Assignment No.5

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Download all python codes from

```
svn co https://github.com/Ganeshyadav712/
    Assignment-5.git
```

Question taken from

```
https://github.com/gadepall/ncert/blob/main/linalg/
    quadratic_forms/gvv_ncert_quadratic_forms.
    pdf-2.26
```

1 Formula

Lemma 1.1. *For ellipse*
Property:

$$|\mathbf{V}| > 0 \tag{1.0.1}$$

$$\lambda_1 > 0, \lambda_2 < 0 \tag{1.0.2}$$

Standard Form:

$$\frac{\mathbf{x}^T\mathbf{D}\mathbf{x}}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f} = 1 \tag{1.0.3}$$

Centre:

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{1.0.4}$$

Axes:

$$\begin{cases} \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}} \\ \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_2}} \end{cases} \tag{1.0.5}$$

Focus:

$$\mathbf{F} = \sqrt{\frac{(\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f)(\lambda_2 - \lambda_1)}{\lambda_1\lambda_2}} \tag{1.0.6}$$

Focal Length:

$$\beta = \frac{1}{2}\left|\frac{(\mathbf{c}^T\mathbf{V} + \mathbf{u}^T)\mathbf{p}_1}{(\lambda_1 + \lambda_2)}\right| \tag{1.0.7}$$

Latus Rectum:

$$(\mathbf{V}\mathbf{c} + \mathbf{u})^T(\mathbf{x} - \beta) + \mathbf{u}^T\mathbf{c} + \mathbf{f} = 0 \tag{1.0.8}$$

End points of latus rectum:

$$\mathbf{u}^T\kappa = -\frac{(\kappa^T\mathbf{V}\kappa + f)}{2} \tag{1.0.9}$$

Length of latus rectum:

$$l = \left\|\beta(\mathbf{V}\mathbf{c} + \mathbf{u})^T\right\| \tag{1.0.10}$$

2 Question

Find the coordinates of the foci, the vertices, the lengths of major and minor axes,the eccentricity and the latus rectum of the ellipse.
$X^T\begin{pmatrix}\frac{1}{25} & 0\\ 0 & \frac{1}{9}\end{pmatrix} = 1$

3 solution

Given ellipse is

$$\mathbf{x}^T\begin{pmatrix}\frac{1}{25} & 0\\ 0 & \frac{1}{9}\end{pmatrix}\mathbf{x} = 36 \tag{3.0.1}$$

On comparing it with standard form we have,

$$\mathbf{V} = \begin{pmatrix}\frac{1}{25} & 0\\ 0 & \frac{1}{9}\end{pmatrix}, \mathbf{u} = 0, f = -1 \tag{3.0.2}$$

$$\implies \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 1 \tag{3.0.3}$$

$$\implies \mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = \begin{pmatrix}0\\0\end{pmatrix} \tag{3.0.4}$$

$$\tag{3.0.5}$$

The eigen vector decomposition of

$$\mathbf{V} = \begin{pmatrix}\frac{1}{25} & 0\\ 0 & \frac{1}{9}\end{pmatrix} \tag{3.0.6}$$

is given by

$$\mathbf{D} = \begin{pmatrix}\frac{1}{25} & 0\\ 0 & \frac{1}{9}\end{pmatrix} \implies \lambda_1 = \frac{1}{25}, \lambda_2 = \frac{1}{9} \tag{3.0.7}$$

$$\mathbf{P} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \implies \mathbf{p}_1 = \begin{pmatrix}1\\0\end{pmatrix}, \mathbf{p}_2 = \begin{pmatrix}0\\1\end{pmatrix} \tag{3.0.8}$$

Since

$$\lambda_2 > \lambda_1 \tag{3.0.9}$$

Eccentricity of the ellipse is,

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} = \frac{4}{5} \tag{3.0.10}$$

Semi major and minor axes of ellipse are,

$$a = \sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} = 5 \tag{3.0.11}$$

$$b = \sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = 3 \tag{3.0.12}$$

The co-ordinates of vertices are,

$$\pm \begin{pmatrix} 0 \\ 5 \end{pmatrix} \tag{3.0.13}$$

The co-ordinates of foci are given by,

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_1} \tag{3.0.14}$$

Where,

$$\mathbf{n} = \sqrt{\lambda_1}\mathbf{p}_2 \tag{3.0.15}$$

$$c = \frac{e\mathbf{u}^\top\mathbf{n} \pm \sqrt{e^2\left(\mathbf{u}^\top\mathbf{n}\right)^2 - \lambda_2\left(e^2-1\right)\left(\|\mathbf{u}\|^2 - \lambda_2 f\right)}}{\lambda_2 e\left(e^2-1\right)} \tag{3.0.16}$$

Substituting we have,

$$\mathbf{n} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} \tag{3.0.17}$$

$$\mathbf{F} = \pm \begin{pmatrix} 0 \\ 4 \end{pmatrix}. \tag{3.0.18}$$

See Fig. 0.

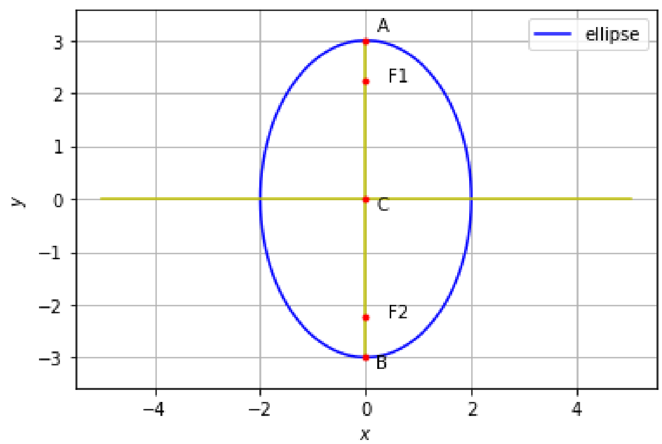

Fig. 0: Plot of the ellipse