# Lecture 3. Competitive Equilibrium in Dynamic Models

· Planner's problem to market based problem → prices.

· The competitive mechanism will involve households taking prices as given, but these prices being set to ensure market-clearing given households' demands.

· 假设：i) 生产要素（资本与劳动）属于家庭

ii) 厂商掌握将生产要素转化为产品的技术

iii) 市场上，家庭决定提供多少资本和劳动给厂商，以及消费多少产品。厂商决定需要多少生产要素以及供给多少产品。

· 市场上，买卖双方以市场价格为基础进行交易。市场均衡是指，给定市~~场~~场均衡价格，市场供给量与需求量相等。市场出清。

· 竞争均衡（A competitive equilibrium）是指一系列价格与数量满足：

1. Households choose quantities $\{c_t, n_t, k_{t+1}, \cdots\}$ to maximize utility given wealth $\{k_t\}$, factor endowments $\{l_t\}$ evaluated at given prices. $\{p_t, w_t, r_t\}$
2. Firms choose production $\{k_t, n_t\}$ to maximize profits at given prices. $\{p_t, w_t, r_t\}$
3. The quantities choosen by households and firms are feasible. (budget constraint) The aggregate quantity of each commodity demanded is produced using the factors supplied. Market clears.

· In dynamic economics we have to describe how trade over time occurs.

(non-storable resources)

(smooth consumption)

(through saving or lending)

- two environments $\begin{cases} \text{date-0 } \cancel{\text{tra}} \text{ trade: all trading is at date 0, financial assets are unnecessary} \\ \text{sequential trade: agents may borrow or save, typically one-period asset.} \end{cases}$

- 3.1 An endowment economy with date-0 trade

  - Let goods be dated and at date 0, trade in all comodities happen once and for all. That is, ~~the~~ all trades are arranged at time zero.

  - An infinitely-lived representative consumer has endowment $\{w_t\}_{t=0}^{\infty}$

  - no production, all agents can do is trade.

  - utility $\sum_{t=0}^{\infty} \beta^t u(c_t)$

  - 记 $P_t$ 为 $c_t$ 在 $t=0$ 时的价格，将 $P_0$ 标准化为 1，从而 $P_t$ 是 $c_t$ 相对于 $c_0$ 的相对价格。

  - 家庭禀赋的价值：$\sum_{t=0}^{\infty} P_t \cdot w_t$

  - 家庭消费：$\sum_{t=0}^{\infty} P_t \cdot c_t$

  - 预算约束：$\sum_{t=0}^{\infty} P_t \cdot c_t \leq \sum_{t=0}^{\infty} P_t \cdot w_t$

  以上市场结构（market structure）在 date-0 交易称为 Arrow Debreu Mckenizie

  - A competitive equilibrium 是一系列价格 $\{P_t\}_{t=0}^{\infty}$ 以及数量 $\{c_t^*\}_{t=0}^{\infty}$，使得：

    1. $\{c_t^*\}_{t=0}^{\infty}$ solves $\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$, s.t. $\sum_{t=0}^{\infty} P_t c_t \leq \sum_{t=0}^{\infty} P_t w_t$, $c_t \geq 0$, $\forall t$

    2. $c_t^* = w_t$, $\forall t$ （因为禀赋不可贮存，总供给=总需求）

· Remark: In an endowment economy, prices must induce consumption to equal demand as there is no way to shift resources overtime.

· $\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda\left(\sum_{t=0}^{\infty} p_t \cdot w_t - \sum_{t=0}^{\infty} p_t \cdot c_t\right)$

FOC: $[c_t]$: $\beta^t u'(c_t^*) = \lambda p_t \quad \forall t$

$\Rightarrow \beta^t u'(w_t) = \lambda p_t \quad \forall t.$

$$\Rightarrow \frac{p_t}{p_{t+1}} = \frac{u'(w_t)}{\beta u'(w_{t+1})}$$

因此: $\frac{p_0}{p_1} = \frac{u'(w_0)}{\beta u'(w_1)}, \frac{p_1}{p_2} = \frac{u'(w_1)}{\beta u'(w_2)}, \cdots, \frac{p_{t-1}}{p_t} = \frac{u'(w_{t-1})}{\beta u'(w_t)}, \cdots$

相乘可得.

$$\frac{p_0}{p_1} \cdot \frac{p_1}{p_2} \cdot \cdots \cdot \frac{p_{t-1}}{p_t} = \frac{u'(w_0)}{\beta u'(w_1)} \cdot \frac{u'(w_1)}{\beta u'(w_2)} \cdot \cdots \cdot \frac{u'(w_{t-1})}{\beta u'(w_t)}$$

$$\Rightarrow \frac{p_0}{p_t} = \frac{u'(w_0)}{\beta^t u'(w_t)}$$

因为 $p_0 = 1$

$$\Rightarrow p_t = \frac{\beta^t u'(w_t)}{u'(w_0)}$$

因此, $\{p_t\}_{t=0}^{\infty}$ 为市场均衡价格

· 以上步骤可总结为两步:

第一步: 通过家庭的一阶条件解出各期的需求, 给定任意价格.

第二步: 将供给等于需求, 解出均衡价格.

## 3.2 Sequential Trade in the endowment economy

· 在 date-0 trade 下，没有资产(assets)，因为所有交易都在第0期安排好了。

· 在 sequential trade 下，我们需要引入资产，称之为 Bonds，记为 $a_t$
利率为 $R_t = 1 + r_t$

· 家庭可以通过储蓄和借贷来调节各期财富（注：禀赋假设不可储存）

· 由于家庭是同质的，总净借贷等于个体借贷。
因此，资本市场出清使得均衡的总资产数量 $a_t^* = 0$, $\forall t$.

· 家庭的预算约束为：

$$c_t + a_{t+1} = a_t R_t + w_t, \ \forall t$$

· no Ponzi Game 条件：

$$\lim_{t\to\infty} \left(\prod_{s=0}^{t} R_{s+1}\right)^{-1} a_{t+1} = 0$$

此条件排除了家庭在每一期拆东补西的可能，因为，在无穷期远的资本的现值为零，从而无人会同意出借资本到无穷期远。

· A competitive equilibrium is a set of sequences $\{c_t^*, a_{t+1}^*\}_{t=0}^{\infty}$ and $\{R_t\}_{t=0}^{\infty}$, s.t.

1. $\{c_t^*, a_{t+1}^*\}_{t=0}^{\infty}$ solves $\max\limits_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$ s.t. $c_t \geq 0$, $a_0 = 0$ and $\lim_{t\to\infty} \left(\prod_{s=0}^{t} R_{s+1}\right)^{-1} a_{t+1} = 0$

2. $c_t^* = w_t \ \forall t$

3. $a_t^* = 0, \ \forall t$

· 我们需要解出市场均衡的资本~~价~~价格 — 利率 $R_t$

$$\mathcal{L}=\sum_{t=0}^{\infty}\beta^t[u(c_t)+\lambda_t(a_tR_t+w_t-c_t-a_{t+1})]$$

FOC: $[c_t]$: $u'(c_t)=\lambda_t$

$[a_{t+1}]$: $-\lambda_t+\beta\lambda_{t+1}R_{t+1}=0$.

$$\Rightarrow R_{t+1}=\frac{u'(c_t)}{\beta u'(c_{t+1})}=\frac{u'(w_t)}{\beta u'(w_{t+1})}$$

Remark: $R_{t+1}$ 对应 date-0 trading model 中的 $\frac{p_t}{p_{t+1}}$

## 3.3 The Neoclassical Growth Model with Date-0 Trade.

· 模型结构:

1. 家庭每期有一单位的时间并且休闲不产生效应

2. 效用为: $U(\{c_t,1-n_t\}_{t=0}^{\infty})=\sum_{t=0}^{\infty}\beta^t u(c_t)$,

   $u(\cdot)$ 严格增和凹

3. 家庭拥有生产资本 $k_t$, 每期折旧为 $\delta$., 资本租赁价格为 $r_t$.

4. 工资率记为 $w_t$

5. 生产函数 $F(k,n)$, $F$ 严格增, 凹, 且 homogenous of degree one.

6. $p_t$ 为 $c_t$ 相对于 $c_0$ 的价格., $p_0=1$

   $r_t, w_t$ 均为相对于 $c_t$ 的价格.

   提问: $p_tr_t$, $p_tw_t$ 代表什么?

- A date-0 competitive equilibrium is a set of sequences:

  (i). prices $\{p_t, r_t, w_t\}_{t=0}^{\infty}$

  (ii). quantities $\{c_t^*, n_t^*, k_{t+1}^*\}_{t=0}^{\infty}$

  使得

1. $\{c_t^*, k_{t+1}^*, n_t^*\}_{t=0}^{\infty}$ solves the households' problem:

$$\max_{\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\sum_{t=0}^{\infty} p_t(c_t + k_{t+1}) \leq \sum_{t=0}^{\infty} p_t(r_t k_t + (1-\delta)k_t + w_t n_t)$$

$$c_t \geq 0, \forall t$$

$$k_0 \text{ given}$$

2. $\{k_t^*, n_t^*\}_{t=0}^{\infty}$ solves the firms problem:

$$\max_{\{k_t, n_t\}_{t=0}^{\infty}} p_t F(k_t, n_t) - p_t r_t k_t - p_t w_t n_t$$

3. 市场出清: $c_t^* + k_{t+1}^* = F(k_t^*, n_t^*) + (1-\delta)k_t^*, \forall t$.

- If $w_t > 0$, then $n_t^* = 1$, because Leisure is not valued.

- 厂商面对的是静态问题:

$$\text{FOC}: [k_t]: \quad F_1(k_t, 1) = r_t$$

$$[n_t]: \quad F_2(k_t, 1) = w_t$$

· 家庭：

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda\left(\sum_{t=0}^{\infty} p_t (r_t k_t + (1-\delta)k_t + w_t n_t) - \sum_{t=0}^{\infty} p_t (c_t + k_{t+1})\right)$$

FOC: $[c_t]: \beta^t u'(c_t^*) = \lambda p_t$

$[k_{t+1}]: \lambda p_t = \lambda p_{t+1}(r_{t+1} + 1 - \delta)$

由消费的一阶条件，可得

$$\underbrace{\frac{p_t}{p_{t+1}}}_{\text{gross real interest rate between } t \text{ and } t+1} = \underbrace{\frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)}}_{\text{marginal rate of substitution of consumption goods between } t \text{ and } t+1}$$

由资本的一阶条件，可得

$$\frac{p_t}{p_{t+1}} = r_{t+1} + 1 - \delta$$

由于 $r_t = F_1(k_t^*, 1)$

$$\Rightarrow \frac{p_t}{p_{t+1}} = \underbrace{F_1(k_{t+1}^*, 1) + 1 - \delta}_{\text{marginal return of saving}}.$$

进一步，$\frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} = F_1(k_{t+1}^*, 1) + 1 - \delta$

$$\Rightarrow u'(c_t^*) = \beta u'(c_{t+1}^*)[F_1(k_t^*, 1) + 1 - \delta] \rightarrow \text{Euler equation}.$$

# 3.4 The neoclassical Growth model with sequential Trade.

- two prices: $\begin{cases} R_t = r_t + 1 - \delta \\ w_t \end{cases}$, both in units of ~~consu~~ current consumption $c_t$.

- 家庭储蓄与借贷，媒介为生产资本 k.

- A competitive equilibrim is a sequence $\{c_t^*, k_{t+1}^*, n_t^*, R_t, w_t\}_{t=0}^{\infty}$ s.t.

1. $\{c_t^*, k_{t+1}^*, n_t^*\}_{t=0}^{\infty}$ solves:

$$\max_{\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s.t.} \quad c_t + k_{t+1} \leq R_t k_t + w_t n_t$$

$$c_t \geq 0, \quad \forall t,$$

$$k_0 \text{ given}$$

$$\lim_{t\to\infty} \left(\prod_{t=0}^{\infty} R_{t+1}\right)^{-1} k_{t+1} = 0.$$

2. $\{k_t^*, n_t^*\}_{t=0}^{\infty}$ solves:

$$\max_{k_t, n_t} F(k_t, n_t) \underbrace{- R_t k_t + (1-\delta) k_t}_{= -r_t k_t} - w_t n_t.$$

3. 市场出清: $c_t^* + k_{t+1}^* = F(k_t^*, n_t^*) + (1-\delta) k_t^*, \ \forall t.$

$$\mathcal{L}=\sum_{t=0}^{\infty}\beta^t[u(c_t)+\lambda_t(R_tk_t+w_tn_t-c_t-k_{t+1})]$$

FOC: $[c_t]$: $u'(c_t^*)=\lambda_t$ , $\forall t$ .

$[k_{t+1}]$: $\lambda_t=\beta\cdot\lambda_{t+1}R_{t+1}$ $\forall t$

$$\Rightarrow \frac{u'(c_t^*)}{u'(c_{t+1}^*)}=\frac{\lambda_t}{\lambda_{t+1}} \quad , \quad \frac{\lambda_t}{\lambda_{t+1}}=\beta R_{t+1}.$$

$$\Rightarrow \frac{u'(c_t^*)}{u'(c_{t+1}^*)}=\beta\cdot R_{t+1}$$

· 从厂商的问题可得,

$$R_t=F_1(k_t^*,1)+1-\delta$$

· 从而, $u'(c_t^*)=\beta u'(c_{t+1}^*)[F_1(k_{t+1}^*,1)+1-\delta] \rightarrow$ Euler equation.

3.5 example: A date-0 economy with N households.

· 之前的 date-0 competitive equilibrium 里假设了代表性家庭,从而在均衡下无借贷发生,且总量 = 个体量.

· 现在,假设有 $N$ 个家庭,每个家庭有 $l_t^i\in[0,1]$ 个单位的时间禀赋以及初始资本 $k_0^i$

· 效用: $U^i(\{c_t^i,1-n_t^i\}_{t=0}^{\infty})=\sum_{t=0}^{\infty}\beta_i^t u^i(c_t^i)$ , $\beta^i\in(0,1)$, $u^i(\cdot)$ 严格增且凹.

· 注意:这里的家庭可以是异质性的,即 $l_t^i$ 和 $\beta^i$, $u^i(\cdot)$ 等均可不同.

· A date-0 competitive equilibrium for the N-agent economy is a set of sequences: (i) prices $\{p_t, r_t, w_t\}_{t=0}^{\infty}$

(ii) quantities $\{(c_t^{i*}, k_{t+1}^{i*}, n_t^{i*})_{i=1}^{N}, c_t^*, n_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ such that.

1. $\{c_t^{i*}, k_{t+1}^{i*}, n_t^{i*}\}_{t=0}^{\infty}$ solves household i's problem for each $i=1,\cdots,N$.

$$\max_{\{c_t^i, k_{t+1}^i, n_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta_i^t u^i(c_t^i)$$

subject to

$$\sum_{t=0}^{\infty} p_t (c_t^i + k_{t+1}^i) \leq \sum_{t=0}^{\infty} p_t (r_t k_t^i + (1-\delta) k_t^i + w_t n_t^i)$$

$$c_t^i \geq 0 \text{ and } 0 \leq n_t^i \leq l_t^i \quad \forall t,$$

$k_0^i$ given.

2. $\{k_t^*, n_t^*\}_{t=0}^{\infty}$ solves

$$\max_{\{k_t, n_t\}_{t=0}^{\infty}} p_t F(k_t, n_t) - p_t r_t k_t - p_t w_t n_t$$

3. 市场出清: (a) $n_t^* = \sum_{i=1}^{N} n_t^{i*}$

(b) $k_t^* = \sum_{i=1}^{N} k_t^{i*}$

(c) $c_t^* = \sum_{i=1}^{N} c_t^{i*}$

(d) $c_t^* + k_{t+1}^* = F(k_t^*, n_t^*) + (1-\delta) k_t^* \quad \forall t.$

· 在均衡中，最优投资决策 $k_{t+1}^{i*}$ 表明

$$p_t = p_{t+1}(r_{t+1}+1-\delta)$$

此条件为无套利条件，这一条件使得家庭对任一资本路径 $\{k_{t+1}^i\}_{t=0}^{\infty}$ 无差异。这一结论也可以通过以下推导表示：

我们将 $\frac{p_t}{p_{t+1}} = r_{t+1}+1-\delta$ 这一条件代入 lifetime budget constraint 中.

$$\sum_{t=0}^{\infty} p_t(c_t^i + k_{t+1}^i) \le \sum_{t=0}^{\infty} p_t\left(\frac{p_{t-1}}{p_t}k_t^i + w_t l_t^i\right)$$

$$\Rightarrow \sum_{t=0}^{\infty} p_t c_t^i + \sum_{t=0}^{\infty} p_t k_{t+1}^i \le \sum_{t=0}^{\infty} p_{t-1}k_t^i + \sum_{t=0}^{\infty} p_t w_t l_t^i$$

$$\Rightarrow \sum_{t=0}^{\infty} p_t c_t^i + \sum_{t=0}^{\infty} p_t k_{t+1}^i - \sum_{t=0}^{\infty} p_t k_{t+1}^i \le p_{-1}k_0^i + \sum_{t=0}^{\infty} p_t w_t l_t^i$$

由于闲暇不产生效用，$\Rightarrow n_t = l_t^i$

由于 $[r_0+(1-\delta)]\cdot k_0 = k_0$ 以及 $p_0 = 1$

$\Rightarrow [r_0+(1-\delta)] = 1$ 以及 $p_{-1} = 1$

$$\Rightarrow \underbrace{\sum_{t=0}^{\infty} p_t c_t^i}_{\text{lifetime spending}} = \underbrace{k_0^i + \sum_{t=0}^{\infty} p_t w_t l_t^i}_{\text{初始财富 + 永久收入的折旧}}$$

$\Rightarrow$ 资本的选择路径不影响消费.

$\Rightarrow$ 进一步，由于 utility function 的严格凹性，只要永久收入 $\sum_{t=0}^{\infty} p_t w_t l_t^i$ 不变，$\{c_t^i\}_{t=0}^{\infty}$ 消费的路径就不变. 即永久收入假说：

消费不是当期收入的函数，而是永久收入的函数。

$\Rightarrow$ 引申：李嘉图等价. $\Rightarrow$ 财政政策有效性问题

· As an illustration，假设 $\beta_i=\beta\in(0,1)$，$u^i(C_t^i)=u(C_t^i)$ for $i=1,\cdots,N$，

FOC for $C_t^i$ is：$\beta^t u'(C_t^i)=\lambda^i P_t$

$$\Rightarrow \frac{u'(C_t^i)}{\beta u'(C_{t+1}^i)}=\frac{P_t}{P_{t+1}}$$，for each $i=1,\cdots,N$ and $\forall t$.

$\Rightarrow$ 任一家庭都有着同样的消费边际替代率.

(marginal rate of substitution of consumption overtime

· 进一步，假定 $u(c)=\frac{c^{1-\sigma}-1}{1-\sigma}$，$\sigma>0$. CRRA

$$\Rightarrow \left(\frac{C_{t+1}^i}{C_t^i}\right)^\sigma=\frac{\beta P_t}{P_{t+1}}$$ for each $i=1,\cdots,N$，$\forall t$.

$\Rightarrow$ 消费增长率 $\frac{C_{t+1}^i}{C_t^i}$ 与个人收入无关，只与相对价格相关.

注意这并不意味着消费的绝对量与个人收入无关

$\Rightarrow$ $C_t^i=\alpha^i C_t^*$，$i=1,\cdots,N$，$\forall t$.

即，家庭消费量与总消费之比为常数 $\alpha^i$

这一常数不随时间变化，与家庭财富相关.

· 注：以上结论均在无信贷约束的情况下得出的.

If utility is iso-elastic，common marginal rates of substitution are associated with identical consumption growth rate.

例 example: A Two-period Endowment Economy. 5-15

本节通过一个简单的例子来介绍一些关键的经济学现象。

$$\max\left(u(c_0)+\beta u(c_1)\right)$$

subject to

$$c_0+a_1\le R_0a_0+w_0$$

$$c_1+a_2\le R_1\cdot a_1+w_1$$

$a_0=0$ given

$c\ge 0$

$u'(c)>0$

$a_2\ge 0$ $\}\Rightarrow$ imply $a_2=0$

There is a representative household, so in equilibrium $c_t=w_t$, $a_t=0$

找到均衡解在这一问题上的含义即为：找到利率 $R_1$，使得上述条件成立。(注意：由于 $a_0=0$ 给定，$R_0$ 与均衡解无关。)

· 拉格朗日函数为：$\mathcal{L}=u(c_0)+\beta u(c_1)+\lambda_0(R_0\cdot a_0+w_0-c_0-a_1)+\beta\cdot\lambda_1(R_1a_1+w_1-c_1-a_2)$

F.O.C: $[c_0]$: $u'(c_0)=\lambda_0$

$[c_1]$: $\beta u'(c_1)=\beta\cdot\lambda_1$

$[a_1]$: $-\lambda_0+R_1\cdot\beta\cdot\lambda_1=0$

$\Rightarrow u'(c_0)=\beta u'(c_1)\cdot R_1$

imposing $c_t=w_t$ for $t=0,1$.

$\Rightarrow u'(w_0)=\beta u'(w_1)\cdot R_1$

注：这里再次强调我们的解模型的步骤：

1). 给定任一系列价格，解出家庭的最优选择方程 (price-taker)

2). 利用市场出清条件，代入上述最优方程，解出均衡价格。

·进一步，假设 $W_1 = W_0 + \varepsilon_1$, where $\varepsilon_1 \leq 0$.

我们有 $u'(W_0) = \beta u'(W_0 + \varepsilon_1) R_1$

·如果 $\varepsilon_1 = 0$，则 $R_1 = \frac{1}{\beta}$，即实际总利率正好补偿时间偏好的折现，消费者实现了平滑消费 (smooth consumption).

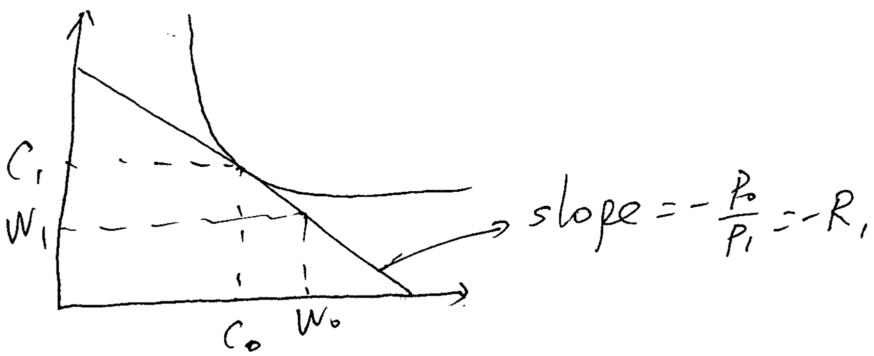
·当我们转而用 date-0 trade 来建模时，我们有如下的终身预算约束.

$$p_0 \cdot c_0 + p_1 c_1 \leq p_0 \cdot W_0 + p_1 \cdot W_1$$

·将 $p_0$ 标准化为 1，计算可得 $p_1 = \cancel{\frac{u'(W_0)}{\beta u'(W_1)}} \frac{\beta u'(W_1)}{u'(W_0)}$

即 $p_1$ 等于 sequential trade 中的 $\frac{1}{R_1}$

·或者说，当我们不标准化 $p_0$ 时，$\frac{p_1}{p_0} = \frac{1}{R_1}$，即实际利率 $R_1$ 为第 0 期消费的机会成本。



· 问题：实际利率（real interest rate）如何影响两期消费的斜率？

为了回答这一问题，我们暂时只考虑消费者面对的问题，而不考虑均衡。

· 设定 $R_1 = R$，假定家庭是一个大经济体中的一个普通的小家庭，其行为不对价格产生任何影响。

· 假设效用函数为 $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$，$\sigma > 0$.

则欧拉方程为

$$C_0^{-\sigma} = \beta R C_1^{-\sigma}$$

$$\Rightarrow C_1 = (\beta R)^{\frac{1}{\sigma}} \cdot C_0$$

· implications：

1). constant elasticity of intertemporal substitution of consumption.

（恒定的消费跨期替代弹性）

这一弹性通常定义为：$\eta = -\dfrac{d\log(\frac{C_1}{C_0})}{d\log(\frac{P_1}{P_0})}$

由于我们已知：$\frac{P_1}{P_0} = \frac{1}{R}$，即 $C_1$ 相对 $C_0$ 的相对价格.

$$\Rightarrow \eta = -\frac{d\log(\frac{C_1}{C_0})}{d\log(\frac{1}{R})}$$

由于 $\log(\frac{C_1}{C_0}) = \frac{1}{\sigma}\log\beta - \frac{1}{\sigma}\log(\frac{1}{R})$

$$\Rightarrow \eta = \frac{1}{\sigma}$$

即，消费与储蓄的反应取决于 $\sigma$ 的大小

进一步,利用预算约束 $C_0+\frac{C_1}{R}=W_0+\frac{W_1}{R}$,我们可解出 $C_0, C_1$

定义: $W=W_0+\frac{W_1}{R}$,终身收入.

$$C_0+\frac{(\beta R)^{\frac{1}{\sigma}}C_0}{R}=W$$

$$C_0=\frac{W}{1+(\beta R^{1-\sigma})^{\frac{1}{\sigma}}}$$

$$C_1=\frac{(\beta R)^{\frac{1}{\sigma}}W}{1+(\beta R^{1-\sigma})^{\frac{1}{\sigma}}}$$

2). our iso-elastic utility function is homothetic.

即当相对价格 $\frac{1}{R}$ 不变时,终身收入 $W$ 的变化会使得 $C_0, C_1$ 成比例的变化。消费曲线的斜率没有变化.

3). $C_0, C_1$ are independent of $W_0$ and $W_1$, depending only on the sum, $W_0+\frac{W_1}{R}$

The marginal propensity to consume from $W$:

$$\frac{dC_0}{dW}=\frac{1}{1+(\beta R^{1-\sigma})^{\frac{1}{\sigma}}}<1$$

implies that the consumer smooths consumption over his lifetime.

(a result of permanent income hypothesis)

4). How about saving or borrowing?

$$\begin{aligned} a_1=W_0-C_0 &= W_0-\frac{W}{1+(\beta R^{1-\sigma})^{\frac{1}{\sigma}}} \\ &= W_0-\frac{W_0+\frac{W_1}{R}}{1+(\beta R^{1-\sigma})^{\frac{1}{\sigma}}} \\ &= \frac{(\beta R^{1-\sigma})^{\frac{1}{\sigma}}W_0-\frac{W_1}{R}}{1+(\beta R^{1-\sigma})^{\frac{1}{\sigma}}} \end{aligned}$$

即 $a_1 \gtrless 0$, as $w_1 \lessgtr (\beta R)^{\frac{1}{\sigma}} w_0$

~~即 $w_0$ 或 $w_1$ 的变化会影~~

即，即使在 $W = w_0 + \frac{w_1}{R}$ 不变（终身收入不变）的情况下，
$w_0$ 和 $w_1$ 的变化虽然不改变 $c_0, c_1$，却使得 saving $a_1$ 变化.

· 问题：$R$ 的变化如何影响 $a_1$？

为使问题简化，令 $w_1 = 0$.

在这一假设下，可以看作一个工人在年轻时工作有收入，退休后无收入.

从而有，
$$a_1 = \frac{(\beta R^{1-\sigma})^{\frac{1}{\sigma}} \cdot w_0}{1 + (\beta R^{1-\sigma})^{\frac{1}{\sigma}}}$$

定义：$X(R) \equiv (\beta R^{1-\sigma})^{\frac{1}{\sigma}}$

$$\frac{da_1}{dX(R)} = \frac{1 + X(R) - X(R)}{(1 + X(R))^2} \cdot w_0 = \frac{1}{(1 + X(R))^2} \cdot w_0 > 0$$

又由于，$X'(R) = \frac{1-\sigma}{\sigma} \cdot \beta^{\frac{1}{\sigma}} R^{\frac{1-2\sigma}{\sigma}} \gtrless 0$ as $\sigma \lessgtr 1$

从而，
$$\frac{da_1}{dR} = \frac{da_1}{dX(R)} \cdot \frac{dX(R)}{dR} = \frac{w_0}{[1+X(R)]^2} \cdot \frac{1-\sigma}{\sigma} \cdot \beta^{\frac{1}{\sigma}} \cdot R^{\frac{1-2\sigma}{\sigma}} \gtrless 0 \text{ as } \sigma \lessgtr 1$$

· 当 $R \uparrow$ 时有何效应？

i) 财富效应：$c_0 \uparrow$, $c_1 \uparrow$, $a_1 \downarrow$

ii) 替代效应：$c_0 \downarrow$, $c_1 \uparrow$, $a_1 \uparrow$

两种效应的相对大小取决于 $\sigma$

当 $\sigma > 1$ 时，财富效应较大，$c_0 \uparrow$, $c_1 \uparrow$, $a_1 \downarrow$

当 $\sigma < 1$ 时，替代效应较大，$c_0 \downarrow$, $a_1 \uparrow$, $c_1 \uparrow$

37 example: Government Debt and Taxes. 

· 政府消费 $\{g_t\}$，人头税 $\{\tau_t\}$，政府债 $\{d_t\}$

· 政府债券 $\{d_t\}$ 在均衡时与资本给出的利率一定相同。

· 均衡：Given $\{g_t, \tau_t\}$, a competitive equilibrium is a set of prices and quantities: $\{R_t, w_t\}_{t=0}^{\infty}$, $\{c_t^*, k_{t+1}^*, n_t^*, d_{t+1}^*\}_{t=0}^{\infty}$, such that.

1. Households solve their problem:

$\{c_t^*, k_{t+1}^*, n_t^*, d_{t+1}^*\}_{t=0}^{\infty}$ solves:

$$\max_{\{c_t, k_{t+1}, d_{t+1}, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to.

$$c_t + k_{t+1} + d_{t+1} + \tau_t = R_t k_t + R_t d_t + w_t n_t, \quad \forall t.$$

$$\lim_{t\to\infty} \frac{d_{t+1}}{\prod_{s=0}^{t} R_s} = 0$$

$$\lim_{t\to\infty} \frac{k_{t+1}}{\prod_{s=0}^{t} R_s} = 0$$

$k_0, d_0$ given.

2. Firms solve their problem, $\{k_t^*, n_t^*\}_{t=0}^{\infty}$ solves:

$$\max_{k_t, n_t} F(k_t, n_t) - R_t k_t + (1-\delta)k_t - w_t n_t$$

3. Government budget constraint holds,

$$g_t + R_t d_t = d_{t+1} + \tau_t, \quad \forall t.$$

4. Markets clear.

$$c_t^* + k_{t+1}^* + g_t^* = F(k_t^*, n_t^*) + (1-\delta)k_t^*, \quad \forall t.$$

推论：1) government bonds are not net wealth in this economy.
(政府债券不是净财富)

家庭的预算约束：

$$c_t = R_t k_t - k_{t+1} + R_t d_t + w_t n_t - d_{t+1} - \tau_t, \ \forall t.$$

定义：$R_{0,t} = \prod_{s=0}^{t} R_s, \ \forall t$ 为 0 到 t 之间的复利.

$$\frac{c_t}{R_{0,t}} = \frac{R_t k_t}{R_{0,t}} - \frac{k_{t+1}}{R_{0,t}} + \frac{R_t d_t}{R_{0,t}} + \frac{w_t n_t}{R_{0,t}} - \frac{d_{t+1}}{R_{0,t}} - \frac{\tau_t}{R_{0,t}}$$

summing over all $t$, and using the government budget constraint.

$$\sum_{t=0}^{\infty} \frac{c_t}{R_{0,t}} = \sum_{t=0}^{\infty} \frac{k_t}{R_{0,t-1}} - \sum_{t=0}^{\infty} \frac{k_{t+1}}{R_{0,t}} + \sum_{t=0}^{\infty} \frac{w_t n_t}{R_{0,t}} - \sum_{t=0}^{\infty} \frac{g_t}{R_{0,t}}$$

$$= k_0 + \sum_{t=0}^{\infty} \frac{w_t n_t}{R_{0,t}} - \sum_{t=0}^{\infty} \frac{g_t}{R_{0,t}}$$ (using $R_{0,-1}=1$, and limit condition on $k_{t+1}$)

- Ricardian Equivalence Theorem (李嘉图等价定理)：
Take any competitive equilibrium $\{R_t, w_t\}$ and $\{c_t^*, k_{t+1}^*, n_t^*\}$ given $\{g_t, \tau_t\}$. Now perturb the path of taxes to $\{\tau_t^a\}_{t=0}^{\infty}$ but don't change $\{g_t\}$. As the households budget constraint isn't affected, there will be no change in prices or $\{c_t^*, k_{t+1}^*, n_t^*\}$.

- Households will understand that current tax cuts, in the absence of any change in spending, will lead to future tax increases. They will save the rise in period s income in bonds.

# 3.8. Recursive Competitive Equilibrium.

## 3.8.1 The Optimal Growth Model.

recall the planner's problem:

$$V(k) = \max_{c,k'} u(c) + \beta V(k')$$

subject to

$$c + k' = F(k,1) + (1-\delta)k$$

定义: gross real interest rate $R = F_1(k,1) + 1 - \delta$

real wage rate $W = F_2(k,1)$

· 在 Recursive competitive equilibrium 中，我们需要区分 aggregate state variables 和 individual state variables.
其中，只有 aggregate state variables 能够决定均衡中的价格量.

· 记 $\bar{k}$ 为 aggregate capital stock.

$$R = R(\bar{k})$$
$$w = w(\bar{k})$$

· 从而，家庭的预算约束为:

$$c + k' = R(\bar{k}) \cdot k + w(\bar{k}) \cdot 1$$

· 家庭面对的问题变为:

$$V(k,\bar{k}) = \max_{c,k'} u(c) + \beta V(k',\bar{k}')$$

subject to

$$c + k' = R(\bar{k}) \cdot k + w(\bar{k})$$

$$\bar{k}' = G(\bar{k}) \rightarrow \text{law of motion of aggregate capital}$$

· 以上问题的解给出一个 policy function $g(k,\bar{k})$

即，$k' = g(k,\bar{k}) = \arg\max_{k' \in [0,\, R(\bar{k})k + w(\bar{k})]} u(R(\bar{k})k + w(\bar{k}) - k') + \beta V(k',\bar{k}')$

· A recursive competitive equilibrium is a set of functions:
quantities $G(\bar{k})$, $g(k,\bar{k})$, Value $V(k,\bar{k})$, prices $R(\bar{k})$, $w(\bar{k})$, such that.

1. $V(k,\bar{k})$ solves households problem.

   $g(k,\bar{k})$ is the associated policy function.

2. Prices are competitive determined.

$$R(\bar{k}) = F_1(\bar{k},1) + 1 - \delta$$

$$w(\bar{k}) = F_2(\bar{k},1)$$

Where is market clear?

zero profit: $F(\bar{k},1) + (1-\delta)\bar{k} = R(\bar{k})\bar{k} + w(\bar{k})$

$$c + k' = R(\bar{k})k + w(\bar{k})$$

$$\Rightarrow F(\bar{k},1) + (1-\delta)\bar{k} = c + k'$$

3. Individual decisions are consistent with aggregates.

$$G(\bar{k}) = g(\bar{k},\bar{k}), \ \forall \bar{k}$$

消费者作最优投资决策时所给定的价格来自于使用同一组资本量所得到的厂商的一阶条件.

3.8.2. An endowment economy with two agents.

· Asset market equilibrium $\Rightarrow$

$$a_t^1 = -a_t^2$$

· A ~~comp~~ Recursive Competitive Equilibrium of the two-agent endowment economy is a set of functions: quantities $G(A_1)$, $g(a_1,A_1)$, $g(a_2,A_1)$, values $V_1(a_1,A_1)$, $V_2(a_2,A_1)$ and prices $q(A_1)$, such that.

1. $V_i(a_i,A_1)$ solves the type i household's problem, $i=1,2$.

$$V_i(a_i,A_1) = \max_{c_i,a_i'} U_i(c_i) + \beta_i V_i(a_i',A_1')$$

subject to.

$$c_i + a_i' q(A_1) = a_i + w_i$$

$$a_i' \geq \underline{a}$$

and $A_1' = G(A_1)$

and the solution to the functional equation implies the policy functions $g_i(a_i,A_1)$

2. Consistency

$$g_1(A_1, A_1) = G(A_1)$$

$$g_2(-A_1, A_1) = -G(A_1)$$

notice that this implies asset market clearing.

$$g_1(A_1, A_1) + g_2(A_1, A_1) = G(A_1) - G(A_1) = 0$$

Furthermore, $q(A_1)$ adjusts with $A_1$ to ensure consistency.