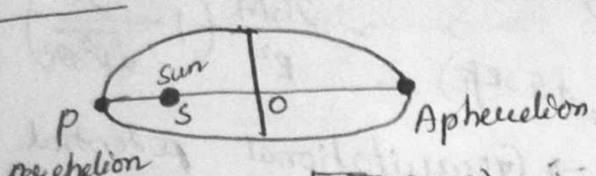
Gravitation

Kepler laws

1st law – Law of orbits



$e = \frac{OS}{OP} = \frac{OS}{OA} < 1$ $F_p > F_A$

$V_p > V_A$

$\boxed{V_{max} r_{min} = V_{min} r_{max}}$

$V_p = \sqrt{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)} = V_{max}$, $V_A = \sqrt{\frac{GM}{a}\left(\frac{1-e}{1+e}\right)} = V_{min}$

2nd law – Law of areas

Line joining sun & planet sweeps equal areas in equal intervals of time.

$V_A = \frac{dA}{dt} = \frac{d}{dt}\left(\tfrac{1}{2} r^2 d\theta\right) = \frac{1}{2} r^2 \omega = L/2m = vr/2$

3rd law – Law of periods

$T^2 \propto d^3$, $v = \sqrt{\frac{GM}{r}}$ $\cancel{T^2 = \frac{(4\pi^2 r)^3}{GM}}$ $T^2 = \frac{4\pi^2 R^3}{GM}$

Newton's law of gravitation

$F = \frac{G m_1 m_2}{r^2}$, $\vec{F} = \frac{GMm}{r^3}\vec{r}$ $G = 6.67 \times 10^{-11}\ m^3/kg\,s^2$

Relation b/w g & G, $g = \frac{GM}{R^2}$

at a height 'h' $= g_h = \frac{GM}{(R+h)^2}$ $g_h = g\left(1 - \frac{2h}{R}\right)$

at a depth 'd' $\Rightarrow g_d = \frac{GM}{R^3}(R-d) \Rightarrow g_d = g(1 - d/R)$

Variation of g on earth

$\boxed{g_{pole} > g_{surface} > g_{equ}}$

$g_\phi = g - R\omega^2\cos^2\phi$

at equator $\phi = 0$: $g_{eq} = g - R\omega^2 < g = 9.78\, m/s^2$

at poles $\phi = 90°$ $\Rightarrow$ $g_{poles} \neq g_{surface} = 9.82\, m/s^2$

→ Gravitational field intensity :

(I) Point mass

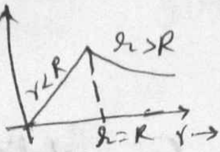
$$E = \frac{F}{m} = \frac{GM}{r^2} \Rightarrow E \propto 1/r^2$$

(II) Solid sphere

(i) $E_{out} = \frac{GM}{r^2}$ $(r > R)$

(ii) $E_{surf} = \frac{GM}{R^2}$ $(r = R)$

(iii) $E_{inside} = \frac{GMr}{R^3}$ $(r < R)$



(III) Spherical shell

$E_{inside} = 0$

Illy to S. Sphere

(IV) Circular ring

$$E_{axial} = \frac{GMx}{(R^2+x^2)^{3/2}}$$

$E_{centre} = 0$

$E_{max} = \frac{2GM}{3\sqrt{3}\,R^2}$ at $(x = R/\sqrt{2})$

(V) Disc

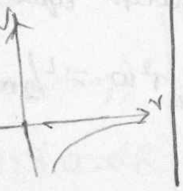
$$E = \frac{2GM}{R^2}(1-\cos\theta)$$
$$= \frac{2GM}{R^2}\left(1 - \frac{x}{\sqrt{R^2+x^2}}\right)$$

→ Gravitational potential :

$$V = W = -\int \vec{E}\cdot d\vec{r} \Rightarrow E = \frac{-dU}{dr}$$

(I) point mass

$V = \frac{-GM}{r}$; $\boxed{E = \frac{V}{r}}$



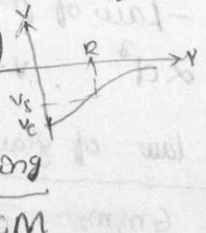
(II) Solid sphere

$V_{out} = \frac{-GM}{r}$ $\quad V_{in} = \frac{-GM}{R}\left(\frac{3}{2} - \frac{r^2}{2R^2}\right)$

centre $= -1.5\frac{GM}{R}$

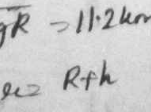
$r = 0$

$V_{surf} = \frac{-GM}{R}$



(III) Spherical shell

$V_{in} = \frac{-GM}{R}$



(IV) Circular ring

$$V_{axial} = \frac{-GM}{\sqrt{R^2+x^2}}$$

→ Gravitational potential energy

$$U = Vm = -\frac{GMm}{r}$$

3 particles system

$U = U_{12} + U_{23} + U_{31}$

$= \frac{-GM_1M_2}{r_{12}} - \frac{GM_2M_3}{r_{23}} - \frac{GM_3M_1}{r_{31}}$

Escape velocity

$V_e = \sqrt{\frac{2GM}{r}} = \sqrt{2gR} = 11.2\, km/s$

at an altitude $r = R+h$

$V_e = \sqrt{\frac{2GM}{R+h}}$

$v > V_e$, $V_{\infty} = \sqrt{v^2 - V_e^2}$

Orbital velocity

$V_o = \sqrt{\frac{GM}{R}} = \sqrt{gR} = 7.92\, km/sec$

$V_o = \sqrt{g(R+h)}$

→ Polar ice caps melts duration of day increases