高宏(上) lecture 1

1.1 The Solow model

predetermined
exogeneous

模型元素：变量(内生，外生，选择)
状态

~~[illegible]~~
约束条件

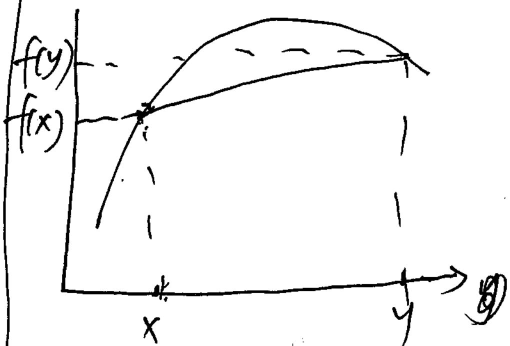
目标函数 or 行为方程 (Behavior function)

运动规律 (Law of motion)

concave function:

$f((1-t)x+ty) \geq (1-t)f(x)+tf(y)$

if $>$, then strict.



convex function:

变量：output y_t (产出，不一定有生产)

用于 C_t (消费)

I_t (投资)

形成 ⇓

$K_t \to$ 折旧 δ

$L \Rightarrow$ 用于生产 $F(K_t, L)$

$$\begin{cases} C_t + I_t = F(K_t, L) & \to \text{约束条件 Resource constraint} \\ K_{t+1} = (1-\delta)K_t + I_t & \to \text{Law of motion of capital (类似客观规律)} \\ I_t = sF(K_t, L) & \to \text{行为方程 (外生)} \end{cases}$$

Given $K_0 \Rightarrow F(K_t, L), I_t \Rightarrow C_t, K_{t+1} \Rightarrow$ next periods variable.

Assumptions:

and twice differentiable in K

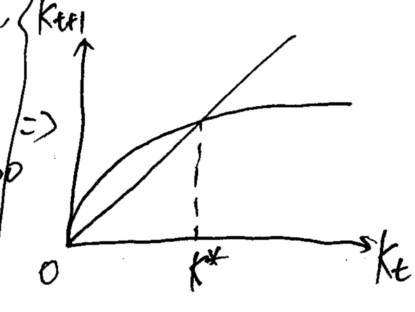
1. F is strictly increasing and concave in K

2. $F(0, L) = 0$

3. $F_K(0, L) > \frac{\delta}{s} \Rightarrow \frac{\partial K_{t+1}}{\partial K_t} > 1$, or $\underset{K_t \to 0}{K_{t+1} - K_t} = F_K(0,L) - \delta > 0$

4. $\lim_{K \to \infty} sF_K(K, L) + 1 - \delta < 1$

$\Rightarrow$



Example 1: $F(K, L) = AK^{\alpha}L^{1-\alpha}$ where $A > 0$ and $\alpha \in (0,1)$

Our assumptions ensure 存在唯一 K^*, s.t. if $K_t = K^*$, then $K_{t+1} = K^*$
i.e the economy remains at this steady state level

解出 K^*: $K^* = (1-\delta)K^* + sF(K^*, L)$

定理: $\exists K^* > 0$, s.t. $\forall K_0 > 0$, $K_t \to K^*$

proof: 1. $K_{t+1} - K_t = sF(K_t, L) - \delta K_t \equiv H(K_t)$

we have: $H(0) = 0$.

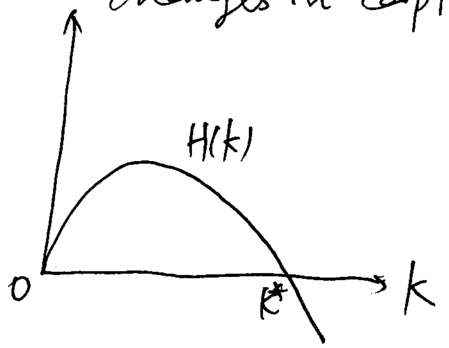
$$H'(k) = sF_k(k, L) - \delta \begin{cases} > 0, & \text{if } k \to 0 \\ < 0, & \text{if } k \to \infty \end{cases}$$

$H''(k) = sF_{kk}(k, L) < 0$, property of concave function if differencable

$H(K^*) = sF(K^*, L) - \delta K^* \equiv 0$.

changes in capital in the solow model

$\Rightarrow$



when $K < K^*$, then $H(k) > 0 \Rightarrow K_{t+1} > K_t$

when $k > K^*$, then $H(k) < 0 \Rightarrow K_{t+1} < K_t$

The solow model is used to understand how output grows over time

· Assumptions:

1. $u(\cdot)$ is strictly increasing $\Rightarrow$ resources will never be wasted. hence budget constraint will bind.

2. $\lim\limits_{c \to 0} u'(0) = +\infty \Rightarrow c_t > 0.$

set up the LaGrangean:

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \left[u(c_t) + \lambda_t \left(f(k_t) - c_t - k_{t+1} \right) + \mu_t k_{t+1} \right]$$

FOC: $[c_t]: \beta^t \left(u'(c_t) - \lambda_t \right) \leq 0, \quad t = 0, \cdots, T$

对选择变量

$$[k_{t+1}]: \begin{cases} -\beta^t \lambda_t + \beta^t \mu_t + \beta^{t+1} \lambda_{t+1} f'(k_{t+1}) \leq 0, \quad t = 0, \cdots, T-1 \\ -\beta^T \lambda_T + \beta^T \mu_T \leq 0 \end{cases}$$

Kuhn-Tucker conditions:

$$\mu_t k_{t+1} = 0, \quad t = 0, \cdots, T, \quad (-\beta^T \lambda_T + \beta^T \mu_T) \cdot k_{T+1} = 0.$$

$\mu_t \geq 0.$

$\cancel{k_{t+1} \geq 0.}$

$\lambda_t \geq 0, \quad t = 0, \cdots, T$

$\lambda_t \left(f(k_t) - c_t - k_{t+1} \right) = 0$

· we assumed $u'(c_t) > 0, \ \forall c \in \mathbb{R}_{++}$

$\Rightarrow$ (i) $u'(c_T) - \lambda_T = 0 \Rightarrow \lambda_T > 0$, since $\mu_T k_{T+1} = 0 \Rightarrow \lambda_T \cdot k_{T+1} = 0$

~~then from $-\beta^T \lambda_T + \beta^T \mu_T = 0 \Rightarrow \mu_T > 0$~~ $\Rightarrow k_{T+1} = 0$

$\Rightarrow$ no saving at T.

(ii), $\begin{cases} f(0) = 0 \\ \lim\limits_{c \to 0} u'(c) = +\infty \end{cases} \Rightarrow k_t > 0$ for all $t = 1, \cdots, T$, b/c if $k_{t+1} = 0 \Rightarrow c_{t+1} = 0$

1.2 Models of optimal growth

- Solow Model : exogenous saving rate → no theory
- we need : { Micro-founded extensions, endogenise savings decision ; explicitly maximizing an objective }

- Start from the finite case, $T+1$ periods
- A representative consumer
- Lifetime utility $U(c_0, c_1, \cdots, c_T)$ → this is an assumption
- we assume $U(c_0, c_1, \cdots, c_T) = \sum_{t=0}^{T} \beta^t u(c_t)$ additive separability.

  $\beta \in (0,1)$ discounting of utility.

  ⇒ a preference for consumption sooner than later

- a finite horizon Neoclassical growth model

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t)$$

→ 强调一下 state variable 和 choice/jump variable

subject to

$$c_t + k_{t+1} \leq F(k_t, L) + (1-\delta)k_t, \quad t = 0, \cdots, T$$

$$c_t \geq 0$$

$$k_{t+1} \geq 0,$$

$$k_0 \text{ given.}$$

- define $f(k) \equiv F(k, L) + (1-\delta)k$
- no markets
- central planner's problem.

from FOCs: we have.

$$-u'(c_t)+\mu_t+\beta u'(c_{t+1})f'(k_{t+1})=0,\ t=0,\cdots,T$$

since $k_{t+1}>0$ for $t=0,\cdots,T-1$

$\Rightarrow \mu_t=0$ for $t=0,\cdots,T-1$

$$\Rightarrow \underbrace{u'(c_t)}_{\substack{\text{marginal cost}\\ \text{of an additional}\\ \text{unit of output}\\ \text{allocated to investment.}}}=\underbrace{\underbrace{\beta u'(c_{t+1})}_{\substack{\text{marginal}\\ \text{value of}\\ \text{a unit rise}\\ \text{in future}\\ \text{consumption}\\ \text{discounted.}}}\underbrace{f'(k_{t+1})}_{\substack{\text{actual increase}\\ \text{in output from}\\ \text{investment.}}}}_{\text{discounted marginal value of investment.}} \rightarrow \text{Euler equation.}$$

given $c_t=f(k_t)-k_{t+1}$

Euler equation can be rewritten as.

$$u'(f(k_t)-k_{t+1})=\beta u'(f(k_{t+1})-k_{t+2})f'(k_{t+1}),\ t=0,\cdots,T-1$$

T such equations,

T+2 variables, $k_0,\cdots,k_{T+1}$

two endpoint conditions: $k_0 \Rightarrow$ predetermined

$k_{T+1}=0$

· have a solution.

and under certain conditions, there is unique solution.

· Assumption:

(i). $u(c)$ be concave

(ii). ~~b/c $f(k_t)$ is a concave function, then~~ the constraint set is convex in $\{c_t,k_{t+1}\}_{t=0}^{T}$

$\Rightarrow$ FOCs are sufficient, if U is strictly concave $\Rightarrow$ unique solution

1.2.1 An example with linear production and logarithmic utility.

- $u(c) = \log c$
- $f(k) \equiv F(k, L) + (1-\delta)k = Ak$. $\begin{cases} \alpha = 1 \\ \text{full depreciation or } A = a + 1 - \delta \end{cases}$

then, under these assumption, the FOCs are sufficient.

$$\frac{1}{Ak_t - k_{t+1}} = \frac{\beta A}{Ak_{t+1} - k_{t+2}}, \text{ where } t = 0, 1, \cdots, T-1$$

k_0 is given, $K_{T+1} = 0$.

$$\Rightarrow \underbrace{Ak_{t+1} - k_{t+2}}_{C_{t+1}} = \beta A (\underbrace{Ak_t - k_{t+1}}_{C_t}), \text{ for } t = 0, 1, \cdots, T-1$$

using the period-by-period budget constraints

$$C_0 + k_1 = Ak_0$$
$$C_1 + k_2 = Ak_1$$
$$\vdots$$
$$C_T + k_{T+1} = Ak_T$$

$$\Rightarrow \left.\begin{array}{l} C_0 + \frac{C_1 + k_2}{A} = Ak_0 \\ \frac{C_2 + k_3}{A} = k_2 \end{array}\right\} \Rightarrow C_0 + \frac{C_1}{A} + \frac{C_2}{A^2} + \frac{k_3}{A^2} = Ak_0$$

By induction, $\sum_{t=0}^{T} \frac{C_t}{A^t} + \frac{K_{T+1}}{A^T} = Ak_0 \rightarrow$ lifetime budget constraint

From Euler equation.

$$C_1 = \beta A C_0$$
$$C_2 = \beta A C_1 = (\beta A)^2 C_0$$
$$\vdots$$
$$C_t = (\beta A)^t C_0$$
$$\vdots$$
$$C_T = (\beta A)^T C_0$$

substituting the Euler equations into the life-time budget constraint

$$\sum_{t=0}^{T} \frac{(\beta A)^t C_0}{A^t} = A k_0$$

$$\Rightarrow C_0 = \frac{1-\beta}{1-\beta^{T+1}} A k_0$$

$$\Rightarrow k_1 = A k_0 - C_0 = \frac{\beta(1-\beta^T)}{1-\beta^{T+1}} A k_0$$

$$\vdots$$

$$k_t = (\beta A)^{t-1} (A k_0) \frac{\beta(1-\beta^{T-(t-1)})}{1-\beta^{T+1}} \quad \text{for } t = 0, 1, \cdots, T-1$$

$$\Rightarrow \frac{k_{t+1}}{k_t} = \beta A \frac{(1-\beta^{T-t})}{1-\beta^{T+1-t}}$$

define $G(t) \equiv \dfrac{1-\beta^{T-t}}{1-\beta^{T+1-t}}$

可证: $\underbrace{G(t+1) - G(t)}_{<0} = \dfrac{1-\beta^{T-t-1}}{1-\beta^{T-t}} - \dfrac{1-\beta^{T-t}}{1-\beta^{T+1-t}} = \dfrac{1-\frac{\beta^{T-t}}{\beta}}{1-\beta^{T-t}} - \dfrac{1-\beta^{T-t}}{1-\beta^{T-t}\cdot\beta}$

$$\frac{1-\beta^{T-t}}{1-\beta^{T-t}\cdot\beta} = \frac{\frac{1}{\beta} - \frac{\beta^{T-t}}{\beta}}{\frac{1}{\beta} - \beta^{T-t}}$$

$$\cancel{\frac{G(t+1)}{G(t)} = \frac{1-\beta^{T-t-1}}{1-\beta^{T-t}} \cdot \frac{1-\beta^{T+1-t}}{1-\beta^{T-t}}}$$

< 0

$\Rightarrow$ The rate of growth of capital falls over time

$$\frac{C_{t+1}}{C_t} = \beta A.$$

A is the real interest rate factor, it is 1+r where r is interest rate.

· if $\beta A > 1$, consumption grows

- if $\beta A < 1$, 〃 declines

· if $\beta A = 1$, 〃 constant.

1.2.2 example 2: strictly concave production with logarithmic utility

· capital share is less than one. $\alpha < 1$

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t)$$

s.t.

$$c_t + k_{t+1} \leq f(k_t), \quad t = 0, \cdots, T$$

$$c_t \geqslant 0$$

$$k_{t+1} \geqslant 0.$$

k_0 given.

as $u(c)$ is strictly increasing, the resource constraint binds: $c_t + k_{t+1} = f(k_t)$

a simpler way of solving the planning problem:

$$\max_{\{k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(f(k_t) - k_{t+1})$$

FOC: $[k_{t+1}]$: $-\beta^t u'(f(k_t)-k_{t+1}) + \beta^{t+1} u'(f(k_{t+1})-k_{t+2}) f'(k_{t+1}) = 0, \quad t=0, \cdots, T-1$

Now assume $u(c) = \log c$, $f(k) = Ak^{\alpha}$, $\alpha \in (0,1)$, $\delta = 1$

$$\Rightarrow \frac{1}{Ak_t^{\alpha} - k_{t+1}} = \frac{\alpha \beta A k_{t+1}^{\alpha-1}}{Ak_{t+1}^{\alpha} - k_{t+2}}, \quad t = 0, 1, \cdots, T-1$$

作业 解出 c_t, k_t, $t = 0, 1, \cdots, T$ 后页部分提前.

we solve for the optimal choice of capital at each date, starting with the last, $t = T-1$

$$\frac{1}{Ak_{T-1}^{\alpha} - k_T} = \frac{\alpha \beta A k_T^{\alpha-1}}{Ak_T^{\alpha} - k_{T+1}}, \text{ as we know } k_{T+1} = 0$$

we have $\frac{1}{Ak_{T-1}^{\alpha}-k_T}=\frac{2\beta}{k_T}$ $\to$ solve for k_T in k_{T-1}

$\Rightarrow k_T = 2\beta(Ak_{T-1}^{\alpha}-k_T)$ ~~$\to$ solve for k_T~~

similarly, we can rearrange the generic FOC.

$Ak_{t+1}^{\alpha}-k_{t+2} = 2\beta A k_{t+1}^{\alpha-1}(Ak_t^{\alpha}-k_{t+1})$ for $t=0,1,\cdots,T-2$

~~when $t=T-2$~~

then $Ak_{T-1}^{\alpha} - \frac{2\beta A k_{T-1}^{\alpha}}{1+2\beta} = 2\beta A k_{T-1}^{\alpha-1}(Ak_{T-2}^{\alpha}-k_{T-1})$

use this to solve for k_{T-1} in k_{T-2}

$\vdots$

$$k_{t+1} = \frac{2\beta+\cdots+(2\beta)^{T-t}}{1+2\beta+\cdots+(2\beta)^{T-t}}Ak_t^{\alpha}$$

$$k_t = \frac{2\beta(1-(2\beta)^{T+1-t})}{1-(2\beta)^{T+2-t}}Ak_{t-1}^{\alpha},\ t=1,\cdots,T.$$

$$c_t = Ak_t^{\alpha}-k_{t+1} = \frac{1-2\beta}{1-(2\beta)^{T+1-t}}Ak_t^{\alpha}.$$

given k_0, we solve for all the paths.

The infinite horizon case

- we have assumed that $T<\infty$, what about the infinite horizon problem as $T\to\infty$?
- The problem can be written as:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

s.t.

$$c_t + k_{t+1} \leq f(k_t),\ t=0,\dots$$

$$c_t \geq 0$$

$$k_{t+1} \geq 0$$

$$k_0 \text{ given}$$

- How can we solve this problem?
- Let's first see what our finite horizon decision rules offer as $T\to\infty$
- Please note that this is a heuristic approach as we have no proof that the limit of the finite horizon problem will be the solution to the infinite horizon problem.

- let $T\to\infty$, we find.

$$k_{t+1} = \alpha\beta A k_t^{\alpha}$$

$$c_t = (1-\alpha\beta)A k_t^{\alpha}$$

- what does this imply for the dynamic path of this economy?

Answer: there is a globally convergent steady state level of capital.

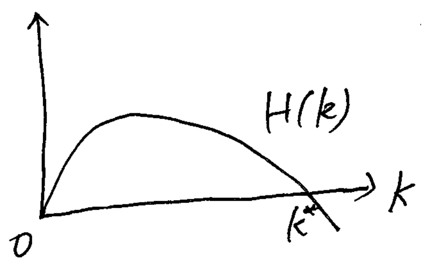
Proof: consider $k_{t+1}-k_t = H(k_t) \equiv \alpha\beta A k_t^{\alpha} - k_t$.

Note that $H(0)=0$, $H'(k) = \alpha^2\beta A k_t^{\alpha-1} - 1$

$\Rightarrow \lim_{k\to 0} H'(k) = \infty$, $\lim_{k\to\infty} H'(k) = -1$

and $H''(k) = \alpha^2(\alpha-1)\beta A k^{\alpha-2} < 0$ since $\alpha\in(0,1)$.

类似之前的结论：



$\Rightarrow \exists$ a unique $k^* > 0$ s.t. $H(k^*) = 0$, i.e. $k_{t+1} = k_t = k^*$

if we solve for k^*, we get

$$k^* = (\alpha\beta A)^{\frac{1}{1-\alpha}}$$

· k^* 有哪些性质？

(i). a rise in β or A increases the long run level of capital meaning larger steady state output and consumption

(ii). 类似 Solow model，我们可以得到一个 saving rate, s. 不同的是，这一 saving rate 是内生的

$$s = \frac{y-c}{y} = \alpha\beta$$

· if $u(c) = \log c$, a rise in A (Total factor productivity) does not change the saving rate.

because, income and substitution effects exactly cancel each other.

· 注意：以上方法并不是求解 infinite horizon 的正式方法。因为我们的解法相当于先解了一个有限期的规划问题，再将 $T \to \infty$，而数学中 lim max 和 max lim 并不一定可以随意调换顺序.

- 什么是可解出 infinite horizon problem 的充分条件呢?
  - Euler equation or first-order condition for capital
  - transversality condition: $\lim_{t \to \infty} \beta^t u'(c_t) f'(k_t) k_t = 0$.

  严格数学证明请参考: section 4.5 of Stokey and Lucas (1989)

  transversality condition: the discounted ~~value~~ shadow value of the state variable must be zero at infinity.

  Intuitively, this prevents overaccumulation of wealth.

- next, we show that the optimal choices we found using the finite horizon problem do satisfy these two conditions.

- 首先, set up the LaGrangean

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t [u(c_t) + \lambda_t (f(k_t) - c_t - k_{t+1})]$$

FOC: $\dfrac{\partial L}{\partial c_t}$: $u'(c_t) = \lambda_t$

$\dfrac{\partial L}{\partial k_{t+1}}$: $-\lambda_t + \beta \lambda_{t+1} f'(k_{t+1}) = 0$.

$$\Rightarrow -u'(c_t) + \beta u'(c_{t+1}) f'(k_{t+1}) = 0.$$

注意: 由于和 finite horizon 类似的原因, 我们可以证明

$c_t > 0$, $k_{t+1} > 0$, $c_t + k_{t+1} = f(k_t)$, for $t = 0, \dots$

因此, 我们在 set up LaGrangean 的时候, 忽略了 $\mu_t k_{t+1}$, $\theta_t c_t$ 等项.

并且, 由于本问题不存在 endpoint, 因此所有解都是内部解, 从而我们省略了库恩-塔克条件.

(i). 将 $u(c)=\log c$，$f(k)=Ak^{\alpha}$ 代入 Euler equation.

$$\frac{1}{Ak_t^{\alpha}-k_{t+1}}-\frac{\alpha\beta Ak_{t+1}^{\alpha-1}}{Ak_{t+1}^{\alpha}-k_{t+2}}=0.$$

将我们之前猜测的解 $k_{t+1}=\alpha\beta Ak_t^{\alpha}$ 代入上式，

可知其满足 Euler equation.

(ii) 第二，我们的解是否满足 transversality condition 呢？

$$\lim_{t\to\infty}\beta^t u'(c_t)f'(k_t)\cdot k_t$$

$$=\lim_{t\to\infty}\beta^t\frac{1}{(1-\alpha\beta)Ak_t^{\alpha}}\cdot(\alpha Ak_t^{\alpha-1})k_t$$

$$=\lim_{t\to\infty}\beta^t\frac{\alpha}{1-\alpha\beta}$$

$$=0$$

满足！

可见，在我们这个问题下，我们通过 $T\to\infty$ 得到的解确实是 infinite horizon problem 的解.

• 这个解的一个重要性质：k_{t+1} 只与 k_t 有关，与 $k_{t-1}, k_{t-2}\cdots$ 均无关.

i.e. k_t captures all past information relevant to the choice facing the planner in period t.