1. (The method of successive approximations). Consider the sequence problem.

$$
\begin{aligned}
V^*(k_0) &= \max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(c) \qquad (1)\\
&\quad \text{subject to}\\
c_t + k_{t+1} &\leq Ak_t,\ t = 0, 1, \ldots,\\
c_t,\ k_{t+1} &\geq 0,\\
&\quad k_0 \text{ given.}
\end{aligned}
$$

Next, define the operator T:$\mathbf{C}(\mathbf{R}) \to \mathbf{C}(\mathbf{R})$,

$$
TV(k) = \max\Big(\log(Ak - k') + \beta V(k')\Big). \qquad (2)
$$

(a) Show $V^*(k_0)$ is finite for any $\beta \in (0,1)$ by establishing that $V^*(k_0) < \overline{V}(k_0) < \infty$ where the latter function is the value of an infeasible policy. Define $\overline{V}(k_0)$ as the value of the following the policy $c_t = Ak_t$ *and* $k_{t+1} = Ak_t$, $\forall t$.

(b) Use the method of successive approximation to solve for V, the fixed point of (2). Let $V^0 = 0$ and use the operator T to define V^1. Next, for each V^N, $N = 1, \ldots$, use V^N in the right-hand side of (2) and solve the maximisation problem to derive the decision rule, $k' = g_N(k)$. Substitution of this rule into (2), with $V^N(g_N(k))$ in the right-hand side, generates $V^{N+1} = TV^N$.

(c) Derive $V \equiv \lim_{N\to\infty} T^N V^0$. What is the savings rate and how does it vary with a change in β? Explain your answer.

2. (The method of undetermined coefficients). Consider the infinite horizon Ak growth model. An infinitely-lived representative household values consumption in each period. Capital, the sole factor of production, is owned by the household. The marginal product of capital is constant and, given its initial stock, the household solves the following problem.

$$
\begin{aligned}
&\max_{\{c_t,k_{t+1}\}} \sum_{t=0}^{\infty} \beta \frac{c_t^{1-\sigma}}{1-\sigma} \qquad . \qquad (3)\\
&\quad \text{subject to}\\
c_t + k_{t+1} &\leq Ak_t,\ t = 0, 1, \ldots,\\
&\quad k_0 \text{ given.}
\end{aligned}
$$

Assume that $\beta A^{1-\sigma} < 1$, which ensures that utility is bounded. Optimal behaviour for this problem may be characterised using the corresponding functional equation,

$$
v(k) = \max_{0 \leq k' \leq Ak} \left(\frac{[Ak - k']^{1-\sigma}}{1-\sigma} + \beta v(k') \right), \qquad (4)
$$

where we have substituted out for c_t using the resource constraint. For this problem there is a unique function v satisfying 4, and it attains the value in 3. Conjecture that the value function takes the form

$$v(k) = E\frac{(k)^{1-\sigma}}{1-\sigma}, \tag{5}$$

where $E > 0$ is a function of the parameters of the problem: A, σ and β.

(a) Using (5) for $v(k')$ in the right hand side of 4, solve the maximisation problem. In other words, use the first-order condition to derive an expression for k'. I'll use $k' = g(k; E)$ to describe your result, which you should label as (A3). Now substitute this function for k' back into (4), and use it to solve for E. That is, solve the following equation for E :

$$E\frac{(k)^{1-\sigma}}{1-\sigma} = \frac{[Ak - g(k;E)]^{1-\sigma}}{1-\sigma} + \beta E\frac{(g(k;E))^{1-\sigma}}{1-\sigma}.$$

(b) Having determined E, provide a solution for $k' = g(k)$ in terms of the primitives of the problem (A, σ, β). Describe the growth rate of output (Ak_t) as γ_y and the savings rate $(\frac{k_{t+1}}{Ak_t})$ as s. When $\sigma < 1$, how does a rise in A change each? Explain your answer.