\binom{n}{r}$

$n > r$

n = no. of dissimilar things in a set

r = no. of things taken from n.

→ 'n' dissimilar things taken 'r' at a time = no. of ways of filling r blank places by n dissimilar things.

$${}^nP_r = n(n-1)(n-2)\cdots\cdots n-(r-1)$$

$${}^nP_r = \frac{n(n-1)(n-2)\cdots\cdots n-(r-1)\ (n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}$$

$${}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

→ ${}^nP_0 = 1$

→ $\dfrac{{}^nP_r}{{}^nP_{r-1}} = n-r+1$

→ $\dfrac{{}^nP_{r-1}}{a} = \dfrac{{}^nP_r}{b} = \dfrac{{}^nP_{r+1}}{c}$ then $b^2 = a(b+c)$

→ No. of permutations of n things p are alike if all are taken at a time then $\dfrac{n!}{p!}$

→ No. of permutations n things, p are alike of 1 kind, q are alike of 2 kind if all are taken at a time then $\frac{n!}{p!q!}$

→ product of 'n' consecutive numbers is divisible by $n!$

→ A number is divisible by 2, if last digit is divisible by 2 (even)

→ A number is divisible by 3, if the sum of digits of number is divisible by 3.

→ A number is divisible by 4, if last 2 digits divisible by 4

→ Divisible by 5, if the last digit is 0 or 5.

→ Divisible by 6, the number divisible by 2 & 3. (even numbers whose sum of digits also divisible by 3).

→ Divisible by 7, if the difference b/w the twice the digits in the unit place and the no. formed by other digits is either 0 or multiple of 7.

Ex: 3675

$2\times5=10 \Rightarrow 367-10 = 357 = 51\times7$

→ Divisible by 8, if the last 3 digits is divisible by 8

→ Divisible by 9, if the sum of the number divisible by 9

→ Divisible by 10, last digit is exactly 0.

→ Divisible by 11 if the sum of the digits in the odd and sum of the digits in the even places are equal

Ex: 3564

$3+6=9$ $\quad 5+4=9$

→ Divisible by 25, if last two digits divisible by 25

→ Permutation when repeatation things are allowed.

→ The no. of permutations of n different things taken r at a time, when repeatation thing are allowed is n^r.

→ No. of permutations of n different things taken 'r' at a time with atleast 1 repeatation is $n^r - {}^nP_r$

→ No. of permutations of "n" different things "taken not more than r" at a time, repeatation things are allowed is

$n\left(\frac{n^r-1}{n-1}\right)$.

→ No. of permutations of 'n' different things taken more than 'r' at a time when repetition is allowed is $n\left(\frac{n^n-n^r}{n-1}\right)$.

Palindrome: A no. or word which reads same from left to right or from right to left

Ex: ATTA, ROTOR, 12321, etc.

Refer notes

→ Sum of no.'s formed by taking all the n-digits (excluding 0) is
(Sum of all n-digits) $\times (n-1)! \times$ (111... n times)

→ Sum of no.'s formed by taking all the n digits (including 0) is
(Sum of all the n digits) [(n−1)! (111... n times) − (n−2)! (111... (n−1) times)]

→ Sum of all the r-digits numbers formed by taking the given n-digits (excluding zero) is
(Sum of all the n digits) $\times {}^{(n-1)}P_{r-1} \times$ (111... r times)

→ Sum of all the 'r' digit numbers formed by taking the given n-digits (including zero) is.
(Sum of all the n-digits) $\left[{}^{(n-1)}P_{(r-1)} \times (111\ldots r \text{ times}) - {}^{(n-2)}P_{(r-2)}\,[111\ldots (r-1) \text{ times}]\right]$

→ Sum of all the digits in any place of all the numbers formed with the help of $a_1, a_2, \ldots, a_n$ (excluding 0) taken all at a time is $(n-1)!\,[a_1+a_2+\cdots+a_n]$

→ When n digits are given excluding zero then the sum of the value of digits in any place of n-digit number is $(n-1)!$ (Sum of all the numbers) (its place value)

→ A permutation is said to circular permutation if the objects are arranged in the form of a circle & closed curve.

→ No. of circular permutations of n - different things taken all at a time is $(n-1)!$

→ No. of circular permutations of n-different things taken r at a time (when clockwise & anticlockwise orders are taken as different) is

$$\frac{^nP_r}{r}$$

→ No. of circular permutations of n-different things taken r at a time in one direction (clockwise & anticlockwise orders are not different) like hanging type is $\frac{^nP_r}{2r}$

→ No. of circular permutations of n-dissimilar thing in clockwise direction that = no. of circular permutations in anticlockwise $= \frac{(n-1)!}{2}$

→ Garlands, chains, necklace are treated as hanging type of arrangement.

Derangements:

1. No. of ways in which exactly 'r' letters can be placed in wrongly addressed envelops when n letters are put in n addressed envelopes is $^nP_r\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^r \frac{1}{r!}\right)$

2. No. of ways in which n different letters can be put in their addressed envelopes so that all the letters are in wrong envelopes is $n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$

3. No. of ways that n letters are in 'n' addressed envelops, then exactly 1 letter will go wrong is 0.

4. No. of ways all the letters will go into correct addressed envelopes is 1.

Combinations

Definition: A selection formed by taking some (or) all of finite set of things or objects.

Ex: Combination formed by taking two at a time from the set $\{A, B, C\}$ are
$\{A,B\}, \{A,C\}, \{B,C\}$

→ Formation of a combination by taking 'r' all elements from a finite set A means picking up an 'r' elements subset of A.

→ No. of combination of n-dissimilar things taken 'r' at a time = no. of 'r' elements subsets of a set containing 'n' elements

→ No. of combinations is denoted by ^nC_r (or) $C\binom{n}{r}$, $\binom{n}{r}$, $C(n,r)$.

→ $^nC_n = 1$, $^nC_r = \frac{n!}{(n-r)!\,r!}$

→ $^nC_r = {}^nC_{n-r}$

→ $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$, $^nC_{r-2} + 2\,{}^nC_{r-1} + {}^nC_r = {}^{n+2}C_r$

→ $^nC_r = {}^nC_s$ then $r = s$, $n = r+s$

→ If no 3 points are collinear from n points then

i) No. of st. lines formed = $^nC_2$

ii) No. of Δle, formed = $^nC_3$

→ No. of diagonals in a regular polygon of n sides is $\frac{n(n-3)}{2}$

→ No. of combinations of n things taken 'r' at a time in which

i) 'S' particular things will always occur is $^{(n-s)}C_{(r-s)}$

ii) 'S' particular things will never occur is $^{(n-s)}C_r$.

iii) 'S' particular things always occur & 'p' particular things will never occur is $^{(n-s-p)}C_{(r-s)}$

→ No. of onto functions $= n^m - {}^nC_1(n-1)^m + {}^nC_2(n-2)^m \ldots + (-1)^{n-1}\,{}^nC_{n-1}$

No. of onto functions $= 2^m - 2$ $\quad n(A) = m, n(B) = n$
$\boxed{m \geq n}$

→ Vandermond's theorem:

$$^{m}C_0\,^{n}C_r + ^{m}C_1\,^{n}C_{r-1} + ^{m}C_2\,^{n}C_{r-2} + \dots + ^{m}C_r\,^{n}C_0 = ^{(m+n)}C_r$$

Geometrical applications

→ If n points are on the circumference of a circle are given then

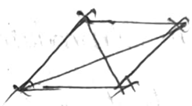
i. No. of st. lines $= ^{n}C_2$

ii. No. of Δles $= ^{n}C_3$

iii. No. of quadrilaterals $= ^{n}C_4$

→ If a polygon has n sides then no. of diagonals in it is $^{n}C_2 - n$

(or) $\dfrac{n(n-3)}{2}$



, interior angle $= \dfrac{(n-2)\pi}{n}$

→ In a plane there are n points & no three of which are collinear except k points which lie on a line then

(i) No. of st. lines that can be formed by joining them $= ^{n}C_2 - ^{k}C_2 + 1$

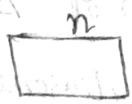
(ii) No. of Δles that can be formed by joining them $= ^{n}C_3 - ^{k}C_3$

→ If a set of m ||el lines are intersected by another set of n ||el lines then the no. of ||elograms that can be formed is $(^{m}C_2)(^{n}C_2)$

→ no. of rectangles of any size in a square of $n \times n$ is $\sum_{r=1}^{n} r^3$

no. of squares of any size is $\sum_{r=1}^{n} r^2$

→ In a rectangle of $n \times p$ $(n<p)$. no. of rectangles of any size is $\dfrac{1}{4}\, n(n+1)\, p(p+1)$



no. of squares of any size is $\sum_{r=1}^{n} (n+1-r)(p+1-r)$

→ no. of rectangles in a chess board including squares is 1296.

no. of squares of all dimensions on a chess boards is 204.

no. of rectangles on a chess board which are not a squares is 1092.

$$\sum_{r=1}^{8} r^3 - \sum_{r=1}^{8} r^2 = 1092$$

→ Groups: No. of ways in which $m+n$ items can be divided into 2 unequal groups containing m & n items is $^{m+n}C_m = \dfrac{(m+n)!}{m!\,n!}$

→ Vandermond's theorem:

$$^mC_0\,^nC_r + {}^mC_1\,^nC_{r-1} + {}^mC_2\,^nC_{r-2} + \dots + {}^mC_r\,^nC_0 = {}^{(m+n)}C_r$$

Geometrical applications

→ If n points are on the circumference of a circle are given then

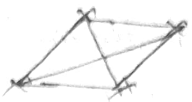
i. No. of st. lines $= {}^nC_2$

ii. No. of Δles $= {}^nC_3$

iii. No. of quadrilaterals $= {}^nC_4$

→ If a polygon has n sides then no. of diagonals in it is ${}^nC_2 - n$

(or) $\dfrac{n(n-3)}{2}$



, interior angle $= \dfrac{(n-2)\pi}{n}$

→ In a plane there are n points & no three of which are collinear except k points which lie on a line then

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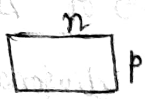

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$$\sum_{r=1}^{8} r^3 - \sum_{r=1}^{8} r^2 = 1092$$

→ Groups: No. of ways in which $m+n$ items can be divided into 2 unequal groups containing m & n items is ${}^{m+n}C_m = \dfrac{(m+n)!}{m!\,n!}$

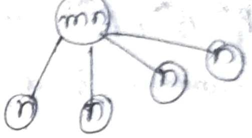
$^{m+n}C_m \; ^{n}C_n = \frac{(m+n)!}{m!\,n!}\,(1) = \frac{m+n!}{m!\,n!}$

→ No of way of distributing m+n different thing to 2 persons so that 1 gets m things, other gets n things is $\frac{(m+n)!}{m!\,n!}\,2!$

→ No of ways of dividing 2n different things into 2 groups each containing n things & order of the groups is not important is $\frac{(2n)!}{2!(n!)^2}$

→ No of ways of dividing 2n different thing into 2 groups each containing n things & order of the group is important is $\frac{(2n)!}{(n!)^2}$

→ No of ways in which mn different items can be divided equally into m groups, each containing n objects & the order of groups is not important is $\frac{(mn!)}{m!(n!)^m}$



→ No of ways in which mn different items can be divided equally into m groups, each containing n objects & the order of groups is important is $\frac{(mn!)}{(n!)^m}$

→ No. of ways in which m+n+p things can divided into 3 different ~~things~~ groups of m, n & p respectively is

$$\frac{((m+n+p)!)}{m!\,n!\,p!}$$

→ Total no of combinations: Tot No of $(P_1+P_2+\dots+P_k)$ things taken

(1) any number at a time when P_1 things are alike of I kind & P_2 are alike of 2nd kind, P_k things are alike of k^{th} kind is $(P_1+1)(P_2+1)\dots\dots(P_k+1)$.

② Total no of combinations $(P_1+P_2+\cdots+P_k)$ things 1 or more at a time when P_1 things are alike of I kind, P_2 things are alike of 2nd kind & P_k things are alike of kth kind is $[(P_1+1)(P_2+1)\cdots(P_k+1)]-1$

③ (i) Total no of combinations of n different things taken any number at a time is 2^n. i.e, $^nC_0+\cdots+^nC_n=2^n$.

(ii) No of ways of answering all n questions when each question has alternative is 2^n.

④ (i) Total no of combination of n different things taken 1 or more at a time is 2^n-1.

(ii) No of ways of answering 1 or more n question is 2^n-1.

Divisors

Let $N=p_1^{\alpha_1}, p_2^{\alpha_2}, p_3^{\alpha_3}, \ldots, p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are different prime numbers & $\alpha_1, \alpha_2, \ldots, \alpha_k$ are natural numbers then

① Total no. of divisors of $N=(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_k+1)$

② Total no. of divisior (proper) (or) non-trivial divisors of N (excluding 1 & N) =

$$(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_k+1)-2$$

(iii) Total no. of divisors of N (excluding either 1 or N) =

$$(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_k+1)-1$$

→ Sum of all divisors $\left(\dfrac{p_1^{\alpha_1+1}-1}{p_1-1}\right)\left(\dfrac{p_2^{\alpha_2+1}-1}{p_2-1}\right)\cdots\cdots\left(\dfrac{p_k^{\alpha_k+1}-1}{p_k-1}\right)$

P is prime no.

No. of ways in which N can be resolved as a product of 2 factors

(i) $\dfrac{(\alpha_1+1)(\alpha_2+1)\cdots\cdots(\alpha_k+1)}{2}$ ($\because$ N is not a perfect square)

(ii) $\dfrac{(\alpha_1+1)\cdots\cdots(\alpha_k+1)+1}{2}$ ($\because$ N is a perfect square)

→ If $N = 2^{\alpha_1} 3^{\alpha_2} 5^{\alpha_3} 7^{\alpha_4} \ldots$ then

(i) Total no of divisors of $N = (\alpha_1+1)(\alpha_2+1)(\alpha_3+1)\ldots$

(ii) Total no of odd divisors of $N = (\alpha_2+1)(\alpha_3+1)\ldots$

(iii) Total no of even divisors of $N = \alpha_1(\alpha_2+1)(\alpha_3+1)\ldots$

(iv) Sum of all divisors of $N = \left(\frac{2^{\alpha_1+1}-1}{2-1}\right)\left(\frac{3^{\alpha_2+1}-1}{3-1}\right)\ldots$

(v) Sum of all odd divisors of $N = \left(\frac{3^{\alpha_2+1}-1}{3-1}\right)\left(\frac{5^{\alpha_3+1}-1}{5-1}\right)\ldots$

(vi) Sum of all even divisors of N = sum of all divisors of N − sum of all odd divisors of N.

NOTE: $n! = 2^{\alpha_1} 3^{\alpha_2} 5^{\alpha_3} 7^{\alpha_4} \ldots$

→ Total no of divisors of $n = 3^{\alpha} 5^{\beta} 7^{\gamma} \ldots$ are of the form $4\lambda+1$ is $\frac{(\alpha+1)(\beta+1)(\gamma+1)\ldots}{2}$

→ Exponent of P in n! :

Exponent of P in n! or highest power of P in n! is denoted by $E_p(n!)$

$$E_p(n!) = \left[\frac{n}{P}\right] + \left[\frac{n}{P^2}\right] + \left[\frac{n}{P^3}\right] + \ldots$$

where P is a prime number & n is a natural number,

$[x]$ is a G.I.F $\leq x$

→ If $n! = 2^{\alpha_1} 3^{\alpha_2} 5^{\alpha_3} 7^{\alpha_4} \ldots$ then n! ends with r-zeroes i.e,

(E_5 - exponent of 5)

Distribution of similar things into groups:

(1) (i) Total no. of ways of dividing "n" identical items among "r" persons, each 1 of whom can receive 0, 1, 2, 3 or more items ($\leq n$) is $\boxed{{}^{n+r-1}C_{r-1}}$ $\quad 0 \leq x_1, x_2, x_3, \ldots, x_r \leq n$

(ii) Total no. of ways of dividing "n" identical things into "r" ordered groups if blank groups are allowed is ${}^{n+r-1}C_{r-1}$

(2) (i) Total no. of ways of dividing "n" identical items among "r" persons, each one of whom receive atleast 1 item is $\boxed{{}^{n-1}C_{r-1}}$

(ii) No. of ways in which "n" identical items can be divided into "r" ordered groups such that blank groups are not allowed is $^{n-1}C_{r-1}$

③ No. of non-negative integral solns of the eqn $x_1+x_2+x_3+\cdots+x_r = n$ is $^{n+r-1}C_{r-1}$, where $0 \le x_1, x_2, \ldots, x_r \le n$.

④ No. of +ve integral solns of the eqn $x_1+x_2+\cdots+x_n$ is $\boxed{^{n-1}C_{r-1}}$, $1 \le x_1, x_2, \ldots, x_r \le n$.

⑤ No. of non-negative integral solns of $x_1+x_2+\cdots+x_r \le n$ is $\boxed{^{n+r}C_r}$

Ex: ① $x+y=6$ $(0,6)\ (6,0)\ (1,5)\ (5,1)\ (2,4)\ (4,2)\ (3,3) \rightarrow 6+1=7$

$0 \le x, y \le 6$

No. of soln $= {}^{n+r-1}C_{r-1} = {}^{6+2-1}C_{2-1} = {}^{7}C_{1} = 7$

② $x+y=6$ $(1,5)\ (5,1),\ (2,4)\ (4,2)\ (3,3) \rightarrow 5$

$1 \le x, y \le 6$

No. of +ve solns $= {}^{n-1}C_{r-1} = {}^{6-1}C_{2-1} = 5$

→ There are n pairs of shoes in a closet. If $r (< n)$ are selected at random then no. of ways that among the selected shoes

① There is no paired is $^{n}C_{r} \cdot 2^{r}$.

(ii) There is atleast one pair $^{2n}C_{r} - {}^{n}C_{r} 2^{r}$

(iii) There is exactly one pair is $^{n}C_{1}\ {}^{n-1}C_{r-2}\ 2^{r-2}$

(iv) There is exactly 2 pairs is $^{n}C_{2}\ {}^{n-2}C_{r-4}\ 2^{r-4}$

→ If ABC is a Δ^{le} then m points on AB, n points on BC, k points on CA then no. of Δ^{les} formed by non collinear points is

$$^{m+n+k}C_3 - \left({}^{m}C_3 \cdot {}^{n}C_3 \cdot {}^{k}C_3\right)$$