Binomial Theorem

If $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\cdots$

then ① $a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + \cdots\cdots = f(1)$

② $a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + \cdots\cdots = f(-1)$

③ ①+② $\rightarrow$ $2(a_0 + a_2 + a_4 + \cdots\cdots) = f(1) + f(-1)$

$a_0 + a_2 + a_4 + \cdots\cdots = \frac{f(1) + f(-1)}{2}$

④ ①−② $\rightarrow$ $2(a_1 + a_3 + a_5 + \cdots\cdots) = f(1) - f(-1)$

$a_1 + a_3 + a_5 + \cdots\cdots = \frac{f(1) - f(-1)}{2}$

⑤ $a_0 + a_1 i - a_2 - a_3 i + a_4 + a_5 i + \cdots\cdots = f(i)$

⑥ $a_0 - a_1 i - a_2 + a_3 i + a_4 - a_5 i + \cdots\cdots = f(-i)$

⑦ ⑤+⑥ $\Rightarrow 2(a_0 - a_2 + a_4 - \cdots\cdots) = f(i) + f(-i)$

$a_0 - a_2 + a_4 \cdots\cdots = \frac{f(i) + f(-i)}{2}$

⑧ ⑤−⑥ $\Rightarrow 2i(a_1 - a_3 + a_5 - \cdots\cdots) = f(i) - f(-i)$

$a_1 - a_3 + a_5 - \cdots = \frac{f(i) - f(-i)}{2i}$

⑨ ③+⑦ $\Rightarrow 4(a_0 + a_4 + \cdots\cdots) = f(1) + f(-1) + f(i) + f(-i)$

$a_0 + a_4 + \cdots\cdots = \frac{1}{4}\left(f(1) + f(-1) + f(i) + f(-i)\right)$

⑩ ④+⑧ $\Rightarrow a_1 + a_5 + \cdots\cdots = \frac{1}{2}\left(\frac{f(1) - f(-1)}{2} + \frac{f(i) - f(-i)}{2i}\right)$

⑪ $a_1 + 2a_2 + 3a_3 + \cdots\cdots = f'(1)$

⑫ $a_1 - 2a_2 + 3a_3 - \cdots\cdots = f'(-1)$

⑬ $f(1) + f(\omega) + f(\omega^2) = (a_0 + a_1 + a_2 + a_3 + \cdots) + (a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + \cdots)$
$+ (a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + \cdots)$

$= 3a_0 + a_1(1 + \omega + \omega^2) + a_2(1 + \omega + \omega^2) + 3a_3 + \cdots\cdots$

$= 3(a_0 + a_3 + a_6 + \cdots\cdots)$

$\omega^3 = 1$, $1 + \omega + \omega^2 = 0$

$\omega^2 = \frac{-1 - \sqrt{3}i}{2}$

$\omega = \frac{-1 + \sqrt{3}i}{2}$

$a_0 + a_3 + a_6 + \cdots\cdots = \frac{1}{3}\left(f(1) + f(\omega) + f(\omega^2)\right)$

→ Expansion contains $(n+1)$ terms

→ General term $= T_{r+1} = {}^nC_r\, x^{n-r} a^r$

→ ${}^nC_0, {}^nC_1, \ldots, {}^nC_n$ are called binomial coefficients

→ p^{th} term from the beginning $= (n-p+2)^{th}$ from end.

$(x+a)^n + (x-a)^n$ even $\frac{n}{2}+1$ odd $\frac{n+1}{2}$

$(x+a)^n - (x-a)^n = \frac{n}{2}$ even, odd $\frac{n+1}{2}$

→ $\left(ax^p + \frac{b}{x^q}\right)^n$

General term $T_{r+1} = {}^nC_r\,(ax^p)^{n-r}\left(\frac{b}{x^q}\right)^r$

$= {}^nC_r\, a^{n-r} b^r x^{np-pr-qr}$

$$r = \frac{np-k}{p+q}$$

→ Middle term of $(x+a)^n$ if n is even $T_{\frac{n}{2}+1}$, n is odd then $T_{\frac{n+1}{2}}, T_{\frac{n+3}{2}}$

→ Numerically greatest term $= \frac{(n+1)|x|}{1+|x|}$

→ $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$, ${}^nC_r = {}^nC_{n-r}$

→ no. of terms in $(x+y+z)^n = \frac{(n+1)(n+2)}{2}$

The 1: $\sum_{r=0}^{n} C_r = 2^n$, $\sum_{r=0}^{n} (-1)^r C_r = 0$

The 2: $\sum_{r=0}^{n} rC_r = n2^{n-1}$, $\sum_{r=0}^{n} (-1)^{r-1} rC_r = 0$

The 3: $aC_0 + (a+d)C_1 + (a+2d)C_2 + \cdots + (a+nd)C_n = (2a+nd)2^{n-1}$

The 4: $C_0 + \frac{C_1}{2}x + \frac{C_2}{3}x^2 + \cdots + \frac{C_n}{n+1}x^n = \frac{(1+x)^{n+1}-1}{(n+1)x}$

The 5: $C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \cdots + C_{n-r}C_n = 2n!/(n-r)!(n+r)!$

→ Multinomial theorem : $(x_1+x_2+\cdots+x_m)^n = \dfrac{n!}{n_1!\,n_2!\cdots n_m!}\, x_1^{n_1} x_2^{n_2} x_3^{n_3}\cdots x_m^{n_m}$

No. of terms $= {}^{n+m-1}C_{m-1}$

$Z = x+iy$

$|Z| = |x+iy| = \sqrt{x^2+y^2}$ $\qquad |1+i| = \sqrt{2}$

$|Z^n| = |Z|^n$

→ No. of rational terms in $\left(x^{1/p} + y^{1/q}\right)^n = \begin{cases} \left[\dfrac{n}{\text{L.C.M of } p,q}\right] + 1 & \text{if } n \text{ is divisible by } p, q \text{ or both} \\ \left[\dfrac{n}{\text{L.C.M of } p,q}\right] & \text{if } n \text{ is not divisible by } p \text{ or } q \end{cases}$

[ ] denotes G.I.F

Some standard results

→ $(x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta$

→ $(x-\alpha)(x-\beta)(x-\gamma) = x^3 - (\alpha+\beta+\gamma)x^2 + \cdots$

→ $(x-\alpha_1)(x-\alpha_2)\cdots\cdots(x-\alpha_n) = x^n - (\alpha_1+\alpha_2+\cdots+\alpha_n)x^{n-1} + \sum \alpha_i\alpha_j\, x^{n-2} \cdots$

→ $(a+b+c)^2 = a^2+b^2+c^2+2\sum ab$

$\sum ab = \dfrac{(a+b+c)^2 - (a^2+b^2+c^2)}{2}$

→ $(\alpha_1+\alpha_2+\cdots\cdots+\alpha_n)^2 = (\alpha_1^2+\alpha_2^2+\cdots\cdots+\alpha_n^2) + 2\sum \alpha_i\alpha_j$

→ $\sum \alpha_i\alpha_j = \dfrac{(\alpha_1+\alpha_2+\cdots\cdots+\alpha_n)^2 - (\alpha_1^2+\alpha_2^2+\cdots\cdots+\alpha_n^2)}{2}$

→ ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

→ If n is not a +ve integer and $|x|<1$ ($\therefore -1<x<1$) then

① $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \cdots + \dfrac{n(n-1)(n-2)\cdots(n-(r-1))}{r!}x^r + \cdots\infty$

② $(1-x)^n = 1 - nx + \dfrac{n(n-1)}{2!}x^2 - \dfrac{n(n-1)(n-2)}{3!}x^3 + \cdots + \dfrac{(-1)^r n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r + \cdots + \infty$

③ $(1-x)^{-n} = 1 + nx + \dfrac{n(n+1)}{2!}x^2 + \dfrac{n(n+1)(n+2)}{3!}x^3 + \cdots + \dfrac{n(n+1)(n+2)\cdots(n+r-1)}{r!}x^r + \cdots + \infty$

④ $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \cdots + \frac{(-1)^r n(n+1)(n+2)\cdots n+r-1}{r!}x^r + \cdots + \infty$

If p, q are +ve integers $(q \neq 1)$ and $|x| < 1$ then

① $(1+x)^{p/q} = 1 + \frac{p}{1!}(x/q) + \frac{p(p-q)}{2!}(x/q)^2 + \frac{p(p-q)(p-2q)}{3!}(x/q)^3 + \cdots + \frac{p(p-q)(p-2q)\cdots p-(r-1)q}{r!}\left(\frac{x}{q}\right)^r + \cdots + \infty$

② $(1-x)^{p/q} = 1 - \frac{p}{1!}\left(\frac{x}{q}\right) + \frac{p(p-q)}{2!}(x/q)^2 - \frac{p(p-q)(p-2q)}{3!}\left(\frac{x}{q}\right)^3 \cdots + (-1)^r \frac{p(p-q)(p-2q)\cdots p-(r-1)q}{r!}\left(\frac{x}{q}\right)^r + \cdots + \infty$

③ $(1-x)^{-p/q} = 1 + \frac{p}{1!}(x/q) + \frac{p(p+q)}{2!}(x/q)^2 + \frac{p(p+q)(p+2q)}{3!}(x/q)^3 + \cdots + \frac{p(p+q)(p+2q)\cdots p+(r-1)q}{r!}(x/q)^r + \cdots + \infty$

④ $(1+x)^{-p/q} = 1 - \frac{p}{1!}(x/q) + \frac{p(p+q)}{2!}(x/q)^2 - \frac{p(p+q)(p+2q)}{3!}(x/q)^3 + \cdots + \frac{(-1)^r p(p+q)(p+2q)\cdots p+(r-1)q}{r!}(x/q)^r + \cdots + \infty$

Binomial Theorem

PART - I

→ If 'n' is +ve $\mathbb{Z}$ then

$$(x+a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} a^r = {}^nC_0 x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \cdots + {}^nC_n a^n$$

i) no. of terms in $(x+a)^n$ = $n+1$ terms

ii) sum of powers of x & a is equal to n.

iii) general term in expansion of $(x+a)^n$ is $T_{r+1} = {}^nC_r\, x^{n-r} a^r$

iv) ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ (or) $C_0, C_1, C_2, \ldots C_n$ are binomial Coeff's of $(x+a)^n$

v) ${}^nC_0 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1}$, ${}^nC_2 = {}^nC_{n-2}$, ...etc. $({}^nC_r = {}^nC_{n-r})$

vi) r^{th} term in expansion of $(x+a)^n$ is $(n-r+2)^{th}$ term from beginning

→ T_{r+1} in $(x-a)^n = (-1)^r\, {}^nC_r\, x^{n-r} a^r$.

→ $(x+a)^n$ & $(a+x)^n$ are equal but respective terms are not equal

→ $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_n x^n$.

⇒ **Number of terms:**

ⓐ No. of non-zero terms in expansion $\{(x+a)^n + (x-a)^n\}$ is

(i) $\frac{n+1}{2}$ if n is odd integer. (ii) $n/2 + 1$ if n is even.

ⓑ No. of non-zero terms in expansion $\{(x+a)^n - (x-a)^n\}$ is

(i) $\frac{n+1}{2}$ if n is odd integer (ii) $\frac{n}{2}$ if n is even

ⓒ no. of terms in $(x+a)^n + (x-a)^n + (x+ai)^n$ is $\left[\frac{n+4}{4}\right]$

→ **Middle terms of $(x+a)^n$**

i) if n is odd then $\left(\frac{n+1}{2}\right)^{th}$ & $\left(\frac{n+3}{2}\right)^{th}$ terms are middle

ii) if n is even then $\left(\frac{n}{2}+1\right)^{th}$ term is middle.

Greatest binomial coefficient :

In $(x+a)^n$

i) if n is odd, there are two greatest binomial coeff which are

$^nC_{\frac{n-1}{2}}$ & $^nC_{\frac{n+1}{2}}$ $\left(\because\ ^nC_{\frac{n-1}{2}} = \ ^nC_{\frac{n+1}{2}}\right)$

ii) if n is even, there is only one i.e, $^nC_{\frac{n}{2}}$

→ coeff of x^k in $\cancel{(x+a)^n}$ $\left(ax^p + b/x^q\right)^n$ is $^nC_r a^{n-r} b^r$ where $r = \frac{np-k}{p+q}$

→ term independent on x (const i.e, x^0) in $\left(ax^p + b/x^q\right)^n$ is $^nC_r a^{n-r} b^r$

where $r = \frac{np}{p+q}$ (middle term is term independent on x if $x \cdot a = 1$ in $(x+a)^n$)

⇒ Multinomial theorem

Generalised theorem is $(x_1+x_2+x_3+\cdots+x_m)^n = \underbrace{\sum \frac{n!}{n_1!\, n_2!\, n_3! \cdots n_m!}}_{\text{assumed}} x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}$

$n_1, n_2, n_3, \ldots, n_m$ are all non negative integers (W) &

$n_1 + n_2 + n_3 + \cdots + n_m = n$.

(a) General terms in expansion $(x_1+x_2+\cdots+x_m)^n$ is $\frac{(n_1+n_2+\cdots+n_m)!}{n_1!\, n_2! \cdots n_m!} (x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m})$

(b) No of terms in expansion is $^{(n+m-1)}C_{(m-1)}$

→ Numerically greatest term (N.G.T) is in $(1+x)^n$

i) if $\frac{(n+1)|x|}{|x|+1} = p+f$ where (p = integer, f = proper fraction, $0<f<1$)

then only 1 (N.G.T) which is $(p+1)^{th}$ term. & It's value is $|T_{p+1}|$.

ii) if $\frac{(n+1)|x|}{|x|+1} = p$ exists 2 numerically G.T which are p^{th} & $(p+1)^{th}$ terms. $|T_p| = |T_{p+1}|$

for $(a+b)^n$ N.G.T is $a^n(1+x)^n = (a+b)^n$ where $x = b/a$

→ Standard results

i) $C_0 + C_1 + C_2 + C_3 + \cdots + C_n = 2^n \Rightarrow \sum_{r=0}^{n} C_r = 2^n$

ii) $C_0 - C_1 + C_2 - C_3 + \cdots + (-1)^n C_n = 0$

iii) $C_0 + C_2 + C_4 + \cdots = C_1 + C_3 + C_5 + \cdots = 2^{n-1}$

iv) $aC_0 + (a+d)C_1 + (a+2d)C_2 + \cdots + (a+nd)C_n = (2a+nd)\,2^{n-1}$

v) $aC_0^2 + (a+d)C_1^2 + (a+2d)C_2^2 + \cdots + (a+nd)C_n^2 = \frac{1}{2}(2a+nd)\,{}^{2n}C_n$

vi) $C_0 + (C_0+C_1) + (C_0+C_1+C_2) + \cdots + (C_0+C_1+\cdots+C_{n-1}) = n \cdot 2^{n-1}$

vii) $C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2 = {}^{2n}C_n = \frac{(2n)!}{(n!)^2}$

→ If ${}^mC_n$ divisible by any +ve integer then check

exp of no. in ${}^mC_n$ if a^0 then it is not divisible by a

$\exp_{p_1}[{}^mC_n]\alpha = \left[\frac{m}{a}\right] + \left[\frac{m}{a^2}\right] + \left[\frac{m}{a^3}\right] + \cdots$ $[\cdot] \rightarrow$ G.I.F

$\beta = \left[\frac{n}{a}\right] + \left[\frac{n}{a^2}\right] + \left[\frac{n}{a^3}\right] + \cdots$

& a is only prime number.

→ Last two digits of x^y is

Ex: ① 31^{35} $35 = y$, $31 = x$

5

35

31

15

1

51

② $221^{50} - 8$

50

221

10

1

$101 - 8 = 93$

→ Applications :

① $(1-x)^{-1} = 1+x+x^2+x^3+\cdots+x^r+\cdots+\infty$

② $(1+x)^{-1} = 1-x+x^2-x^3+\cdots+(-1)^r x^r+\cdots+\infty$

③ $(1-x)^{-2} = 1+2x+3x^2+\cdots+(r+1)x^r+\cdots+\infty$

④ $(1+x)^{-2} = 1-2x+3x^2+\cdots+(-1)^r(r+1)x^r+\cdots+\infty$

⑤ $(1-x)^{-3} = 1+3x+6x^2+\cdots+\dfrac{(r+1)(r+2)}{2}x^r+\cdots+\infty$

⑥ $(1+x)^{-3} = 1-3x+6x^2-\cdots+(-1)^r\dfrac{(r+1)(r+2)}{2}x^r+\cdots+\infty$

→ First $-$ve term in $(1+x)^{p/q}$ is $T_{[p/q]+3}$ where $[\cdot]$ GIF

$0<x<1$. also p is not multiple of q.

Approximations :

→ If x is very small so that x^2 and higher powers are eliminated then

$$(1+x)^n \cong (1+nx)$$

→ If x is very small, x^3 is neglected and higher powers also then

$$(1+x)^n \cong 1+nx+\frac{n(n-1)}{2}x^2$$

$${}^{2n}C_n = ({}^nC_0)^2+({}^nC_1)^2+\cdots+({}^nC_n)^2$$

→ $\sum_{r=2}^{n} r(r-1)\cdot C_r = n(n-1)\,2^{n-2}$

→ $\sum_{r=1}^{n} r^2\cdot C_r = n(n+1)\,2^{n-2}$

→ Important results

(i) $aC_0-(a+d)C_1+(a+2d)C_2-(a+3d)C_3+\cdots = 0$

(ii) $C_1+2C_2+3C_3+\cdots+nC_n = n2^{n-1} \Rightarrow \sum_{r=1}^{n} r\cdot{}^nC_r = n2^{n-1}$

(iii) $C_1-2C_2+3\cdot C_3-\cdots = 0 \Rightarrow \sum_{r=1}^{n}(-1)^{r-1}\,r\,{}^nC_r = 0$

(iv) $C_0C_r+C_1C_{r+1}+\cdots+C_{n-r}C_n = {}^{2n}C_{n-r}$ (or) ${}^{2n}C_{n+r}$

(v) $C_0^2-C_1^2+C_2^2-\cdots+(-1)^nC_n^2 = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{n/2}\,{}^nC_{n/2}, & \text{if } n \text{ is even} \end{cases}$

(vi) $C_0+\frac{C_1}{2}x+\frac{C_2}{3}x^2+\cdots+\frac{C_n}{n+1}x^n = \frac{(1+x)^{n+1}-1}{(n+1)x}$

(vii) $\frac{C_1}{2}+\frac{C_3}{4}+\cdots = \frac{2^n-1}{n+1}$

(viii) $\frac{C_0}{1}+\frac{C_2}{3}+\frac{C_4}{5}+\cdots = \frac{2^n}{n+1}$

(ix) $\frac{C_0}{1}-\frac{C_1}{2}+\frac{C_2}{3}-\cdots+\frac{(-1)^nC_n}{n+1} = \frac{1}{n+1}$

→ if $S = 1+\frac{a}{b}+\frac{a(a+d)}{b\cdot(2b)}+\frac{a(a+d)(a+2d)}{b(2b)(3b)}+\cdots$

$$S = \left(1-\frac{d}{b}\right)^{-a/d}$$