Advanced Microeconomics

Lecture 1: Utility and decision theory under certainty

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Outline

1. The consumer's problem
   - Overview
   - Behavioral Assumption
   - Questions
   - Solution
2. Marshallian and Hicksian demand functions
   - Utility maximisation
   - Roy's identity
   - Expenditure minimisation
   - Shepard's lemma
   - Duality relationship
3. Slutsky substitution
   - Slutsky matrix
   - Properties

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1. Overview

- We now have a good description of preferences. We assume the consumer's preference relation $\succeq$ is represented by a utility function u, that is differentiable, strictly increasing, and strictly quasi-concave.
- and have already mentioned our model of the feasible set: $B = \{\mathbf{x} \in \mathbb{R}^n_+ | \mathbf{p} \cdot \mathbf{x} < y\}$, where $y > 0$ and $\mathbf{p} \gg \mathbf{0}$.
- What is missing for a theory of consumer behavior is the behavioral assumption.

2. Behavioral Assumption

- We assume that the consumer chooses the alternative from her budget set that is best according to her preferences.
- This can be stated directly with preferences or with the utility function.
- The consumer chooses $\mathbf{x}^* \in B$ so that $\mathbf{x}^* \succeq \mathbf{x}$ for all $\mathbf{x} \in B$.
  $\Rightarrow$ Choose an alternative from the budget set that is at least as good as all other alternatives (there is no strictly better alternative).

2. Behavioral Assumption

- Given a utility function $u(\mathbf{x}) : \mathbb{R}^n_+ \rightarrow \mathbb{R}$, the consumer chooses a Walrasian demand correspondence $\mathbf{x}^*$ that solves

$$\mathbf{x}^*(\mathbf{p}, y) = \arg\max_{x \in \mathbb{R}^n_+} u(\mathbf{x})$$

$$\text{s.t. } \mathbf{p} \cdot \mathbf{x} \leq y$$

  $\Rightarrow$ Choose an alternative from the budget set that maximises utility subject to the constraint that total expenditure does not exceed income.

- Think of the Walrasian demand correspondence as the choice rule, induced by the preference $\succeq$ represented by u, for the budget set $B_{\{\mathbf{p},y\}} = \{\mathbf{x} : \mathbf{p} \cdot \mathbf{x} \leq y \text{ and } x_i \geq 0\}$, i.e. $\mathbf{x}^*(\mathbf{p}, y) = C_{\succeq}(B_{\{\mathbf{p},y\}})$.

2. Behavioral Assumption

- Hidden assumptions:
  - each commodity is perfectly divisible;
  - each commodity is consumed in weakly positive quantities;
  - prices are linear in consumption;
  - prices are strictly positive;
  - income is weakly positive;
  - and the total price of consumption cannot exceed income.
- But there are obvious scenarios that violate each of these assumptions.

3. Questions

- Is there a solution?
    - Yes - Weierstrass theorem.
- Is the solution unique?
    - Yes - Because B is convex and the utility function has been assumed to be strictly quasi-concave.
- How do we find the solution?

4. Solution

- Define the Lagrangian: $L(\mathbf{x}, \lambda) = u(\mathbf{x}) + \lambda(y - \mathbf{p} \cdot \mathbf{x})$.
- Necessary and sufficient condition for $\mathbf{x}^* \gg \mathbf{0}$ to solve the consumer's problem is that there exists $\lambda^* \geq 0$ such that:

$$\frac{\partial L}{\partial x_i} = \frac{\partial u(\mathbf{x}^*)}{\partial x_i} - \lambda^* p_i = 0, i = 1, 2, \ldots, n$$

  and

$$\frac{\partial L}{\partial \lambda} = y - \mathbf{p} \cdot \mathbf{x}^* = 0$$

- If the solution does not satisfy $\mathbf{x}^* \gg \mathbf{0}$ then the Kuhn-Tucker rather than the Lagrange approach must be used.

4. Solution

- The condition $y - \mathbf{p} \cdot \mathbf{x}^* = 0$ simply says that the consumer spends all her income.
  - This follows from the assumption of strict monotonicity.
- If the Lagrange parameter λ^* is strictly positive, the remaining conditions can be rewritten as

$$\frac{\partial u(\mathbf{x}^*)/\partial x_j}{\partial u(\mathbf{x}^*)/\partial x_k} = \frac{p_j}{p_k}$$

  for all $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, n$.
  - This says that at the optimum, the marginal rate of substitution between any two goods must be equal to the ratio of the goods' prices.

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1. Utility maximisation

- The solution to the consumer's problem depends on the parameters $\mathbf{p}$ and y of that problem.
- The corresponding function $\mathbf{x}^* = \mathbf{x}(\mathbf{p}, y)$ is known as the ordinary, or Marshallian demand function.
- This demand function summarises all relevant information about the consumer's demand behaviour.
- Question: What can be said about the Marshallian demand function?
  - In addressing this question we assume that the Marshallian demand function is differentiable. See Theorem 1.5 in JR for suitable conditions.
- If our model of consumer behaviour is correct, then demand behaviour must display certain observable characteristics.
  - These can be used to test the theory on the one hand and to improve empirical estimates when the theory is taking for granted on the other hand.

1. Utility maximisation

1. Budget Balancedness (Walras' Law (non-satiation)): $\mathbf{p} \cdot \mathbf{x}(\mathbf{p}, y) = y$ holds for all $\mathbf{p}, y$.
2. Homogeneity of Degree Zero: $\mathbf{x}(\mathbf{p}, y)$ is homogenous of degree zero in all prices and income: for all $\alpha > 0$, $\mathbf{x}(\alpha\mathbf{p}, \alpha y) = \mathbf{x}(\mathbf{p}, y)$. (Could you list other functions in Economics that must suggest similar property, e.g., HD 1?)

1. Utility maximisation

- Walras' Law indicates that there is no satiation and that at least one good is desirable (consumer spending exhausts income).
- Homogeneity of Degree Zero states that the consumer does not suffer from money illusion.
  - If prices and wealth are increased in the same proportion, then choices should be unchanged. This is so because you can easily notice that $B_{\{\mathbf{p},y\}} = B_{\{\alpha\mathbf{p},\alpha y\}}$.
  - Only relative prices and real income affect demand behaviour.

1. Utility maximisation

Recall the Envelop theorem:

- $v(p_x, p_y, I) = \max_{p_x x + p_y y \leq I} u(x, y)$ - the *indirect utility function*;
- $(x(p_x, p_y, I), y(p_x, p_y, I)) = \arg\max_{p_x x + p_y y \leq I} u(x, y)$ - the *Marshallian demand functions*;
- $e(p_x, p_y, I) = \min_{u(x,y) \geq v} p_x x + p_y y$ - the *expenditure function*;
- $(h(p_x, p_y, v), k(p_x, p_y, v)) = \arg\min_{u(x,y) \geq v} p_x x + p_y y$ - the *Hicksian (compensated) demand functions*.

1. Utility maximisation

Recall the Envelop theorem:

- We now have the following consequences of the envelope theorem: Roy's Identity, Marshallian Demand & Hicksian Demand

$$\frac{\partial v}{\partial p_x}(p_x, p_y, I) = -x(p_x, p_y, I) \cdot \frac{\partial v}{\partial I}(p_x, p_y, I)$$
$$\frac{\partial v}{\partial p_y}(p_x, p_y, I) = -y(p_x, p_y, I) \cdot \frac{\partial v}{\partial I}(p_x, p_y, I)$$

Proof?

$$\frac{\partial e}{\partial p_x}(p_x, p_y, v) = h(p_x, p_y, v)$$
$$\frac{\partial e}{\partial p_y}(p_x, p_y, v) = k(p_x, p_y, v)$$

Proof?

2. Roy's identity

- The indirect utility function specifies the highest level of utility a consumer with utility function u can obtain when faced with the budget set $B = \{\mathbf{x} \in \mathbb{R}^n_+ | \mathbf{p} \cdot \mathbf{x} \leq y\}$.
- The indirect utility function v is given by

$$v(\mathbf{p}, y) = u(\mathbf{x}(\mathbf{p}, y)) = \arg \max_{\mathbf{x} \in \mathbb{R}^n_+} u(\mathbf{x})$$

$$\text{s.t. } \mathbf{p} \cdot \mathbf{x} \leq y$$

- Suppose someone provides you with an indirect utility function. Question: Is this enough information to derive the Marshallian demand function?
- Yes, it is. The Marshallian demand function can be determined by Roy's Identity:

$$x_i(\mathbf{p}, y) = -\frac{\partial v(\mathbf{p}, y)/\partial p_i}{\partial v(\mathbf{p}, y)/\partial y}$$

3. Expenditure minimisation

- Given a utility function $u(\mathbf{x})$ representing a consumer's preference relation we may ask: what is the minimal expenditure that allows the consumer to obtain utility level u if prices are given by $\mathbf{p}$?
- To answer that question we consider the expenditure minimisation problem:

$$\arg\min_{\mathbf{x}\in\mathbb{R}^n_+} \mathbf{p}\cdot\mathbf{x} \quad \text{s.t. } u(\mathbf{x}) \geq u$$

- The solution to this problem $\mathbf{x}^* = \mathbf{x}^h(\mathbf{p}, u)$ is known as the Hicksian demand function.
- The corresponding minimal expenditure is given by the expenditure function

$$e(\mathbf{p}, u) = \mathbf{p}\mathbf{x}^h(\mathbf{p}, u) = \arg\min_{\mathbf{x}\in\mathbb{R}^n_+} \mathbf{p}\cdot\mathbf{x} \quad \text{s.t. } u(\mathbf{x}) \geq u$$

4. Shepard's lemma

- Given an expenditure function, the Hicksian demand function can be derived without solving the expenditure minimisation problem. This counterpart to Roy's Identity is known as Shepard's Lemma:

$$x_i(\mathbf{p}, u) = \frac{\partial e(\mathbf{p}, u)}{\partial p_i}$$

- Furthermore, it can be shown that the expenditure function is concave, which implies that the matrix with the entries $\partial x_i(\mathbf{p}, u)/\partial p_j$ is negative semidefinite and symmetric.
  - This means, in particular, that Hicksian demand functions have the property that the demand for every good is decreasing in its own price.

5. Duality relationship

- The Marshallian demand function and the Hicksian demand function are related as follows:

$$\mathbf{x}(\mathbf{p}, y) = \mathbf{x}^h(\mathbf{p}, v(\mathbf{p}, y))$$
$$\mathbf{x}^h(\mathbf{p}, u) = \mathbf{x}(\mathbf{p}, e(\mathbf{p}, y))$$

  Can you say this in words?

- The indirect utility function and the expenditure function are related as follows:

$$e(\mathbf{p}, v(\mathbf{p}, y)) = y$$
$$v(\mathbf{p}, e(\mathbf{p}, u)) = u$$

  Can you say this in words?

5. Duality relationship

- These relations imply that the expenditure function contains all the information required to obtain the Marshallian demand function.
- Furthermore, using the duality relations and Shepard's Lemma, it is not (too) difficult to show that with $u = v(\mathbf{p}, y)$ the relationship

$$s_{ij}(\mathbf{p}, y) = \frac{\partial x_i(\mathbf{p}, u)}{\partial p_j}$$

  holds
- Together with the observation that the matrix with entries $\partial x_i(\mathbf{p}, u)/\partial p_j$ is negative semidefinite and symmetric this establishes the properties of the Slutsky matrix.

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1. Slutsky matrix

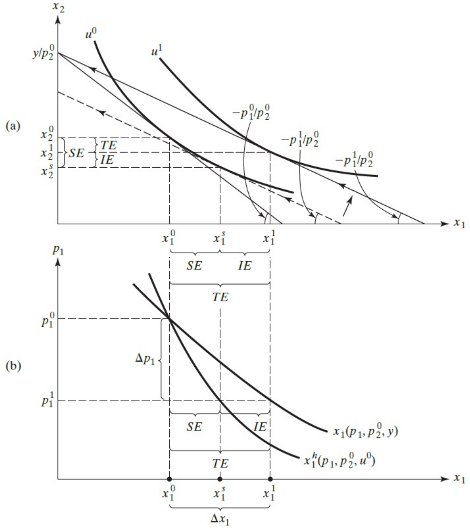
The Slutsky matrix $s(\mathbf{p}, y)$ with elements:

$$s_{ij}(\mathbf{p}, y) = \frac{\partial x_i(\mathbf{p}, y)}{\partial p_j} + x_j(\mathbf{p}, y)\frac{\partial x_i(\mathbf{p}, y)}{\partial y} \tag{1}$$

is symmetric and negative semidefinite.

2. Properties

Hicksian decomposition of an own-price change:



2. Properties

- Slutsky substitution effects: rewrite Eq. (1) as

$$\underbrace{\frac{\partial x_i(\mathbf{p}, y)}{\partial p_j}}_{TE} = \underbrace{s_{ij}(\mathbf{p}, y)}_{SE} \underbrace{- x_j(\mathbf{p}, y)\frac{\partial x_i(\mathbf{p}, y)}{\partial y}}_{IE} \tag{2}$$

  - TE: Total effect of an increase in price p_j on the demand for good i.
  - IE: Income effect of an increase in price p_j on the demand for good j.
  - SE: Substitution effect of an increase in price p_j on the demand for good i.
- s_{ii} is the (familiar?) substitution effect of a change in price p_i on the demand for good i.

2. Properties

Negative semidefiniteness and symmetry of the Slutsky matrix $s(\mathbf{p}, y)$: What does it mean?

- An $n \times n$ matrix A is negative semidefinite if for all vectors $z \in \mathbb{R}^n$, $\mathbf{z}^T A\mathbf{z} = 0$.
  - In particular, if A is negative semidefinite its diagonal elements satisfy $a_{ii} \leq 0$.
  - So negative semidefiniteness of the Slutsky matrix is a generalisation of the observation that the own-price substitution effects $s_{ii}(\mathbf{p}, y)$ are negative.
- An $n \times n$ matrix A is symmetric if $a_{ij} = a_{ji}$ holds for all i and j.
  - Hence, symmetry of the Slutsky matrix means that the substitution effect of an increase in the price of good j on the demand for good i is identical to the substitution effect of an increase in the price of good i on the demand for good j.

2. Properties

Negative semidefiniteness and symmetry of the Slutsky matrix $s(\mathbf{p}, y)$: What does it mean?

- Negativity of the own-price substitution effects $s_{ii}(\mathbf{p}, y)$ implies the law of demand.
- Recall the following definitions:
  - A good is normal if its consumption increases when income increases, holding prices constant.
  - A good is inferior if its consumption decreases when income increases, holding prices constant.
- Law of Demand: A decrease in the own price of a normal good causes the quantity demanded of the good to increase.

2. Properties

Negative semidefiniteness and symmetry of the Slutsky matrix $s(\mathbf{p}, y)$: Why is it true?

- There is no easy way to derive the properties of the Slutsky Matrix.
- The standard derivation (as given in JR) uses the duality between
  - the Marshallian demand function and indirect utility function (obtained from the utility maximisation problem) on the one hand and
  - the Hicksian demand function and the expenditure function (obtained from the expenditure minimisation problem) on the other hand.
- As this duality and the associated concepts are very useful in more applied work, we go through the key concepts and some of the results...
  - but will skip the proofs.