4: Uncertainty and the Neoclassical growth model

U assume that households and firms have rational expectations.
: subjective probability over future events are consistent with the actual probabilities for these events.

4.1 Maximization under Uncertainty

- example: 2-period economy, $t=0,1$
  - a household in a large economy.
  - consumes in both period.
  - faces earnings risk in his second period when he works.
- in period 1: n possible states of the world, the real wage varies across these states, $w \in \{\tilde{w}_1, \dots, \tilde{w}_n\}$
  The probability of state i is $\pi_i = Pr(w = \tilde{w}_i), i=1,\cdots,n$
  with $\sum_{i=1}^{n} \pi_i = 1$
- The household maximizes expected lifetime utility, that is he has a Von Neumann-Morgensten utility function:

$$U = \sum_{i=1}^{n} \pi_i u(c_0, c_{1i}, n_i) \equiv E[u(c_0, c_{1i}, n_i)]$$

where $u(c_0, c_{1i}, n_i)$ is his utility given a specific bundle of consumption in period 0 and 1, and work in period 1 when the state of the world is i.

- 理解：可以将 c_{1i} 和 c_{1j}，$\forall i \neq j$ 想象成不同的产品，即可以想象家庭面对一个静态问题，选择 $n+1+n$ 种产品.

不完全市场 (incomplete market).

household's choice of c_0 and $\{c_{1i}, n_i\}_{i=1}^{n}$ will depend on what assets are available to the household.

· One extreme example of incomplete markets is: Only a single risk-free asset, a, with discount price $q = \frac{1}{R}$.

· ~~date~~ period 0 budget constraint:

$$c_0 + qa = \underbrace{I}_{\text{initial income}}$$

initial income (初始收入)

· There are n separate period 1 budget constraints:

$$c_{1i} = a + w_i n_i, \quad i = 1, \cdots, n$$

· markets are incomplete, because we don't have as many assets as state-of-nature, therefore we cannot use a (the asset) to derive a single ~~line~~ lifetime budget constraint.

· 换句话说，如果我们用 $i=1$ 时的 period 1 budget constraint，有 $a = c_{11} - w_1 n_1$ 并将 a 代入 period 1 budget constraint when $i \neq 1$，那么，当 $i \neq 1$ 时，$c_{1i} = c_{11} - w_1 n_1 + w_i n_i$，即对 c_{11} 的选择影响了 c_{1i}，$i \neq 1$。我们说，对 c_{1i} 的选择不是独立的。这即造成了市场的不完全性。

· 假设：$u(c_0, c_{1i}, n_i) = u(c_0) + \beta u(c_{1i}) + \beta v(n_i)$, where $v'(n_i) < 0$

The household solves:

$$\max_{C_0, a, \{C_{1i}, n_i\}_{i=1}^n} \quad u(C_0) + \beta \sum_{i=1}^n \pi_i [u(C_{1i}) + v(n_i)]$$

subject to

$$C_0 + qa = I$$

$$C_{1i} = a + w_i n_i \ , \ i = 1, \cdots, n$$

let $\lambda, \{\lambda_i\}_{i=1}^n$ be the multipliers.

FOC: $[C_0]$: $\quad u'(C_0) = \lambda$

$[a]$: $\quad \lambda q = \sum_{i=1}^n \lambda_i$

$[C_{1i}]$: $\quad \beta \cdot \pi_i u'(C_{1i}) = \lambda_i$

$[n_i]$: $\quad \beta \cdot \pi_i \cdot v'(n_i) = -\lambda_i \cdot w_i$

$$\Rightarrow \underbrace{w_i \cdot u'(C_{1i})}_{\text{the marginal value of another unit of time devoted to work}} = \underbrace{-v'(n_i)}_{\text{marginal utility of Leisure}}, \ i = 1, \cdots, n \ , \text{ labor-leisure condition.}$$

also,

$$\Rightarrow \underbrace{q u'(C_0)}_{\text{marginal cost of savings, } q\text{, valued in units of period 0 consumption}} = \underbrace{\beta \sum_{i=1}^n \pi_i u'(C_{1i})}_{\text{discounted expected marginal benefit.}} \ , \text{ Euler equation.}$$

· A unit more of saving raises consumption in every state-of-the-world next period, no matter the shock is high or low, which is not ~~optimal~~ ideal because consumption is not smoothed.

· 假设 $u(c)=\log c$, $v(n)=\log(1-n)$

labor-Leisure condition $\Rightarrow$

$$\frac{w_i}{c_{1i}}=\frac{1}{1-n_i}$$

$$\Rightarrow c_{1i}=w_i(1-n_i)\quad , i=1,\cdots,n$$

period 1 budget constraint becomes:

$$c_{1i}=a+w_i\cdot n_i=a+w_i-c_{1i}$$

$$\Rightarrow c_{1i}=\frac{a+w_i}{2}$$, consumption in each state i fluctuates with the wage shock. Thus, not fully insured against consumption risk.

This is a consequence of incomplete market.

· from Euler equation:

$$\frac{q}{I-qa}=\beta\sum_{i=1}^{n}\pi_i\frac{2}{a+w_i}$$

we can solve for a.

1.2 完全市场

- Instead of a single risk-free asset, there are state-contingent claims, n separate assets traded at date 0. Name: Arrow securities. The j-th asset is bought at price q_j, where $j \in \{1, \cdots, n\}$, and pays 1 unit of consumption only if in date 1 the state of the world is j. If in date 1 the event j does not occur, the household receives 0 per unit of the j-th asset that it holds. It is in this sense that these are ~~contig~~ contingent-claims.

- period 0 budget constraint is

$$C_0 + \sum_{i=1}^{n} q_i a_i = I.$$

period 1 budget constraint is

$$C_{1i} = a_i + w_i n_i$$

- The risk-free asset can be reconstructed 1 unit of each a_i.

$$\Rightarrow q = \sum_{i=1}^{n} q_i$$

- Now, we can derive a single lifetime budget constraint.

$$\Rightarrow C_0 + \sum_{i=1}^{n} q_i C_i = I + \sum_{i=1}^{n} q_i w_i n_i$$

- Now, we have a single ~~Larg~~ Lagrange multiplier λ.

FOCs: $[C_0]$: $u'(C_0) = \lambda$

$[C_{1i}]$: $\beta \pi_i u'(C_{1i}) = \lambda q_i$

$[n_i]$: $\beta \pi_i v'(n_i) = -\lambda q_i w_i$

Labor-Leisure condition:

$$w_i u'(c_{1i}) = -v'(n_i)$$

- Euler equations:

$$q_i u'(c_0) = \beta \pi_i u'(c_{1i}), \quad i = 1, \cdots, n$$

- Assume actuarially fair asset prices, $q_i = \pi_i q$, $i = 1, \cdots, n$

Then, $u'(c_0) \cdot q = \beta \cdot u'(c_{1i})$, $i = 1, \cdots, n$.

$\Rightarrow c_{1i} = c_1$, a constant over states of nature

- This is the result that a risk-averse consumer with fair insurance will fully insure himself.

- Complete markets gives a marginal rate of substitution between c_0 and each c_{1i} as q_i

- The marginal rate of substitution between c_{1i} and c_{1j} is.

$$\frac{u'(c_{1i})}{u'(c_{1j})} = \frac{q_i}{q_j}$$

- assume $u(c_0) = \log(c_0)$, $V(n) = \log(1-n)$. we have

$$c_0 = \frac{1}{\lambda}$$
$$\beta \pi_i c_0 = q_i c_{1i}$$
$$\beta \pi_i c_0 = q_i w_i (1 - n_i)$$

- The lifetime budget constraint.

$$c_0 + \beta \sum_{i=1}^{n} \pi_i c_0 = I + \sum_{i=1}^{n} (q_i w_i - \beta \pi_i c_0)$$

$$\Rightarrow C_0(1+2\beta) = I + \sum_{i=1}^{n} q_i w_i$$

and from Euler equations.

$$C_{1i} = \beta \cdot \frac{\pi_i}{q_i} \cdot C_0$$

- If the prices of securities do not reflect the probability of payment the higher is q_i relative to π_i, the less is a_i and therefore C_{1i}.
- 因此，完全市场并不足以得出 full insurance.

4.2 Markov Chains

定义: A stochastic process that the probability distribution of the random variable next period depends only on its current value.

定义: $x_t \in X$. A stationary Markov Chain is a stochastic process $\{x_t\}_{t=0}^{\infty}$ defined by X, P, π_0, such that there exist a stationary (invariant) distribution ~~for P~~ (a probability vector) $\pi = \pi P$.

where, X is $1 \times n$ set of values., $X = \{\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_i, \cdots, \tilde{x}_n\}$

P is a $n \times n$ transition matrix

π_0 is a $1 \times n$ initial probability distribution for x_0.

where, $P_{i,j}$ = probability $\{x_{t+1} = \tilde{x}_j \mid x_t = \tilde{x}_i\}$

This implies probability $\{x_{t+2} = \tilde{x}_j \mid x_t = \tilde{x}_i\} = \sum_{k=1}^{n} P_{ik} \cdot P_{kj} \equiv [P^2]_{i,j}$.

Given π_0, the probability distribution of x_1 is π_1, given by.

$$\pi_1 = \pi_0 P$$

Analogously,

$$\pi_2 = \pi_1 P = \pi_0 P^2$$

$$\vdots$$

$$\pi_t = \pi_{t-1} P = \pi_0 P^t$$

- A stationary distribution satisfies

$$\pi I = \pi P, \quad I \text{ is the identity matrix.}$$

$$\Rightarrow \pi (I - P) = 0.$$

This defines π as an eigenvector of P associated with eigenvalue $\lambda = 1$

例：$P=\begin{pmatrix}0.7 & 0.3\\ 0.6 & 0.4\end{pmatrix}$

$$\pi=(\pi_1,\pi_2)=(\pi_1,\pi_2)\begin{pmatrix}0.7 & 0.3\\ 0.6 & 0.4\end{pmatrix}$$

$$=(0.7\pi_1+0.6\pi_2,\ 0.3\pi_1+0.4\pi_2)$$

And $\pi_1+\pi_2=1$

$\Rightarrow (\pi_1,\pi_2)=(2/3,\ 1/3)$

4.3 The neoclassical Growth model with uncertainty

下面我们将讨论如何在新古典增长模型下引用不确定性.

· 技术冲击 ($\underline{T}$otal $\underline{F}$actor $\underline{P}$roductivity shock)

· 以 z 来代表外生的随机的技术

· 以大写 $\mathcal{Z}$ 来代表技术 z 所有可能的取值，即 $\mathcal{Z}$ 是 $\mathbb{R}_+$ 下的一个可数子集

· $z_t\in\mathcal{Z}$, $\forall t$ be a random variable following a stationary stochastic process.

· $\mathcal{Z}^t=\overbrace{\mathcal{Z}\times\mathcal{Z}\times\mathcal{Z}\times\cdots\times\mathcal{Z}}^{t}$ 为 t 重 $\underbrace{\text{笛卡尔积}}$.

例：$X\times Y$，第一个对象是 X 中元素，第二个对象是 Y 中元素.

例 $X=\{a,b\}$，$Y=\{1,2,3\}$

· $z^t=\{z_0,z_1,\cdots,z_t\}\in\mathcal{Z}^{t+1}$ 为 $t+1$ 期的冲击历史，

记其概率 $Pr\{z^t\}\equiv\pi(z^t)$

即 $\pi(z^t)$ 表示在 $t+1$ 期观察到历史 $z^t=\{z_0,z_1,\cdots,z_t\}$ 的概率

· 我们记 $z^t \prec z^{t+s}$，iff $z^{t+s} = (z^t, z_{t+1}, z_{t+2}, \cdots, z_{t+s})$ for some $s>0$.
即 z^{t+s} 的前 $t+1$ 个元素是由 z^t 来表示的.

例 若 $z^t \prec z^{t+1}$，则 $z^{t+1} = (z^t, z_{t+1})$

· 下面，我们来讨论一下状态的概率.

$$Pr\{z_{t+1} | z^t\} = \pi\{z_{t+1} | z^t\}$$

由 $\underbrace{Pr\{z^{t+1} | z^t\}}_{\text{给定已经观察到历史 } z^t \text{ 后，能够观察到历史 } z^{t+1} \text{ 的概率}} = Pr\{z_{t+1} | z^t\}$。

给定已经观察到
历史 z^t 后，能够
观察到历史 z^{t+1} 的
概率.

如果我们~~假~~假定~~随~~随机过程是 Markov Process，我们有

$$Pr\{(z_{t+1}, z^t) | z^t\} = Pr\{z_{t+1} | z^t\} = Pr\{z_{t+1} | z_t\}$$

· 由条件期望概率：有，若 $z^t \prec z^{t+1}$，$\pi(z^{t+1}) = \pi(z_{t+1} | z^t) \cdot \pi(z^t)$

4.3.1. The planner's problem.

· 产出 at z^t，$y_t(z^t) = z_t F(k_t(z^t), n_t(z^t))$

· $U(\{C(z^t), 1-n_t(z^t)\}_{z^t \in Z^{t+1}}) = \sum_{t=0}^{\infty} \beta^t \sum_{z^t \in Z^{t+1}} \pi(z^t) u(C_t(z^t))$

· $k_{t+1}(z^t) = (1-\delta) k_t(z^{t-1}) + i_t(z^t)$

· $C_t(z^t) + i_t(z^t) \le y_t(z^t)$

· $n_t(z^t) \in [0,1]$

· $U(\cdot)$, $F(\cdot)$ are strictly increasing, concave and twice continuously differentiable

· $\beta \in (0,1)$, $\delta \in (0,1)$

· 由于闲暇不产生效用，$\Rightarrow n_t(z^t)=1$, for each z^t,

· 令 $\lambda_t(z^t)$ 为资源约束在 z^t 状态下的拉格朗日乘子：

$$c_t(z^t)+k_{t+1}(z^t)\le z_t F(k_t(z^{t-1}),n_t(z^t))+(1-\delta)k_t(z^{t-1})$$

· given $k(z^{-1})=k_0\ (z^{-1}=\{\phi\})$

· planner chooses $c_t(z^t)$, $n_t(z^t)$, $k_{t+1}(z^t)$ at each $z^t\in Z^{t+1}$, for $t=0,1,\dots$

$$\mathcal{L}=\sum_{t=0}^{\infty}\beta^t\sum_{z^t\in Z^{t+1}}\pi(z^t)\big[U(c_t(z^t))+\lambda_t(z^t)\big(z_tF(k_t(z^{t-1}),1)+(1-\delta)k_t(z^{t-1})-c_t(z^t)-k_{t+1}(z^t)\big)\big]$$

F.O.C for some specific event $\bar{z}^t$ are:

$$[c_t(\bar{z}^t)]:\quad U'(c_t(\bar{z}^t))=\lambda_t(\bar{z}^t) \qquad (1)$$

$$[k_{t+1}(\bar{z}^t)]:\ \pi(\bar{z}^t)\cdot\lambda_t(\bar{z}^t)=\beta\sum_{z^{t+1}\in Z^{t+2}}\pi(z^{t+1})\lambda_{t+1}(z^{t+1})\big(z_{t+1}F_1(k_{t+1}(\bar{z}^t),1)+1-\delta\big) \qquad (2)$$

以及市场出清：$\bar{z}_tF(k_t(\bar{z}^{t-1}),1)+(1-\delta)k_t(\bar{z}^{t-1})-c_t(\bar{z}^t)-k_{t+1}(\bar{z}^t)=0$

注意：$[k_{t+1}(\bar{z}^t)]$ 的一阶条件只对包含历史 $\bar{z}^t$ 的 z^{t+1} 成立，$\bar{z}^t<z^{t+1}$

利用：$\dfrac{\pi(z^{t+1})}{\pi(\bar{z}^t)}=\pi(z_{t+1}\mid\bar{z}^t)$

我们将 (2) 式化为

$$\lambda_t(\bar{z}^t)=\beta\sum_{z^{t+1}\in Z^{t+2}}\pi(z_{t+1}\mid\bar{z}^t)\cdot\lambda_{t+1}(z^{t+1})\big(z_{t+1}F_1(k_{t+1}(\bar{z}^t,1)+1-\delta\big)$$

$$=\beta\sum_{z_{t+1}\in Z}\pi(z_{t+1}\mid\bar{z}^t)\cdot\lambda_{t+1}(z^{t+1})\big(z_{t+1}F_1(k_{t+1}(\bar{z}^t,1)+1-\delta\big).$$

ady state and linearization

steady-state: is a "constant solution"

$$k_t = k^*, \ \forall t$$
$$c_t = c^*, \ \forall t.$$

· ~~Exa~~ example: Neoclassical Growth model

$$\sum_{t=0}^{\infty} \beta^t \log c_t$$

s.t. $c_t + k_{t+1} - (1-\delta)k_t = k_t^{\alpha}$

$\Rightarrow$ Euler: $\begin{cases} \dfrac{1}{c_t} - \beta \dfrac{1}{c_{t+1}} [\alpha k_{t+1}^{\alpha-1} + 1 - \delta] = 0. \\ \end{cases}$

Budget Constraint $c_t + k_{t+1} - (1-\delta)k_t = k_t^{\alpha}$

· steady state:

$$\begin{cases} \dfrac{1}{c^*} - \beta \cdot \dfrac{1}{c^*} (\alpha k^{*\alpha-1} + 1 - \delta) = 0 \\ c^* + k^* - (1-\delta)k^* = k^{*\alpha} \end{cases}$$

$$\Rightarrow \begin{cases} 1 = \beta(\alpha k^{*\alpha-1} + 1 - \delta) \Rightarrow k^* = \left[\dfrac{1}{\alpha}\left[\dfrac{1}{\beta} - (1-\delta)\right]\right]^{\frac{1}{\alpha-1}} \\ c^* = k^{*\alpha} - \delta k^* \end{cases}$$

· Log-linearization:

for a variable X_t, its log-deviation from steady state is

$\hat{X}_t = \log \dfrac{X_t}{X_{ss}}$, so, $X_t = X_{ss} \cdot e^{\hat{X}_t} \approx X_{ss}(1 + \hat{X}_t)$

earization the simple model

er equation:

$$C_{t+1} = \beta C_t [\alpha k_{t+1}^{\alpha-1} + 1 - \delta] = \beta C_t X_t \text{, where } X_t \equiv \alpha k_{t+1}^{\alpha-1} + 1 - \delta$$

$$C_{ss} \cdot e^{\hat{C}_{t+1}} = \beta C_{ss} \cdot X_{ss} \cdot e^{\hat{C}_t + \hat{X}_t}$$

since from steady-state relations.

$$C_{ss} = \beta C_{ss} \cdot X_{ss} \Rightarrow X_{ss} = \frac{1}{\beta}$$

$$\Rightarrow e^{\hat{C}_{t+1}} = e^{\hat{C}_t + \hat{X}_t}$$

$$\Rightarrow 1 + \hat{C}_{t+1} = 1 + \hat{C}_t + \hat{X}_t$$

$$\Rightarrow \hat{C}_{t+1} = \hat{C}_t + \hat{X}_t$$

then $X_t = \alpha k_{t+1}^{\alpha-1} + 1 - \delta$, steady state $\underset{\frac{1}{\beta}}{\overset{\shortparallel}{X_{ss}}} = \alpha k_{ss}^{\alpha-1} + 1 - \delta \Rightarrow \alpha k_{ss}^{\alpha-1} = \frac{1}{\beta} - 1 + \delta$

$$X_{ss} \cdot e^{\hat{X}_t} = \alpha k_{ss}^{\alpha-1} \cdot e^{(\alpha-1)\hat{k}_{t+1}} + 1 - \delta$$

$$\Rightarrow \frac{1}{\beta}(1 + \hat{X}_t) = (\frac{1}{\beta} - 1 + \delta) \cdot (1 + (\alpha-1)\hat{k}_{t+1}) + 1 - \delta$$

$$\Rightarrow \frac{1}{\beta}\hat{X}_t = (\frac{1}{\beta} - 1 + \delta) \cdot (\alpha-1)\hat{k}_{t+1}$$

$$\Rightarrow \hat{C}_{t+1} = \hat{C}_t + \beta(\frac{1}{\beta} - 1 + \delta) \cdot (\alpha-1)\hat{k}_{t+1} \qquad (1)$$

· Budget constraint.

$$C_t + k_{t+1} - (1-\delta) k_t = k_t^{\alpha}$$

$$C_{ss} \cdot e^{\hat{C}_t} + k_{ss} \cdot e^{\hat{k}_{t+1}} - (1-\delta) k_{ss} e^{\hat{k}_t} = k_{ss}^{\alpha} \cdot e^{\alpha \hat{k}_t}$$

$$\Rightarrow C_{ss} \cdot \hat{C}_t + k_{ss} \cdot \hat{k}_{t+1} - (1-\delta) \cdot k_{ss} \cdot \hat{k}_t = k_{ss}^{\alpha} \cdot \alpha \cdot \hat{k}_t$$

$$\Rightarrow C_{ss} \cdot \hat{C}_t = [\alpha k_{ss}^{\alpha} + (1-\delta) k_{ss}] \cdot \hat{k}_t - k_{ss} \cdot \hat{k}_{t+1} \qquad (2)$$

S-S

$$C_{ss} + k_{ss} - (1-\delta)k_{ss} = k_{ss}^{\alpha}$$

$$C_{ss} = k_{ss}^{\alpha} - \delta k_{ss} = (k_{ss}^{\alpha-1} - \delta)\cdot k_{ss} = \left[\frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right) - \delta\right]\cdot k_{ss}$$

$$\Rightarrow \frac{k_{ss}}{C_{ss}} = \frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right) - \delta$$

$$C_{ss}\cdot \hat{C}_t = [\alpha k_{ss}^{\alpha-1} + (1-\delta)]\cdot k_{ss}\cdot \hat{k}_t - k_{ss}\cdot \hat{k}_{t+1}$$

$$C_{ss}\cdot \hat{C}_t = \frac{1}{\beta}\cdot k_{ss}\cdot \hat{k}_t - k_{ss}\cdot \hat{k}_{t+1}$$

$$\hat{C}_t = \frac{1}{\beta}\cdot \frac{k_{ss}}{C_{ss}}\cdot \hat{k}_t - \frac{k_{ss}}{C_{ss}}\hat{k}_{t+1} \qquad (3).$$

combine (1), (3)

$$\frac{1}{\beta}\cdot\frac{k_{ss}}{C_{ss}}\cdot\hat{k}_{t+1} - \frac{k_{ss}}{C_{ss}}\cdot\hat{k}_{t+2} = \frac{1}{\beta}\cdot\frac{k_{ss}}{C_{ss}}\cdot\hat{k}_t - \frac{k_{ss}}{C_{ss}}\cdot\hat{k}_{t+1} + \beta\left(\frac{1}{\beta} - 1 + \delta\right)(\alpha-1)\cdot\hat{k}_{t+1}$$

$$\Rightarrow \frac{k_{ss}}{C_{ss}}\cdot\left[\left(\frac{1}{\beta}+1\right)\cdot\hat{k}_{t+1} - \hat{k}_{t+2} - \frac{1}{\beta}\cdot\hat{k}_t\right] = \beta\left(\frac{1}{\beta} - 1 + \delta\right)(\alpha-1)\,\hat{k}_{t+1}$$

$\Rightarrow k_{t+2} - \phi k_{t+1} + \frac{1}{\beta}\hat{k}_t = 0$, where ϕ is a function of α, β, δ.

2. solving difference equations:

$$\hat{k}_{t+2} - \phi\hat{k}_{t+1} + \frac{1}{\beta}\hat{k}_t = 0. \qquad (4).$$

· assume, $\hat{k}_0 \neq 0$, $0 < \beta < 1$, $\phi > 1 + \frac{1}{\beta}$

$f(\lambda) = \lambda^2 - \phi\lambda + \frac{1}{\beta}$, two roots: λ_1, λ_2

(4)式的通解为: $\hat{k}_t = (\hat{k}_0 - a)\lambda_1^t + a\lambda_2^t$

a为某一常数.

· minimal state variable (MSV) solution: a solution with the least number of state variables

- a solution is "non-explosive" if $\hat{k}_t \to 0$ as $t \to \infty$

(i) $|\lambda_1| > 1, |\lambda_2| < 1$: 唯一 non-explosive 解, $a = \hat{k}_0$

(ii) $|\lambda_1| > 1, |\lambda_2| > 1$: 无 non-explosive 解.

(iii) $|\lambda_1| < 1, |\lambda_2| < 1$: all solutions are non-explosive

2.1. Deterministic Case

$$\alpha_0 X_{t+2} + \alpha_1 X_{t+1} + \alpha_2 X_t = 0, \quad t = 0, 1, 2, \dots$$

X_{t+1} is $n \times 1$ vector of time t endogeneous variables.

α_i are known $n \times n$ matrices.

0 is $n \times 1$ vector of zeros.

X_0 are given

记 $Y_t = \begin{pmatrix} X_{t+1} \\ X_t \end{pmatrix}$, $a = \begin{pmatrix} \alpha_0 & 0 \\ 0 & I \end{pmatrix}$, $b = \begin{pmatrix} \alpha_1 & \alpha_2 \\ -I & 0 \end{pmatrix}$

则有,

$$a Y_{t+1} + b Y_t = 0, \quad t \geq 0$$

2.1.1. a is invertible

$$\Rightarrow Y_t = \Pi^t Y_0, \quad \Pi = -a^{-1} b$$

we assume the eigenvalues of Π are distinct, which guarantees that Π has the following eigenvector-eigenvalue decomposition.

$$\Pi = P \Lambda P^{-1}$$

where $P = (P_1 \cdots P_{2n})$, $P^{-1} = \begin{pmatrix} \tilde{P}_1 \\ \vdots \\ \tilde{P}_{2n} \end{pmatrix}$, $\Lambda = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{2n} \end{pmatrix}$

where, the elements λ_i are the eigenvalues of Π.

The column vectors, P_i are the right eigenvectors of Π, 即

$$\Pi P_i = \lambda_i P_i, \quad i=1,\cdots,2n.$$

$\tilde{P}_i$ are the left eigenvectors of Π, 即

$$\tilde{P}_i \cdot \Pi = \lambda_i \cdot \tilde{P}_i, \quad i=1,\cdots,2n$$

· $\Pi^j = P\Lambda^j P^{-1}$

then if we define $\tilde{Y}_t = P^{-1}\cdot Y_t$.

we have, $\underbrace{P^{-1}\cdot Y_t}_{\tilde{Y}_t} = P^{-1}\cdot \Pi^t \cdot Y_0 = P^{-1}P\cdot\Lambda^t\cdot \underbrace{P^{-1}Y_0}_{\tilde{Y}_0}$

$$\Rightarrow \tilde{Y}_t = \Lambda^t \cdot \tilde{Y}_0$$

$$\Rightarrow \tilde{Y}_{i,t} = \lambda_i^t \tilde{Y}_{i,0} \quad \text{for } i=1,2,\cdots,2n \qquad (5)$$

where $\tilde{Y}_t = \begin{pmatrix} \tilde{Y}_{1,t} \\ \vdots \\ \tilde{Y}_{2n,t} \end{pmatrix}$

· Now, find the non-explosive solutions.

· Note: $\tilde{Y}_t \to 0$ if and only if $Y_t \to 0$.

· 当某一个$|\lambda_i|>1$时，我们需要选择Y_0，使得λ_i对应的解被"extinguished" "熄灭"

从(5)式得知，当$\tilde{P}_i Y_0 = 0$时，λ_i对解无影响.

· 记q为 explosive eigenvalue 的数量.

· 注意：$\underset{(2n\times1)}{Y_0} = \begin{pmatrix} X_1 \\ X_0 \end{pmatrix}$，由于$X_0$给定，我们只能选择$X_1$，即有$n$个变量的自由度

· case 1: $n=q$，有唯一 non-explosive 解.

即找出这 n 个 eigenvalue 对应的 left-eigenvectors.

组成 $n\times 2n$ 的矩阵 D.

令 $D Y_0 = 0$.

再将 D 分块：$D = [\underset{n\times q}{D_1} \; \vdots \; \underset{n\times(2n-q)}{D_2}]$

则有 $[D_1 \vdots D_2]\begin{pmatrix} X_1 \\ X_0 \end{pmatrix} = D_1 \cdot X_1 + D_2 \cdot X_0 = 0$.

若 D_1 invertible, $A = -D_1^{-1} \cdot D_2$ real

则 $X_1 = A X_0$ 为 (actual) MSV.

· case 2: $n < q$，自由度不够，无 non-explosive solution.

· case 3: $n > q$，有多个 non-explosive solution.

2.2. Stochastic Case, Invertible a.

$$E[\alpha_0 x_{t+2} + \alpha_1 x_{t+1} + \alpha_2 x_t + \beta_0 s_{t+1} + \beta_1 s_t | s_t] = 0$$

$$s_t = \rho s_{t-1} + \varepsilon_t.$$

可以将以上表达式记为

$$\alpha_0 x_{t+2} + \alpha_1 x_{t+1} + \alpha_2 x_t + \beta_0 s_{t+1} + \beta_1 s_t = \xi_{t+1\,(n\times 1)}$$

where $E_t \xi_{t+1} = 0$，注意满足这一条件的 ξ_{t+1} 有很多.

· 记 $\underset{(2n+n_s)\times 1}{Y_t} = \begin{pmatrix} x_{t+1} \\ x_t \\ s_t \end{pmatrix}$, $a = \begin{pmatrix} \alpha_0 & 0 & \beta_0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$, $b = \begin{pmatrix} \alpha_1 & \alpha_2 & \beta_1 \\ -I & 0 & 0 \\ 0 & 0 & -\rho \end{pmatrix}$, $w_t = \begin{pmatrix} \xi_{t+1} \\ 0 \\ \varepsilon_{t+1} \end{pmatrix}$

$\Rightarrow a Y_{t+1} + b Y_t = w_{t+1}$

when a is invertible.

记 $\Pi = -a^{-1}b$

$\Rightarrow Y_t = \Pi^t Y_0 + a^{-1}w_t + \Pi a^{-1} w_{t-1} + \cdots + \Pi^{t-1} a^{-1} w_1$

the space of solution is characterized by the choice of Y_0, $\{\xi_{t+1}\}$, where Y_0 has n free elements and $\{\xi_{t+1}\}$ also has n free elements

~~case~~ · Define non-explosive solution:

$$E_0 Y_t \to 0$$

$$Var_0(Y_t) \text{ bounded.}$$

记 q 为 explosive eigenvalue 的数量.

· case 1: $q=n$, 唯一 non-explosive 解.

即找出这 n 个 eigenvalue 对应的 left-eigenvectors.

组成 $n \times (2n+n_s)$ 矩阵 D. s.t.

$$D Y_t = 0 \quad , t=1,2,\cdots$$

然后 construct ξ_t, s.t.

$$D a^{-1} w_t = 0_{n\times 1}.$$

· case 2: $q>n$ 无 non-explosive 解

· case 3: $q<n$. multiple non-explosive 解.

2.3. The Non-Invertible a Case

$$a Y_{t+1} + b Y_t = 0 \ , \ t \geq 0 \qquad (6)$$

now a is not invertible.

Y_t 为 $2n\times 1$ vector, 记 $m=2n$, 记 $a_{2n\times 2n}$ 矩阵的秩为 $l<m$

· The QZ decomposition of a and b is:

$$QaZ = H_0 \ , \ QbZ = H_1$$

where Q, Z are unitary matrices

H_0, H_1 are upper triangular matrices

· unitary matrics: Q is unitary if its conjugate transpose Q^* is its inverse

i.e. $QQ^* = I$

· conjugate transpose: $Q^* = (\bar{Q})^T = \overline{(Q^T)}$
(共轭转置)

即转置之后每个元素取其复数的共轭数, 即
若元素为 $a+bi$, ~~则~~ 则取 $a-bi$

· It is possible to order the rows of H_0 so that the l zeros on its diagonal are located in the ~~right~~ lower right part of H_0, Denote the upper $(m-l)\times(m-l)$ block of H_0 by G_0. denote the corresponding upper left $(m-l)\times(m-l)$ ~~bec~~ block in H_1 be G_1., 注意 G_0 is invertible

$$H_0 = \begin{pmatrix} G_{0\,(m-l)\times(m-l)} & & \\ & 0 & \\ & & \ddots \\ & & & 0 \end{pmatrix} , \quad H_1 = \begin{pmatrix} G_1 & \\ & \boxed{\ddots} \end{pmatrix} \rightarrow \text{non-zero}.$$

· 令 Z的共轭转置 $Z^* = \begin{pmatrix} L_{1\,(m-l)\times m} \\ L_{2\,l\times m} \end{pmatrix}$

· 在(6)式中插入 $ZZ^*(=I)$ before Y_{t+1} and Y_t，然后左乘 Q.

$$\underbrace{QaZ}_{H_0}Z^*Y_{t+1} + \underbrace{QbZ}_{H_1}Z^*Y_t = 0, \quad t \geq 0$$

记 $\gamma_t \equiv Z^* Y_t$

$$\Rightarrow H_0 \cdot \gamma_{t+1} + H_1 \gamma_t = 0 \qquad (7)$$

对 γ_t 分块，得

$$\gamma_t = \begin{pmatrix} \gamma_t^1 \\ \gamma_t^2 \end{pmatrix}$$ where γ_t^1 is $(m-l)\times 1$, γ_t^2 is $l \times 1$

· it can be verified that (7) 式 implies $\gamma_t^2 = 0, t \geq 0$ 即

$L_2 Y_t = 0, t \geq 0 \rightarrow$ represents l restrictions on Y_0 (8)

$$\Rightarrow G_0 \cdot \gamma_{t+1}^1 + G_1 \cdot \gamma_t^1 = 0, \quad t \geq 0$$

由于 G_0 is invertible

$$\Rightarrow \gamma_t^1 = (-G_0^{-1}G_1)^t \gamma_0^1, \quad t \geq 0$$

记 $-G_0^{-1}G_1 = P \Lambda P^{-1}$

$$\Rightarrow P^{-1} \cdot \gamma_t^1 = \Lambda^t P^{-1} \cdot \gamma_0^1$$

由于(8)式中包含了对 Y_0 的 l 个约束条件，我们还有 $n-l$ 个自由变量.

所以，记 q 为 $-G_0^{-1}G_1$ 的 explosive eigenvalue 的个数

· case 1: $q = n-l$，有唯一 non-explosive 解.

记 $\hat{P}$ 为这 q 个 explosive eigenvalue 的左特征向量.

则构造 $D = \begin{pmatrix} \hat{P} L_1 \\ L_2 \end{pmatrix}$，$DY_0 = 0 \Rightarrow DY_0 = [D_1 : D_2]\begin{pmatrix} X_1 \\ X_0 \end{pmatrix} = 0$.

$$\Rightarrow X_1 = -D_1^{-1} \cdot D_2 X_0$$

· case 2: $q > n-l$，无 non-explosive 解

· case 3: $q < n-l$，multiple non-explosive 解

· 4.3 Recursive formulation of the Neoclassical Growth Model with Uncertainty

· The planner's problem in recursive version is:

$$V(k,z) = \max_{k'} \left\{ u[zf(k) - k' + (1-\delta)k] + \beta \sum_{z' \in Z} \pi(z'|z) V(k', z') \right\}.$$

where we have assumed a first order Markov process for $\{z_t\}_{t=0}^{\infty}$.

· The solution to this problem involves the policy rule:

$$k' = g(k, z)$$

· 如果我们进一步假设 Z 有有限个元素，

$$Z = \{z_1, \cdots, z_n\}$$

则上述问题可以写成如下形式：

$$V_i(k) = \max_{k'} \left\{ u[z_i f(k) - k' + (1-\delta)k] + \beta \sum_{j=1}^{n} \pi_{ij} V_j(k') \right\}.$$

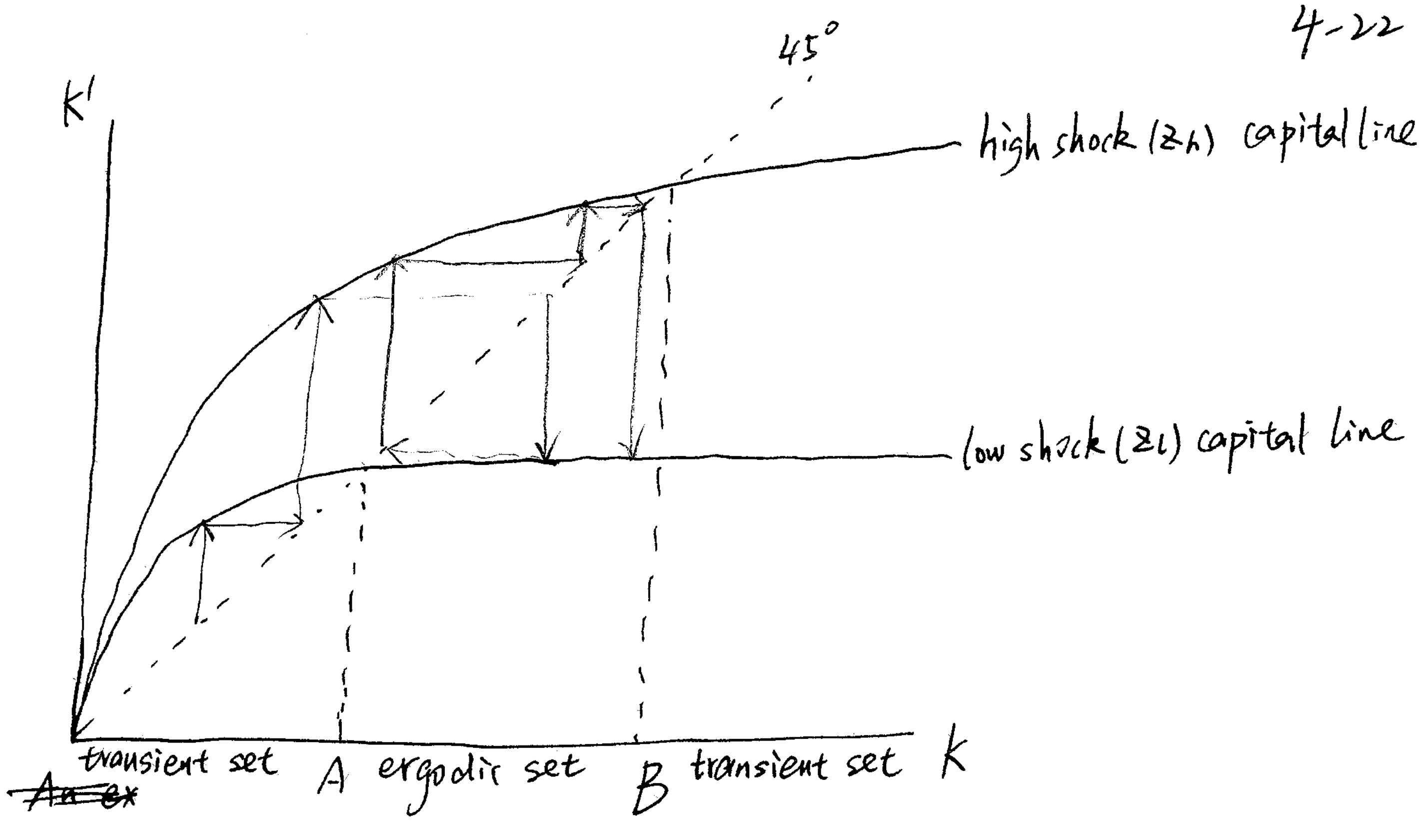
where $\pi_{ij} \equiv \pi(z_{t+1} = z_j \mid z_t = z_i)$

- 4.3.1 Stationary stochastic process for (k, z)

· We want to find the probability distribution over values of (k, z), which is preserved at $t+1$ if applied at time t.

· Example: z takes two values, $z \in \{z_L, z_H\}$

概念：
- transient set: a set of values of capital, which cannot occur in the long run
- ergodic set: a set, that the capital will never leave once it is there.



· An example of (k,z) stochastic process when $z \in (z_l, z_h)$

· 记 $P(k,z)$ 为 joint density，平稳的随机过程要求这一密度函数不随时间变化

· 由于 z 只有两个可能的取值，可记 $P(k,z) = (P_h(k), P_l(k))$

· 由于 $P(k,z)$ 为平稳状态下的密度函数，只有在资本 k 属于上述讨论的 ergodic set 中，$P(k,z)$ 的取值不为零

· $P(k,z)$ 满足如下属性：

1. $\int (P_h(k)+P_l(k))dk = 1$，即在整个 (z,k) 空间中的概率为 1

2. $\int P_h(k)dk = \pi_h$

   $\int P_l(k)dk = \pi_l$

   where π_l and π_h are invariant probabilities of z_l and z_h

3. $\text{Prob}\{k \le \bar{k}, z = z_h\} = \int_{k\le\bar{k}} P_h(k)dk = [\int_{k:g_h(k)\le\bar{k}} P_h(k)dk]\pi_{hh} + [\int_{k:g_l(k)\le\bar{k}} P_l(k)dk]\pi_{lh}$

$\text{Prob}\{k \le \bar{k}, z = z_l\} = \int_{k\le\bar{k}} P_h(k)dk = [\int_{k:g_h(k)\le\bar{k}} P_h(k)dk]\cdot\pi_{hl} + [\int_{k:g_l(k)\le\bar{k}} P_l(k)dk]\cdot\pi_{ll}$

4.3.2.

· Solving the model: linearization of the Euler equation

The planner's problem can give us the following Euler equation.

$$\underbrace{u'[zf(k)+(1-\delta)k-k']}_{LHS}=\underbrace{\beta E_z\left[u'[z'f(k')+(1-\delta)k'-k'']\cdot[z'f'(k')+1-\delta]\right]}_{RHS}$$

· example: suppose that $\{z_t\}_{t=0}^{\infty}$ follows an AR(1) process.

$$z_{t+1}=\rho z_t+(1-\rho)\bar{z}+\varepsilon_{t+1}$$

where $|\rho|<1$, $E(\varepsilon_t)=0$, $E(\varepsilon_t^2)=\sigma^2<\infty$, $E(\varepsilon_t\varepsilon_{t+j})=0\ \forall j\geq 1$

~~记 plim z_t~~

则 $\text{plim}\, z_t=\bar{z}$, i.e. the long run value of z_t

· 给定 z_t 的 long run value, the associated steady state level of capital $\bar{k}$ is solved from the usual deterministic Euler equation.

$$u'(\bar{c})=\beta u'(\bar{c})[\bar{z}f(\bar{k})+1-\delta]$$

$$\Rightarrow \frac{1}{\beta}=\bar{z}f(\bar{k})+1-\delta$$

$$\Rightarrow \bar{k}=f^{-1}\left(\frac{\beta^{-1}-(1-\delta)}{\bar{z}}\right)$$

$$\Rightarrow \bar{c}=\bar{z}f(\bar{k})-\delta\bar{k}$$

· 记 $\hat{k}\equiv k-\bar{k}$, $\hat{z}\equiv z-\bar{z}$ 即 deviations from steady state values.

(注意,这里我们用线性化而不是之前的对数线性化。事实上,这两种方法在实践中都很常用。)

· 我们利用一阶泰勒展开：

$$LHS \approx a_L \cdot \hat{z} + b_L \cdot \hat{k} + c_L \hat{k}' + d_L$$

$$RHS \approx E_z[a_R \cdot \hat{z}' + b_R \cdot \hat{k}' + c_R \cdot \hat{k}''] + d_R$$

where, the coefficients a_L, a_R, b_L, etc. are the derivatives of the expressions LHS and RHS with respect to the corresponding variables, evaluated at the steady state, for example. $a_L = u''(\bar{c}) \cdot f(\bar{k})$

另外，LHS = RHS 在 $\hat{z} = \hat{z}' = \hat{k} = \hat{k}' = \hat{k}'' = 0$ (the steady state) 时成立

所以有 $d_L = d_R$

· 我们可以利用差分方程的方法解，也可以采用 guess and verify 的方法，即，猜测

$$\hat{k}' = g_k \cdot \hat{k} + g_z \cdot \hat{z}$$

则 $LHS \approx a_L \cdot \hat{z} + b_L \cdot \hat{k} + c_L g_k \cdot \hat{k} + c_L \cdot g_z \cdot \hat{z} + d_L$

$$RHS \approx a_R \cdot E_z[\hat{z}'] + b_R \cdot g_k \cdot \hat{k} + b_R \cdot g_z \cdot \hat{z} + c_R \cdot g_k^2 \cdot \hat{k} + c_R \cdot g_k \cdot g_z \cdot \hat{z} + c_R \cdot g_z \cdot E_z[\hat{z}'] + d_R$$

LHS = RHS 对任意 $\hat{k}, \hat{z}$ 都成立，则

$$\hat{z} \cdot A + E_z[\hat{z}'] \cdot B + \hat{k} \cdot C = 0 \qquad (*)$$

where $A = a_L + c_L \cdot g_z - b_R g_z - c_R \cdot g_k \cdot g_z$

$B = -a_R - c_R \cdot g_z$

$C = b_L + c_L \cdot g_k - b_R g_k - c_R \cdot g_k^2$

由于我们假设：$z_{t+1} = \rho z_t + (1-\rho)\bar{z} + \varepsilon_{t+1}$

then,
$$\begin{aligned}\hat{z}' &\equiv z' - \bar{z} \\ &= \rho z + (1-\rho)\bar{z} + \varepsilon' - \bar{z} \\ &= \rho \underbrace{(z-\bar{z})}_{\hat{z}} + \varepsilon'\end{aligned}$$

$\Rightarrow$ $E_z(\hat{z}') = \rho \hat{z}$

利用上式，我们把(*)式写为：

$$\hat{z} A + \hat{k} B = 0$$

where
$$A = a_L + c_L \cdot g_z - a_K \cdot \rho - b_K \cdot g_z - c_K \cdot g_K \cdot g_z - c_K \cdot g_z \cdot \rho$$
$$B = b_L + c_L \cdot g_K - b_K \cdot g_K - c_K \cdot g_K^2$$

则解满足：$A = 0$
$B = 0$

4.3.3. Simulation and impulse response

· 给定我们解出 g_K, g_z 后，我们可以通过随机抽取$\{z_t\}$对模型进行数值模拟

· 假设我们对一个一次性的技术冲击感兴趣，则 $\hat{z}_0 = \Delta$，$\hat{z}_{t+1} = \rho \hat{z}_t$，

则我们有：
$$\hat{k}_0 = 0$$
$$\hat{k}_1 = g_z \cdot \Delta$$
$$\hat{k}_2 = g_K \cdot \hat{k}_1 + g_z \cdot \rho \cdot \Delta = (g_K \cdot g_z + g_z \rho) \cdot \Delta$$
$$\vdots$$
$$\hat{k}_t = (g_K^{t-1} + g_K^{t-2} \cdot \rho + \cdots + g_K \cdot \rho^{t-2} + \rho^{t-1}) \cdot g_z \cdot \Delta$$

and $|g_K| < 1, |\rho| < 1 \Rightarrow \lim_{t\to\infty} \hat{k}_t = 0$

· Recursive formulation issue

· There is one more issue to discuss: it involves the choice of state variable in recursive formulation. 选择状态变量的技巧.

考虑如下问题:

$$\max_{\{c_t(z^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi(z^t) u(c_t(z^t))$$

s.t. $z^t = (z_l, z^{t-1})$: $c_t(z^t) + q_{h,t}(z^t) a_{h,t+1}(z^t) + q_{l,t}(z^t) a_{l,t+1}(z^t) = w_t(z^t) + a_{l,t}(z^{t-1})$, $\forall t, z^t$

$z^t = (z_h, z^{t-1})$: $c_t(z^t) + q_{h,t}(z^t) \cdot a_{h,t+1}(z^t) + q_{l,t}(z^t) \cdot a_{l,t+1}(z^t) = w_t(z^t) + a_{h,t}(z^{t-1})$, $\forall t, z^t$

and no-Ponzi-game condition

where z_t follows a first order Markov process and $z_t \in \{z_h, z_l\}$.

To simplify matters, suppose that

$$z_t = z_l: \ w_t(z^t) = w_l$$
$$z_t = z_h: \ w_t(z^t) = w_h$$

那么,哪些变量是状态变量? z_t 自然是其中之一。

另一个状态变量则为 wealth, 记为 x, 我们将其定义为:

$$z_t = z_l: \ x_t(z^t) = w_l + a_{l,t}(z^{t-1})$$
$$z_t = z_h: \ x_t(z^t) = w_h + a_{h,t}(z^{t-1})$$

即, the sum of the endowment and the income from asset holdings.

The recursive formulation is now.

$$V(x, z_i) \equiv V_i(x) = \max_{a_l', a_h'} \{ u(x - q_{ih} a_h' - q_{il} a_l') + \beta [\pi_{ih} V_h(\underbrace{w_h + a_h'}_{x_h'}) + \pi_{il} V_l(\underbrace{w_l + a_l'}_{x_l'})] \}$$

where the policy rules are now

$$a_h' = g_{ih}(w)$$
$$a_l' = g_{il}(w), \quad i = l, h$$

是否可以用 a 做为状态变量? 是! 但是在那种情况下我们将不得不使用两个状态变量: a_l 和 a_h, 所以 最恰当的状态变量将是 wealth, x

4.3 Competitive equilibrium under uncertainty

4.3.1 The neoclassical growth model with complete markets

· Arrow-Debreu date-0 trading

The Arrow-Debreu date-0 competitive equilibrium is

$$\{c_t(z^t), k_{t+1}(z^t), l_t(z^t), P_t(z^t), r_t(z^t), w_t(z^t)\}_{t=0}^{\infty}$$

such that

1. Consumer's problem is to find $\{c_t(z^t), k_{t+1}(z^t), l_t(z^t)\}_{t=0}^{\infty}$ which solve

$$\max_{\{c_t(z^t), k_{t+1}(z^t), l_t(z^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi(z^t) u(c_t(z^t), 1-l_t(z^t))$$

$$\text{s.t. } \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} P_t(z^t)[c_t(z^t) + k_{t+1}(z^t)] \leq \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} P_t(z^t)[(r_t(z^t)+1-\delta)k_t(z^{t-1}) + w_t(z^t) \cdot l_t(z^t)]$$

2. First-order conditions from firm's problem are:

$$r_t(z^t) = z_t F_k(k_t(z^{t-1}), l_t(z^t))$$

$$w_t(z^t) = z_t F_L(k_t(z^{t-1}), l_t(z^t))$$

3. Market clearing is

$$c_t(z^t) + k_{t+1}(z^t) = (1-\delta) k_t(z^{t-1}) + z^t F(k_t(z^{t-1}), l_t(z^t)), \forall t, \forall z^t$$

You should be able to show the Euler equation in this problem is identical to the Euler equation in the planner's problem.

· 我们将上述问题中,所有关于 $k_{t+1}(z^t)$ 的项整理到一起:

$$\sum_{t=0}^{\infty} \sum_{z^t \in Z^t} k_{t+1}(z^t)\left[-P_t(z^t) + \sum_{z_{t+1} \in Z} P_{t+1}(z_{t+1}, z^t)[r_{t+1}(z_{t+1}, z^t) + 1 - \delta]\right]$$

If $-P_t(z^t) + \sum_{z_{t+1} \in Z} P_{t+1}(z_{t+1}, z^t)[r_{t+1}(z_{t+1}, z^t) + 1 - \delta] \gtrless 0$, then $k_{t+1}(z^t) = \infty$ or $-\infty$ gives unbounded wealth. 所以,在均衡中,上述式子等于零。这一条件称为 no-arbitrage condition, stating in equilibrium the price of a unit of capital must equal its future value summed across all states.

· Sequantial trade

· We assume the existence of one-period assets, and Z is a finite set with n possible shock values:

$$
t \to \begin{cases} z_{t+1} = z_1 \in Z \\ z_{t+1} = z_2 \in Z \\ \quad \vdots \\ z_{t+1} = z_n \in Z \end{cases}
$$

· Assume there are q assets, with asset j paying off r_{ij} consumption units in $t+1$ if the realized state is z_i. The following matrix shows the payoff of each asset for every realization of z_{t+1}:

$$
\begin{array}{c|c}
 & \begin{array}{cccc} a_1 & a_2 & \cdots & a_q \end{array} \\
\begin{array}{c} z_1 \\ z_2 \\ \vdots \\ z_n \end{array} & \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1q} \\ r_{21} & r_{22} & \cdots & r_{2q} \\ \vdots & \vdots & & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nq} \end{pmatrix} \equiv R
\end{array}
$$

· Then the portfolio $a = (a_1, a_2, \cdots, a_q)'$ pays p (in terms of consumption goods at $t+1$), where

$$
\underset{n\times 1}{p} = \underset{n\times q}{R} \cdot \underset{q\times 1}{a}
$$

and each component $p_i = \sum_{j=1}^{q} r_{ij} a_j$ is the amount of consumption goods obtained in state i from holding portfolio a.

· If $\text{rank}(R) = n$, then the market structure is complete.

· example, Arrow security with $q < n \Rightarrow$ incomplete

$$
\begin{array}{c|c}
 & \begin{array}{cccc} a_1 & a_2 & \cdots & a_q \end{array} \\
\begin{array}{c} z_1 \\ z_2 \\ \vdots \\ z_q \\ \vdots \\ z_n \end{array} & \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}
\end{array}
$$

4.3.2 General equilibrium under uncertainty: the case of two agents types in a two-period setting

Assumptions:

1) Random shock: $z \in \{z_1, z_2, \cdots, z_n\}$

$$\pi_j = Pr(z = z_j)$$

$$\bar{z} = \sum_{j=1}^{n} \pi_j z_j \text{, the expected value of } z$$

2) preference. $U_i = u_i(c_0^i) + \beta \sum_{j=1}^{n} \pi_j u_i(c_j^i)$, $i = 1, 2$

where $u_1(x) = x$, $u_2(x)$ is strictly concave ($u_2' > 0$, $u_2'' < 0$). we also assume that $\lim_{x \to 0} u_2'(x) = \infty$.

That is, Agent 1 is risk neutral and Agent 2 is risk averse.

3) Endowments: No consumption goods in period 0.

one unit of labor in period 1.

4) Technology: $y_j = z_j K^\alpha \left(\frac{n}{2}\right)^{1-\alpha}$

we know that $n = 2$, so

$$y_j = z_j K^\alpha$$

$\Rightarrow$ in period 1, if state j is realized, we have

$$r_j = z_j \alpha K^{\alpha - 1}$$

$$w_j = z_j \frac{(1-\alpha)}{2} K^\alpha$$

· Structure 1 - one asset. (incomplete market)

· Capital is the only asset that is traded in this setup.

With K denotes the aggregate capital stock, a_i denotes the capital stock held by agent i, so the asset market clearing requires that:

$$a_1 + a_2 = K$$

the budget constraints for each agent is:

$$c_0^i + a_i = w_0 \to \text{endowment}$$

$$c_j^i = a_i r_j + w_j \to \text{wage rate}$$

· Agent 1: The maximized utility function and the constraints are linear in this case. We therefore use the no-arbitrage condition to express optimality. (注:亦可通过Euler方程得到类似结论)

$$\left[-1 + \beta \sum_{j=1}^{n} \pi_j r_j\right] a_i = 0$$

$$\Rightarrow \quad 1 = \beta \sum_{j=1}^{n} \pi_j \alpha z_j K^{\alpha-1} = \alpha \beta K^{\alpha-1} \underbrace{\sum_{j=1}^{n} \pi_j z_j}_{\bar{z}}$$

⇒ The optimal choice of aggregate capital K, from Agent 1's preferences is given by:

$$K^* = (\bar{z} \alpha \beta)^{\frac{1}{1-\alpha}}$$

注:只有随机冲击的均值有关,这是因为 Agent 1 是 risk neutral.

· Agent 2: Euler equation:

$$u_2'(w_0 - a_2) = \beta \sum_{j=1}^{n} \pi_j u_2'(a_2 r_j^* + w_j^*) r_j^*$$

Given K^* from Agent 1's problem, we have the values of r_j^* and w_j^*.

we can ~~so~~ solve for a_2^*, and $a_1^* = K^* - a_2^*$, $c_0^i = w_0 - a_i^*$

More importantly, Agent 2 will face a stochastic consumption prospect for period 1,

$$c_j^2 = a_2^* \cdot r_j^* + w_j^*, \text{ where } r_j^* \text{ and } w_j^* \text{ are stochastic.}$$

This implies that Agent 1 has NOT provided full insurance to Agent 2

- Structure 2 – Arrow securities (complete market)
  - It is allowed to trade in n different Arrow securities in this setup.
  - Let a_j denote the Arrow security paying off one unit of the realized state is z_j and zero otherwise. Let q_j denote the price of a_j.

  - Total savings are thus given by

$$S \equiv \sum_{j=1}^{n} q_j (a_{1j} + a_{2j})$$

  - Investment is the accumulation of capital, K. Then clearing of the savings-investment market requires that:

$$\sum_{j=1}^{n} q_j (a_{1j} + a_{2j}) = K$$

  - Total remuneration to capital services in state j is $r_j K$

    clearing of (all of) the Arrow security markets requires that

$$a_{1j} + a_{2j} = K r_j, \quad j = 1, \cdots, n$$

$$\Rightarrow \sum_{j=1}^{n} q_j (a_{1j} + a_{2j}) = K \sum_{j=1}^{n} q_j r_j$$

$$\Rightarrow K = K \sum_{j=1}^{n} q_j r_j$$

$$\Rightarrow \sum_{j=1}^{n} q_j r_j = 1, \text{ a no-arbitrage condition.}$$

· The budget constraint of each Agent i is

$$C_0^i + \sum_{j=1}^{n} q_j a_{ij} = w_0$$

$$C_j^i = a_{ij} + w_j, \quad \text{for } j = 1, \cdots, N$$

using the first-order conditions of Agent 1's problem, the equilibrium prices are:

$$q_j = \beta \pi_j$$

and $K^* = (\bar{z} \alpha \beta)^{\frac{1}{1-\alpha}}$ (~~check~~ derive it yourself)

· Agent 2's problem yields the Euler equation.

$$u_2'(c_0^2) = \lambda = q_j^{-1} \beta \pi_j u_2'(c_j^2)$$

since $q_j = \beta \pi_j$

$$\Rightarrow u_2'(c_0^2) = u_2'(c_j^2), \quad j = 1, \cdots, n$$

Therefore, with the new market structure, Agent 2 is able to obtain full insurance from Agent 1.

· General equilibrium under uncertainty: multiple-period model with two agent types.

· Structure 1 – one asset

Agent 1's problem is

$$\cancel{\max \sum_{z^t \in Z^t} \sum_{t=0}^{\infty} \beta^t}$$

$$\max \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi(z^t) c_t(z^t)$$

$$\text{s.t. } c_{1t}(z^t) + a_{1t+1}(z^t) = r_t(z^t) a_{1t}(z^{t-1}) + w_t(z^t)$$

Firm's problem yields

$$r_t(z^t) = z_t \alpha k_t^{\alpha-1}(z^{t-1}) + 1 - \delta$$

$$w_t(z^t) = z_t \left(\frac{1-\alpha}{2}\right) k_t^{\alpha}(z^{t-1})$$

Market clearing condition is (for capital)

$$a_{1t+1}(z^t) + a_{2t+1}(z^t) = k_{t+1}(z^t)$$

F.O.C w.r.t. $a_{1t+1}(z^t)$ gives us

$$1 = \beta \sum_{z_{t+1}} \frac{\pi(z_{t+1}, z^t)}{\pi(z^t)} r_{t+1}(z_{t+1}, z^t)$$

$$\Rightarrow 1 = \beta E_{z_{t+1}|z^t}(r_{t+1})$$

Using the formula for r_{t+1} from firm's first-order conditions, we get

$$1 = \beta \sum_{z_{t+1}} \pi(z_{t+1}|z^t)\left(z_{t+1} \alpha k_{t+1}^{\alpha-1}(z^t) + 1 - \delta\right)$$

$$= \alpha \beta k_{t+1}^{\alpha-1}(z^t) \underbrace{\sum_{z_{t+1}} \pi(z_{t+1}|z^t) z_{t+1}}_{E(z_{t+1}|z^t)} + \beta(1-\delta)$$

$$\Rightarrow k_{t+1}(z^t) = \left[\frac{1/\beta - 1 + \delta}{\alpha E(z_{t+1}|z^t)}\right]^{\frac{1}{\alpha-1}} \quad (1)$$

Agent 2's utility function is $u(c_{2t}(z^t))$ and his first-order conditions yield:

$$u'(c_{2t}(z^t)) = \beta E_{z_{t+1}|z^t}\left[u'(c_{2,t+1}(z^{t+1}))(1-\delta + \alpha z_{t+1} k_{t+1}^{\alpha-1}(z^t))\right]. \quad (2)$$

利用(1),(2)式以及 Agent 2 的预算约束,可以解出 $c_{2t}(z^t)$ 以及 $a_{2,t+1}(z^t)$。进一步,利用市场出清可以解出 $c_{1t}(z^t)$

Structure 2 – Arrow securities

这一建模方式下,只有预算约束有相应改变:

$$c_t^i(z^t) + \sum_{j=1}^{n} q_j(z^t) a_{j,t+1}^i(z^t) = a_{i,t}^i(z^{t-1}) + w_t(z^t)$$

由于我们这里有多种资产,无套利条件需要成立(no-Arbitrage)

$$\sum_{j=1}^{n} q_j(z^t) a_{j,t+1}(z^t) = k_{t+1}(z^t)$$

(注:这一式中都没有上标,即 $a_{j,t+1}$ 和 k_{t+1} 均表示总量,是 aggregate variables.)

(上式亦可看作资本市场出清)

$$a_{j,t+1}(z^t) = [1-\delta + z_j \alpha k_{t+1}^{\alpha-1}(z^t)] \cdot k_{t+1}(z^t)$$

$$\Rightarrow 1 = \sum_{j=1}^{n} q_j(z^t)[1-\delta + z_j \alpha k_{t+1}^{\alpha-1}(z^t)]$$

solving the first-order condition of Agent 1 w.r.t. $q_{j,t+1}(z^t)$ yields

$$q_{j,t}(z^t) = \beta \frac{\pi(z_j, z^t)}{\pi(z^t)} = \beta \pi(z_j|z^t) \quad (3)$$

Agent 2:

The first-order condition w.r.t. $a^2_{j,t+1}(z^t)$ yields

$$0 = -\beta^t \pi(z^t) q_{j,t}(z^t) u'(c^2_t(z^t)) + \beta^{t+1} \pi(z_j, z^t) u'(c^2_{t+1}(z_j, z^t))$$

把(3)式代入，得

$$0 = -\beta^t \pi(z^t) \beta \frac{\pi(z_j, z^t)}{\pi(z^t)} u'(c^2_t(z^t)) + \beta^{t+1} \pi(z_j, z^t) u'(c^2_{t+1}(z_j, z^t))$$

$$\Rightarrow u'(c^2_t(z^t)) = u'(c^2_{t+1}(z_j, z^t))$$

$$\Rightarrow c^2_t(z^t) = c^2_{t+1}(z_j, z^t)$$

Agent 2 insures completely and his consumption does not vary across states.