Advanced Microeconomics

Lecture 2: Dynamic optimisation and recursive methods in economic analysis

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Outline

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1. Dynamic optimisation in discrete time
   - Two-period optimisation
   - N-period optimisation

2. Dynamic optimisation in continuous time
   - Bellman's equation
   - Hamiltonian

1. Two-period optimisation
(1) The problem

Consumptions in periods 1 & 2 are C_1 and C_2. Incomes are Y_1 and Y_2. r is the real interest rate. Note that you can save in period 1 and earn interest at a rate of r , or you can borrow against future income by paying interest r on borrowing.

- Maximise a well-behaved utility function defined over present and future consumption, $u(C_1, C_2)$, subject to the lifetime budget constraint:

$$\max_{C_1, C_2} u(C_1, C_2) \text{ s.t. } C_1 + \frac{C_2}{1+r} = A_0(1+r) + Y_1 + \frac{Y_2}{1+r}$$

- A_0 initial wealth or debts at the beginning of period 1 (or the end of period 0). $A_2 = 0$

1. Two-period optimisation

(2) Solutions

- Lagrangean

$$\mathcal{L} = u\left(C_1, C_2\right) + \lambda\left[A_0\left(1+r\right) + Y_1 + \frac{Y_2}{1+r} - C_1 - \frac{C_2}{1+r}\right]$$

- FOCs:

$$\begin{aligned} \mathcal{L}'_{C_1} &= \frac{\partial u\left(C_1, C_2\right)}{\partial C_1} - \lambda = 0 \\ \mathcal{L}'_{C_2} &= \frac{\partial u\left(C_1, C_2\right)}{\partial C_2} - \frac{\lambda}{1+r} = 0 \\ \mathcal{L}'_{\lambda} &= A_0\left(1+r\right) + Y_1 + \frac{Y_2}{1+r} - C_1 - \frac{C_2}{1+r} = 0 \end{aligned}$$

1. Two-period optimisation

(2) Solutions

- The first two eqs. give

$$\frac{\frac{\partial u(C_1,C_2)}{\partial C_1}}{\frac{\partial u(C_1,C_2)}{\partial C_2}} = 1 + r$$

- Once we know the function form for utility, we can obtain the optimal consumption over periods.

2. N-period optimisation

(1) The problem

- Consumer's problem becomes

$$\max_{C_t} U = \sum_{t=1}^{T} \frac{u\left(C_t\right)}{\left(1+\rho\right)^t} \text{ s.t. } \sum_{t=1}^{T} \frac{C_t}{\left(1+r\right)^t} \leq A_0 + \sum_{t=1}^{T} \frac{Y_t}{\left(1+r\right)^t}$$

where $u'\left(\cdot\right) > 0, u''\left(\cdot\right) < 0, \rho$ is the discount rate or time preference rate.

- Assume a utility function form

$$\max_{C_t} U = \sum_{t=1}^{T} \frac{1}{\left(1+\rho\right)^t} \frac{C_t^{1-\theta}}{1-\theta} \text{ s.t. } \sum_{t=1}^{T} \frac{C_t}{\left(1+r\right)^t} \leq A_0 + \sum_{t=1}^{T} \frac{Y_t}{\left(1+r\right)^t}$$

- Lagrangean

$$\mathcal{L} = \sum_{t=1}^{T} \frac{1}{\left(1+\rho\right)^t} \frac{C_t^{1-\theta}}{1-\theta} + \lambda \left[A_0 + \sum_{t=1}^{T} \frac{Y_t}{\left(1+r\right)^t} - \sum_{t=1}^{T} \frac{C_t}{\left(1+r\right)^t} \right]$$

2. N-period optimisation

(2) Solutions

- FOCs

$$\mathcal{L}'_{C_t} = \frac{C_t^{-\theta}}{(1+\rho)^t} - \lambda \frac{1}{(1+r)^t} = 0 \text{ for every } C_t$$

$$\frac{(1+r)^t \, C_t^{-\theta}}{(1+\rho)^t} = \lambda \text{ where } t = 1, \ldots, T$$

- This implies that for periods t and $t+1$

$$\frac{(1+r)^t \, C_t^{-\theta}}{(1+\rho)^t} = \frac{(1+r)^{t+1} \, C_t^{-\theta}}{(1+\rho)^{t+1}}$$

$$\frac{C_{t+1}}{C_t} = \left(\frac{1+r}{1+\rho}\right)^{\frac{1}{\theta}}$$

Outline

Bellman's equation

- The intertemporal problem

$$\max_{C_t} U = \sum_{t=0}^{T} \frac{u\left(C_t\right)}{\left(1-\rho\right)^t} \text{ s.t. } A_{t+1} - A_t = rA_t + Y_t - C_t$$

- Bellman's equation

$$V_t\left(A_t\right) \equiv \max_{C_t} \left\{ u\left(C_t\right) + \frac{1}{1+\rho} V_{t+1}\left(A_{t+1}\right) \right\}$$

Hamiltonian

- The intertemporal problem

$$\max_{c}\left[\int_0^T (\ln c_t)\, e^{-\theta t} dt\right] \text{ s.t. } \frac{dA}{dt} = rA_t + y_t - c_t$$

- Hamiltonian

$$H\left(A(t), c(t), \lambda(t)\right) = (\ln c_t)\, e^{-\theta t} + \lambda(t)\left(rA_t + y_t - c_t\right)$$