Couette Flow with slip boundary condition

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1 Couette Flow with Slip Boundary Condition:

The velocity profile for an incompressible Couette flows with slip can be obtained by considering a two dimensional channel extending between $\mathbf{x = 0}$ to $\mathbf{x = L}$ and $\mathbf{y = 0}$ to $\mathbf{y = H}$, with top surface moving with a prescribed velocity $\mathbf{U_\infty}$.

Assumptions

- Steady, incompressible flow, and no mass transport.
- No end effects, $\frac{\partial}{\partial \mathbf{x}}() = \mathbf{0}$.
- No edge effects, $\frac{\partial}{\partial \mathbf{z}}() = \mathbf{0}$.
- Only gravity force.

Continuity equation:

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \frac{\partial \mathbf{v}}{\partial \mathbf{y}} + \frac{\partial \mathbf{w}}{\partial \mathbf{z}} = \mathbf{0} \tag{1}$$

By neglecting edge effects and end effects, the above continuity equation(1) can be reduced to

$$\frac{\partial \mathbf{v}}{\partial \mathbf{y}} = \mathbf{0} \tag{2}$$

X-momentum equation: At $\mathbf{y = 0, v = 0, w = 0}$ everywhere

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} + \mathbf{u}\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \mathbf{v}\frac{\partial \mathbf{u}}{\partial \mathbf{y}} + \mathbf{w}\frac{\partial \mathbf{u}}{\partial \mathbf{z}} = -\frac{\mathbf{1}}{\rho}\frac{\partial \mathbf{P}}{\partial \mathbf{x}} + \boldsymbol{\nu}\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x^2}} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y^2}} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{z^2}}\right] + \mathbf{F_x} \tag{3}$$

After considering steady state, neglecting edge effects and end effects, and other forces the above x-momentum equation(3) can be reduced to

$$\mathbf{0} = \boldsymbol{\nu}\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{y^2}}\right] \tag{4}$$

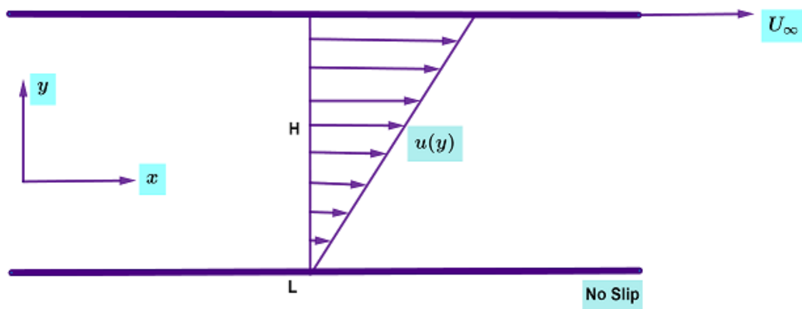

Figure 1: Analysis of Couette Flow with slip boundary condition

or

$$\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2}\right] = \mathbf{0} \tag{5}$$

Now non- dimensionalizing the above equation

$$\mathbf{u}^* = \frac{\mathbf{u}}{\mathbf{u}_\infty}, \mathbf{y}^* = \frac{\mathbf{y}}{\mathbf{h}} \tag{6}$$

$$\mathbf{u} = \mathbf{u}_\infty \mathbf{u}^*, \mathbf{y} = \mathbf{h}\mathbf{y}^* \tag{7}$$

$$\frac{\mathbf{d}^2(\mathbf{u}_\infty \mathbf{u}^*)}{\mathbf{d}(\mathbf{h}\mathbf{y}^*)^2} = \mathbf{0} \tag{8}$$

$$\frac{\mathbf{u}_\infty}{\mathbf{h}^2}\frac{\mathbf{d}^2\mathbf{u}^*}{\mathbf{d}\mathbf{y}^{*2}} = \mathbf{0} \tag{9}$$

$$\frac{\mathbf{d}^2\mathbf{u}^*}{\mathbf{d}\mathbf{y}^{*2}} = \mathbf{0} \tag{10}$$

Now integrating the above equation (Eq:10), then we get

$$\mathbf{u}^* = \mathbf{C_1}\mathbf{y}^* + \mathbf{C_2} \tag{11}$$

Slip boundary condition

$$\mathbf{u_s} - \mathbf{u_w} = \left(\frac{\mathbf{2} - \boldsymbol{\sigma}_\mathbf{v}}{\boldsymbol{\sigma}_\mathbf{v}}\right)\frac{\mathbf{1}}{\boldsymbol{\rho}}\sqrt{\frac{\pi}{\mathbf{2RT}_\infty}}\boldsymbol{\tau}_\mathbf{s} + \frac{\mathbf{3}}{\mathbf{4}}\frac{\mathbf{Pr}}{\boldsymbol{\rho}\mathbf{RT_w}}\left(\frac{\gamma - \mathbf{1}}{\gamma}\right)(-\mathbf{q_s}) \tag{12}$$

$$\mathbf{u_s} - \mathbf{u_w} = \left(\frac{\mathbf{2} - \boldsymbol{\sigma}_\mathbf{v}}{\boldsymbol{\sigma}_\mathbf{v}}\right)\frac{\boldsymbol{\mu}}{\boldsymbol{\rho}}\sqrt{\frac{\pi}{\mathbf{2RT}_\infty}}\left(\frac{\partial \mathbf{u_s}}{\partial \mathbf{n}}\right)_{\mathbf{wall}} + \frac{\mathbf{3}}{\mathbf{4}}\frac{\mathbf{Pr}}{\boldsymbol{\rho}\mathbf{RT_w}}\left(\frac{\gamma - \mathbf{1}}{\gamma}\right)\left(\mathbf{k}\frac{\partial \mathbf{T}}{\partial \mathbf{x}}\right)_{\mathbf{wall}} \tag{13}$$

Where, $\boldsymbol{\tau_s} = \boldsymbol{\mu}\left(\frac{\partial \mathbf{u_s}}{\partial \mathbf{y}}\right)_{\mathbf{wall}}$ and $-\mathbf{q_s} = \left(\mathbf{k}\frac{\partial \mathbf{T}}{\partial \mathbf{x}}\right)_{\mathbf{wall}}$
After non-dimensionalizing the above slip B.C:

$$\mathbf{u_s^* - u_w^*} = \left(\frac{\mathbf{2} - \boldsymbol{\sigma_v}}{\boldsymbol{\sigma_v}}\right)\mathbf{Kn}\left(\frac{\partial \mathbf{u_s^*}}{\partial \mathbf{n^*}}\right)_{\mathbf{wall}} + \frac{\mathbf{3}}{\mathbf{2}\boldsymbol{\pi}}\frac{\mathbf{Kn^2Re}}{\mathbf{E_c}}\left(\frac{\boldsymbol{\gamma} - \mathbf{1}}{\boldsymbol{\gamma}}\right)\left(\frac{\partial \mathbf{T^*}}{\partial \mathbf{x^*}}\right)_{\mathbf{wall}} \tag{14}$$

or

$$\mathbf{u_s^* - u_w^*} = \boldsymbol{\alpha}\left(\frac{\partial \mathbf{u_s^*}}{\partial \mathbf{n^*}}\right)_{\mathbf{wall}} + \boldsymbol{\beta} \tag{15}$$

Where,

$$\boldsymbol{\alpha} = \left(\frac{\mathbf{2} - \boldsymbol{\sigma_v}}{\boldsymbol{\sigma_v}}\right)\mathbf{Kn} \tag{16}$$

$$\boldsymbol{\beta} = \frac{\mathbf{3}}{\mathbf{2}\boldsymbol{\pi}}\frac{\mathbf{Kn^2Re}}{\mathbf{E_c}}\left(\frac{\boldsymbol{\gamma} - \mathbf{1}}{\boldsymbol{\gamma}}\right) \tag{17}$$

Applying boundary conditions:
a) At lower wall :
$\mathbf{y = 0, y^* = \frac{y}{h} \Rightarrow = \frac{0}{h}, \Rightarrow y^* = 0}$
$\mathbf{u^* = u_s^*, u_w = 0}$(**stationary wall**)

$$\mathbf{u_s^* = u_w^*} + \boldsymbol{\alpha}\frac{\boldsymbol{\partial}\mathbf{u_s^*}}{\boldsymbol{\partial}\mathbf{n^*}} + \boldsymbol{\beta} \tag{18}$$

$\mathbf{u_w^* = \frac{u_w}{u_\infty} \Rightarrow \frac{0}{u_\infty} = 0}$
Also,

$$\boldsymbol{\alpha}\frac{\boldsymbol{\partial}\mathbf{u_s^*}}{\boldsymbol{\partial}\mathbf{n^*}} = \boldsymbol{\alpha}\frac{\boldsymbol{\partial}\mathbf{u_s^*}}{\boldsymbol{\partial}\mathbf{y^*}} \tag{19}$$

From equation (11),

$$\mathbf{u^* = C_1y^* + C_2} \tag{20}$$

$$\frac{\boldsymbol{\partial}\mathbf{u^*}}{\boldsymbol{\partial}\mathbf{y^*}} = \mathbf{C_1} \tag{21}$$

Now, from equation(18)

$$\mathbf{u_s^* = 0} + \boldsymbol{\alpha}\mathbf{C_1} + \boldsymbol{\beta} \tag{22}$$

$$\mathbf{u_s^*} = \boldsymbol{\alpha}\mathbf{C_1} + \boldsymbol{\beta} \tag{23}$$

From equation(11),

$$\mathbf{u_s^* = C_1y^* + C_2} \tag{24}$$

$$\boldsymbol{\alpha}\mathbf{C_1} + \boldsymbol{\beta} = \mathbf{C_1(0) + C_2} \tag{25}$$

$$\boldsymbol{\alpha}\mathbf{C_1} + \boldsymbol{\beta} = \mathbf{C_2} \tag{26}$$

b) At upper wall:
$\mathbf{u_w = u_\infty, u^* = u_s^*, u_w^* = \frac{u_w}{u_\infty} = \frac{u_\infty}{u_\infty} = 1, \Rightarrow u_w^* = 1}$

$\mathbf{y = h, y^* = \frac{y}{h} = \frac{h}{h} = 1 \Rightarrow y^* = 1}$
Now,

$$\mathbf{u_s^* = u_w^* + \boldsymbol{\alpha}\frac{\partial u_s^*}{\partial n^*} + \boldsymbol{\beta}} \tag{27}$$

Here,

$$\boldsymbol{\alpha}\mathbf{\frac{\partial u_s^*}{\partial n^*}} = -\boldsymbol{\alpha}\mathbf{\frac{\partial u_s^*}{\partial y^*}} = -\mathbf{\frac{\partial(C_1 y^* + C_2)}{\partial y^*}} \tag{28}$$

$$\boldsymbol{\alpha}\mathbf{\frac{\partial u_s^*}{\partial n^*}} = -\boldsymbol{\alpha}\mathbf{\frac{\partial u_s^*}{\partial y^*}} = -\mathbf{C_1} \tag{29}$$

$$\mathbf{u_s^* = 1 + \boldsymbol{\alpha}(-\boldsymbol{C_1}) + \boldsymbol{\beta}} \tag{30}$$

$$\mathbf{u_s^* = 1 - \boldsymbol{\alpha C_1} + \boldsymbol{\beta}} \tag{31}$$

Again,

$$\mathbf{u_s^* = C_1 y^* + C_2} \tag{32}$$

$$\mathbf{1 - \boldsymbol{\alpha C_1} + \boldsymbol{\beta} = C_1(1) + C_2} \tag{33}$$

$$\mathbf{1 - \boldsymbol{\alpha C_1} + \boldsymbol{\beta} = C_1 + C_2} \tag{34}$$

From equation (26), put the value of $\mathbf{C_2}$ in equation 33

$$\mathbf{1 - \boldsymbol{\alpha C_1} + \boldsymbol{\beta} = C_1 + \boldsymbol{\alpha C_1} + \boldsymbol{\beta}} \tag{35}$$

$$\mathbf{1 = C_1 + \boldsymbol{\alpha C_1} + \boldsymbol{\alpha C_1} = C_1 + 2\boldsymbol{\alpha C_1}} \tag{36}$$

$$\mathbf{1 = (1 + 2\boldsymbol{\alpha} C_1} \tag{37}$$

$$\mathbf{C_1 = \frac{1}{(1 + 2\boldsymbol{\alpha})}} \tag{38}$$

From equation 26,

$$\mathbf{\boldsymbol{\alpha} C_1 + \boldsymbol{\beta} = C_2} \tag{39}$$

$$\mathbf{\frac{\boldsymbol{\alpha}}{1 + 2\boldsymbol{\alpha}} + \boldsymbol{\beta} = C_2} \tag{40}$$

$$\mathbf{C_2 = \left(\frac{\boldsymbol{\alpha}}{1 + 2\boldsymbol{\alpha}}\right) + \boldsymbol{\beta}} \tag{41}$$

$$\implies \mathbf{u^* = C_1 y^* + C_2} \tag{42}$$

$$\implies \mathbf{u^* = \left(\frac{1}{1 + 2\boldsymbol{\alpha}}\right) y^* + \left(\frac{\boldsymbol{\alpha}}{1 + 2\boldsymbol{\alpha}}\right) + \boldsymbol{\beta}} \tag{43}$$

$$\implies \mathbf{u^* = \left(\frac{y^* + \boldsymbol{\alpha}}{1 + 2\boldsymbol{\alpha}}\right) + \boldsymbol{\beta}} \tag{44}$$

$$\implies \mathbf{\frac{u}{u_\infty} = \left(\frac{\frac{y}{h} + \boldsymbol{\alpha}}{1 + 2\boldsymbol{\alpha}}\right) + \boldsymbol{\beta}} \tag{45}$$

Now, replacing the values of $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$, then the above equation (45) can be reduced to,

$$\Longrightarrow \boxed{\frac{u}{u_\infty} = \left[\frac{\frac{y}{h} + \left(\frac{2-\boldsymbol{\sigma}_v}{\boldsymbol{\sigma}_v}\right) Kn}{1 + 2\left(\frac{2-\boldsymbol{\sigma}_v}{\boldsymbol{\sigma}_v}\right) Kn} + \frac{3}{2\boldsymbol{\pi}} \frac{Kn^2 Re}{E_c} \left(\frac{\boldsymbol{\gamma}-1}{\boldsymbol{\gamma}}\right) \left(\frac{\boldsymbol{\partial} T_s}{\boldsymbol{\partial} x}\right)\right]} \tag{46}$$

This is the expression of slip velocity for couette flow.