TRIGONOMETRY

	Right angle	
→ Sexagesimal system	90°	1 minute $1'$ = 60 seconds ($60''$)
Centesimal system	100^g	100 seconds ($100''$)
Radian	$\pi/2$	

i) $\dfrac{D}{90} = \dfrac{G}{100} = \dfrac{R}{\pi/2}$ ii) $\pi = 180^\circ$, $\pi = \dfrac{22}{7} \simeq 3.141$

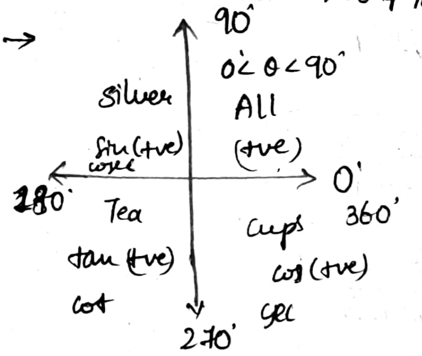
iii) 1 radian $\cong 57^\circ 17' 45''$ iv) $1^\circ \cong 0.01745^c$ (radians)

→ $\sin^2\theta + \cos^2\theta = 1$, $\sin\theta = \sqrt{1-\cos^2\theta}$, $\cos\theta = \sqrt{1-\sin^2\theta}$

$\sec^2\theta - \tan^2\theta = 1$, $\sec\theta = \sqrt{1+\tan^2\theta}$, $\tan\theta = \sqrt{\sec^2\theta - 1}$

$\operatorname{cosec}^2\theta - \cot^2\theta = 1$, $\operatorname{cosec}\theta = \sqrt{1+\cot^2\theta}$, $\cot\theta = \sqrt{\operatorname{cosec}^2\theta - 1}$

$\sec\theta \pm \tan\theta = \dfrac{1}{\sec\theta \mp \tan\theta}$ Similarly $\operatorname{cosec}\theta \pm \cot\theta = \dfrac{1}{\operatorname{cosec}\theta \mp \cot\theta}$

→ 

→ Componendo & dividendo

$\dfrac{a}{b} = \dfrac{c}{d} \Rightarrow \dfrac{a-b}{a+b} = \dfrac{c-d}{c+d}$

→ $\sin\theta + \sin(\pi+\theta) + \cdots + \sin(n\pi+\theta) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \sin\theta, & \text{if } n \text{ is even} \end{cases}$

$\cos\theta + \cos(\pi+\theta) + \cdots + \cos(n\pi+\theta) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \cos\theta, & \text{if } n \text{ is even} \end{cases}$

Compound angles

→ $\sin(A+B) = \sin A\cos B + \sin B\cos A$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$ $\cot(A+B) = \dfrac{\cot A\cot B - 1}{\cot B + \cot A}$

$\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$ $\cot(A-B) = \dfrac{\cot A\cot B + 1}{\cot B - \cot A}$

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\operatorname{cosec}\theta$
$0°$	0	1	0	∞	1	∞
$30°$	$1/2$	$\sqrt{3}/2$ (0.866)	$1/\sqrt{3}$	$\sqrt{3}$	$2/\sqrt{3}$	2
$45°$	$1/\sqrt{2}$	$1/\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$60°$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	$1/\sqrt{3}$	2	$2/\sqrt{3}$
$90°$	1	0	∞	0	∞	1
$15°$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$2-\sqrt{3}$	$2+\sqrt{3}$		
$75°$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$2+\sqrt{3}$	$2-\sqrt{3}$		
$22\frac{1}{2}°$	$\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$	$\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$	$\sqrt{2}-1$	$\sqrt{2}+1$	$\sin 22\frac{1}{2}°$ $\frac{\sqrt{2-\sqrt{2}}}{2}$	$\cos 22\frac{1}{2}°$ $\frac{\sqrt{2+\sqrt{2}}}{2}$
$67\frac{1}{2}°$	$\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$	$\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$	$\sqrt{2}+1$	$\sqrt{2}-1$		
$18°$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$	$\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}$		
$72°$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}$	$\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$		
$36°$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}$	$\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}$		
$54°$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}$	$\frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}$		
$9°$	$\frac{\sqrt{3-\sqrt{5}}-\sqrt{5-\sqrt{5}}}{4}$	$\frac{\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}}}{4}$				
$81°$	$\frac{\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}}}{4}$	$\frac{\sqrt{3-\sqrt{5}}-\sqrt{5-\sqrt{5}}}{4}$				
$27°$	$\frac{\sqrt{5+\sqrt{5}}-\sqrt{3-\sqrt{5}}}{4}$	$\frac{\sqrt{5+\sqrt{5}}+\sqrt{3-\sqrt{5}}}{4}$				
$63°$	$\frac{\sqrt{5+\sqrt{5}}+\sqrt{3-\sqrt{5}}}{4}$	$\frac{\sqrt{5+\sqrt{5}}-\sqrt{3-\sqrt{5}}}{4}$				
$7\frac{1}{2}°$			$(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$ $=\sqrt{2}-\sqrt{3}-\sqrt{4}+\sqrt{6}$	$(\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$ $=\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$		

6

→ $\sin(A+B) + \sin(A-B) = 2\sin A \cos B$

$\sin(A+B) - \sin(A-B) = 2\cos A \sin B$

$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$

$\cos(A+B) - \cos(A-B) = -2\sin A \sin B$

$\cos(A-B) - \cos(A+B) = 2\sin A \sin B$

→ $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

$\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

$\tan(A+B)\tan(A-B) = \dfrac{\tan^2 A - \tan^2 B}{1 - \tan^2 A \tan^2 B}$

→ $\sin(A+B+C) = \Sigma \sin A \cos B \cos C - \Pi \sin A$

$\cos(A+B+C) = \Pi \cos A - \Sigma \cos A \sin B \sin C$

$\tan(A+B+C) = \dfrac{\Sigma \tan A - \Pi \tan A}{1 - \Sigma \tan A \tan B}$

$\cot(A+B+C) = \dfrac{\Sigma \cot A - \Pi \cot A}{1 - \Sigma \cot A \cot B}$

→ If $A+B+C = n\pi,\ n \in Z$

i) $\Sigma \tan A = \Pi \tan A$

ii) $\Sigma \cot A \cot B = 1$

→ If $A+B+C = (2n+1)\pi/2,\ n \in Z$

i) $\Sigma \tan A \tan B = 1$

ii) $\Sigma \cot A = \Pi \cot A$

<u>Multiple & submultiple angles</u>

→ $\sin 2A = 2\sin A \cos A = \dfrac{2\tan A}{1+\tan^2 A}$

$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

$\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

$\cot 2A = \dfrac{\cot^2 A - 1}{2\cot A}$

$\sin 3A = 3\sin A - 4\sin^3 A$

$\cos 3A = 4\cos^3 A - 3\cos A$

$\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

$$\rightarrow \sin^2 A = \frac{1-\cos 2A}{2} \Rightarrow \sin A = \pm\sqrt{\frac{1-\cos 2A}{2}}$$

$$\cos^2 A = \frac{1+\cos 2A}{2} \Rightarrow \cos A = \pm\sqrt{\frac{1+\cos 2A}{2}}$$

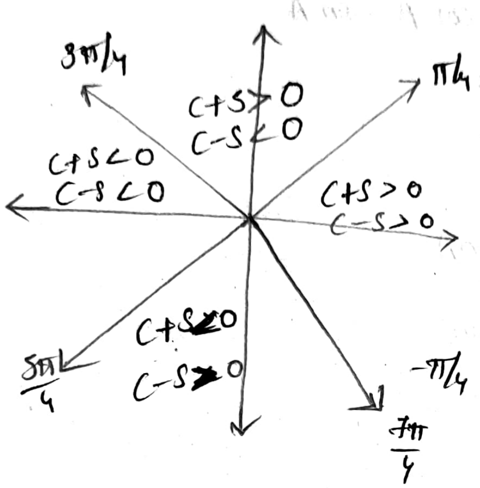
$$\tan^2 A = \frac{1-\cos 2A}{1+\cos 2A} \Rightarrow \tan A = \pm\sqrt{\frac{1-\cos 2A}{1+\cos 2A}}$$

$$\rightarrow \cot A - \tan A = 2\cot 2A$$

$$\cot A + \tan A = 2\operatorname{cosec} 2A$$

$$\cot A - \tan A - 2\tan 2A - \cdots - 2^{n-1}\tan 2^{n-1}A = 2^n \cot 2^n A$$

$\rightarrow$



$$\cos A + \sin A = \pm\sqrt{1+\sin 2A}$$

$$\cos A - \sin A = \pm\sqrt{1-\sin 2A}$$

$$\rightarrow \sqrt{2+\sqrt{2+\sqrt{2+\cdots\sqrt{2+2\cos\theta}}}} = 2\cos\left(\frac{\theta}{2^n}\right) \qquad 0<\theta<\pi$$

$$\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}} = 2\cos\left(\frac{\pi}{2^{n+1}}\right)$$

$$\rightarrow \tan\theta/_2\,(1+\sec\theta)(1+\sec 2\theta)\cdots(1+\sec 2^n\theta) = \tan 2^n\theta$$

$$(2\cos\theta-1)(2\cos 2\theta-1)(2\cos 4\theta-1)\cdots(2\cos 2^{n-1}\theta-1) = \frac{2\cos 2^n\theta+1}{2\cos\theta+1}$$

$$\rightarrow \cos\theta + \cos(120^\circ+\theta) + \cos(120^\circ-\theta) = 0$$

$$\sin\theta - \sin(120^\circ-\theta) + \sin(120^\circ+\theta) = 0$$

$$\tan\theta + \tan(1\times 60^\circ+\theta) + \tan(2\times 60^\circ+\theta) = 3\tan 3\theta \quad \left[\therefore 1\le n\le 5 \quad n\neq 3\right]$$

$$\tan\theta + \tan(n\times 60^\circ+\theta) + \tan((n+1)\times 60^\circ+\theta) = 3\tan 3\theta$$

$$\sin A\,\sin(x+A)\,\sin(x-A) = 1/_4\,\sin 3A$$

$$\cos A\,\cos(x+A)\,\cos(x-A) = 1/_4\,\cos 3A \qquad x = 60^\circ \text{(or) } 120^\circ \text{(or) } 240^\circ \text{(or) } 300^\circ$$

$$\tan A\,\tan(x+A)\,\tan(x-A) = \tan 3A$$

→ $\cos\theta \cdot \cos 2\theta \cdot \cos 4\theta \cdots \cos(2^{n-1}\theta) = \dfrac{\sin(2^n\theta)}{2^n \sin\theta}$

If $\theta = \dfrac{\pi}{2^n \pm 1}$ then $\cos\theta \cdot \cos 2\theta \cdot \cos 2^2\theta \cdots \cos 2^{n-1}\theta = \pm \dfrac{1}{2^n}$

Transformations

→ $\sin(A+B) + \sin(A-B) = 2\sin A \cos B$

$\sin(A+B) - \sin(A-B) = 2\cos A \sin B$

$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$

$\cos(A-B) - \cos(A+B) = 2\sin A \sin B$

$\sin^2 A + \sin^2(A+d) + \sin^2(A+2d) + \cdots = \frac{1}{2}\left(1 + \frac{l-a}{d}\right)$

→ $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

$\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

→ $A+B+C = 180^\circ$

$\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

$\sin 2A + \sin 2B - \sin 2C = 4\cos A \cos B \sin C$

$\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$

$\cos 2A + \cos 2B - \cos 2C = 1 - 4\sin A \sin B \cos C$

$\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$\cos A + \cos B - \cos C = -1 + 4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

→ $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan\left(\dfrac{A+B}{2}\right)$; $\dfrac{\sin A_1 + \sin A_2 + \sin A_3 + \cdots + \sin A_n}{\cos A_1 + \cos A_2 + \cos A_3 + \cdots + \cos A_n} = \tan\left(\dfrac{\sum_{i=1}^{n} A_i}{n}\right)$

→ $\sin(y+z-x), \sin(z+x-y), \sin(x+y-z)$ are in AP then $\tan x, \tan y, \tan z$ are in AP

→ period of $\sin(ax+b) = \dfrac{2\pi}{|a|}$

period of $\cos(ax+b) = \dfrac{2\pi}{|a|}$

period of $\operatorname{cosec}(ax+b) = \dfrac{2\pi}{|a|}$

period of $\sec(ax+b) = \dfrac{2\pi}{|a|}$

period of $\cot(ax+b) = \dfrac{\pi}{|a|}$

period of $\tan(ax+b) = \dfrac{\pi}{|a|}$

→ period of $|\sin x|, |\cos x|, |\tan x|, |\cot x|, |\operatorname{cosec} x|, |\sec x|$ is π

period of $a|\sin x| + b|\cos x|$ & $a|\operatorname{cosec} x| + b|\sec x|$ is $\pi/2$ if $(a=b)$, π if $a \neq b$

period of $|\tan x - \cot x|$ & $|\tan x + \cot x| = \pi/2$

period of $|\sin x + \cos x|$ & $|\sin x - \cos x| = \pi$

→ Min value of $a\cos x + b\sin x + c$ is $c - \sqrt{a^2+b^2}$

Max. value is $c + \sqrt{a^2+b^2}$

→ Trigonometric eqn

Trigonometric eqn	General solution	Principle soln
$\sin\theta = 0$	$\theta = n\pi, n \in Z$	$\theta = 0$
$\cos\theta = 0$	$\theta = (2n+1)\frac{\pi}{2}, n \in Z$	$\theta = \pi/2$
$\tan\theta = 0$	$\theta = n\pi, n \in Z$	$\theta = 0$
$\sin\theta = \sin\alpha$	$\theta = n\pi + (-1)^n\alpha, n \in Z$	$\theta = \alpha, \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos\theta = \cos\alpha$	$\theta = 2n\pi \pm \alpha, n \in Z$	$\theta = \alpha, \alpha \in [0, \pi]$
$\tan\theta = \tan\alpha$	$\theta = n\pi + \alpha, n \in Z$	$\theta = \alpha, \alpha \in (-\pi/2, \pi/2)$

$\sin^2\theta = \sin^2\alpha$ (or) $\cos^2\theta = \cos^2\alpha$ (or) $\tan^2\theta = \tan^2\alpha$ then

$\theta = n\pi \pm \alpha$, $n \in Z$ where $\alpha \in (0, \pi/2)$

<u>Inverse trigonometric functions:</u>

→

function	Domain	Range
$\sin^{-1}x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	$(-\infty, \infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1}x$	$(-\infty, \infty)$	$(0, \pi)$
$\sec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$
$\operatorname{cosec}^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, 0) \cup (0, \pi/2]$

→ $\sin(\sin^{-1}x) = x$, $\cos(\cos^{-1}x) = x$ Illy $\sin^{-1}(\sin\theta) = \theta$, $\cos^{-1}(\cos\theta) = \theta$, $\theta \in$ Range

$x \in$ domain

$\sin^{-1}(-x) = -\sin^{-1}x$, $\cos^{-1}(-x) = \pi - \cos^{-1}x$, $\tan^{-1}(-x) = -\tan^{-1}x$

$\cot^{-1}(-x) = \pi - \cot^{-1}x$, $\sec^{-1}(-x) = \pi - \sec^{-1}x$, $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$

→ $\sin^{-1}x = \operatorname{cosec}^{-1}(1/x)$, $\cos^{-1}x = \sec^{-1}(1/x)$, $\tan^{-1}x = \cot^{-1}(1/x)$

$= -\pi + \cot^{-1}(1/x)$

→ $\sin^{-1}x + \cos^{-1}x = \pi/2$, $\tan^{-1}x + \cot^{-1}x = \pi/2$, $\sec^{-1}x + \operatorname{cosec}^{-1}x = \pi/2$

$\boxed{0 \le x \le 1,\ 0 \le y \le 1}$

→ $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$ $x^2+y^2 \le 1$

$= \pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$ $x^2+y^2 > 1$

$\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$

$\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$ $x^2+y^2 \le 1$

$= \pi - \cos^{-1}\left(-xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$ $x^2+y^2 > 1$

$\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$

→ $\boxed{x > 0,\ y > 0}$

$$\tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & \text{if } xy < 1 \\ \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & \text{if } xy > 1 \\ \pi/2 & \text{if } xy = 1 \end{cases}$$

Max. & Min values of $(\sin^{-1}x)^2$ & $(\cos^{-1}x)^2$ are always $\pi^2/8$, $\frac{5\pi^2}{4}$

→ $2\sin^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$2\cos^{-1}x = \cos^{-1}(2x^2-1)$

$2\tan^{-1}x = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$, $-1 < x < 1$ $= \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ $-1 \le x \le 1$ $= \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ $0 \le x \le \infty$

→ $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$

$3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$

$3\tan^{-1}x = \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right)$

→ i) $\sin^{-1}(a/x) + \sin^{-1}(b/x) = \pi/2$ (or) $\cos^{-1}(a/x) + \cos^{-1}(b/x) = \pi/2$

then $x = \sqrt{a^2+b^2}$

ii) $\tan^{-1}(a/x) + \tan^{-1}(b/x) = \pi/2$ then $x = \sqrt{ab}$

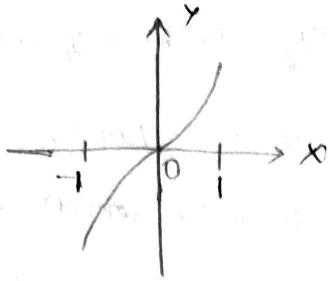
iii) $\tan^{-1}\left(\dfrac{S_1 - S_3 + S_5 + \cdots}{1 - S_2 + S_4 - S_6 + \cdots}\right) = \tan^{-1}x_1 + \tan^{-1}x_2 + \cdots + \tan^{-1}x_n$

→ $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right) = \tan^{-1}\left(\dfrac{S_1 - S_3}{1 - S_2}\right)$

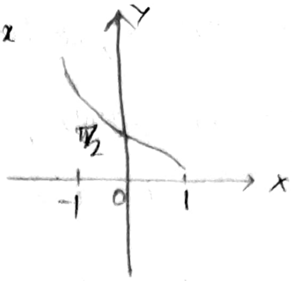
$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi/2$ then $xy + yz + zx = 1$

$= \pi$ then $x + y + z = xyz$

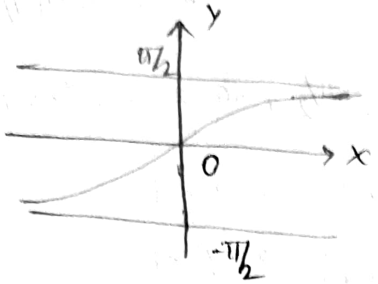
→ $y = \sin^{-1}x$



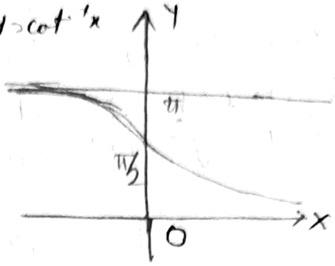
$y = \cos^{-1}x$



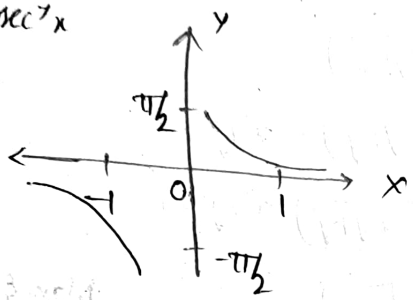
$y = \tan^{-1}x$



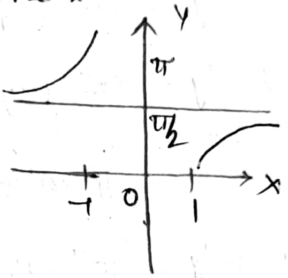
$y = \cot^{-1}x$



$y = \operatorname{cosec}^{-1}x$

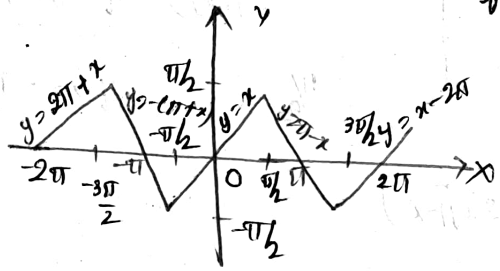


$y = \sec^{-1}x$



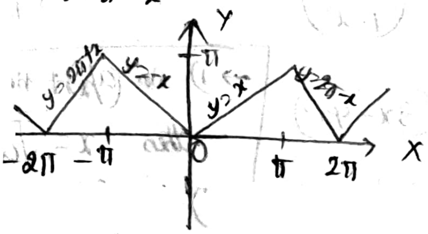
→ $y = \sin^{-1}(\sin x) = \begin{cases} -2n\pi + x \\ (2n+1)\pi - x \end{cases} = \operatorname{cosec}^{-1}(\operatorname{cosec} x)$

fecom $=$ G.S $= n\pi \pm (-1)^n x$



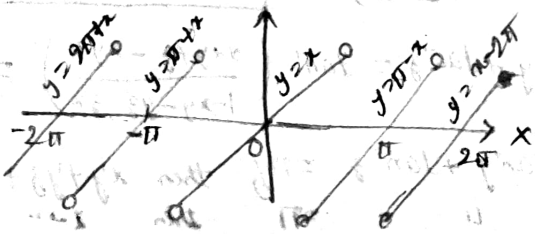
$y = \cos^{-1}(\cos x) = \begin{cases} -2n\pi + x \\ 2n\pi - x \end{cases} = \sec^{-1}(\sec x)$

G.S $= 2n\pi \pm x$



$y = \tan^{-1}(\tan x) = -n\pi + x,\ n\pi - \pi/2 = \cot^{-1}(\cot x)$

$n\pi + 2$



→ i) $\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right) = \begin{cases} 3\tan^{-1}x & ; \; -1/\sqrt{3} < x < 1/\sqrt{3} \\ \pi + 3\tan^{-1}x & ; \; -\infty < x < -1/\sqrt{3} \\ -\pi + 3\tan^{-1}x & ; \; 1/\sqrt{3} < x < \infty \end{cases}$

$1/\sqrt{3} = 0.56$

$1/\sqrt{2} = 0.77$

ii) $\sin^{-1}(2x\sqrt{1-x^2}) = \begin{cases} 2\sin^{-1}x & \text{if } |x| \le 1/\sqrt{2} \\ \pi - 2\sin^{-1}x & \text{if } 1/\sqrt{2} < x \le 1 \\ -(\pi + 2\sin^{-1}x) & \text{if } -1 \le x < -1/\sqrt{2} \end{cases}$

iii) $\cos^{-1}(2x^2-1) = \begin{cases} 2\cos^{-1}x & \text{if } 0 \le x \le 1 \\ 2\pi - 2\cos^{-1}x & \text{if } -1 \le x < 0 \end{cases}$

iv) $\tan^{-1}\dfrac{2x}{1-x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| < 1 \\ \pi + 2\tan^{-1}x & \text{if } x < -1 \\ -(\pi - 2\tan^{-1}x) & \text{if } x > 1 \end{cases}$

v) $\sin^{-1}\dfrac{2x}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| \le 1 \\ \pi - 2\tan^{-1}x & \text{if } x > 1 \\ -(\pi + 2\tan^{-1}x) & \text{if } x < -1 \end{cases}$

vi) $\cos^{-1}\dfrac{1-x^2}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } x \ge 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$

Hyperbolic functions

		D	R			Domain	Range
→ $\sinh x$	$= \dfrac{e^x - e^{-x}}{2}$	R	R	$\operatorname{cosech} x$	$= \dfrac{2}{e^x - e^{-x}}$	$R-\{0\}$	$R-\{0\}$
$\cosh x$	$= \dfrac{e^x + e^{-x}}{2}$	R	$[1,\infty)$	$\operatorname{sech} x$	$= \dfrac{2}{e^x + e^{-x}}$	R	$(0,1]$
$\tanh x$	$= \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$	R	$(-1,1)$	$\coth x$	$= \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$	$R-\{0\}$	$(-\infty,-1)\cup(1,\infty)$

→ $\sinh(-x) = -\sinh x$ $\qquad \operatorname{cosech}(-x) = -\operatorname{cosech} x$

$\cosh(-x) = \cosh x$ $\qquad \operatorname{sech}(-x) = \operatorname{sech} x$

$\tanh(-x) = -\tanh x$ $\qquad \coth(-x) = -\coth x$

→ $\cosh^2 x - \sinh^2 x = 1$, $\operatorname{sech}^2 x + \tanh^2 x = 1$, $\coth^2 x - \operatorname{cosech}^2 x = 1$

→ $\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \cosh(x)\sinh(y)$

$\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)$

$\tanh(x \pm y) = \dfrac{\tanh(x) \pm \tanh(y)}{1 \pm \tanh(x)\tanh(y)}$

(iv) $\coth(x \pm y) = \dfrac{\coth(x)\coth(y) \pm 1}{\coth(y) \pm \coth(x)}$

→ $\sinh(2x) = 2\sinh(x)\cosh(x) = \dfrac{2\tanh(x)}{1-\tanh^2 x}$

$\tanh(2x) = \dfrac{2\tanh x}{1+\tanh^2 x}$

$\cosh(2x) = \cosh^2 x + \sinh^2 x = 1 + 2\sinh^2 x = 2\cosh^2 x - 1$

$= \dfrac{1+\tanh^2 x}{1-\tanh^2 x}$

→ $\sinh(3x) = 3\sinh x + 4\sinh^3 x$ $\qquad \tanh(3x) = \dfrac{3\tanh x + \tanh^3 x}{1+3\tanh^2 x}$

$\cosh(3x) = 4\cosh^3 x - 3\cosh x$

→ $\sinh x$, $\tanh x$, $\sinh^{-1} x$, $\tanh^{-1} x$ are odd functions

$\cosh x$, $\cosh^{-1} x$ are even functions

→ $\sinh^{-1} x = \log_e\left[x + \sqrt{x^2+1}\right] = \cosh^{-1}\left[\sqrt{x^2+1}\right] = \operatorname{cosech}^{-1}(1/x)$

$\cosh^{-1} x = \log_e\left[x + \sqrt{x^2-1}\right] = \sinh^{-1}\left[\sqrt{x^2-1}\right] = \operatorname{sech}^{-1}(1/x)$

$\tanh^{-1} x = \frac{1}{2}\log_e\left[\dfrac{x+1}{1-x}\right]$

$\coth^{-1} x = \frac{1}{2}\log_e\left[\dfrac{x+1}{x-1}\right]$

$\operatorname{sech}^{-1} x = \log_e\left[\dfrac{1+\sqrt{1-x^2}}{x}\right]$

$\operatorname{cosech}^{-1} x = \log_e\left[\dfrac{1-\sqrt{1+x^2}}{x}\right], x < 0$ & $\log_e\left[\dfrac{1+\sqrt{1+x^2}}{x}\right], x > 0$

$\sinh^{-1} x = \tanh^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right)$, $\cosh^{-1} x = \tanh^{-1}\left(\dfrac{\sqrt{x^2-1}}{x}\right)$

Properties of Δ^{le}

→ $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$ $\qquad$ R = circumradius

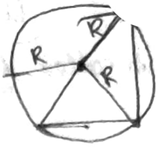
$\cos A = \dfrac{b^2+c^2-a^2}{2bc}$, $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$, $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

Projection rule

→ $a = b\cos C + c\cos B$

$b = c\cos A + a\cos C$

$c = a\cos B + b\cos A$



→ $\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\frac{A}{2}$, $\tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\cot\frac{C}{2}$

$\tan\left(\frac{C-A}{2}\right) = \left(\frac{c-a}{c+a}\right)\cot\frac{B}{2}$

→ $\frac{r_1-r}{a} + \frac{r_2-r}{b} = \frac{c}{r_3}$

→ $\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$, $\frac{b+c}{a} = \frac{\cos\left(\frac{B-C}{2}\right)}{\sin\frac{A}{2}}$

→ $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$, $\tan\left(\frac{A}{2}\right) = \frac{\Delta}{s(s-a)}$

$\cot\frac{A}{2} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} = \frac{s(s-a)}{\Delta}$

$a^2 = (r_1-r)(r_2+r_3)$
$b^2 = (r_2-r)(r_1+r_3)$
$c^2 = (r_3-r)(r_2+r_1)$

→ Area of Δ^{le}

$\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C = \sqrt{s(s-a)(s-b)(s-c)}$

$\Delta = \frac{abc}{4R} = 2R^2\sin A\sin B\sin C = rs$

r = radius of incle.

⇒ $\cot A = \frac{b^2+c^2-a^2}{4\Delta}$

$\cot A, \cot B, \cot C$ are in AP then a^2, b^2, c^2 are in AP

→ $r = \Delta/s = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

$r = \frac{a}{\cot\frac{B}{2} + \cot\frac{C}{2}} = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

$r_2 = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$

$r_3 = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

r_1, r_2, r_3 → $r_1+r_2+r_3-r = 4R$

$r_1 = r_2 = r_3 = \frac{3R}{2}$

→ $r_1 = \Delta/(s-a)$, $\frac{b^2c}{R}\cos C + \frac{c^2b}{R}\cos B = \frac{abc}{R} = 4\Delta$

$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$, $r_1 r_2 r_3 r = \Delta^2$, $r_1r_2 + r_2r_3 + r_3r_1 = s^2$

Nature of Δ^{le} –

$a\cos B = b\cos A$, then Δ^{le} is isosceles

→ Length of median through C is $\frac{1}{2}\sqrt{a^2+b^2+2ab\cos C}$ ||ly for B, A

→ $r + r_2 + r_3 - r_1 = 4R\cos A$

$r + r_1 + r_3 - r_2 = 4R\cos B$