Limits

→ If $\lim\limits_{x\to a} f(x) = l$ & $\lim\limits_{x\to a} g(x) = m$ then

(i) $\lim\limits_{x\to a} \{f(x) \pm g(x)\} = l \pm m$ (ii) $\lim\limits_{x\to a} (\log f(x)) = \log\left(\lim\limits_{x\to a} f(x)\right)$

(iii) $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = l/m, \; m \neq 0$

→ $0/0$, ∞/∞, $\infty - \infty$, $0 \times \infty$, 0^0, ∞^0 & 1^∞ are indeterminate forms

→ Standard limits

(i) $\lim\limits_{x\to a} \frac{x^n - a^n}{x-a} = n a^{n-1}$ (ii) $\lim\limits_{x\to a} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} a^{m-n}$

If $0 \leq |x| < \pi/2$ and x in radians

$\lim\limits_{x\to 0} \frac{\sin x}{x} = 1 = \lim\limits_{x\to 0} \frac{\tan x}{x}$, $\lim\limits_{x\to 0} \frac{\sin ax}{x} = a = \lim\limits_{x\to 0} \frac{\tan ax}{x}$

(iii) $\lim\limits_{x\to 0} \frac{\sin x^\circ}{x} = \frac{\pi}{180} = \lim\limits_{x\to 0} \frac{\tan x^\circ}{x}$ (iv) $\lim\limits_{x\to \infty} \frac{\sin x}{x} = 0 = \lim\limits_{x\to \infty} \frac{\cos x}{x}$

(v) $\lim\limits_{x\to 0} \frac{\sin^{-1} x}{x} = 1 = \lim\limits_{x\to 0} \frac{\tan^{-1} x}{x}$ (vi) $\lim\limits_{x\to 0} \frac{\sin(hx)}{x} = 1 = \lim\limits_{x\to 0} \frac{\tan(hx)}{x}$

(vii) $\lim\limits_{x\to 0} \left(\frac{e^x - 1}{x}\right) = 1$ (viii) $\lim\limits_{x\to 0} \left(\frac{a^x - 1}{x}\right) = \log_e a$ $(a>0)$

(ix) $\lim\limits_{x\to 0} \frac{a^x - b^x}{x} = \log_e (a/b)$ (x) $\lim\limits_{x\to a} \frac{|x-a|}{x-a}$ does not exist

(xi) $\lim\limits_{x\to 0} (1+x)^{1/x} = e$, $\lim\limits_{x\to 0} (1+ax)^{1/x} = e^a = \lim\limits_{x\to \infty} \left(1+\frac{a}{x}\right)$, $\lim\limits_{x\to \infty} \frac{[x]}{x} = 1$

(OR) $\lim\limits_{x\to 0} (1+ax)^{b/x} = e^{ab} = \lim\limits_{x\to \infty} (1+a/x)^{bx}$

Standard results

i) $\lim\limits_{x\to 0^+} e^{1/x} = \infty$, $\lim\limits_{x\to 0^-} e^{\frac{1}{x}} = 0$ ii) $\lim\limits_{x\to 0^+} e^{-1/x} = 0$, $\lim\limits_{x\to 0^-} e^{-1/x} = \infty$

iii) $\lim\limits_{n\to \infty} x^n = 0$ if $|x| < 1$ iv) $\lim\limits_{n\to \infty} x^n = \infty$ if $|x| > 1$

v) $\lim\limits_{x\to 0} \sin(1/x) = \lim\limits_{x\to 0} \cos(1/x) =$ does not exist vi) $\lim\limits_{x\to 0} x \cos(1/x) = \lim\limits_{x\to 0} x \sin(1/x) = 0$

if $\lim\limits_{x\to \infty} \frac{(f(x))^m}{(g(x))^n} = \frac{\text{coeff of } f(x)^m}{\text{coeff of } g(x)^n}$ if $m=n$

vii) $\lim\limits_{x\to 0^+} x^x = 1$

→ Brahmastra of Limits – L'Hospital's rule

If $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$ then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, so on

→ $\lim_{x\to 0} \frac{\sin ax}{\tan bx} = \frac{a}{b}$, $\lim_{x\to 0} \frac{1-\cos ax}{x^2} = \frac{a^2}{2}$

→ $\lim_{x\to a} f(x) = 1$, $\lim_{x\to a} g(x) = \infty$ then $\lim_{x\to a} [f(x)]^{g(x)} = e^{\lim_{x\to a} g(x)[f(x)-1]}$

→ $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = 0$ then $\lim_{x\to a} [f(x)]^{g(x)} = e^{\lim_{x\to a} g(x)\log f(x)}$

→ Sandwich theorem (or) Squeeze principle

$f(x) \le g(x) \le h(x)$ then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x) \le \lim_{x\to a} h(x)$ and

$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = l$ then $\lim_{x\to a} g(x) = l$

→ $\lim_{x\to 0} (1+x)^{1/x} = e$, $f(x) \to m$ & $g(x) = n$

i) $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$ if $n > m$ ii) $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$ if $m > n$ & coeff of $x^m > 0$

iii) $\lim_{x\to\infty} \frac{f(x)}{g(x)} = -\infty$ if $m > n$, coeff of $x^m < 0$ (iv) $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\text{coeff of } x^m \text{ in Nr}}{\text{coeff of } x^n \text{ in Dr}}$ if $m = n$

→ $\lim_{x\to\infty} a^x = \begin{cases} 0 & \text{if } 0 \le a < 1 \\ 1 & \text{if } a = 1 \\ \infty & \text{if } a > 1 \\ \text{DNE} & \text{if } a < 0 \end{cases}$

→ $\lim_{x\to 0} \left(\frac{a_1^x + a_2^x + \cdots + a_n^x}{n}\right)^{1/x} = \sqrt[n]{a_1 a_2 a_3 \cdots a_n}$

$\lim_{x\to\infty} \left(\frac{a_1^{1/x} + a_2^{1/x} + \cdots + a_n^{1/x}}{n}\right)^{x} = \sqrt[n]{a_1 a_2 a_3 \cdots a_n}$

→ $\lim_{n\to\infty} \frac{[1^k x] + [2^k x] + \cdots + [n^k x]}{n^{k+1}} = \frac{x}{k+1}$

→ $\lim_{x\to 0} \frac{\tan^n x - \sin^n x}{x^{n+2}} = \frac{n}{2}$

→ If $0 < |x| < \pi/2$ then $|\sin x| < |x| < |\tan x|$

$\sin^{-1} x > x > \tan^{-1} x$

→ form limits

$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ then $\lim_{x\to a} \{1+f(x)\}^{1/g(x)} = e^{\lim_{x\to a} \frac{f(x)}{g(x)}}$

→ if $\lim_{n\to\infty} f(x) = 1$, $\lim_{n\to\infty} g(x) = \infty$ then $\lim_{n\to\infty} (f(x))^{g(x)} = e^{\lim_{n\to\infty} g(x)\log f(x)}$

→ $\lim_{n\to\infty} \log(1 + 1/an)^n = 1/a$ → $\lim_{x\to\infty} [x] = \lim_{x\to\infty} x$

Expansions

1. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \infty$

2. $a^x = 1 + \frac{x}{1!} \cdot \log_e a + \frac{x^2}{2!} (\log_e a)^2 + \cdots \infty$

3. $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \infty$

4. $\log_e(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots \infty\right)$

Expansions

1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \infty$

2. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \infty$

3. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} \cdots \infty$

4. $\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \cdots \infty$

5. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \infty$