Assignment 4

1. This involves the neoclassical stochastic growth model augmented with variable labour effort. Following the notation defined in lecture, let total factor productivity, $z_t \in Z$, be a stochastic process corresponding to total factor productivity of a representative firm. $Z = \{\widetilde{z}_1, \ldots, \widetilde{z}_N\} \in \mathbf{R}_+^N$. The probability of the event $z^t \in Z^{t+1}$ is defined to be $\pi\left(z^t\right)$; $0 \leq \pi\left(z^t\right) \leq 1$ with $\sum_{z^t \in Z^{t+1}} \pi\left(z^t\right) = 1$. Production is determined by a constant returns to scale technology $y_t\left(z^t\right) = z_t F\left(k_t\left(z^{t-1}\right), n_t\left(z^t\right)\right)$ where F is strictly increasing, concave and twice continuously differentiable, y_t is output, k_t is capital and n_t is hours of work. If $i_t\left(z^t\right)$ is investment in date-event $\left(t, z^t\right)$, then next period's capital stock is $k_{t+1}\left(z^t\right) = (1-\delta) k_t\left(z^{t-1}\right) + i_t\left(z^t\right)$ where $\delta \in (0,1)$.[1] A representative household has preferences over consumption and leisure, $U\left(\left\{c_t\left(z^t\right), l_t\left(z^t\right)\right\}_{t=0}^{\infty}\right) = \sum_{t=0}^{\infty} \beta^t \sum_{z^t \in Z^{t+1}} \pi\left(z^t\right) u\left(c_t\left(z^t\right), l_t\left(z^t\right)\right)$ where u is strictly increasing in both consumption and leisure, and strictly concave; $\beta \in (0,1)$. The household's time constraint is $l_t\left(z^t\right) + n_t\left(z^t\right) = 1$ while the aggregate resource constraint is $y_t\left(z^t\right) \geq c_t\left(z^t\right) + i_t\left(z^t\right)$.

   (a) Let $p_t\left(z^t\right)$ denote the relative price of output in date-event $\left(t, z^t\right)$ relative to output in $(0, \widetilde{z}_1)$. (Since there can only be one numeraire, I'm defining all intertemporal prices relative to output in date 0 when TFP is the first level in Z.) Let $r_t\left(z^t\right)$ be the rental price of capital, and $w_t\left(z^t\right)$ describe the real wage, relative to output in $\left(t, z^t\right)$. Answer the following: (i) define date-0 Arrow-Debreu-McKenzie competitive equilibrium, (ii) show that the household's Euler equation and the firm's profit maximisation together imply the Planner's Euler equation for this problem.

   (b) Let $R_t\left(z^t\right)$ be the purchase price of capital, and $q_t\left(\widetilde{z}_{t+1}, z^t\right)$ be the date-event $\left(t, z^t\right)$ price of an Arrow-security which pays one unit of output tomorrow iff $\widetilde{z}_{t+1} \in Z$ is the actual state-of-the-world. (That is, if $z^{t+1} = \left(\widetilde{z}_{t+1}, z^t\right)$.) Answer the following: (i) define sequential competitive equilibrium, (ii) derive the Euler equations for Arrow-securities and the labour-leisure condition.

   (c) Let z_t be a first-order Markov Process with $\Pr\left(z_{t+1} = z_j \mid z_t = z_i\right) = P_{ij} \geq 0$ and $\sum_{j=1}^{N} P_{ij} = 1$ for each $i = 1, \ldots, N$. Let ω be the representative household's beginning of period wealth. Assuming the household can hold both Arrow securities and capital, define recursive competitive equilibrium.

[1] Here $z^t = \left(z_t, z^{t-1}\right)$ where z^{t-1} is the partial history $(z_0, \ldots, z_{t-1})$ *that determined* k_t. Thus z^{t-1} preceeded z^t.

2. Reconsider the technology and preferences in (1), eliminating capital from the model and assuming a production function $y_t = z_t n_t$. Further, assume the technology shock takes on only 2 values, $Z = \{z_l, z_h\}$ with $P_{11} = P_{22} = p > 0.5$. Finally, let $u(c, l) = \frac{\left(c - \theta \frac{n^{1+\nu}}{1+\nu}\right)^{1-\sigma}}{1-\sigma}$ where $n = 1 - l$ with $\theta > 0$, $\nu > 0$ and $\sigma > 0$. Questions (a) and (b) involve a complete markets economy with a full set of state-contingent bonds. In contrast, question (c) involves the study of an economy with incomplete markets where there is a single non-contingent bond that pays one unit in every state-of-the-world tomorrow. As there is no capital or storage, all bonds, state-contingent or not, are in zero net supply. Since the aggregate stock of bonds is always 0 in equilibrium, and there is no capital or way to store output, z is the aggregate state of the economy.

   (a) Allowing for state-contingent claims, the household's state variable is their beginning of period total wealth, $s = a(z_i)$ where $a(z_i)$ is the quantity of Arrow securities he purchase last period that pay off if the state today is actually z_i. Define recursive competitive equilibrium and solve for the equilibrium quantities of employment and production. Use the result that $s = 0$ in equilibrium, and solve for c and n as functions of the aggregate state.

   (b) Compute the equilibrium price of each Arrow security. A risk-free bond is one that pays one unit of output tomorrow regardless of the state-of-the-world. The household can construct such an asset as the sum of Arrow securities. The risk-free real interest rate is the inverse of the price of this bond, minus one. Why is the real interest rate countercyclical in this economy? (Why is it negatively correlated with production?)

   (c) Assume now that there is a single non-contingent bond with a discount price q. Please answer each of the following questions: (i) why is there no effect on consumption and employment as we move to a model where markets are incomplete?