$V^*(k_0) = \max \sum_{t=0}^{\infty} \beta^t \log c_t$

subject to $c_t + k_{t+1} \leq Ak_t, t = 0, 1....$

$c_t, k_{t+1} \geqslant 0$

k_0 given

a Consider the infeasible plan

$c_t = Ak_t = A^2 k_{t-1} = ... = A^{t+1} k_0$

$k_{t+1} = Ak_t = A^2 k_{t-1} = ... = A^{t+1} k_0$

$\overline{V}(k_0) \equiv \sum_{t=0}^{\infty} \beta^t \log c_t = \sum_{t=0}^{\infty} \beta^t \log A^{t+1} k_0 = \sum_{t=0}^{\infty} \beta^t [(t+1) \log A + \log k_0] = \frac{\log A}{(1-\beta)^2} + \frac{\log k_0}{1-\beta}$

As $V^*(k_0) < \overline{V}(k_0)$, we know $V^*(k_0)$ is finite.

b $TV(k) = \max_{0 \leq k' \leq Ak} (\log(Ak - k') + \beta V(k'))$

Let $V^0(k) = 0$, then $k' = 0 \quad \forall k \in \mathbb{R}$ and $V^1 = TV^0 = \log Ak$

$V^2(k) = TV^1(k) = \max_{0 \leq k' \leq Ak} (\log(Ak - k') + \beta \log Ak')$

First order condition

$\frac{1}{Ak-k'} = \frac{\beta}{k'} \Rightarrow k' = \frac{\beta}{1+\beta} Ak$

Plug k' into the above equation, we can get $V^2(k) = \log \frac{1}{1+\beta} Ak + \beta \log \frac{\beta}{1+\beta} A^2 k = \beta \log \beta + (1+\beta) \log \frac{1}{1+\beta} + (1+2\beta) \log A + (1+\beta) \log k$

Similarly, $V^3(k) = TV^2(k) = \max_{0 \leq k' \leq Ak} (\log(Ak - k') + \beta V^2(k'))$

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$\frac{1}{Ak-k'} = \frac{\beta+\beta^2}{k'} \Rightarrow k' = \frac{\beta+\beta^2}{1+\beta+\beta^2} Ak$

So, $V^3(k) = \log \frac{1}{1+\beta+\beta^2} Ak + \beta[\log \frac{1}{1+\beta} A \frac{\beta+\beta^2}{1+\beta+\beta^2} Ak + \beta \log \frac{\beta}{1+\beta} A^2 \frac{\beta+\beta^2}{1+\beta+\beta^2} Ak] = (\beta + 2\beta^2)\log \beta + (1+\beta+\beta^2)\log \frac{1}{1+\beta+\beta^2} + (1+2\beta+3\beta^2)\log A + (1+\beta+\beta^2)\log k$

.....$\Rightarrow V^N(k) = (\sum_{s=1}^{N-1} s\beta^s)\log\beta - (\sum_{s=1}^{N-1}\beta^s)\log(\sum_{s=1}^{N-1}\beta^s) + (\sum_{s=1}^{N} s\beta^{s-1})\log A + (\sum_{s=1}^{N-1}\beta^s)\log k$ and $k' = \frac{(\sum_{s=1}^{N}\beta^s)}{\sum_{s=0}^{N}\beta^s} Ak$

Since the double geometric series $\sum_{s=1}^{N} s\beta^{s-1}$ has a limit $\frac{1}{(1-\beta)^2}$ (i.e. $\lim_{N\to\infty} \sum_{s=1}^{N} s\beta^{s-1} = \frac{1}{(1-\beta)^2}$)

$\lim_{N\to\infty} V^N(k) = \frac{\beta}{(1-\beta)^2}\log\beta + \frac{1}{(1-\beta)^2}\log A - \frac{1}{(1-\beta)}\log(1-\beta) + \frac{1}{(1-\beta)}\log k$

c $k' = \beta Ak$

The saving rate is β, and it raises linearly with the subjective discount factor. Higher values of β represent preference of a more patient household that discounts the future less. A relatively higher weight on future utility implies more saving.