2nd Unit Problems

1) For the servomechanism with open loop transfer function given below, what type of input signal given rise to a constant steady state error and calculate their values

$$G(S) = \frac{10}{S^2(S+2)(S+3)}.$$

Given : $G(S) = \dfrac{10}{S^2(S+2)(S+3)}$

Let us assume unity feed back system

$$H(S) = 1$$

The open loop system has two poles at origin. Hence it is a type-2 system.

$S^2 = 0$, $S+2=0$, $S+3=0$

$S=0, S=0,$ $S=-2$, $S=-3$

In systems with type number-2 the acceleration (parabolic) input will give a constant steady state error.

The steady state error with unit acceleration input,

$$\boxed{e_{ss} = \frac{1}{K_a}}$$

Acceleration error constant,

$$K_a = \underset{S \to 0}{Lt}\ S^2 G(S) H(S)$$

$$= \underset{S \to 0}{Lt}\ \cancel{S^2} \frac{10}{\cancel{S^2}(S+2)(S+3)} \quad (1)$$

$$= \frac{10}{(0+2)(0+3)}$$

$$= \frac{10}{(2)(3)} = \frac{10}{6} = 1.67$$

$$\boxed{K_a = 1.67}$$

Steady state error,

$$e_{ss} = \frac{1}{K_a} = \frac{1}{1.67} = 0.6$$

In the given system, with unit acceleration input,

Steady state error = 0.6

Q) A unity feedback control system has an open loop transfer function, $G(s) = \frac{10}{s(s+2)}$. Find the time domain specifications for a step i/p of 12 units.

The unity feedback system is shown in fig.

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

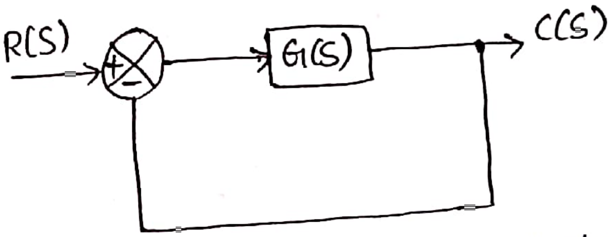

Fig: Unity Feedback System

The closed loop transfer function,

G.T, $G(s) = \frac{10}{s(s+2)}$

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{10}{s(s+2)}}{1+\frac{10}{s(s+2)}} = \frac{\frac{10}{s(\cancel{s+2})}}{\frac{s(s+2)+10}{s(\cancel{s+2})}}$$

$$\frac{C(s)}{R(s)} = \frac{10}{s^2+2s+10} \longrightarrow ①$$

The values of damping ratio and natural frequency of oscillation ω_n are obtained by comparing the system transfer function with standard form of second order transfer function,

Standard form of second order transfer function is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \longrightarrow ②$$

On comparing eq ① & ② we get,

$\omega_n^2 = 10$

$\omega_n = \sqrt{10}$

$\omega_n = 3.162$ rad/sec

$2\zeta\omega_n = 2$

$\zeta = \frac{2}{2\omega_n}$

$= \frac{1}{3.162}$

$\boxed{\zeta = 0.316}$

$\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \tan^{-1}\frac{\sqrt{1-0.316^2}}{0.316} = 1.249$ rad

$\omega_d = \omega_n\sqrt{1-\zeta^2} = 3.162\sqrt{1-0.316^2} = 3$ rad/sec

(i) Rise time, $t_r = \frac{\pi-\theta}{\omega_d}$

$t_r = \frac{\pi - 1.249}{3} = 0.63$ sec

$\boxed{t_r = 0.63 \text{ sec}}$

(ii) Percentage overshoot, %Mp

$\% M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$

$= e^{\frac{-0.316\pi}{\sqrt{1-0.316^2}}} \times 100$

$= 0.3512 \times 100 = 35.12\%$

(iii) Peak overshoot $= \frac{35.12}{100} \times 12$ units

$= 4.2144$ units

(iv) Peak time,

$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3} = 1.047$ sec

$\boxed{t_p = 1.047 \text{sec}}$

(V) Time constant

$$T = \frac{1}{\xi \omega_n}$$

$$= \frac{1}{0.316 \times 3.162} = 1 sec$$

$$\boxed{T = 1 sec}$$

$\therefore$ For 5% error, settling time, $t_s = 3T = 3sec$

$\therefore$ For 2% error, setting time, $t_s = 4T = 4sec.$

vi) Settling time t_s = 3 sec for 5% error

= 4 sec for 2% error.

Q A unity feedback system has the forward transfer function $G(s) = \frac{k_1(2s+1)}{s(5s+1)(1+s)^2}$. The input $r(t) = 1+6t$ is applied to the system. Determine the value of k_1 if the steady error is to be less than 0.1.

Given that, input $r(t) = 1+6t$

on taking laplace transform of $r(t)$ we get $R(s)$.

$$\therefore R(s) = L\{r(t)\}$$

$$= L\{1+6t\}$$

$$R(s) = \frac{1}{s} + \frac{6}{s^2}$$

Ther error signal in s-domain $E(s)$ is given by

$$\therefore E(s) = \frac{R(s)}{1+G(s)H(s)} = \frac{\frac{1}{s}+\frac{6}{s^2}}{1+\frac{k_1(2s+1)}{s(5s+1)(1+s)^2}} \quad [\because H(s)=1]$$

$$= \frac{\frac{1}{s}+\frac{6}{s^2}}{\frac{s(5s+1)(1+s)^2+k_1(2s+1)}{s(5s+1)(1+s)^2}}$$

$$E(S) = \frac{1/s}{\frac{s(5s+1)(1+s)^2 + K_1(2s+1)}{s(5s+1)(1+s)^2}} + \frac{\frac{6}{s^2}}{\frac{s(5s+1)(1+s)^2 + K_1(2s+1)}{s(5s+1)(1+s)^2}}$$

$$= \frac{1}{s}\left[\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2 + K_1(2s+1)}\right] + \frac{6}{s^2}\left[\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2 + K_1(2s+1)}\right]$$

The steady state error ess can be obtained from final value theorem.

$$e_{ss} = \underset{t\to\infty}{Lt}\; e(t) = \underset{s\to 0}{Lt}\; s\,E(s)$$

$$= \underset{s\to 0}{Lt}\; s\left\{\frac{1}{s}\left[\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2 + K_1(2s+1)}\right] + \frac{6}{s^2}\left[\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2 + K_1(2s+1)}\right]\right\}$$

$$e_{ss} = \underset{s\to 0}{Lt}\left\{\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2 + K_1(2s+1)} + \frac{6(5s+1)(1+s)^2}{s^2(5s+1)(1+s)^2 + K_1(2s+1)}\right\}$$

$$e_{ss} = 0 + \frac{6}{K_1} = \frac{6}{K_1}$$

Given that, $e_{ss} < 0.1$

$$\therefore\; 0.1 = \frac{6}{K_1} \quad (\text{or}) \quad K_1 = \frac{6}{0.1} = 60$$

For steady state error, $e_{ss} < 0.1$, the value of K_1 should be greater than 60.

Q A unity feed back control system is characterized by the following open loop transfer function

$G(S) = \dfrac{0.4S+1}{S(S+0.6)}$. Determine its transient response for unit step input. Evaluate the maximum overshoot and corresponding peak time.

The closed loop transfer function,

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)\,H(S)}$$

Given that, $G(S) = \frac{(0.4S+1)}{S(S+0.6)}$

For unity feedback system, $H(S) = 1$

$$\therefore \frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)} = \frac{\frac{0.4S+1}{S(S+0.6)}}{1+\frac{0.4S+1}{S(S+0.6)}} = \frac{0.4S+1}{S(S+0.6)+0.4S+1}$$

$$= \frac{0.4S+1}{S^2+0.6S+0.4S+1}$$

$$\frac{C(S)}{R(S)} = \frac{0.4S+1}{S^2+S+1}$$

The s-domain response, $C(s) = R(S) \times \frac{0.4S+1}{S^2+S+1}$

For step, input, $R(S) = 1/S$.

$$\therefore C(s) = \frac{1}{S}\,\frac{0.4S+1}{S^2+S+1} = \frac{0.4S+1}{S(S^2+S+1)}$$

By partial fraction expansion $C(s)$ can be expressed as

$$C(S) = \frac{0.4S+1}{S(S^2+S+1)} = \frac{A}{S} + \frac{BS+C}{S^2+S+1}$$

The residue A is solved by multiplying $C(s)$ by S and letting $S=0$

$$\therefore A = C(S) \times S|_{S=0}$$

$$A = \frac{0.4S+1}{\cancel{S}(S^2+S+1)}(\cancel{S})\Big|_{S=0}$$

$$A = \frac{0+1}{0+0+1}$$

$$\boxed{A=1}$$

B and C are solved by cross Multiplication and equating coefficients

$$\frac{0.4S+1}{S(S^2+S+1)} = \frac{A}{S} + \frac{BS+C}{S^2+S+1}$$

$$0.4S+1 = A(S^2+S+1) + (BS+C)(S)$$

$$0.4S+1 = A(S^2+S+1) + BS^2+CS$$

Equating S^2 coefficients

$$A+B=0$$

$$1+B=0$$

$$\boxed{B=-1}$$

S- coefficients

$$A+C=0.4$$

$$C=0.4-A$$

$$C=0.4-1$$

$$\boxed{C=-0.6}$$

$$\therefore \quad C(S) = \frac{1}{S} + \frac{-S-0.6}{S^2+S+1} = \frac{1}{S} - \frac{S+0.6}{S^2+S+1}$$

$$C(S) = \frac{1}{S} - \frac{S+0.5+0.1}{S^2+2\times0.25S+0.5^2+1-0.25}$$

$$C(S) = \frac{1}{S} - \frac{S+0.5+0.1}{(S+0.5)^2+0.75} = \frac{1}{S} - \frac{S+0.5}{(S+0.5)^2+0.75} - \frac{0.1}{\sqrt{0.75}} \frac{\sqrt{0.75}}{(S+0.5)^2+0.75}$$

The time domain response is obtained by taking inverse Laplace transform.

$$C(t) = L^{-1}\{C(S)\} = L^{-1}\left\{\frac{1}{S} - \frac{S+0.5}{(S+0.5)^2+0.75} - \frac{0.1}{\sqrt{0.75}} \frac{\sqrt{0.75}}{(S+0.5)^2+0.75}\right\}$$

$$C(t) = 1-e^{-0.5t}\cos\sqrt{0.75}\,t - \frac{0.1}{\sqrt{0.75}} e^{-0.5t} \sin\sqrt{0.75}\,t$$

$$C(t) = 1-e^{-0.5t}\left[0.1155\sin(\sqrt{0.75}\,t) + \cos(\sqrt{0.75}\,t)\right]$$

The transient response is the part of the output which vanishes as t tends to infinity. Here as t tends to infinity the exponential component $e^{-0.5t}$ tends to zero. Hence the transient response is given by damped sinusoidal component.

The transient response of $c(t) = e^{-0.5t}[0.1155 \sin\sqrt{0.75}t + \cos\sqrt{0.75}t]$

Maximum overshoot M_p:

Standard form of second order characteristic equation

$$s^2 + 2\xi\omega_n s + \omega_n^2 = s^2 + s + 1$$

on comparing we get

$\omega_n^2 = 1$ | $2\xi\omega_n = 1$

$\omega_n = 1$ rad/sec | $2\xi(1) = 1$

$\xi = 1/2 = 0.5$

$$M_p = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} = e^{\frac{-0.5\pi}{\sqrt{1-0.5^2}}} = 0.163$$

% Maximum overshoot $\%M_p = M_p \times 100$

$= 0.163 \times 100$

$\%M_p = 16.3\%$

Peak time, $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{1 \times \sqrt{1-0.5^2}}$

$$\boxed{t_p = 3.628 \sec}$$

The response of system is underdamped.

Q: For a unity feedback control system the open loop transfer function $G(s) = \frac{10(s+2)}{s^2(s+1)}$.

Find : a) Position, velocity and acceleration error constants

b) Steady state error when input $R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$

Given, $G(s) = \frac{10(s+2)}{s^2(s+1)}$, $R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$

For, unity feedback system, $H(s) = 1$

Position error constant,

$$K_p = \underset{s\to 0}{Lt}\ G(s)\ (H(s)) \qquad [H(s)=1]$$

$$K_p = \underset{s\to 0}{Lt}\ \frac{10(s+2)}{s^2(s+1)} = \infty$$

$\boxed{K_p = \infty}$

Velocity Error Constant,

$$K_v = \underset{s\to 0}{Lt}\ s.G(s)\ H(s)$$

$$= \underset{s\to 0}{Lt}\ s.\ \frac{10(s+2)}{s^2(s+1)}\ (1)$$

$\boxed{K_v = \infty}$

Acceleration Error constant:

$$K_a = \underset{s\to 0}{Lt}\ s^2\ G(s)\ H(s)$$

$$= \underset{s\to 0}{Lt}\ s^2\ \frac{10(s+2)}{s^2(s+1)}\ (1)$$

$$= \frac{10(0+2)}{0(1)}$$

$\boxed{K_a = 20}$

b) To find steady state error:-

The error signal in s-domain, $E(s) = \dfrac{R(s)}{1+G(s)\ H(s)}$

GT, $R(s) = \dfrac{3}{s} - \dfrac{2}{s^2} + \dfrac{1}{3s^3}$; $G(s) = \dfrac{10(s+2)}{s^2(s+2)}$; $H(s) = 1$

$$\therefore\ E(s) = \frac{\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}}{1 + \frac{10(s+2)}{s^2(s+1)}} = \frac{\frac{3}{s} + \frac{2}{s^2} + \frac{1}{3s^3}}{\frac{s^2(s+1)+10(s+2)}{s^2(s+1)}}$$

$$\therefore\ E(s) = \frac{3}{s}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] - \frac{2}{s^2}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right]$$

$$+ \frac{1}{3s^3}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right]$$

The steady state error e_{ss} can be obtained from final value theorem.

Steady state error,

$$e_{ss} = \underset{t \to \infty}{Lt}\, e(t) = \underset{s \to 0}{Lt}\, s\,E(s)$$

$$\therefore e_{ss} = \underset{s \to 0}{Lt}\, s\left\{\frac{3}{s}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] - \frac{2}{s^2}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] + \frac{1}{3s^3}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right]\right\}$$

$$e_{ss} = \underset{s \to 0}{Lt}\left\{\frac{3s^2(s+1)}{s^2(s+1)+10(s+2)} - \frac{2s(s+1)}{s^2(s+1)+10(s+2)} + \frac{(s+1)}{3s^2(s+1)+30(s+2)}\right\}$$

$$= 0 - 0 + \frac{1}{60}$$

Steady state error, $\boxed{e_{ss} = \frac{1}{60}}$

Q: For the servo mechanisms with open loop transfer function given below explain what type of input signal give rise to a constant steady state error and calculate their values.

i) $G(s) = \dfrac{20(s+2)}{s(s+1)(s+3)}$ (ii) $G(s) = \dfrac{10}{(s+2)(s+3)}$

(iii) $G(s) = \dfrac{10(s+2)}{s^2(s+1)(s+2)}$

(i) $G(s) = \dfrac{20(s+2)}{s(s+1)(s+3)}$

Let us assume unity feedback system $\therefore H(s) = 1$

The ~~loop~~ open loop system has a pole at origin. Hence it is a type-1 system. In systems with type number-1, the velocity (ramp) input will give a constant steady state error.

The steady state error with unit velocity input,

$$e_{ss} = \frac{1}{K_v}$$

velocity error constant, $K_v = \underset{s \to 0}{Lt}\ s\,G(s)\,H(s) = \underset{s \to 0}{Lt}\ s\,G(s)$

$$K_v = \underset{s \to 0}{Lt}\ s\,\frac{20(s+2)}{s(s+1)(s+3)}$$

$$= \frac{20(0+2)}{(0+1)(0+3)} = \frac{40}{3}$$

$$K_v = \frac{40}{3}$$

steady state error, $e_{ss} = \frac{1}{K_v} = \frac{3}{40} = 0.075$.

(ii) $G(s) = \frac{10}{(s+2)(s+3)}$

$H(s) = 1$

The open loop system has no pole at origin. Hence it is a type-0 system. In systems with type number-0, the step input will give a constant steady state error.

The steady state error with unit step input

$$e_{ss} = \frac{1}{1+K_p}$$

position error constant, $K_p = \underset{s \to 0}{Lt}\ G(s)\,H(s)$

$$= \underset{s \to 0}{Lt}\ G(s)$$

$$K_p = \underset{s \to 0}{Lt}\ \frac{10}{(s+2)(s+3)} = \frac{10}{2 \times 3} = \frac{5}{3}$$

steady state error, $e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\frac{5}{3}} = \frac{3}{3+5} = \frac{3}{8}$

$$\boxed{e_{ss} = 0.375}$$

(iii) $G(s) = \frac{10(s+2)}{s^2(s+1)(s+2)}$

$H(s) = 1$

The open loop system has two poles at origin. Hence it is a type-2 system. In systems with type no. 2, the acceleration (parabolic) input will give a constant steady state error.

The steady state error with unit acceleration input,

$$e_{ss} = \frac{1}{K_a}$$

$$K_a = \lim_{s \to 0} s^2 G(s)H(s) = \lim_{s \to 0} \cancel{s^2} \frac{10(s+2)}{\cancel{s^2}(s+1)(s+2)}$$

$$= \frac{10(2)}{(1)(2)} = 10$$

Steady state error, $e_{ss} = \frac{1}{K_a} = \frac{1}{10} = 0.1$