HW 3

1.

(a)

A competitive equilibrium is a set of prices $\{p_t, r_t, w_t\}_{t=0}^{\infty}$ and quantities $\{c_t^*, c_{0,t}^*, c_{1,t}^*, k_{t+1}^*, k_{0,t+1}^*, k_{1,t+1}^*, n_t^*, n_{0,t}^*, n_{1,t}^*\}_{t=0}^{\infty}$ such that

(1)$\{c_{h,t}^*, k_{h,t+1}^*, n_{h,t}^*\}_{t=0}^{\infty} = \arg\max \sum \beta^t u(c_{h,t})$

subject to $\sum p_t(c_{h,t} + k_{h,t+1}) = \sum p_t(r_t k_{h,t} + (1-\delta)k_{h,t}) + \sum p_t w_t n_{h,t}$

$c_{h,t} \geqslant 0, k_{h,0}$ given for $h = 0, 1$

(2)$\{k_t^*, n_t^*\}_{t=0}^{\infty} = \arg\max(p_t F(k_t, n_t) - p_t r_t k_t - p_t w_t n_t)$

(3) $c_t^* = c_{0,t}^* + c_{1,t}^*$, $k_t^* = k_{0,t}^* + k_{1,t}^*$, $n_t^* = l$ and $c_t^* + k_{t+1}^* = F(k_t^*, l) + (1-\delta)k_t^*$

(b)

Household $h$ Lagrange multiplier for $\sum p_t(r_t k_{h,t} + (1-\delta)k_{h,t} + w_t n_{h,t} - c_{h,t} - k_{h,t+1})$ is $\lambda_h$, $h = 0, 1$

FOC for $c_{h,t}$ and $k_{h,t+1}$ are

$$\beta^t u'(c_{h,t}) = \lambda_h p_t \tag{1}$$

$$\lambda_h p_t = \lambda_h p_{t+1}(r_{t+1} + 1 - \delta) \tag{2}$$

Equation (2) implies $\frac{p_t}{p_{t+1}} = r_{t+1} + 1 - \delta$

Equation (1) led one period is $\beta^{t+1} u'(c_{h,t+1}) = \lambda_h p_{t+1}$,, divide this by equation (1), we can get

$$\frac{\beta u'(c_{h,t+1})}{u'(c_{h,t})} = \frac{p_{t+1}}{p_t} = \frac{1}{r_{t+1} + 1 - \delta} \tag{3}$$

First order condition for firm from condition (2) is $p_t D_1 F(k_t, n_t) = p_t r_t$

In equilibrium $n_t^* = l$ and $k_t^* = k_{0,t}^* + k_{1,t}^*$

Thus

$$D_1 F(k_t^*, l) = r_t \tag{4}$$

Leading (4) by one period and rearranging (3) gives

$$\frac{u'(c_{h,t})}{u'(c_{h,t+1})} = \beta[D_1 F(k_{t+1}^*, l) + 1 - \delta] \tag{5}$$

If consumption shares are time invariant (i.e. $c_{h,t} = \theta_h c_t$ with $\theta_0 + \theta_1 = 1$) and marginal rate of substitution satisfies homogenous of degree 0, then there is an aggregate Euler equation. For example, $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$. In general, each household is optimizing against the technical rate of substitution $\frac{p_{t+1}}{p_t}$. This doesn't imply a representative household with the same marginal rate of substitution.

(c)

A sequential equilibrium is a sequences $\{R_t, w_t, c_t^*, c_{0,t}^*, c_{1,t}^*, k_{t+1}^*, k_{0,t+1}^*, k_{1,t+1}^*, n_t^*, n_{0,t}^*, n_{1,t}^*\}_{t=0}^{\infty}$ such that

(1)$\{c_{h,t}^*, k_{h,t+1}^*, n_{h,t}^*\}_{t=0}^{\infty} = \arg\max \sum \beta^t u(c_{h,t})$

subject to $c_{h,t}+k_{h,t+1}=R_t k_{h,t}+w_t n_{h,t}$

$c_{h,t}\geqslant 0, k_{h,0}$ given

$\lim_{t\to\infty}(\Pi_{t=0}^{\infty}R_{t+1})^{-1}k_{h,t+1}=0$ for $h=0,1$

(2)$\{k_t^*,n_t^*\}_{t=0}^{\infty}=\arg\max(F(k_t,n_t)-R_t k_t+(1-\delta)k_t-w_t n_t)$

(3) $c_t^*=c_{0,t}^*+c_{1,t}^*, k_t^*=k_{0,t}^*+k_{1,t}^*, n_t^*=l$ and $c_t^*+k_{t+1}^*=F(k_t^*,l)+(1-\delta)k_t^*$

The household sequential budget constraints are

$c_{h,t}+k_{h,t+1}=R_t k_{h,t}+w_t n_{h,t} \quad t=0,1,....$

Define $R_{0,t}=\Pi_{s=0}^{t}R_s$

Divide the sequential bedget constraints by this compound interest rate and sum them up

$\sum\limits_{t=0}^{\infty}\frac{c_{h,t}}{R_{0,t}}=k_{h,0}+\sum\limits_{t=0}^{\infty}\frac{w_t n_{h,t}}{R_{0,t}}$

Condition (2) of sequential equilibrium implies $R_t=D_1F(k_t^*,l)+1-\delta\Rightarrow R_{t+1}=r_{t+1}+1-\delta=\frac{p_t}{p_{t+1}}$

Without loss of generality, set $p_0=1$

$p_{t+1}=\frac{p_t}{R_{t+1}}=\frac{1}{R_{t+1}R_t}p_{t-1}=...=\frac{1}{R_{0,t+1}}$

$\sum\limits_{t=0}^{\infty}p_t(c_{h,t}+k_{h,t+1})=\sum\limits_{t=0}^{\infty}p_t(r_t k_{h,t}+(1-\delta)k_{h,t})+\sum\limits_{t=0}^{\infty}p_t w_t n_{h,t}$

$\sum\limits_{t=0}^{\infty}\frac{1}{R_{0,t}}(c_{h,t}+k_{h,t+1})=\sum\limits_{t=0}^{\infty}\frac{1}{R_{0,t}}(R_t k_{h,t}+w_t n_{h,t})$

Rearrange, we can get $\sum\limits_{t=0}^{\infty}\frac{c_{h,t}}{R_{0,t}}=k_{h,0}+\sum\limits_{t=0}^{\infty}\frac{w_t n_{h,t}}{R_{0,t}}$

(d)

Define $\overline{s}\in R_+^2$ as $\overline{s}=(\overline{k}_0,\overline{k}_1)$

A recursive equilibrium is a set of functions

quantities $G(\overline{s},e), g(k_h,\overline{s},e), h=0,1$ values $V_h(k_h,\overline{s},e)$ prices $R(\overline{s}), w(\overline{s})$ such that

(1)$V_h(k_h,\overline{s},e)$ solves $V_h(k_h,\overline{s},e)=\max(u(c_h)+\beta V_h(k_h',\overline{s}',\widetilde{e}))$

subject to $\quad c_h+k_h'=R(\overline{s})k_h+w(\overline{s})I_h(e)$

$\overline{s}'=G(\overline{s},e)$

where $e=0$ for an odd period

$=1$ for an even date

if $h=0$, then $I_h(0)=0, I_h(1)=l$

$h=1$, then $I_h(0)=l, I_h(1)=0$

$\widetilde{e},e\in\{0,1\}$ and $\widetilde{e}\neq e$

$g(k_h,\overline{s},e)$ is the policy function

(2) Let $\overline{k}=\overline{k}_0+\overline{k}_1$ where $\overline{s}=(\overline{k}_0,\overline{k}_1)$

$R(\overline{s})=D_1F\left(\overline{k},l\right)+(1-\delta)$

$w(\overline{s})=D_2F\left(\overline{k},l\right)$

(3)$\overline{s}'=G(\overline{s},e)=(g_0(k_0,\overline{s},e),g_1(k_1,\overline{s},e))$

(e)

From equation (5) $u'(c_{h,t})=\beta[D_1F\left(k_{t+1}^*,l\right)+1-\delta]u'(c_{h,t+1})$

In steady stae, $c_{h,t}^*=c_{h,t+1}^*=c_h^*\Rightarrow u'(c_{h,t})=u'(c_{h,t+1})$

Thus $\beta[D_1F\left(k_{t+1}^*,l\right)+1-\delta]=1\Rightarrow k_{t+1}^*=k^*$

Also $p_{t+1}=\frac{\beta u'(c_{h,t+1})}{u'(c_{h,t})}p_t=\beta^{t+1}p_0=\beta^{t+1}$

$D_1 F\left(k_t^*, l\right) + 1 - \delta = r_t + 1 - \delta = \frac{1}{\beta}$

The household's date-0 budget constraint is

$\sum p_t(c_{h,t} + k_{h,t+1}) = \sum p_t(r_t k_{h,t} + (1-\delta)k_{h,t}) + \sum p_t w_t n_{h,t}$

$\sum \beta^t(c_{h,t} + k_{h,t+1}) = \sum \beta^t(r_t + (1-\delta))k_{h,t} + \sum \beta^t w_t n_{h,t}$

$\sum\limits_{t=0}^{\infty} \beta^t(c_{h,t} + k_{h,t+1}) = \sum\limits_{t=0}^{\infty} \beta^t \frac{1}{\beta} k_{h,t} + \sum \beta^t w_t I_h(e)$

$\sum\limits_{t=0}^{\infty} \beta^t c_h^* = \frac{k_{h,0}}{\beta} + \sum\limits_{t=0}^{\infty} \beta^t D_2 F\left(k_t^*, l\right) I_h(e)$

where $h = 0$, then $I_h(0) = 0, I_h(1) = l$

$h = 1$, then $I_h(0) = l, I_h(1) = 0$

At $t = 0$, for $h = 0$ we have labor income

$w^* l + \beta^2 w^* l + \beta^4 w^* l + .... = \frac{w^* l}{1-\beta^2}$

At $t = 0$, for $h = 1$ we have labor income

$\beta w^* l + \beta^3 w^* l + \beta^5 w^* l + .... = \frac{\beta w^* l}{1-\beta^2}$

This solves for consumption as follows:

$c_0^* = (\frac{k_{0,0}}{\beta} + \frac{w^* l}{1-\beta^2})(1-\beta)$

$c_1^* = (\frac{k_{1,0}}{\beta} + \frac{\beta w^* l}{1-\beta^2})(1-\beta)$

The marginal propensity to consume from wealth whould be $(1-\beta)$

2.

The aggregate state will depend on $e = 0, 1$. Let $\widetilde{e}, e \in \{0, 1\}$ and $\widetilde{e} \neq e$

(a) A recursive competitive equilibrium is a set of functions

$V(a,e)$, $g(a,e), R(e)$

such that

(1)$V(a,e)$ solves the household functional equation

$V(a,e) = \max_{c \geqslant 0, a'}(u(c) + \beta V(a', \widetilde{e}))$

subject to

$c + a' = R(e)a + w(e)$

where $w(e) = w_h$ for an even period

$= w_l$ for an odd date

(2) consistency

$g(0,e) = 0$

(b)Notice that in equilibrium $a = 0$ and $c = w(e)$ The first order condition for $a'$ is

$-u'(c) + \beta D_1 V(a', \widetilde{e}) = 0$

The Benveniste-Scheinkman condition is $D_1 V(a,e) = u'(c)R(e)$

In equilibrium $c = w(e)$

Combining these results $u'(w(e)) = \beta R(\widetilde{e}) u'(w(\widetilde{e}))$

When $e = 0$,

$u'(w_h) = \beta R_l u'(w_l)$

When $e = 1$,

$u'(w_l) = \beta R_h u'(w_h)$

As $w_h > w_l \Rightarrow u'(w_h) < u'(w_l)$ and $R_l < R_h$

In periods with high endowment, households want to smooth the rise in income over their lifetimes. In equilibrium there can be no saving, so the gross return on saving must fall. This reduces the relative price of current consumption and thus the desire to save.