Matrices

→ Order = $m \times n$ (m ↓ rows, n ↓ columns)

→ diagonal matrix ⇒ $\text{diag}(d_1, d_2, d_3, \cdots, d_n) = A$, $A^n = \text{diag}(d_1^n, d_2^n, \cdots, d_n^n)$

→ identity matrix ⇒ $a_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

→ Triangular matrix

Upper Δ^{le} matrix = $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$, lower Δ^{le} matrix = $\begin{bmatrix} a & 0 & 0 \\ c & b & 0 \\ d & e & f \end{bmatrix}$

→ Trace = sum of diagonal elements ($tr(A)$). Also called as spur.

Properties

i) $\lambda\, tr(A) = tr(\lambda A)$

ii) $tr(AB) = tr(BA)$

iii) $tr(A \pm B) = \cancel{tr(B \pm A)}$ $tr(A) \pm tr(B)$

iv) $tr(I_n) = n$

v) $tr(AB) \neq tr(A) \cdot tr(B)$

vi) $tr(ABC) = tr(B\cancel{C}A) = tr(CAB)$

→ A & B are 2 square matrices of same order

i) $(A+B)^2 = A^2 + B^2 + AB + BA$

ii) $(A-B)^2 = A^2 + B^2 - AB - BA$

iii) $A^m A^n = A^{m+n}$

iv) $(A^m)^n = A^{mn}$

v) $I^n = I$

→ Properties of transpose, (A^T):

(i) $(A^T)^T = A$

(ii) $(A \pm B)^T = A^T \pm B^T$

(iii) $(kA)^T = kA^T$

(iv) $(AB)^T = B^T A^T$

(v) $(A_1 A_2 A_3 \cdots A_{n-1} A_n)^T = A_n^T A_{n-1}^T \cdots A_3^T A_2^T A_1^T$

(vi) $A = B$ then $A^T = B^T$

→ symmetric matrix $A^T = A$.

Skew-symmetric matrix $A^T = -A$, $tr(A) = 0$

→ Orthogonal matrix — $AA^T = A^T A = I$ i.e, $A^T = A^{-1}$ & also works for $|A| = \pm 1$

→ sum of squares of any row or column $= 1$

→ Idempotent matrix — $A^2 = A$

Properties — i) $AB = BA = 0$ then $A+B$ is idempotent

ii) $AB = A$, $BA = B$, $A^n + B^n = A + B$

→ Involutary matrix — $A^2 = I$

→ Nilpotent matrix — $A^p = 0$, $p \in N$

Determinants

→ **Minor**: The minor of an element a_{ij} in A is det of [A] after deleting the i^{th} row & j^{th} column. (M_{ij})

→ **Cofactor**: minor multiplied by $(-1)^{i+j} M_{ij}$

Properties:
→ det changes its sign when any two rows are interchanged.
→ If any 2 rows are proportional then det $= 0$

→ $$\begin{vmatrix} a_1+x & a_2 & a_3 \\ b_1+y & b_2 & b_3 \\ c_1+z & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} x & a_2 & a_3 \\ y & b_2 & b_3 \\ z & c_2 & c_3 \end{vmatrix}$$

→ $|A| = |A^T|$, $|AB| = |A||B|$, $\boxed{|KA| = K^n|A|}$ n = order of A

→ det of cofactor is Δ^{n-1}

Product of determinants:

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \& \quad \Delta_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$ then row by row multiplication.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \Delta^T = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$ then row by column multiplication

Derivative of det

$$\Delta(x) = \begin{vmatrix} f_1(x) & g_1(x) \\ f_2(x) & g_2(x) \end{vmatrix} \Rightarrow \Delta'(x) = \begin{vmatrix} f_1'(x) & g_1'(x) \\ - & - \end{vmatrix} + \begin{vmatrix} - & - \\ f_2'(x) & g_2'(x) \end{vmatrix}$$

Integration of det:

$$\Delta(x) = \begin{vmatrix} f(x) & g(x) \\ \lambda_1 & \lambda_2 \end{vmatrix} \quad \text{then} \quad \int_a^b \Delta(x)\,dx = \begin{vmatrix} \int_a^b f(x)\,dx & \int_a^b g(x)\,dx \\ \lambda_1 & \lambda_2 \end{vmatrix}$$

only applicable for det like this otherwise expand & integrate

⇒ **Inverse of matrices**

→ If $|A| = 0$, it is singular matrix otherwise non-singular

→ **Adjoint matrix**: The matrix obtained by transposing cofactor matrix.

Properties:

i) $A(\text{adj } A) = |A| I_n = (\text{adj } A)A$

ii) $|\text{adj } A| = |A|^{n-1}$

iii) $|\text{adj}(\text{adj}(\text{adj } A))\ldots\ldots r \text{ times}| = |A|^{(n-1)^r}$

iv) $adj(adj A) = |A|^{n-2} A$

v) $adj(A^T) = (adj A)^T$

vii) $adj(AB) = (adj B)(adj A)$

viii) $adj(A^m) = (adj A)^m$

ix) $adj(KA) = K^{n-1}(adj A)$

$adj(adj(adj A)) = (|A|^{(n-1)})^{n-2}$

Inverse – $A^{-1} = \frac{adj A}{\det A}$

for 2X2 $A^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad-bc}$

Properties

i) $(A^{-1})^{-1} = A$

ii) $(A^T)^{-1} = (A^{-1})^T$

iii) $(AB)^{-1} = B^{-1}A^{-1}$

iv) $adj(A^{-1}) = (adj A)^{-1}$

v) $|A^{-1}| = \frac{1}{|A|} = |A|^{-1}$

Linear Equations

⇒ Characteristic eqn – $|A-\lambda I| = 0$

Every square matrix satisfies its characteristic eqn.

→ Add '$-\lambda$' to all diag (elements) & expand |det|.

$$\boxed{\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0}$$

$S_1 = a_{11} + a_{22} + a_{33}$ tr(A)

S_2 = sum of minors of diag(a_{11}, a_{22}, a_{33}) $= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$

$S_3 = |A|$

⇒ Non homogenous in 3 unknowns

$a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$, $a_3x + b_3y + c_3z = d_3$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow AX = D$$

i) $|A| \neq 0$ then system has unique soln. (consistent)

ii) $|A| = 0$ then system has infinitely many solns (or) no soln.

Homogenous

$a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$, $a_3x + b_3y + c_3z = 0$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow AX = 0$$

i) $|A| \neq 0$ system has 1 soln. (trival) $x=y=z=0$

ii) $|A| = 0$ system has infinitely many solns (non-trival)

Conditions for consistency :

i) $\Delta \neq 0$, $\Delta_1, \Delta_2, \Delta_3$ are non zero, system has a unique soln.

ii) $\Delta = 0$, $\Delta_1, \Delta_2, \Delta_3$ if any one of them is non zero, system has no soln.

iii) $\Delta = 0$, $\Delta_1 = \Delta_2 = \Delta_3 = 0$ then system has infinitely many soln.

In two variables

$\Rightarrow$ Non - homogenous - $\quad a_1x + b_1y = c_1 \,,\; a_2x + b_2y = c_2$

$\boxed{AX = D}$

$\rightarrow$ 3 variables (x,y,z) can be converted into 2 variables (x,y) by eliminating a variable

i) $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2} \Rightarrow$ system has unique soln.

ii) $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2} \Rightarrow$ system has no soln.

iii) $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} \Rightarrow$ system has many solns.

$\Rightarrow$ Homogenous - $\quad a_1x + b_1y = 0, \; a_2x + b_2y = 0$

i) $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2} \Rightarrow$ system has unique soln. $(x = y = z = 0)$ (trivial soln)

ii) $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \Rightarrow$ system has infinitely many soln. (non trivial)

$\rightarrow$ Standard results in det

① $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc) \Rightarrow$ circulant matrix

② $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & bc \\ 1 & b & ac \\ 1 & c & ab \end{vmatrix} = (a-b)(b-c)(c-a)$

③ $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

$\rightarrow$ if $AB - BA = X$ then $tr(X^3) - 3|AB - BA| = 0$