Stats 545: Midterm exam

This is a 75-minute exam for 32 points. Write your name and PUID on each sheet, and also include the number of answer sheets. Attempt all questions.

1 Miscellaneous [6 pts]

1. Briefly explain what Bayesian inference is. Give one advantage and one disadvantage of it. [2pts]
2. Give an example of a conjugate prior, justifying you choice. [2pts]
3. Write down the update rule for Newton's method and explain its intuition. [2pts]

2 Monte Carlo estimation [8 pts]

1. You want to sample from a distribution $p(x) \propto \exp(-x^2/2)\mathbb{1}(x \in [0, c])$ (that equals 0 outside the interval $[0, c]$). For rejection sampling, two possible proposal distributions are the Gaussian and the uniform distributions. For each, give the upper bound, and the acceptance probability. Briefly explain which you might prefer for different settings of c. [4pts]
2. Show that importance sampling estimates are unbiased. Recall that the importance weight $w(x)$ equals $p(x)/q(x)$. Often, $p(x) = f(x)/Z$ where we cannot calculate Z. Explain how to get around this problem. [2pts]
3. You want to estimate $\int_{-\infty}^{\infty} \log(|x|)/x^2 \mathrm{d}x$ using Monte Carlo. You only have access to Gaussian and uniform random number generators. Provide a Monte Carlo scheme for a consistent estimate of this. [2pts]

3 Metropolis-Hastings [8 pts]

1. For a Metropolis-Hastings algorithm with proposal distribution $q(x_{new}|x_{old})$ that targets $\pi(x)$, write down the acceptance probability. What is the transition probability? Show that this satisfies detailed balance. [3pts]
2. Consider a Metropolis-Hastings (MH) algorithm where you propose x_{new} from $\mathtt{U}(0,1)$, a uniform distribution on $[0, 1]$, and accept with probability $\min(1, \frac{2+x_{old}^2}{2+x_{new}^2})$. What is the stationary distribution? Is this unique? Explain. [3pts]
3. What does it mean to mix two MCMC kernels? Give a situation where you might want to mix MH kernels with proposal variances equal to 1 and 10. [2pts]

4 Gibbs [3 pts]

Recall the Poisson distribution is given by $p(x) = \lambda^x \exp(-\lambda)/x!$. Let x be sampled from this distribution, and y from a standard normal. You want to sample from $p(x, y|y > x)$ via Gibbs.

1. What are the conditional distributions $p(x|y, y > x)$ and $p(y|x, y > x)$? What families do these belong to? How will you sample from these given only uniform and Gaussian random number generators? [3pts]

5 Miscellaneous MCMC [7 pts]

1. Give a Markov chain that has no stationary distribution. [1pt]
2. Give an example of a Markov chain that has exactly two stationary distributions (stating what these are). [1pt]
3. Briefly explain what Rao-Blackwellization and data-augmentation are in the context of MCMC. [2pt]
4. For parallel tempering to sample from $p(x)$, briefly explain a) what problem it is trying to overcome, b) what we mean by 'tempering', c) what the target distribution of the overall algorithm is, and d) what the MCMC update steps are. [3pt]