...ectur 2. Preliminary

· The infinite horizon optimal growth problem is:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to $c_t + k_{t+1} \leq f(k_t)$,

$c_t, k_{t+1} \geq 0, t = 0, 1 \dots$

k_0, given.

· The functional equation is:

$$V(k) = \max_{c, k'} (u(c) + \beta V(k'))$$

subject to

$c + k' \leq f(k)$

$c, k' \geq 0.$

我们将会说明，以上两个问题是等价的.

Lecture 2. Recursive Analysis.

- Euler equation and the transversality condition for the infinite horizon problem:

$$\begin{cases} u'(f(k_t)-k_{t+1}) = \beta u'(f(k_{t+1})-k_{t+2}) f'(k_{t+1}), \quad t=0,\cdots \\ \lim_{t\to\infty} \beta^t u'(f(k_t)-k_{t+1})\cdot f'(k_t)\cdot k_t = 0 \end{cases}$$

with $u(c)=\log c$, $f(k)=Ak^{\alpha}$

the above equations can be:

$$\begin{cases} \dfrac{1}{Ak_t^{\alpha}-k_{t+1}} = \dfrac{\alpha\beta A k_{t+1}^{\alpha-1}}{Ak_{t+1}^{\alpha}-k_{t+2}}, \quad t=0,\cdots \\ \lim_{t\to\infty} \beta^t \dfrac{1}{Ak_t^{\alpha}-k_{t+1}}\cdot \alpha A k_t^{\alpha-1}\cdot k_t = 0 \end{cases}$$

- we have guessed a solution $k_{t+1}=\alpha\beta A k_t^{\alpha}$ and verified it satisfied the above conditions
- Other methods that can solve this infinite horizon problem?
  Yes.. (i) Solve the difference equation system. (Euler)
  (ii) dynamic programming.
- Recall the property of our solution: $k_{t+1}=\alpha\beta A k_t^{\alpha}$
  the savings decision between t and $t+1$ is decided at t, not at 0 or any other period.
- Definition: a problem is stationary whenever the structure of the choice problem that a decision maker faces is identical at every point in time.

· Go back to our example.

a consumer placed at the beginning of time choosing his infinite future consumption stream given an initial capital stock k_0. As a result, we solved for a sequence of real numbers $\{k^*_{t+1}\}_{t=0}^{\infty} = \{k_0, k^*_1, k^*_2, \cdots, k^*_T, k^*_{T+1}, k^*_{T+2}, \cdots\}$

if now, we consider at time $t=T$, he will find himself ~~we~~ with a capital stock k^*_T. If at that moment, we let the consumer to forget about his initial plan and ask him to decide on his consumption stream again, using k^*_T as the new initial level of capital, that is $k_0 = k^*_T$, what sequence of capital would ~~be~~ he choose?

· If the problem is stationary then for any two periods $t \neq s$, $k_t = k_s$ implies $k_{t+j} = k_{s+j}$ for all $j > 0$.

$\Rightarrow$

That is, he would not change his mind if he could decide all over again.

· $\Rightarrow$ we can think of a function that, for every period t, assigns to each possible initial level of capital k_t ~~and~~ an optimal level for next period's capital: $k_{t+1} = g(k_t)$.

stationary means that the function $g(\cdot)$ has no other argument than current capital. In particular, the function does not vary with time. We call $g(\cdot)$ as the policy function.

· 例: finite horizon problem is not stationary.

- with an infinite time horizon, the remaining horizon is the same at each point in time. The only matter is the current capital stock, k_t, which is the only state variable.

- A more formal way:

  idea: $\max_{x,y} f(x,y) = \max_{y} \{\max_{x} f(x,y)\}$

- If we do this overtime, the idea would be to maximize over $\{k_{s+1}\}_{s=t}^{\infty}$ first by choice of $\{k_{s+1}\}_{s=t+1}^{\infty}$ conditional on k_{t+1}, and then to choose k_{t+1}.

- denote $V(k_t)$ as the value of the optimal program from period t for an initial condition k_t:

$$V(k_t) \equiv \max_{\{k_{s+1}\}_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} F(k_s, k_{s+1}), \text{ s.t. } k_{s+1} \in \Gamma(k_s), \forall s \geq t,$$

where $\Gamma(k_t)$ represents the feasible choice set for k_{t+1} given k_t.

Remember: $F(k_s, k_{s+1})$ could be think of $U(f(k_s) - k_{s+1})$.

That is, $V(\cdot)$ is an indirect utility function.

- using the maximization-by-steps idea, we can write:

$$V(k_t) = \max_{k_{t+1} \in \Gamma(k_t)} \{F(k_t, k_{t+1}) + \max_{\{k_{s+1}\}_{s=t+1}^{\infty}} \sum_{s=t+1}^{\infty} \beta^{s-t} F(k_s, k_{s+1}) \text{ (s.t. } k_{s+1} \in \Gamma(k_s), \forall s \geq t+1)\}$$

rewrite:

$$\Rightarrow V(k_t) = \max_{k_{t+1} \in \Gamma(k_t)} \{F(k_t, k_{t+1}) + \beta \max_{\{k_{s+1}\}_{s=t+1}^{\infty}} \sum_{s=t+1}^{\infty} \beta^{s-(t+1)} F(k_s, k_{s+1}) \text{ (s.t. } k_{s+1} \in \Gamma(k_s), \forall s \geq t+1)\}$$

$$= \max_{k_{t+1} \in \Gamma(k_t)} \{F(k_t, k_{t+1}) + \beta V(k_{t+1})\}$$

So, we have

$$V(k_t) = \max_{k_{t+1} \in \Gamma(k_t)} \{F(k_t, k_{t+1}) + \beta V(k_{t+1})\}$$

This is the dynamic programing / recursive formulation.

· two variables, k_t current capital

k_{t+1} next period capital.

· In other words, we find a V that, using k to denote current capital and k' next period's capital, then.

$$V(k) = \max_{k' \in \Gamma(k)} \{F(k, k') + \beta V(k')\} \rightarrow \text{Bellman equation, 贝尔曼方程}$$

and $k'^{*} = g(k) = \underline{\arg\max_{k' \in \Gamma(k)} \{F(k, k') + \beta V(k')\}}$

the Bellman equation is a functional equation: the unknown is a function.

· Fact: dynamic programming equation $\Longleftrightarrow$ the sequential problem. (infinite horizon problem)

· Note: since the maximization that needs to be done is finite-dimensional, ordinary Kuhn-Tucker methods can be used, without reference to extra conditions.

Facts: Contraction mapping Theorem (3.2 of ~~Lucas~~ Stokey and Lucas (1988))

- F is continuously differentiable in its two arguments, it is strictly increasing in its first argument and decreasing in the second, strictly concave and bounded.

- $\beta \in (0,1)$.

- Γ is a nonempty, compact-valued, monotone and continuous correspondence with a convex graph.

Then,

1. There exists a function $V(\cdot)$ that solves the Bellman equation. This solution is unique.
2. V is strictly concave
3. V is strictly increasing.
4. V is differentiable.
5. It is possible to find $V(\cdot)$ by the following interative process
   i. Pick any initial V_0 function, for example $V_0(k)=0, \forall k$
   ii. find $V_1(k) = \max_{k'} \{F(k,k') + \beta V_0(k')\}$

   then
   $$V_2(k) = \max_{k'} \{F(k,k') + \beta V_1(k')\}$$
   $$\vdots$$
   $$V_{n+1}(k) = \max_{k'} \{F(k,k') + \beta V_n(k')\}$$
   $$\vdots$$
   we get a sequence of functions $\{V_j\}_{j=0}^{\infty}$ which converges to V.
6. optimal behavior can be characterized by a function g, with $k'=g(k)$, that is increasing so long as F_2 is increasing in k.

example 2.1: solving a parametric dynamic ~~problem~~ programming problem.

$$u(c) = \log c, \quad f(k) = Ak^{\alpha}$$

that is, $V(k) = \max\limits_{c,k'} \{\log c + \beta V(k')\}$

$$\text{s.t.} \quad c = Ak^{\alpha} - k'$$

将上述问题重述为：

$$V(k) = \max_{k' \geq 0} \{\log(Ak^{\alpha} - k') + \beta V(k')\}$$

我们用递归方法解上述问题：(value function iteration).

· 首先，猜测一个解：$V_0(k) = 0$.

$$V_1(k) = \max_{k' \geq 0} \{\log(Ak^{\alpha} - k') + \beta V_0(k')\}$$

$$= \max_{k' \geq 0} \{\log(Ak^{\alpha} - k') + \beta \cdot 0\}$$

$$= \max_{k' \geq 0} \{\log(Ak^{\alpha} - k')\}$$

$k' = 0$ 取得最大值，所以

$$V_1(k) = \log A + \alpha \log k$$

· 第二步：$V_2(k) = \max\limits_{k' \geq 0} \{\log(Ak^{\alpha} - k') + \beta V_1(k')\}$

$$= \max_{k' \geq 0} \{\log(Ak^{\alpha} - k') + \beta(\log A + \alpha \log k')\}$$

一阶条件：$\dfrac{1}{Ak^{\alpha} - k'} = \dfrac{\beta\alpha}{k'} \Rightarrow k' = \dfrac{\alpha\beta A k^{\alpha}}{1 + \alpha\beta}$

$$\Rightarrow V_2(k) = \log\left(Ak^{\alpha} - \frac{\alpha\beta A k^{\alpha}}{1+\alpha\beta}\right) + \beta\left[\log A + \alpha \log\left(\frac{\alpha\beta A k^{\alpha}}{1+\alpha\beta}\right)\right]$$

$$V_2(k) = (\alpha + \alpha^2\beta)\log k + \log\left(A - \frac{\alpha\beta A}{1+\alpha\beta}\right) + \beta \log A + \alpha\beta \log\left(\frac{\alpha\beta A}{1+\alpha\beta}\right)$$

我们可以进一步解出 $V_3(k)$，也可以用以下一种更直接的方法：

记 $a_2 \equiv \log\left(A - \frac{\alpha\beta A}{1+\alpha\beta}\right) + \beta\log A + \alpha\beta\log\frac{\alpha\beta A}{1+\alpha\beta}$

$b_2 \equiv (\alpha + \alpha^2\beta)$.

Then, $V_2(k) = a_2 + b_2\log(k)$,

回忆一下，$V_1(k) = \underbrace{\log A}_{a_1} + \underbrace{\alpha}_{b_1}\log(k)$.

and $V_2(k) = a_2 + b_2\log(k)$.

$\vdots$

$V_n(k) = a_n + b_n\log(k)$ for all n.

因此，我们可以~~猜测~~猜测最终的解有如下形式：

$$V(k) = a + b\cdot\log(k)$$

当解出 a, b，即解出了 $V(\cdot)$.

$$V(k) = a + b\cdot\log(k) = \max_{k' \geq 0}\{\log(Ak^\alpha - k') + \beta(a + b\log k')\}, \forall k$$

一阶条件：$\frac{1}{Ak^\alpha - k'} = \frac{\beta b}{k'} \Rightarrow k' = \frac{\beta b}{1+\beta b}Ak^\alpha$

$$\Rightarrow V(k) = \log\left(Ak^\alpha - \frac{\beta b}{1+\beta b}Ak^\alpha\right) + \beta\left[a + b\log\left(\frac{\beta b}{1+\beta b}Ak^\alpha\right)\right]$$

$$= (1+b\beta)\log A + \log\left(\frac{1}{1+b\beta}\right) + \alpha\beta + b\beta\log\left(\frac{\beta b}{1+\beta b}\right) + (\alpha + \alpha\beta b)\log k$$

$$\Rightarrow \begin{cases} a = (1+b\beta)\log A + \log\left(\frac{1}{1+b\beta}\right) + \alpha\beta + b\beta\log\left(\frac{\beta b}{1+\beta b}\right) \\ b = \alpha + \alpha\beta b \end{cases}$$

$$\Rightarrow \quad b = \frac{\alpha}{1-\alpha\beta}$$

and $a = \frac{1}{1-\beta}\left[(1+b\beta)\log A + b\beta\log(b\beta) - (1+b\beta)\log(1+b\beta)\right]$

$$\Rightarrow a = \frac{1}{1-\beta}\cdot\frac{1}{1-\alpha\beta}\left[\log A + (1-\alpha\beta)\log(1-\alpha\beta) + \alpha\beta\log(\alpha\beta)\right].$$

回到 $k' = \frac{b\beta}{1+b\beta}Ak^{\alpha}$

我们得到 $k' = \alpha\beta A k^{\alpha}$ 与上一节课结果相同.

~~Example 3.2~~

· The Functional Euler equation.

With the recursive strategy, an Euler equation can be derived as well.

利用 policy function $k' = g(k)$.

$$V(k) = F(k, g(k)) + \beta V[g(k)].$$

$g(k)$ 满足一阶条件:

$$F_2(k, k') + \beta V'(k') = 0$$

即. $F_2(k, g(k)) + \beta V'[g(k)] = 0$

$V(\cdot)$ 未知, 但可以得出 $V'(\cdot)$.

$$V'(k) = F_1(k, g(k)) + \underbrace{g'(k)\{\overbrace{F_2(k, g(k)) + \beta V'[g(k)]}^{=0}\}}_{\text{indirect effect through optimal } k'}$$

$$\Rightarrow V'(k) = F_1(k, g(k))$$

我们知道，$V'(g(k)) = F_1[g(k), g(g(k))]$ 也成立。

因此，一阶条件 $F_2(k, g(k)) + \beta V'[g(k)] = 0$

可以表述为：$F_2(k, g(k)) + \beta F_1[g(k), g(g(k))] = 0, \forall k$.

现在，未知的变为 $g(\cdot)$.

That is, under the recursive formulation, the Euler equation turned into a functional equation.

· 上述步骤给出了第三种解无穷期问题的方法：

(i) 求出 Euler equation.

(ii) solve it for the function $g(\cdot)$.

· Example 2.2.

$F(k, k') = u(f(k) - g(k))$, $u(c) = \log(c)$, $f(k) = Ak^{\alpha}$

Euler equation:

$$F_2(k, g(k)) + \beta F_1[g(k), g(g(k))] = 0, \forall k$$

$$\Rightarrow \frac{1}{Ak^{\alpha} - g(k)} = \frac{\beta \alpha A (g(k))^{\alpha-1}}{A(g(k))^{\alpha} - g(g(k))}, \forall k.$$

上述即为 $g(k)$ 的一个 functional equation.

Guess: $g(k) = sAk^{\alpha}$, 即 the savings are a constant fraction of output.

(同Solow model)

代入，可解出 $s = \alpha\beta$.