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Assignment 2.4

Derive the *Maximum-Likelihood* estimates for the *Univariate Gaussian Distribution*! The *Probability Density Function* is defined as followed:

$$P(x|\theta) = \mathcal{N}(x|\mu, \sigma) := \frac{1}{\sqrt{2\pi\sigma^2}} exp\{-\frac{(x-\mu)^2}{2\sigma^2}\}$$

Use the *independent and identical distribution* (i.i.d.) assumption about the sample set X:

$$\mathcal{L}(X := \{x_1, ...x_N\}) = P(X|\theta) \\ = \prod_{x_i \in X} P(x_i|\theta)$$

$$L(\theta, X_1, \ldots X_N) \simeq P(X = x_1, \ldots x_N | \theta)$$

$$(i.i.d) = \prod_{i \in \{1, ..N\}} N(X_i = x_i \,|\, \mu, \sigma^2)$$

$$= \prod_{i \in \{1, ..N\}} \frac{1}{(\sigma \cdot \sqrt{2\pi})} \cdot e^{-\frac{(x_i - \mu)^2}{2 \cdot \sigma^2}}$$

$$= \left(\frac{1}{(\sigma \cdot \sqrt{2\pi})}\right)^N \cdot e^{\left(\sum_{i \in \{1, ..N\}} - \frac{(x_i - \mu)^2}{2 \cdot \sigma^2}\right)}$$

Taking the logarithm of $P(X|\theta)$

$$= \ln\left(\left(\frac{1}{(\sigma \cdot \sqrt{2\pi})}\right)^N\right) + \sum_{i \in \{1, ..N\}} - \frac{(x_i - \mu)^2}{2 \cdot \sigma^2}$$

Differentiating wrt μ,

$$\frac{\delta}{\delta\mu} \rightarrow \ln\left(\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N\right) - 0.5 \sum_{i \in \{1,..N\}} \frac{\left(x_i - \mu\right)^2}{\sigma^2} = 0$$

$$0 - 0.5 \sum_{i \in \{1,..N\}} \frac{1}{\sigma^2} \cdot 2 \cdot \left(x_i - \mu^*\right) \cdot (-1) = 0$$

$$\sum_{i \in \{1,..N\}} \frac{1}{\sigma^2} \cdot \left(x_i - \mu^*\right) = 0$$

$$\sum_{i \in \{1,..N\}} \frac{x_i}{\sigma^2} - \sum_{j \in \{1,..N\}} \frac{\mu^*}{\sigma^2} = 0$$

$$\sum_{i = \{1,..N\}} \frac{x_i}{\sigma^2} = \sum_{j = \{1,..N\}} \frac{\mu^*}{\sigma^2}$$

$$\sum_{i \in \{1,..N\}} x_i = N \cdot \mu^*$$

$$\boxed{\mathbf{\frac{1}{N} \sum_{i \in \{1,..N\}} x_i = \mu^* = \mu^{MLE}}}$$

Differentiating wrt σ,

$$\frac{\delta}{\delta\sigma} \rightarrow ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N - \sum_{i \in \{1,..N\}} \frac{\left(x_i - \mu\right)^2}{2 \cdot \sigma^2} = 0$$

$$\frac{\delta}{\delta\sigma} \rightarrow ln(1)^N - ln\left(\sqrt{2\pi\sigma^2}\right)^N - \sum_{i \in \{1,..N\}} \frac{\left(x_i - \mu\right)^2}{2 \cdot \sigma^2} = 0$$

$$\frac{\delta}{\delta\sigma} \rightarrow 0 - ln\left(\sqrt{2\pi\sigma^2}\right)^N - \frac{\left(x_i - \mu\right)^2}{2 \cdot \sigma^2} = 0$$

$$\frac{\delta}{\delta\sigma} \rightarrow -ln\left(2\pi\sigma^2\right)^{\frac{N}{2}} - \sum_{i \in \{1,..N\}} \frac{\left(x_i - \mu\right)^2}{2 \cdot \sigma^2} = 0$$

$$\frac{\delta}{\delta\sigma} \rightarrow -\frac{N}{2} ln\left(2\pi\sigma^2\right) - \sum_{i \in \{1,..N\}} \frac{\left(x_i - \mu\right)^2}{2 \cdot \sigma^2} = 0$$

$$-\frac{N}{2} \frac{\delta}{\delta\sigma}\left(ln\left(2\pi\sigma^2\right)\right) - \sum_{i \in \{1,..N\}} \frac{\left(x_i - \mu\right)^2}{2} \frac{\delta}{\delta\sigma}\left(\sigma^{-2}\right) = 0$$

$$-\frac{N}{2} \cdot \frac{1}{2\pi\sigma^2} \cdot 2\pi \cdot \frac{\delta}{\delta\sigma}\left(\sigma^2\right) - \sum_{i \in \{1,..N\}} \frac{\left(x_i - \mu\right)^2}{2} \left(-2 \cdot \sigma^{-3}\right) = 0$$

$$-\frac{N}{2} \cdot \frac{1}{2\pi\sigma^2} \cdot (4\pi\sigma) - \sum_{i = \{1,..N\}} \frac{\left(x_i - \mu\right)^2}{2} \left(\frac{-2}{\sigma^3}\right) = 0$$

$$-\frac{N}{\sigma} + \frac{1}{\sigma^3} \cdot \sum_{i = \{1,..N\}} \left(x_i - \mu\right)^2 = 0$$

$$\frac{1}{\sigma^3} \sum_{i \in \{1,..N\}} \left(x_i - \mu\right)^2 = \frac{N}{\sigma}$$

$$\sum_{i \in \{1,..N\}} \left(x_i - \mu\right)^2 = N \cdot \sigma^2$$

$$\boxed{\sqrt{\mathbf{\frac{1}{N} \sum_{i \in \{1,..N\}} (x_i - \mu)^2}} = \boldsymbol{\sigma}^{\mathbf{MLE}}}$$