1.

a) Arrow-Debreu-Mckenzie date 0 competitive equilibrium is $\{c_t^*(z^t), k_{t+1}^*(z^t), n_t^*(z^t), l_t^*(z^t), p_t(z^t), r_t(z^t), w_t$
such that

1. $\{c_t^*(z^t), k_{t+1}^*(z^t), n_t^*(z^t), l_t^*(z^t)\}_{t=0}^{\infty} = \arg\max \sum_{t=0}^{\infty} \beta^t \sum_{z^t \in Z^{t+1}} \pi(z^t) u(c_t(z^t), l_t(z^t))$

   subject to $\sum_{t=0}^{\infty} \sum_{z^t \in Z^{t+1}} p_t(z^t)(c_t(z^t)+k_{t+1}(z^t)) \leq \sum_{t=0}^{\infty} \sum_{z^t \in Z^{t+1}} p_t(z^t)\{[r_t(z^t)+ (1-\delta)]k_t(z^{t-1}) + w_t(z^t)n_t(z^t)\}$

   $$n_t(z^t) + l_t(z^t) = 1,\ c_t(z^t) \geqslant 0, k_0 \text{ given}$$

2. $\{k_t^*(z^{t-1}), n_t^*(z^t)\} = \arg\max p_t(z^t)(z_t F(k_t, n_t) - r_t(z^t)k_t - w_t(z^t)n_t)$

3. $c_t^*(z^t) + k_{t+1}^*(z^t) = z_t F\left(k_t^*(z^{t-1}), n_t^*(z^t)\right) + (1-\delta)k_t^*(z^{t-1})$

Let λ be the lagrange multiplier for the date 0 budget constraint. The first order condition w.r.t.

$c_t(z^t) : \beta^t \pi(z^t) D_1 u(c_t(z^t), l_t(z^t)) = \lambda p_t(z^t)$

$k_{t+1}(z^t) : p_t(z^t) = \sum_{z^{t+1} \in Z^{t+2}} p_{t+1}(z^{t+1})[r_{t+1}(z^{t+1}) + (1-\delta)]$

where $z^{t+1} = (z_{t+1}, z^t)$ for the exact z^t indexing $p_t(z^t)$. Combining these equations, we have

$$\pi(z^t) D_1 u(c_t(z^t), l_t(z^t)) = \sum_{z^{t+1} \in Z^{t+2}} \beta \pi(z^{t+1}) D_1 u(c_{t+1}(z_{t+1}, z^t), l_{t+1}(z_{t+1}, z^t))[r_{t+1}(z_{t+1}, z^t) + (1-\delta)]$$

The law of conditional probability yields $\pi(z^{t+1}) = \pi(z_{t+1}|z^t)\pi(z^t)$, which then gives

$$D_1 u(c_t(z^t), l_t(z^t)) = \sum_{z_{t+1} \in Z} \beta \pi(z_{t+1}|z^t) D_1 u(c_{t+1}(z_{t+1}, z^t), l_{t+1}(z_{t+1}, z^t))[r_{t+1}(z_{t+1}, z^t) + (1-\delta)]$$

The summation is only over z_{t+1} as z^t is predetermined or fixed.

The firm's profit maximization implies

$r_{t+1}(z_{t+1}, z^t) = z_{t+1} D_1 F\left(k_{t+1}(z^t), n_{t+1}(z_{t+1}, z^t)\right)$

substituting this into the Euler equation gives the planner's Euler equation

$$D_1 u(c_t(z^t), l_t(z^t)) = \sum_{z_{t+1} \in Z} \beta \pi(z_{t+1}|z^t) D_1 u(c_{t+1}(z_{t+1}, z^t), l_{t+1}(z_{t+1}, z^t))[z_{t+1} D_1 F\left(k_{t+1}(z^t), n_{t+1}(z_{t+1}, z^t)\right) + (1-\delta)]$$

b)

A sequential competitive equilibrium is $\{c_t^*(z^t), k_{t+1}^*(z^t), n_t^*(z^t), l_t^*(z^t), a_t^*(z_{t+1}, z^t), R_t(z^t), w_t(z^t), q_t(z_{t+1}, z^t$
such that

(1)$\{c_t^*(z^t), k_{t+1}^*(z^t), n_t^*(z^t), l_t^*(z^t), a_{t+1}^*(z_{t+1}, z^t)\}_{t=0}^{\infty} = \arg\max \sum_{t=0}^{\infty} \beta^t \sum_{z^t \in Z^{t+1}} \pi(z^t) u(c_t(z^t), l_t(z^t))$

subject to $c_t(z^t) + k_{t+1}(z^t) + \sum_{z^{t+1} \in Z} q_t(z_{t+1}, z^t) a_{t+1}(z_{t+1}, z^t) \leq R_t(z^t) k_t(z^{t-1}) + w_t(z^t) n_t(z^t) + a_t(z^t)$

$n_t(z^t)+l_t(z^t)=1,\ c_t(z^t)\geqslant 0, k_0, a_0$ given

$\lim_{t\to\infty}(\Pi_{s=0}^t R_s(z^t))^{-1}k_{t+1}(z^t)=0$

$\lim_{t\to\infty}(\Pi_{s=0}^t R_s(z^t))^{-1}a_{t+1}(z_{t+1},z^t)=0\quad \forall z_{t+1}\in Z$

(2)$\{k_t^*(z^{t-1}), n_t^*(z^t)\} = \arg\max(z_tF\,(k_t,n_t)+(1-\delta)k_t-R_t(z^t)k_t-w_t(z^t)n_t)$

(3)market clears

$c_t^*(z^t)+k_{t+1}^*(z^t)=z_tF\left(k_t^*(z^{t-1}),n_t^*(z^t)\right)+(1-\delta)k_t^*(z^{t-1})$

$a_{t+1}(z_{t+1},z^t)=0 \qquad\qquad \forall z_{t+1}\in Z$

Let $\lambda_t(z^t)$ be the (t,z^t) lagrange multiplier for the household budget constraint. The first order conditions for $c_t(z^t), n_t(z^t), k_{t+1}(z^t), a_{t+1}(z_{t+1},z^t)$ (there are N of these) are

$\beta^t\pi(z^t)D_1u(c_t(z^t),1-n_t(z^t))=\lambda_t(z^t)\beta^t\pi(z^t)$

$\beta^t\pi(z^t)D_2u(c_t(z^t),1-n_t(z^t))=\lambda_t(z^t)\beta^t\pi(z^t)w_t(z^t)$

$\beta^t\pi(z^t)\lambda_t(z^t)=\sum\limits_{z_{t+1}\in Z}\beta^{t+1}\pi(z_{t+1},z^t)\lambda_{t+1}(z^{t+1})R_{t+1}(z^{t+1})$

$\beta^t\pi(z^t)\lambda_t(z^t)q_t(z_{t+1},z^t)=\beta^{t+1}\pi(z_{t+1},z^t)\lambda_{t+1}(z_{t+1},z^t)$

The Euler equation for Arrow security if z_{t+1} realized

$q_t(z_{t+1},z^t)D_1u(c_t(z^t),1-n_t(z^t))=\beta\pi(z_{t+1}|z^t)D_1u(c_{t+1}(z_{t+1},z^t),1-n_{t+1}(z_{t+1},z^t))$

The labor-leisure condition

$D_2u(c_t(z^t),1-n_t(z^t))=w_t(z^t)D_1u(c_t(z^t),1-n_t(z^t))$

c)

A recursive competitive equilibrium is a set of functions

quantities $g(\varpi,z,\overline{k}), G(z,\overline{k}), a(z',\varpi,z,\overline{k}), h(\varpi,z,\overline{k}), H(z,\overline{k})$

value $V(\varpi,z,\overline{k})$

prices $R(z,\overline{k}), w(z,\overline{k}), q(z',z,\overline{k})$

such that

(1) $V(\varpi,z,\overline{k})$ solves the household functional equation

$V(\varpi,z_i,\overline{k})=\max_{c,k',a'(z'),n}(u(c,1-n)+\beta\sum\limits_{z'\in Z}P_{i,j}V(\varpi'(z_j),z_j,\overline{k}'))$

subject to

$c+k'+\sum\limits_{z'\in Z}q(z',z,\overline{k})a'(z')=\varpi+w(z,\overline{k})n$

$\varpi'(z')=R(z',\overline{k}')k'+a'(z')$

$\overline{k}'=G(z,\overline{k})$

$a'(z')\geqslant \underline{a}$

$k'\geqslant 0$

$c\geqslant 0$

$0\leqslant n\leqslant 1$

(2) Prices are determined competitively

$R(z,\overline{k})=zD_1F(\overline{k},H(z,\overline{k}))+1-\delta$

$w(z,\overline{k})=zD_2F(\overline{k},H(z,\overline{k}))$

(3) consistency

$g(\overline{k},z,\overline{k})=G(z,\overline{k})$

$h(\overline{k},z,\overline{k})=H(z,\overline{k})$

$a(z',\overline{k},z,\overline{k})=0 \qquad\qquad \forall \overline{k},z,z'\in Z$

2.

a) Recursive Competitive Equilibrium is a set of functions

quantities $a(z', z, \varpi), h(\varpi, z), H(z)$

value $V(\varpi, z)$

prices $w(z), q(z', z)$

such that

(1) $V(\varpi, z)$ solves the household functional equation

$$V(\varpi, z_i) = \max_{c, a'(z'), n} \left(\frac{\left(c - \theta \frac{n^{1+\gamma}}{1+\gamma}\right)^{1-\sigma}}{1-\sigma} + \beta \sum_{z' \in Z} P_{i,j} V(\varpi'(z_j), z_j) \right)$$

subject to

$$c + \sum_{j=l,h} q(z_j, z_i) a'(z_j) = w(z_i) n + \varpi$$

$a'(z_j) \geqslant \underline{a}$

$\varpi'(z_j) = a'(z_j)$

$c \geqslant 0$

$0 \leqslant n \leqslant 1$

(2) Prices are determined competitively

$w(z_i) = z_i$

(3) Individual decisions are consistent with aggregate outcomes

$h(0, z) = H(z)$

$a'(z', 0, z) = 0 \qquad \forall z' \in Z$

Calculating equilibrium quantities of consumption and wealth. The household's first-order condition for consumption

$$\left(c - \theta \frac{n^{1+\gamma}}{1+\gamma}\right)^{-\sigma} = \lambda$$

where λ is lagrange for budget constraint

FOC for employment

$$\left(c - \theta \frac{n^{1+\gamma}}{1+\gamma}\right)^{-\sigma} \theta n^{\gamma} = \lambda w(z_i)$$

Combined with consumption first-order condition

$$\theta n^{\gamma} = w(z_i) \qquad n = \left(\frac{w(z_i)}{\theta}\right)^{\frac{1}{\gamma}}$$

In equilibrium $a'(z_j) = 0$

This implies $c = y = wn$

In equilibrium consumption $c = y = zn = z \left(\frac{z}{\theta}\right)^{\frac{1}{\gamma}} = \left(\frac{z^{1+\gamma}}{\theta}\right)^{\frac{1}{\gamma}}$

Note $\eta_w^n \equiv \frac{\partial n}{\partial w} \frac{w}{n} = \frac{1}{\gamma}$

This is the wage elasticity of labor supply. Higher γ makes labor supply unresponsive to changes in the real wage. As z rises, c, y and n increase. The relative price of leisure rises with z as it increase w. This causes a substitution from leisure to consumption. For this utility function, there are no wealth effects on labor supply.

b) In equilibrium $c = zn, n = \left(\frac{z}{\theta}\right)^{\frac{1}{\gamma}}$

$$c - \theta \frac{n^{1+\gamma}}{1+\gamma} = \left(\frac{z^{1+\gamma}}{\theta}\right)^{\frac{1}{\gamma}} - \frac{\theta}{1+\gamma} \left(\frac{z}{\theta}\right)^{\frac{1+\gamma}{\gamma}} = \frac{\gamma}{1+\gamma} \left(\frac{z^{1+\gamma}}{\theta}\right)^{\frac{1}{\gamma}}$$

The first order condition for the j-th Arrow-Security $q(z_j, z)$, where z is the current aggregate state and this claim will pay off next period if $z' = z_j$ is as follows.

$q(z_j, z_i) D_1 u(c(z_i), 1 - n(z_i)) = \beta P_{i,j} D_1 u(c(z_j), 1 - n(z_j))$

Since $D_1 u(c(z_i), 1 - n(z_i)) = \left(c - \theta \frac{n^{1+\gamma}}{1+\gamma}\right)^{-\sigma} = \left(\frac{\gamma}{1+\gamma}\left(\frac{z^{1+\gamma}}{\theta}\right)^{\frac{1}{\gamma}}\right)^{-\sigma}$

$q(z_j, z_i) = \beta P_{i,j} \left(\left(\frac{z_j}{z_i}\right)^{\frac{1+\gamma}{\gamma}}\right)^{-\sigma}$

A risk free bond has cost $q = \sum\limits_{j=1,2} q(z_j, z_i)$

If $z_i = z_1$ then $q(z_1) = \beta(p + (1-p)\left(\frac{z_2}{z_1}\right)^{\frac{-\sigma(1+\gamma)}{\gamma}})$

If $z_i = z_2$ then $q(z_2) = \beta(p + (1-p)\left(\frac{z_1}{z_2}\right)^{\frac{-\sigma(1+\gamma)}{\gamma}})$

As $z_1 < z_2$, and $\frac{\sigma(1+\gamma)}{\gamma} > 0, q(z_1) < \beta < q(z_2)$

This implies $\frac{1}{q(z_i)} \equiv R(z_i)$ has the property

$R(z_2) < \beta < R(z_1)$

When $z = z_1$, households are poorer than they will be in the future when z rises. To smooth consumption they would like to borrow from the future. The equilibrium real interest rate rises to offset this desire for debt and prevent $A(z_1) < 0$, where $A(z_i)$ is the aggregate stock of bonds which must equal zero in all periods. The converse is true when $z = z_2$, the fall in the real interest rate prevents saving for consumption in the future when income will be lower.

c)

$V(\varpi, z_i) = \max_{c,a',n}\left(\frac{\left(c - \theta\frac{n^{1+\gamma}}{1+\gamma}\right)^{1-\sigma}}{1-\sigma} + \beta \sum\limits_{z' \in Z} P_{i,j} V(\varpi', z_j)\right)$

subject to

$c + qa\prime = w(z_i)n + \varpi$

$\varpi' = a'$

$a\prime \geqslant \underline{a}$

$c \geqslant 0$

$0 \leqslant n \leqslant 1$

The first order condition for consumption and employment are unchanged. Further, in equilibrium $a = 0$ so $c = zn$. Again, $c - \theta\frac{n^{1+\gamma}}{1+\gamma} = \frac{\gamma}{1+\gamma}\left(\frac{z^{1+\gamma}}{\theta}\right)^{\frac{1}{\gamma}}$

The equilibrium price of non-contingent bonds again ensures $c = zn$ and $a' = 0$ when $a = 0$.

$q(z_i) D_1 u(c(z_i), 1 - n(z_i)) = \beta \sum P_{i,j} D_1 u(c(z_j), 1 - n(z_j))$

As before $D_1 u(c(z_i), 1 - n(z_i)) = \left(c - \theta \frac{n^{1+\gamma}}{1+\gamma}\right)^{-\sigma} = \left(\frac{\gamma}{1+\gamma}\left(\frac{z^{1+\gamma}}{\theta}\right)^{\frac{1}{\gamma}}\right)^{-\sigma}$

Thus, $q(z_i) = \beta \sum P_{i,j} \left(\left(\frac{z_j}{z_i}\right)^{\frac{1+\gamma}{\gamma}}\right)^{-\sigma}$. This is exactly the price of the risk-free bond. Incomplete market yields the same real allocation of consumption, outputand employment because there is only a representative household.

The single household implies a lack of actual risk-sharing opportunities. In equilibrium asset prices adjust to ensure the same real allocation under complete and incomplete markets. Thus, even with non-contingent claims, all possible sharing of risk has been implemented.