Definite Integration

→ $\int_a^b f(x)\,dx = F(b) - F(a)$

Geometerical interpretation of definite integration

→ $\int_a^b f(x)\,dx = A_1 - A_2 + A_3 - A_4 + A_5$

Properties of definite integration:

① $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(t)\,dt$

② $\int_a^a f(x)\,dx = 0$

③ $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

④ If $a < c < b$ then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

⑤ $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

⑥ $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

⑦ $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$

⑧ $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$

→ $I = \frac{U\cdot L - L\cdot L}{2}$ if $\int_0^{\pi/2} \frac{f(\sin x)}{f(\sin x) + f(\cos x)}\,dx$, $\int_0^{\pi/2} \frac{f(\tan x)}{f(\tan x) + f(\cot x)}\,dx$, $\int_0^{\pi/2} \frac{f(\sec x)}{f(\sec x) + f(\operatorname{cosec} x)}\,dx$

→ $\int \frac{a + b\cos x}{(b + a\cos x)^2}\,dx \rightarrow \div$ by $\sin^2 x$ both Nr & Dr

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→ $\int_0^{n^2} [\sqrt{x}]\,dx = \frac{n(n-1)(4n+1)}{6}$

→ $[x] + [-x] = \begin{cases} -1 & \text{if } x \notin Z \\ 0 & \text{if } x \in Z \end{cases}$

$\rightarrow \int_0^{nT} f(x)dx = n\int_0^{T} f(x)\,dx \quad, n\in \mathbb{Z}$

$\rightarrow \mathrm{sgn}(x) = \frac{|x|}{x}$ (or) $\frac{x}{|x|}$

$\begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -1 & \text{if } x<0 \end{cases}$

$\int_a^{a+nT} f(x)dx = n\int_0^{T} f(x)dx$

period of $x-[x] = 1$

$\int_{a+nT}^{b+nT} f(x)dx = \int_a^b f(x)dx$

Leibnitz rule:

$\frac{d}{dx}\left(\int_{\phi(x)}^{\varphi(x)} f(t)dt\right) = f(\varphi(x))\varphi'(x) - f(\phi(x))\phi'(x)$

$\int_{mT}^{nT} f(x)dx = (n-m)\int_0^{T} f(x)dx$

→ Reduction formulae:

$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{1}{2}\left(\frac{\pi}{2}\right)$ if n is even

$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{2}{3}(1)$ if n is odd

$\int_0^{\pi/4} \tan^n x\,dx = \frac{1}{n-1} - I_{n-2}$, $\int_0^{\pi/4} (\tan^n x + \tan^{n-2}x)\,dx = \frac{1}{n-1}$

$I_n + I_{n-2} = 1/n-1$

$\int_0^{\pi/4} \sec^n x\,dx = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}$

$\int_0^{\pi/2} \sin^m x\,\cos^n x\,dx = \frac{[(m-1)(m-3)(m-5)\cdots 2(\text{or})1][(n-1)(n-3)\cdots 2(\text{or})1]}{[(m+n)(m+n-2)(m+n-4)\cdots 2(\text{or})1]}\cdot K$

where $K = \pi/2$ if m,n are even

$K = 1$ if m,n are not even.

$\rightarrow \lim_{n\to\infty} \frac{1}{n}\sum_{r=0}^{n-1} f(r/n) = \int_0^1 f(x)dx$

Replace r/n by x, $1/n$ by dx & $\lim_{n\to\infty}\Sigma$ by $\int$

$\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} f(r/n) = \int_0^1 f(x)dx$

$\lim_{n\to\infty} \frac{1}{n}\sum_{r=0}^{a(n-1)} f(r/n) = \int_0^a f(x)\,dx$

$\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{an} f(r/n) = \int_0^a f(x)dx$

$\rightarrow I_n = \int_0^{\pi/2} x^n \sin x\,dx = n(\pi/2)^{n-1} - n(n-1)I_{n-2}$

$\int_0^{\pi/2} x^n \cos x\,dx = (\pi/2)^n - n(n-1)I_{n-2}$

→① $\int_a^b \sqrt{\frac{x-a}{b-x}}\,dx = \frac{\pi}{2}(b-a)$, $\int_a^b \sqrt{(x-a)(b-x)}\,dx = \frac{\pi}{8}(b-a)^2$, $\int_a^b \frac{1}{\sqrt{(x-a)(b-x)}}\,dx = \pi$

② $\int_0^a \sqrt{\frac{a+x}{a-x}}\,dx = a\left(\frac{\pi}{2}+1\right)$, $\int_0^a \sqrt{\frac{a-x}{a+x}}\,dx = a\left(\frac{\pi}{2}-1\right)$

③ $\int_0^1 x^p(1-x)^q\,dx = \frac{p!\,q!}{(p+q+1)!}$

④ $\int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin x + \cos x}\,dx = (a+b)\pi/4$, $\int_0^{\pi/2} \frac{a\tan x + b\cot x}{\tan x + \cot x}\,dx = (a+b)\pi/4$,

$\int_0^{\pi/2} \frac{a\sec x + b\operatorname{cosec} x}{\sec x + \operatorname{cosec} x}\,dx = (a+b)\pi/4$

⑤ $a>0$ (i) $\int_0^\infty e^{-ax}\cos(bx)\,dx = \frac{a}{a^2+b^2}$, (ii) $\int_0^\infty e^{-ax}\sin(bx)\,dx = \frac{b}{a^2+b^2}$

⑥ $\int_0^{\pi/2} \ln(\sin x)\,dx = \int_0^{\pi/2} \ln(\cos x)\,dx = -\frac{\pi}{2}\ln 2$

⑦ $\int_0^{\pi/2} \ln \tan x\,dx = \int_0^{\pi/2} \ln\cot x\,dx = 0$

⑧ $\int_0^{\pi/4} \ln(1+\tan x) = \frac{\pi}{8}\ln 2 = \int_{\pi/4}^{\pi/2} \ln(1+\cot x)\,dx$

⑨ $\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{2ab}$

⑩ $\int_0^\infty \frac{1}{\left(x+\sqrt{x^2+1}\right)^n}\,dx = \int_0^{\pi/2} \frac{\sec^2\theta}{(\sec\theta+\tan\theta)^n}\,d\theta = \frac{n}{n^2-1}$

⑪ $\int_0^n [x]\,dx = \frac{n(n-1)}{2}$, $\int_0^n \{x\}\,dx = \int_0^n x-[x]\,dx = \frac{n}{2}$

$\int_0^{n^2} [\sqrt{x}]\,dx = \frac{n(n-1)(4n+1)}{6}$

⑫ $\int_{-\pi/2}^{\pi/2} \frac{\log(1+a\sin x)}{\sin x}\,dx \Rightarrow \pi\sin^{-1}(a)$