Vectors

→ **Unit vector**: vector of length 1 unit is called unit vector

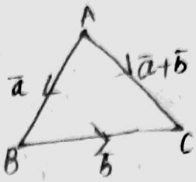
(i) unit vector in the direction of $\bar{a} = \frac{\bar{a}}{|\bar{a}|}$

(ii) unit vector in the opposite direction of $\bar{a} = \frac{-\bar{a}}{|\bar{a}|}$

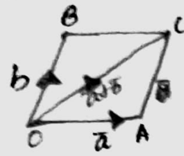
(iii) unit vector ||el to $\bar{a} = \pm\frac{\bar{a}}{|\bar{a}|}$

→ **Addition of vectors:**

$\bar{AC} = \bar{AB} + \bar{BC} = \bar{a} + \bar{b}$

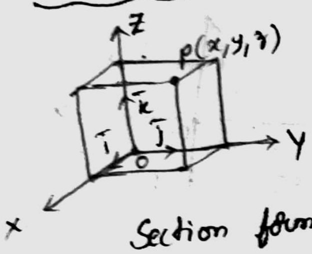


Parallelogram law of vector addition:



$\bar{OC} = \bar{OA} + \bar{OB} = \bar{a} + \bar{b}$

Components of a space vector:



$\bar{i}, \bar{j}, \bar{k}$ are unit vectors, x, y, z are scalars, $\bar{OP} = x\hat{i} + y\hat{j} + z\hat{k}$

$|\bar{OP}| = \sqrt{x^2+y^2+z^2}$

Section formula ⇒ $\bar{OC} = \frac{m\bar{b} \pm n\bar{a}}{m \pm n}$ (+ if internal, − if external)

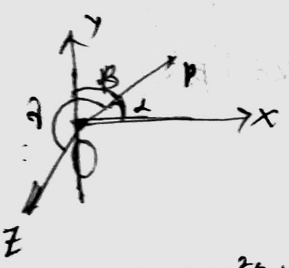
Like vectors & ||el vectors – vectors having same directions are like vectors. vectors having same (or) opposite directions are called ||el vectors.

→ $\bar{OA} = \bar{a}, \bar{OB} = \bar{b}, \bar{OC} = \bar{c}$ are non-coplanar. Viewing from C point, if $\bar{OA}$ to $\bar{OB}$ doesnot exceed 180° in ACW then $\bar{a}, \bar{b}, \bar{c}$ are set to form a right triad system otherwise left triad system.

→ Collinear vectors : 1) $\bar{a}, \bar{b}$ are collinear ⇔ $\bar{a} = \lambda\bar{b}$, λ is scalar

ii) $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$, $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$ collinear ⇔ $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

→ **D.C**:



$\alpha = (\bar{r}, \bar{i}), \beta = (\bar{r}, \bar{j}), \gamma = (\bar{r}, \bar{k})$

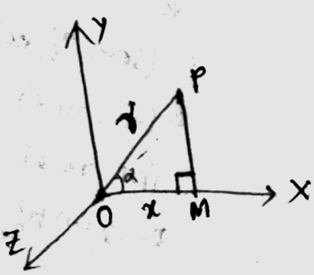
$\cos\alpha, \cos\beta, \cos\gamma$ are d.c's.

$\bar{a} + \bar{b} + \bar{c} = 0$ forms Δ^{le}

DR:

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$



$\cos\alpha = \frac{x}{r}$

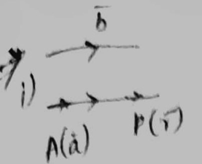
$x = r\cos\alpha = lr$

$y = mr$

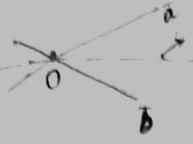
$z = nr$

Ex. if $\bar{a}$ & $\bar{b}$ are non collinear

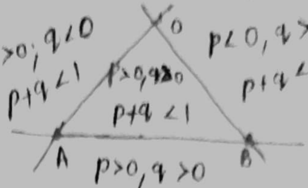
coeff $\bar{a} = 0$, coeff $\bar{b} = 0$

i)  $\overline{AP} = \lambda\bar{b}$, $\bar{r} = \bar{a} + \lambda\bar{b}$

ii)  $\overline{AP} = \lambda\overline{AB}$, $\bar{r} = (1-\lambda)\bar{a} + \lambda\bar{b}$

iii)  $\dfrac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$

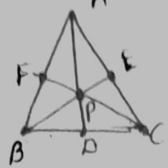
iv) $\overline{OA} = \bar{a}$, $\overline{OB} = \bar{b}$, ($\overline{OC} = p\bar{a} + q\bar{b}$ - point it's position)



$p>0, q<0$, $p+q<1$; $p>0, q>0$, $p+q<1$; $p<0, q>0$, $p+q<1$; $p>0, q>0$, $p+q>1$

v) outside ΔOAB if $p<0, q>0$ & $p+q>1$

⇒ Ceva's theorem:



$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = 1 \qquad \overline{AD}+\overline{BE}+\overline{CF} = \bar{0}$$

⇒ Menelaus theorem:

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = -1$$

Vectors $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$ & $\bar{c} = c_1\bar{i} + c_2\bar{j} + c_3\bar{k}$

i) If $\Delta \neq 0$, then they are linearly independent

ii) If $\Delta = 0$, then they are linearly dependent.

Vector eqn of plane:

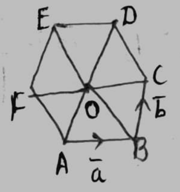
→ eqn of plane passing through a point $A(\bar{a})$ & ||el to non-collinear vectors

$\bar{r} = \bar{a} + s\bar{b} + t\bar{c}$

→ non collinear points $A(\bar{a}), B(\bar{b}), C(\bar{c})$ is $\bar{r} = (1-s-t)\bar{a} + s\bar{b} + t\bar{c}$

→ passing through $A(\bar{a}), B(\bar{b})$ & ||el to $C(\bar{c})$ is $\bar{r} = (1-s)\bar{a} + s\bar{b} + t\bar{c}$

Regular Hexagon



$\overline{AB} + \overline{AC} + \overline{AD} + \overline{AE} + \overline{AF} = 3\overline{AD} = 6\overline{AO}$

Dot product:

→ $\bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|\cos\theta$ $\theta = (\bar{a},\bar{b})$, $0 \le \theta \le 180$

→ $(\bar{a},\bar{b}) = \theta$, $\cos\theta = \dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}$

→ $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$, $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}$$

→ $(\bar{a},\bar{b}) = \theta$, i) $\theta < 90 \Leftrightarrow \bar{a}\cdot\bar{b} > 0$, ii) $\theta > 90 \Leftrightarrow \bar{a}\cdot\bar{b} < 0$, iii) $\theta = 90$, $\bar{a}\cdot\bar{b} = 0$

→ $\bar{a}, \bar{b}$ are two unit vectors, $\bar{a}\cdot\bar{b} = \cos\theta$

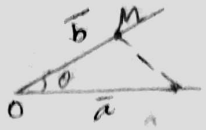
→ $\bar{a}\cdot\bar{b} = 0$ if $\bar{a} = 0$ (or) $\bar{b} = 0$ (or) $\bar{a} \perp \bar{b}$, $\bar{a},\bar{b} \neq 0$

→ $\bar{a}\cdot\bar{a} = |\bar{a}|^2 = \bar{a}^2$

→ Component of $\bar{a}$ on $\bar{b}$ (or) projection (scalar) $= \dfrac{(\bar{a}\cdot\bar{b})}{|\bar{b}|}$

Component vector of $\bar{a}$ on $\bar{b}$ (or) orthogonal projection (vector) $= \dfrac{(\bar{a}\cdot\bar{b})\bar{b}}{|\bar{b}|^2}$

Component vector of $\bar{a}$ ⊥ to $\bar{b}$ $= \bar{a} - \dfrac{(\bar{a}\cdot\bar{b})\bar{b}}{|\bar{b}|^2}$



$= \dfrac{\bar{b}\times(\bar{a}\times\bar{b})}{|\bar{b}|^2}$

Component vector of $\bar{b}$ on $\bar{a}$ $= \dfrac{(\bar{b}\cdot\bar{a})\bar{a}}{|\bar{a}|^2}$

Orthogonal projection $\bar{b}\perp\bar{a}$ $= \bar{b} - \dfrac{(\bar{b}\cdot\bar{a})\bar{a}}{|\bar{a}|^2}$

length of orthogonal projection of $\bar{b}$ on $\bar{a}$ is $\left|\dfrac{(\bar{a}\cdot\bar{b})}{|\bar{a}|}\right|$

$\bar{a}$ on $\bar{b}$ is. $\left|\dfrac{(\bar{a}\cdot\bar{b})}{|\bar{b}|}\right|$

→ $|\bar{i}| = |\bar{j}| = |\bar{k}| = 1$, $\bar{i}\cdot\bar{i} = \bar{j}\cdot\bar{j} = \bar{k}\cdot\bar{k} = 1$

$\bar{i}\cdot\bar{j} = \bar{j}\cdot\bar{k} = \bar{k}\cdot\bar{i} = 0$ (converse is same)

→ i) $\bar{n}$ of d.c's (l,m,n) plane eqn is $lx+my+nz=p$

ii) eqn of plane passing through origin ⊥ar to $\hat{n}$ is $\vec{r}\cdot\bar{n}=0$

its cartesian eqn is $lx+my+nz=0$

iii) if $(\bar{r}-\bar{a})\hat{n} = 0 \Rightarrow \bar{r}\cdot\bar{n} = \bar{a}\cdot\bar{n}$ $\Rightarrow \bar{r}-\bar{a} = \lambda\bar{n}$

$(\bar{r}-\bar{a})\bar{b} = 0 \Rightarrow \bar{r}\cdot\bar{b} = \bar{a}\cdot\bar{b}$ $\Rightarrow \bar{r}-\bar{a} = \lambda\bar{b}$

→ angle b/w 2 diagonals of cube is $\cos^{-1}(1/3)$, face & diagonal is $\cos^{-1}\left(\sqrt{2/3}\right)$

cube's diagonal & edge is $\cos^{-1}(1/\sqrt{3})$

→ angle b/w planes $\bar{r}\cdot\bar{n}_1 = p_1$, $\bar{r}\cdot\bar{n}_2 = p_2$ $\cos\theta = \dfrac{\bar{n}_1\cdot\bar{n}_2}{|\bar{n}_1||\bar{n}_2|}$

Vector (or) Cross product :

→ $\bar{a}\times\bar{b} = |\bar{a}||\bar{b}|\sin(\bar{a},\bar{b})\hat{n}$

$\sin(\bar{a},\bar{b}) = \dfrac{|\bar{a}\times\bar{b}|}{|\bar{a}||\bar{b}|}$

$\bar{a}\times\bar{c} = \bar{b}\times\bar{c}$ $\Rightarrow \bar{a}=\bar{b}$ (or) $\bar{a}-\bar{b}$ $\parallel$ $\bar{c}$

→ $\sin\theta = \dfrac{\sqrt{\Sigma(a_1b_2-a_2b_1)^2}}{\sqrt{\Sigma a_1^2}\sqrt{\Sigma b_1^2}}$, $\cos\theta = \dfrac{\Sigma a_1b_1}{\sqrt{\Sigma a_1^2}\sqrt{\Sigma b_1^2}}$ $\bar{a} = (a_1,a_2,a_3)$ $\bar{b} = (b_1,b_2,b_3)$

→ $\lambda(\bar{a}\times\bar{b})$ is ⊥ $\bar{a}$ & $\bar{b}$, unit vector $= \pm\dfrac{(\bar{a}\times\bar{b})}{|\bar{a}\times\bar{b}|}$ is $\bar{a}\perp\bar{b}$

$\overline{ABC}$ are $\pm\dfrac{(\overline{AB}\times\overline{AC})}{|\overline{AB}\times\overline{AC}|}$ ~~are~~ unit vector ⊥ ABC plane

→ $|\bar{a}\times\bar{b}|^2 = |\bar{a}|^2|\bar{b}|^2 - (\bar{a}\cdot\bar{b})^2 = \begin{vmatrix} \bar{a}\cdot\bar{a} & \bar{a}\cdot\bar{b} \\ \bar{b}\cdot\bar{a} & \bar{b}\cdot\bar{b} \end{vmatrix}$

→ vector area of Δ^{le} is $\frac{1}{2}(\bar{a}\times\bar{b})$, area $= \frac{1}{2}|\bar{a}\times\bar{b}|$

$A(\bar{a}), B(\bar{b}), C(\bar{c})$ are vertices of Δ^{le} ABC then

vector area $= \frac{1}{2}(\overline{AB}\times\overline{AC}) = \frac{1}{2}(\overline{BC}\times\overline{BA}) = \frac{1}{2}(\overline{CA}\times\overline{CB})$ Area $= \frac{1}{2}|\overline{AB}\times\overline{AC}|$

→ vector area of ||logram (i) vector area $= \bar{a}\times b$ (ii) Area $= |\bar{a}\times\bar{b}|$

if diagonals are given then vector area $= \frac{1}{2}(\bar{a}\times\bar{b})$, area $= \frac{1}{2}|\bar{a}\times\bar{b}|$

→ ⊥ar distance from a point 'P' to the line joining A, B is $\dfrac{|\overline{AP}\times\overline{AB}|}{|\overline{AB}|}$

→ $(\bar{a}\times\bar{i})^2 + (\bar{a}\times\bar{j})^2 + (\bar{a}\times\bar{k})^2 = 2|\bar{a}|^2$

Triple product:

→ $(\bar{a}\times\bar{b})\cdot\bar{c} = [\bar{a}\ \bar{b}\ \bar{c}] = \bar{a}\cdot(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}$

→ $[\bar{a}\ \bar{b}\ \bar{c}] = [\bar{b}\ \bar{c}\ \bar{a}] = [\bar{c}\ \bar{a}\ \bar{b}] = -[\bar{b}\ \bar{a}\ \bar{c}] = -[\bar{c}\ \bar{b}\ \bar{a}] = -[\bar{a}\ \bar{c}\ \bar{b}]$

→ $[\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

→ $[\bar{a}\ \bar{b}\ \bar{c}] = 0$ if $\bar{a}=\bar{b}$, $\bar{b}=\bar{c}$, $\bar{c}=\bar{a}$ | A, B, C, D are coplanar $\Leftrightarrow [\overline{AB}\ \overline{AC}\ \overline{AD}] =$

→ $[\bar{a}\ \bar{b}\ \bar{c}] > 0$, $\bar{a}, \bar{b}$ & $\bar{c}$ forms a traid of right-handed system.

$[\bar{a}\ \bar{b}\ \bar{c}] < 0$, $\bar{a}, \bar{b}$ & $\bar{c}$ forms a traid of left-handed system.

→ $\bar{a}, \bar{b}, \bar{c}$ are coterminous edges of ||lelopiped then volume is given as $V = |[\bar{a}\ \bar{b}\ \bar{c}]|$

→ $\overline{OA} = \bar{a}$, $\overline{OB} = \bar{b}$, $\overline{OC} = \bar{c}$ then the volume of tetrahedron OABC $= \frac{1}{6}|[\bar{a}\ \bar{b}\ \bar{c}]|$

→ A, B, C & D are the vertices $\frac{1}{6}|[\overline{AB}\ \overline{AC}\ \overline{AD}]|$

$$V = \frac{1}{6}|[\bar{a}-\bar{b}\ \ \bar{a}-\bar{c}\ \ \bar{a}-\bar{d}]|$$

→ $[\bar{a}\times\bar{b}\ \ \bar{b}\times\bar{c}\ \ \bar{c}\times\bar{a}] = 0$ if $\bar{a}, \bar{b}, \bar{c}$ are coplanar

→ $[\bar{i}\ \bar{j}\ \bar{k}] = [\bar{j}\ \bar{k}\ \bar{i}] = [\bar{k}\ \bar{i}\ \bar{j}] = 1$, $[\bar{a}\ \bar{b}\ \bar{c}][\bar{l}\ \bar{m}\ \bar{n}] = \begin{vmatrix} \bar{a}\cdot\bar{l} & \bar{b}\cdot\bar{l} & \bar{c}\cdot\bar{l} \\ \bar{a}\cdot\bar{m} & \bar{b}\cdot\bar{m} & \bar{c}\cdot\bar{m} \\ \bar{a}\cdot\bar{n} & \bar{b}\cdot\bar{n} & \bar{c}\cdot\bar{n} \end{vmatrix}$

→ Skew lines: two st. lines in space do not intersect and are also not ||el, then two lines are called skew lines.

→ l & m are two skew lines shortest distance b/w $\bar{r} = \bar{a} + s\bar{b}$, $\bar{r} = \bar{c} + t\bar{d}$ is

$$\left|\frac{[\bar{a}-\bar{c}\ \ \bar{b}\ \ \bar{d}]}{|\bar{b}\times\bar{d}|}\right| \text{ (or) } \left|(\bar{a}-\bar{c})\cdot\frac{\bar{b}\times\bar{d}}{|\bar{b}\times\bar{d}|}\right|$$

A, B, C, D are $\bar{a}, \bar{b}, \bar{c}, \bar{d}$. shortest distance $= \dfrac{[\overline{AC}\ \overline{AB}\ \overline{AD}]}{|\overline{AB}|}$

→ $\bar{a}, \bar{b}, \bar{c}$ be any three vectors such that $[\bar{a}\ \bar{b}\ \bar{c}] \neq 0$ then

$$\bar{a}' = \frac{\bar{b}\times\bar{c}}{[\bar{a}\ \bar{b}\ \bar{c}]}, \quad \bar{b}' = \frac{\bar{c}\times\bar{a}}{[\bar{a}\ \bar{b}\ \bar{c}]}, \quad \bar{c}' = \frac{\bar{a}\times\bar{b}}{[\bar{a}\ \bar{b}\ \bar{c}]}$$

$$\bar{a}\cdot\bar{a}' = \frac{\bar{a}\cdot(\bar{b}\times\bar{c})}{[\bar{a}\ \bar{b}\ \bar{c}]} = \bar{b}\cdot\bar{b}' = \bar{c}\cdot\bar{c}' = 1$$

$$\bar{a}\cdot\bar{b}' = \bar{a}\cdot\bar{c}' = \bar{b}\cdot\bar{a}' = \bar{b}\cdot\bar{c}' = \bar{c}\cdot\bar{a}' = \bar{c}\cdot\bar{b}' = 0$$

$$[\bar{a}'\ \bar{b}'\ \bar{c}'] = \frac{1}{[\bar{a}\ \bar{b}\ \bar{c}]}$$

→ $\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}$

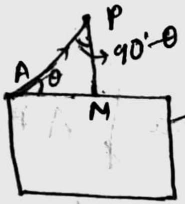
$[\bar{a}\times\bar{b}\ \ \bar{b}\times\bar{c}\ \ \bar{c}\times\bar{a}] = [\bar{a}\ \bar{b}\ \bar{c}]^2$, $[\bar{a}+\bar{b},\ \bar{b}+\bar{c}\ \ \bar{c}+\bar{a}] = 2[\bar{a}\ \bar{b}\ \bar{c}]$

$$(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d}) = (\bar{a}\cdot\bar{c})(\bar{b}\cdot\bar{d}) - (\bar{a}\cdot\bar{d})(\bar{b}\cdot\bar{c}) = \begin{vmatrix} \bar{a}\cdot\bar{c} & \bar{a}\cdot\bar{d} \\ \bar{b}\cdot\bar{c} & \bar{b}\cdot\bar{d} \end{vmatrix}$$

$$(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = [\bar{a}\ \bar{b}\ \bar{d}]\bar{c} - [\bar{a}\ \bar{b}\ \bar{c}]\bar{d} = [\bar{a}\ \bar{c}\ \bar{d}]\bar{b} - [\bar{b}\ \bar{c}\ \bar{d}]\bar{a}$$

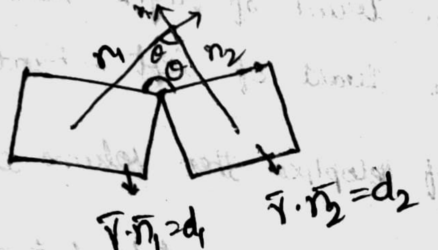
$(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = 0$ coplanar

→ angle b/w planes



$\bar{r}\cdot\bar{m} = \bar{d}$ $\cos(90°-\theta) = \frac{\bar{b}\cdot\bar{m}}{|\bar{b}|\,|\bar{m}|}$

$$\sin\theta = \frac{\bar{b}\cdot\bar{m}}{|\bar{b}|\,|\bar{m}|}$$



$$\cos\theta = \frac{\bar{n}_1\cdot\bar{n}_2}{|\bar{n}_1|\,|\bar{n}_2|}$$

S.C

→ vector along bisector of $\boxed{\bar{a}\ \&\ \bar{b} = \lambda(\hat{a}+\hat{b})}$ $\hat{a}$ — unit vector.

another vector be unit vector ie. $x\hat{i}+y\hat{j}+z\hat{k}$ = and find λ.