A MOVING EQUILIBRIUM OF DEMAND AND SUPPLY

SUMMARY

Elasticity of demand and flexibility of prices. — Relative cost of production and relative efficiency of organization, 359. — Laws of relative cost and relative return as contrasted with laws of cost and laws of return, 360. — Typical equations to the law of supply, 364. — The statistical derivation of the law of supply, 367. — A moving equilibrium of demand and supply, 368.

THE fundamental symmetry with which demand and supply coöperate in the determination of price suggests the possibility and indicates the desirability that the typical equation to the supply curve may be of the same general form as the equation to the curve of demand. In an earlier paper [1] I have shown that three types of equations to the law of demand may be derived from certain very simple hypotheses with regard to elasticity of demand. The definition of elasticity of demand is $\eta = \frac{y}{x}\frac{dx}{dy}$, where y = the price per unit of commodity, and x = the amount of commodity that is taken at price y. If ϕ, the flexibility of prices, is defined as the reciprocal of η, then three types of equations of demand may be deduced by placing

$$\phi = a,$$
$$\phi = a + \beta x,$$
$$\phi = a + \beta x + \gamma x^2.$$

In the subsequent development of this paper we shall

1 "Elasticity of Demand and Flexibility of Prices," Journal of the American Statistical Society, March, 1922.

find that, in fact, the actual practice of business and the exigencies of economic theory concur in leading to the conclusion that the same typical equations reproduce the essential characteristics of both demand and supply.

The flexibility of prices ϕ, or its reciprocal the elasticity of demand η, is a summary description of the equation of demand. The manner of its variation is all we need to know in order to solve most economic problems as far as their solution turns upon the law of demand. In the law of supply the analogue of ϕ is κ, the coefficient of relative cost of production, which, we shall find, is a summary description of constant, increasing and diminishing relative cost. The empirical determination of ϕ and κ, which will be exemplified in the sequel, makes possible the practical, concrete treatment of problems of demand and supply.

While ϕ and κ epitomize the information that is employed in the solution of most questions relating to demand and supply, in some cases it is necessary to resort to other coefficients. Long ago Cournot gave criteria of laws of cost that lead to types of supply curves different from those that are derived by means of κ. If the total demand curve is treated in a manner similar to that in which Cournot treated the total cost curve, it is possible to obtain demand curves corresponding in type to the supply curves derived from the Cournot criteria of cost.

The traditional, statical theory of economic equilibrium acquires a new value in the light of these results. The concrete determination of the laws of supply and demand leads to the conception of a moving equilibrium of demand and supply, and this novel point of view will necessitate a revaluation and, I hope, a higher appreciation, of the problems and technique of statical equilibrium.

Relative Cost of Production (κ) and Relative Efficiency of Organization (ω)

Economic phenomena must be treated realistically, and a realistic treatment considers the phenomena in a state of flux. In consequence of the ceaseless changes in the conditions of business the representative entrepreneur is constantly asking, and constantly answering, the question whether he shall increase or diminish the amount of his physical output. His decisions are made from the point of view of the probable movement of demand, which lies beyond his control, and from the point of view of the efficiency of his own organization, which he is capable of modifying. This latter phase of the problem relates to supply, and the criterion upon which his decision is made should have a technical name. I propose to call it either the coefficient of relative cost of production (κ), or the coefficient[2] of relative efficiency of organization (ω). The two quantities are related by the equation $\kappa = \frac{1}{\omega}$.

Relative cost of production, κ, may be defined as the ratio of the relative change of the total cost to the relative change in the total production. If y equals the total cost of production and x, the total amount of production, the symbolic representation of relative cost of production is $\kappa = \frac{\Delta y}{y} \Big/ \frac{\Delta x}{x}$, or at the limit, $\kappa = \frac{x}{y}\frac{dy}{dx}$. If, using Cournot's symbol, $y = \phi(x)$, $\kappa = \frac{x\,\phi'(x)}{y}$. This criterion gives the information desired by the

2. The quantity ω, the coefficient of relative efficiency of organization, is the same as the quantity which Professor Bowley, following W. E Johnson, has called the "efficiency of money." A. L. Bowley, The Mathematical Groundwork of Economics, p. 22.

entrepreneur. He wishes to know, if he increase his output, whether the relative increase in total cost will be greater than, equal to, or less than the relative increase in the output. That is, he wishes to know whether $\kappa = \dfrac{x\,\phi'(x)}{y} \gtreqless 1$.

Relative efficiency of organization, ω, gives, under a different form, the same information as the relative cost of production, κ. The criterion ω is defined as the ratio of the relative change in total production to the relative change in total cost. Symbolically $\omega = \dfrac{dx}{x} \Big/ \dfrac{dy}{y} = \dfrac{y}{x\,\phi'(x)}$. The information desired by the entrepreneur is whether $\omega \gtreqless 1$.

Since $\kappa = \dfrac{1}{\omega}$, it is obvious that if the value of either κ or ω is known, the other may be determined.

Laws of Relative Cost and Relative Return as Contrasted with Laws of Cost and Laws of Return

The description of laws of return in mathematical form was first given by Cournot. Cournot [3] shows that if $y = \phi(x)$ is the expression for the total cost of production, then there are three types of laws of cost or return according as $\phi''(x) \gtreqless 0$. He avoided many difficulties by contenting himself with a mathematical definition of the laws without passing on to identify them by name as the law of diminishing return, the law of constant return, and the law of increasing return. Without using the customary unprecise designations he amply made good his claim that the condition whether

3. Cournot, Recherches sur les principes mathématiques de la théorie des richesses, p. 66, §§ 29–30. He considers, in addition to the above three cases, a fourth, where $\phi(x)$ is a constant.

$\phi''(x) \gtreqless 0$ exerts very great influence on the principal problems of economic science.[4] It would be conducive to clearness and accuracy if the Cournot criterion $\phi''(x)$ were regarded as the criterion of laws of cost or laws of return.

In the preceding section the coefficient of relative return, or of relative cost, was defined as

$$\kappa = \frac{x}{y}\frac{dy}{dx} = \frac{x\,\phi'(x)}{y}.$$

According as $\kappa \gtreqless 1$ we have to do with the law of increasing relative cost, or of diminishing relative return; the law of constant relative cost, or of constant relative return; the law of decreasing relative cost, or of increasing relative return. Just as Cournot's criterion $\phi''(x)$ is the criterion of laws of cost or of return, so the coefficient κ may be regarded as the criterion of laws of relative cost or of relative return.

The distinction between the two conceptions may be illustrated. Suppose, for example, that it is required to show the difference between the law of diminishing return and the law of diminishing relative return.

The two criteria are

$$\phi''(x) > 0 \quad .. \quad \text{(i)}$$
$$\kappa > 1 \quad .. \quad \text{(ii)}.$$

But, by definition, $\kappa = \dfrac{x\,\phi'(x)}{y}$, and, as a result of the

4. Cournot, Recherches, p. 65 "Dans la suite de nos recherches, nous auront rarement occasion de considérer directement la fonction ϕ (D) [the ϕ (x) of the text], mais seulement son coefficient différentiel $\dfrac{d\,\phi\,(D)}{d\,D}$ que nous désignerons par la caractéristique ϕ' (D). Ce coefficient différentiel est une nouvelle fonction de D, dont la form exerce la plus grande influence sur la solution des principaux problèmes de la science économique

La fonction ϕ' (D) est, selon la nature des forces productrices et des denrées produites, susceptibles de croître ou de décroître quand D augmente."

inequality (ii), the law of diminishing relative return gives the information that

$$\phi'(x) > \frac{y}{x}, \text{ or } \phi'(x) > \frac{\phi(x)}{x} \quad \ldots \text{ (iii).}$$

That is to say, in stating the law of diminishing relative return we assume that the marginal cost of production is greater than the average cost of production.

The inequality in (iii) may be written

$$x\,\phi'(x) > \phi(x) \ldots \text{ (iv).}$$

This states the well-known proposition in economic theory that where the law of diminishing relative return prevails, the ordinate of the integral supply curve is greater than the corresponding ordinate of the total cost curve. Economic theory has also led to the conclusion that under these conditions the slope of the supply curve is greater than the slope of the cost curve. This means that

$$\frac{d}{dx}\left\{x\,\phi'(x)\right\} > \phi'(x), \;.\;.\; \text{(v) or}$$

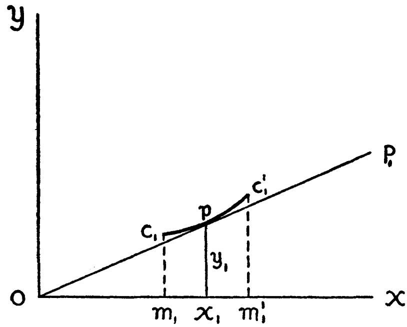
$$\phi'(x) + x\,\phi''(x) > \phi'(x), \text{ or}$$

$$\phi''(x) > 0 \ldots \text{ (vi).}$$

Here we reach a conclusion in case of diminishing return and diminishing relative return which, by similar reasoning with regard to constant and increasing relative return, may be proved to be general, namely, where a given condition of κ holds, so likewise does the corresponding condition with reference to $\phi''(x)$ hold. In this particular case we started with $\kappa > 1$ in (ii) and reached $\phi''(x) > 0$, in (vi). Where the law of diminishing relative return exists, there likewise does the law of diminishing return exist. The latter conception is at least as broad as the former: we shall now proceed to show that it is broader.

Auspitz and Lieben have described how a total cost curve is made up of bits of individual expense curves. Following their method of exposition,[5] let us suppose that an individual producer has such a plant that he can produce, with moderate variations of outlay, an amount of commodity ranging between m_1 and m_1' (See Fig. 1), and let us assume, further, that the total cost

FIGURE 1



of producing amounts between m_1 and m_1' is given by the respective ordinates of the curve c_1c_1', which, by hypothesis, is assumed to be throughout its extent convex to the axis of x. If $\phi_1(x)$ is put for the integral cost curve to this producer (1), the convexity of the curve requires that $\phi_1''(x) > 0$, and, consequently, that the business should be subject to the law of diminishing return between the limits $x = m_1$, $x = m_1'$. Suppose now that, still following the method of Auspitz and Lieben, the price of the commodity is given by the price line oP_1. At the price *tan* P_1ox producer (1) could not afford to produce less nor more than ox_1, for which the total cost would be y_1.

5. Auspitz und Lieben, Untersuchungen uber die Theorie des Preises, p. 112, Fig. 27 a.

But when ox_1 units of the commodity are produced we have $\frac{dy}{dx} = \frac{y_1}{x_1}$. For any point on the cost curve between c_1 and p we have $\frac{dy}{dx} = \phi_1'(x) < \frac{y}{x}$, and, consequently, by the criterion κ the industry is subject between these limits to the law of increasing relative return. For any point on the cost curve between p and c_1', we have $\frac{dy}{dx} = \phi_1'(x) > \frac{y}{x}$, and, consequently, by the criterion κ, the business between these limits is subject to the law of diminishing relative return.[6]

The finding in this particular case of diminishing return is general: The criterion $\phi''(x)$ is more inclusive than the criterion κ: where κ occurs in a particular form the corresponding form of $\phi''(x)$ always occurs, but where $\phi''(x)$ exists in any particular form the corresponding condition of κ may or may not be fulfilled.

Typical Equations to the Law of Supply

It is desirable to select the typical equation to the law of supply so that the constants in the equation shall reflect the connection of supply with the laws of cost. The types of the supply equations will, therefore, obviously vary according as the values of $\phi''(x)$ or of κ are taken as the criteria of laws of cost.

We shall first derive the equations of supply by means of $\phi''(x)$. The simplest possible assumptions as to the character of $\phi''(x)$ are summarized in (vii)

$$\left.\begin{aligned} \phi''(x) &= a \\ \phi''(x) &= a + bx \\ \phi''(x) &= a + bx + cx^2 \end{aligned}\right\} \quad .. \text{(vii)}.$$

6. This point, as far as I am aware, was first made by Professor Edgeworth. Economic Journal, June 1899, p 294.

If $\phi''(x) = a$, then $\phi'(x) = ax + b$, and the variation of marginal cost is described by a straight line. When the law of diminishing return dominates industry, $\phi'(x) = ax + b$ is the equation to the supply curve. In that case, if p_s be put for the supply price per unit of commodity, the supply equation is $p_s = ax + b$. When the law of constant return dominates the industry, $a = o$, $\phi'(x) =$ a constant, and the supply price is $p_s = \frac{\phi(x)}{x}$. When the law of increasing return prevails, the supply price [7] is likewise $p_s = \frac{\phi(x)}{x}$.

More complex supply equations could be derived in a similar manner from the other assumptions in (vii). But in the first attempts to deal concretely with the law of supply very great advantages will be secured by retaining the simple hypothesis that $\phi''(x) = a$, and, in case of diminishing return, that the law of supply is a straight line, $p_s = ax + b$.

The criterion κ leads to another useful type of supply curves. The simplest possible assumptions with regard to κ are summarized in (viii)

$$\left.\begin{array}{l}\kappa = \alpha \\ \kappa = \alpha + \beta x \\ \kappa = \alpha + \beta x + \gamma x^2\end{array}\right\} \ldots \text{(viii)}.$$

Suppose that $\kappa = \alpha$, a constant.

Since by definition $\kappa = \frac{x}{y}\frac{dy}{dx}$, the hypothesis becomes $\frac{x}{y}\frac{dy}{dx} = \alpha$, or $\frac{dy}{y} = \alpha\frac{dx}{x}$. Integrating, we have

$$y = \text{A}x^{\alpha} \equiv \phi(x) \ldots \text{(ix)},$$

7. Marshall, Principles of Economics, 4th edition, p. 539, note 1 Fig. 36.

which is the law of the variation of total cost with the amount of commodity that is produced.

The derivation of the equation to the supply curve from the equation to the cost curve will vary according as $\kappa \gtreqless 1$, that is, according as the business is subject to increasing, constant, or diminishing relative cost.

When the industry is subject to increasing relative cost, a is greater than unity. The supply price p_s is equal to the marginal cost of production, $\phi'(x)$, and the equation to the supply curve is

$$p_s = \phi'(x) = a\,Ax^{a-1} \ldots \text{(x)}.$$

When the industry is subject to constant relative cost, $a = 1$, and the supply price per unit of commodity is equal to the mean cost of production

$$p_s = \frac{\phi(x)}{x} = Ax^{a-1} = A, \text{ since } a = 1 \ldots \quad \text{(xi)}.$$

When the industry is subject to decreasing relative cost, $\kappa < 1$, and the supply price per unit of commodity is equal to the mean cost of production

$$p_s = \frac{\phi(x)}{x} = Ax^{a-1} \quad \ldots \text{(xii)}.$$

In the preceding discussion the laws of supply have been deduced from the laws of cost, and the equations (x) (xi) (xii) show that an expression of the type $y = Ax^a$ is an appropriate form for the law of supply whatever may be the constant value of κ. It will therefore be allowable to take this type of curve to describe the law of supply directly and then to deduce from it the corresponding law of cost. Further on we shall find a case where it is possible to obtain concretely the supply curve, from which the corresponding cost curve may be deduced, when it would be impossible to obtain directly the equation to the cost curve.

Just as the typical equation $y = Ax^a$ has been derived from the simplest hypothesis in (viii) with regard to the value of κ, so, by similar reasoning, more complex equations may be derived from the other hypotheses in (viii). If, for example, it be assumed that the variation of κ is linear we have $\kappa = \frac{x}{y}\frac{dy}{dx} = \alpha + \beta x$ and the typical equation to the cost curve, from which the supply curve may be deduced is

$$y = \phi(x) = A\, x^{\alpha} e^{\beta x} \ldots \text{(xiii)}.$$

The Statistical Derivation of the Law of Supply

In discussing the law of demand I have shown elsewhere that when the typical equation to the curve has been agreed upon, the empirical law of demand may be ascertained either by the method of trend ratios or by the method of link ratios.[8] The same methods could be used in deriving the empirical law of supply, but in the following illustration the method of trend ratios alone will be employed.

Table I contains the data that are sufficient to deduce the curves both of demand and of supply. The raw material is given in columns II and III. To these figures parabolas of the second degree were fitted by the method of least squares in order to obtain the respective secular trends. Columns IV and V give the ratios of the observations to their corresponding trends, and these two columns suffice to ascertain the law of demand for potatoes. The law of supply is derived from columns VI and VII. In column VI the price trend ratios that are given in column IV are advanced

8. The method of trend ratios was described in an article on "Elasticity of Demand and Flexibility of Prices," in the Journal of the American Statistical Association, March, 1922.

The method of link ratios was first presented in Economic Cycles, 1914, and subsequently in Forecasting the Yield and the Price of Cotton, 1917, and in "Empirical Laws of Supply and Demand," Political Science Quarterly, December 1919.

TABLE I. — DATA FOR COMPUTING THE CURVES OF DEMAND AND SUPPLY. THE ANNUAL PRODUCTION OF POTATOES AND THEIR FARM PRICES, IN THE UNITED STATES

I Year	II December Farm prices (cents per bushel)	III Production (millions of bushels)	IV Price trend ratio	V Production trend ratio	VI Price ratio of preceeding year	VII Production ratio of current year
1900	43.1	211	0.794	0.988	.	.
1901	76 7	188	1.397	0.810	0.794	0.810
1902	47.1	285	0.850	1.144	1.397	1.144
1903	61.4	247	1.094	0 932	0 850	0 932
1904	45 3	333	0.798	1 191	1 094	1.191
1905	61 7	261	1 073	0 891	0.798	0.891
1906	51.1	308	0.877	1.011	1.073	1.011
1907	61 8	298	1.044	0 945	0.877	0 945
1908	70.6	279	1.179	0 860	1.044	0.860
1909	54 1	389	0.887	1.170	1.179	1.170
1910	55 7	349	0.898	1.029	0.887	1.029
1911	79 9	293	1.268	0 850	0.898	0 850
1912	50 5	421	0.788	1.207	1.268	1.207
1913	68.7	332	1.054	0.945	0.788	0.945
Mean		. .	1 000	0.998	0.996	0.999

one year; the production trend ratios in column VII are the same, year for year, as those given in column V.

The correlation of the data for the law of demand for potatoes (columns IV and V) is $r = -.95$; the correlation of the data for the law of supply (columns VI and VII) is $r = +.80$. In the following section graphs are drawn for two types of curves.

A Moving Equilibrium of Demand and Supply

In the preceding section we have seen how the equation to the law of supply when it is deduced from the criterion $\phi''(x) = a$ is $p_s = a + bx$, and when it is deduced from $\kappa = \alpha$ is of the type $p_s = Ax^{\alpha}$. Exactly corresponding types of equations may be used to describe demand. We know that when the flexibility of prices, ϕ, which is the reciprocal of the elasticity of demand, η, is placed equal to α the type of the demand curve is $p_d = Ax^{\alpha}$.

A linear equation to the law of demand may be obtained in a manner corresponding to that in which, by the use of Cournot's criterion $\phi''(x)$, the linear equation to supply was obtained from the equation to the cost curve. For example, let $F(x)$ be put for the money measure of the total utility of the commodity. Types of demand curves corresponding to the supply curves derived from $\phi''(x)$ will then be obtained by making the following hypotheses as to the variation of $F''(x)$

$$\left.\begin{aligned} F''(x) &= a_1 \\ F''(x) &= a_1 + b_1 x \\ F''(x) &= a_1 + b_1 x + c_1 x^2 \end{aligned}\right\} \ldots \text{(xiv).}$$

The demand equations deduced from (xiv) will then correspond with the supply equations derived from (xv)

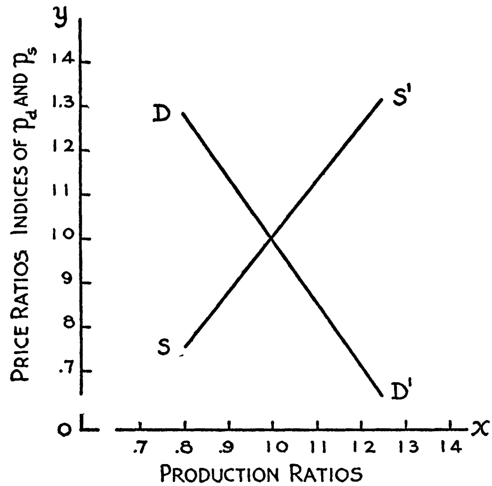
$$\left.\begin{aligned} \phi''(x) &= a_2 \\ \phi''(x) &= a_2 + b_2 x \\ \phi''(x) &= a_2 + b_2 x + c_2 x^2 \end{aligned}\right\} \;. \text{(xv).}$$

If, for example, the assumption is made that $F''(x) = a_1$ and $\phi''(x) = a_2$, the equation to the law of demand is $p_d = a_1 x + b_1$, and the equation to the supply curve is $p_s = a_2 x + b_2$. Figure 2 shows these types of equations fitted to the data given, respectively, in columns IV and V and in columns VI and VII of Table I.

In addition to the great technical advantage of having the equations to demand and supply in the simple linear form, there is the theoretical gain of having the graphs of supply and demand pass exactly through the mean of the system of points on the scatter diagram. When the straight lines are fitted by the method of least squares, they must pass through the mean of the system. And when the production ratios and price ratios are deduced from parabolic trends fitted to the

FIGURE 2. A MOVING EQUILIBRIUM OF DEMAND AND SUPPLY. POTATOES.

DEMAND: $p_d = -1.425x + 2.425$, origin (0,0);
SUPPLY: $p_s = 1.2224x - .2224$, origin (0,0).



data by the method of least squares, the means of these ratios are, within the limit of error, equal to unity. In the particular case of the data referring to potatoes, Table I shows that this latter statement is true.

In consequence of the graphs of supply and demand passing through the point whose coördinates are (1.0, 1.0) the demand for the commodity and the supply of the commodity are in a moving equilibrium about the trends of prices and production. When, for example, the supply price ratio of a given year was unity, the production ratio of the following year was unity; and when the production ratio of that following year was unity the demand price ratio of the same year was unity. These facts are illustrated [9] in Figure 2.

9. By the use of a simple device, described in the article "Elasticity of Demand and Flexibility of Prices," Journal of the American Statistical Society, March 1922, it is possible to pass from curves in the ratio form to corresponding curves referring to absolute quantities.

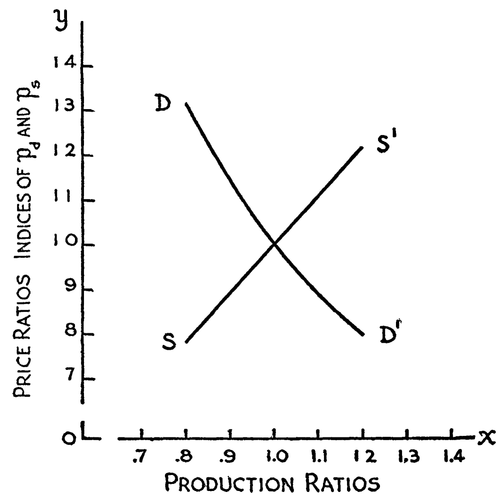
The second type of curves for the laws of demand and supply is, as we have seen,

$$p_d = A_1 x^{\alpha_1} \qquad\qquad p_s = A_2 x^{\alpha_2}$$

If these curves are fitted to the trend ratios directly by the method of least squares, values of A_1 and A_2 will be obtained that will vary slightly from unity. But if the assumption is made that when $x = 1.0$, $p_d = 1.0$ and $p_s = 1.0$, which we found to be true practically and to

FIGURE 3. A Moving Equilibrium of Demand and Supply. Potatoes

Demand. $p_d = x^{-1\,2310}$, origin (0,0);
Supply: $p_s = x^{1\,0828}$, origin (0,0).



be fulfilled theoretically in the linear types of curves, the above equations take the simple forms

$$p_d = x^{\alpha_1} \qquad\qquad p_s = x^{\alpha_2}$$

The graphs of these equations are given in Figure 3. Again there is a moving equilibrium of demand and supply about the secular trends of prices and production.

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