Assignment 2 : Manipulator Kinematics

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1 Question 1 : Prove that once the co-ordinate frames are assigned according to DH conventions, there exists unique DH parameters such that the homogenous transformation can be expressed as a combination of 2 rotation and 2 translation matrices

From the discussion in class, we know that 4 fundamental operations are involved in making $(k-1)^{th}$ frame coincident with k^{th} frame :

- Rotate L_{k-1} about z_{k-1} by θ_k
- Translate L_{k-1} along z_{k-1} by d_k
- Translate L_{k-1} along x_{k-1} by a_k
- Rotate L_{k-1} about x_{k-1} by α_k

This gives rise to the transformation matrix given by,

$$T = R(\theta, z)Trans(d, z)Trans(a, x)R(\alpha, x)$$

i.e.,

$$T_{k-1}^{k}(\theta_k, d_k, a_k, \alpha_k) = R(\theta_k, 3)Trans(d_k, 3)Trans(a_k, 1)R(\alpha_k, 1)$$

We can write T as,

$$\begin{aligned} T_{k-1}^{k} &= \begin{bmatrix} \cos\theta_k & -\cos\alpha_k \sin\theta_k & \sin\alpha_k \sin\theta_k & a_k \cos\theta_k \\ \sin\theta_k & \cos\alpha_k \cos\theta_k & -\sin\alpha_k \cos\theta_k & a_k \sin\theta_k \\ 0 & \sin\alpha_k & \cos\alpha_k & d_k \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} R_{k-1}^{k} & P_{k-1}^{k} \\ 0 & 1 \end{bmatrix} \end{aligned}$$

where,

$$R_{k-1}^{k} = \begin{bmatrix} \cos\theta_k & -\cos\alpha_k \sin\theta_k & \sin\alpha_k \sin\theta_k \\ \sin\theta_k & \cos\alpha_k \cos\theta_k & -\sin\alpha_k \cos\theta_k \\ 0 & \sin\alpha_k & \cos\alpha_k \end{bmatrix} = \begin{bmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{bmatrix}$$

We can see that

$$
\begin{aligned}
r_{00}^2 + r_{10}^2 + r_{20}^2 &= 1 \\
r_{01}^2 + r_{11}^2 + r_{21}^2 = r_{00}^2 + r_{10}^2 + r_{20}^2 &= 1
\end{aligned}
$$

Giving us 2 unknowns θ_k and α_k and sufficient equations to solve for unique values of θ_k and α_k that would satisfy them.

Now, we know that

$$
\begin{aligned}
P_{k-1}^{k} &= \begin{bmatrix} a_k \cos\theta_k \\ a_k \sin\theta_k \\ d_k \end{bmatrix} \\
&= a_k \begin{bmatrix} \cos\theta_k \\ \sin\theta_k \\ 0 \end{bmatrix} + d_k \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\end{aligned}
$$

Here, $(a_k \cos\theta_k)^2 + (a_k \sin\theta_k)^2 = 1$ Hence, we have 2 unknowns a_k and d_k and 2 equations to solve them giving us unique solutions for a_k and d_k.

2 Question 2 : Figure below shows a 6-dof manipulator

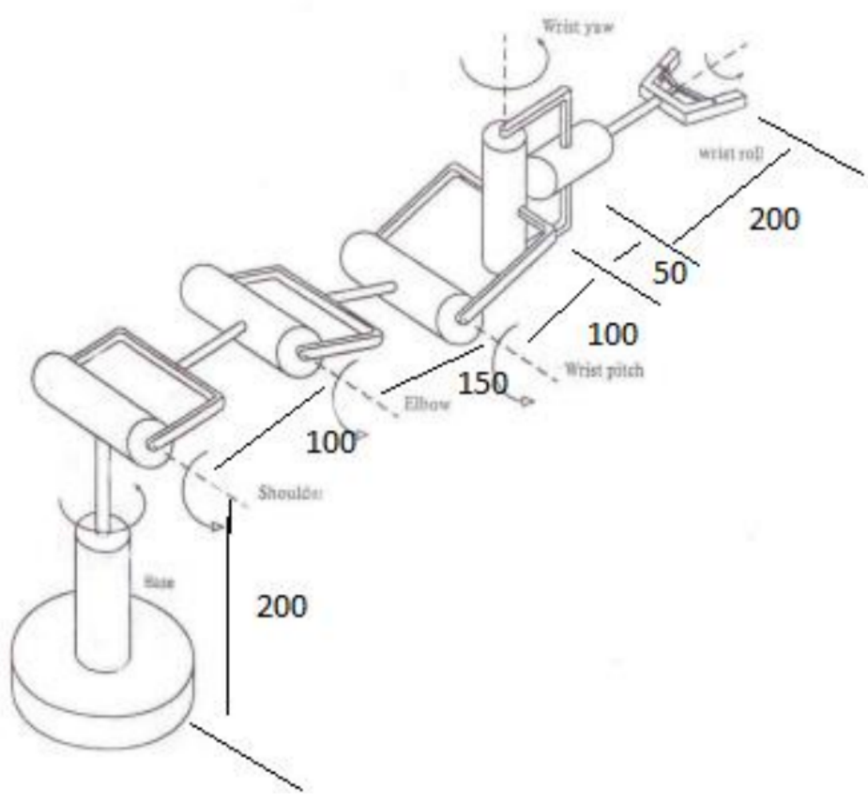

Figure 1: PUMA

2.1 Assign coordinate frames to all the links

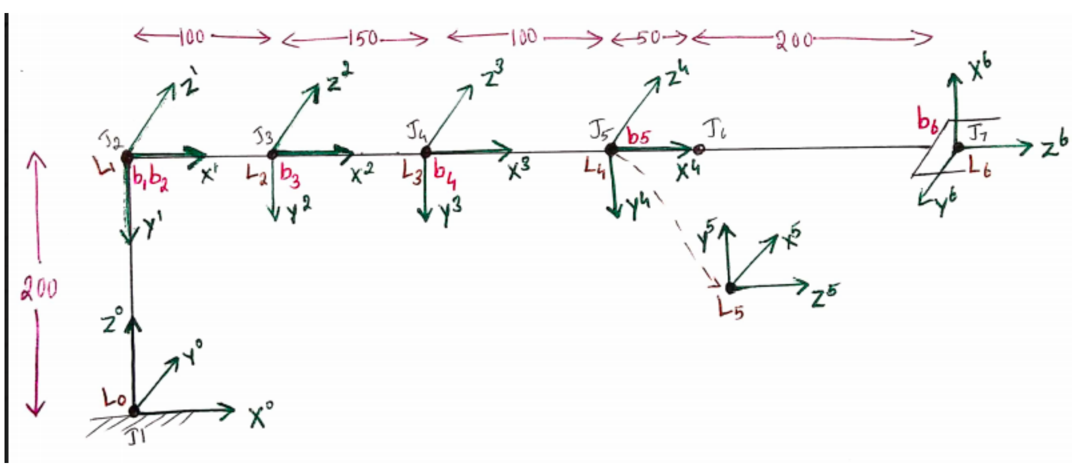

Figure 2: Home position

2.2 Determine DH parameters

Sl. No	$\boldsymbol{\theta}$	$\boldsymbol{d}$	$\boldsymbol{a}$	$\boldsymbol{\alpha}$
1	$\theta_1 = 0^0$	200	0	-90^0
2	$\theta_2 = 0^0$	0	100	0^0
3	$\theta_3 = 0^0$	0	150	0^0
4	$\theta_4 = 0^0$	0	100	90^0
5	$\theta_5 = 90^0$	0	0	90^0
6	$\theta_6 = -90^0$	250	0	0^0

2.3 Using the numerical values of the DH parameters, get the Transformation matrix T_{base}^{elbow}

The general Link Coordinate transformation matrix is given by,

$$T_{k-1}^{k} = \begin{bmatrix} \cos\theta_k & -\cos\alpha_k\sin\theta_k & \sin\alpha_k\sin\theta_k & a_k\cos\theta_k \\ \sin\theta_k & \cos\alpha_k\cos\theta_k & -\sin\alpha_k\cos\theta_k & a_k\sin\theta_k \\ 0 & \sin\alpha_k & \cos\alpha_k & d_k \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence, by substituting the above values with $\theta)k$ as variable gives us,

$$T_0^1 = \begin{bmatrix} \cos\theta_1 & -\cos\alpha_1\sin\theta_1 & \sin\alpha_1\sin\theta_1 & a_1\cos\theta_1 \\ \sin\theta_1 & \cos\alpha_1\cos\theta_1 & -\sin\alpha_1\cos\theta_1 & a_1\sin\theta_1 \\ 0 & \sin\alpha_1 & \cos\alpha_1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -1 & 0 & 200 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_1^2 = \begin{bmatrix} \cos\theta_2 & -\cos\alpha_2\sin\theta_2 & \sin\alpha_2\sin\theta_2 & a_2\cos\theta_2 \\ \sin\theta_2 & \cos\alpha_2\cos\theta_2 & -\sin\alpha_2\cos\theta_2 & a_2\sin\theta_2 \\ 0 & \sin\alpha_2 & \cos\alpha_2 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & 100\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & 100\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here, $T_0^1 = T_{base}^{shoulder}$ and $T_1^2 = T_{shoulder}^{elbow}$

Hence,

$$\begin{aligned} T_{base}^{elbow} &= T_{base}^{shoulder} \times T_{shoulder}^{elbow} \\ &= \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -1 & 0 & 200 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & 100\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & 100\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta_1\cos\theta_2 & -\cos\theta_1\sin\theta_2 & -\sin\theta_1 & 100\cos\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 & -\sin\theta_1\sin\theta_2 & \cos\theta_1 & 100\sin\theta_1\sin\theta_2 \\ -\sin\theta_2 & -\cos\theta_2 & 0 & -100\sin\theta_2 + 200 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

2.4 Verify your results for an assumed soft home position of arm

For assumed home position,

Sl. No	$\boldsymbol{\theta}$
1	$\theta_1 = 0^0$
2	$\theta_2 = 0^0$

Substituting the values of $\boldsymbol{\theta}$ in the above equation of T_{base}^{elbow} gives us,

$$\begin{aligned} T_{base}^{elbow} &= T_{base}^{elbow}(\theta_1, \theta_2) \\ &= T_{base}^{elbow}(0, 0) \\ &= \begin{bmatrix} 1 & 0 & 0 & 100 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 200 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Here we have P = [100, 0, 200] which is correct w.r.t to our base frame and hence verified :) .

Here

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

3 Question 3 : Write a computer program to solve the forward kinematics of a manipulator. Specifically, given the joint variables and DH parameters, your program should be able to:

3.1 Calculate a T_{i-1}^{i} matrix

3.2 Calculate the manipulator transformation matrix, i.e., T_0^n

3.3 Calculate the Cartesian space coordinates of the end-effector, i.e., the position vector and the orientation relative to the base of the manipulator

3.4 Your program should be written in a generic way independent of any specific manipulator configuration. Use C / C++ /Python programming language. Use the data from the PUMA manipulator to test your program. Assume non-zero joint angles for the arm. Submit your source codes (well-documented) and the output of calculation for the PUMA

The code is available in the folder as ***fw_kinematics.py***.

For home position,

```
[Gangas-Air:Downloads gangameghanath$ python fw_kinematics.py

Choose 1 or 2 :

         1. T {0} to {n}

         2. T {i} to {j}

         3. T {k-1} to {k}
1

Enter value of theta as : theta1, theta2, etc
0,0,0,0,90,-90

Enter value of 'a' as : a1, a2, etc
0,100,150,100,0,0

Enter value of 'd' as : d1, d2, etc
200,0,0,0,0,250

Enter value of alpha as : alpha1, alpha2, etc
-90,0,0,90,90,0

The transformation matrix is given as T = [[  6.12e-17   6.12e-17   1.00e+00   6.00e+02]
 [  6.12e-17   1.00e+00  -6.12e-17  -1.53e-14]
 [ -1.00e+00   6.12e-17   6.12e-17   2.00e+02]
 [  0.00e+00   0.00e+00   0.00e+00   1.00e+00]]

The position coordinate is given as P = [  6.00e+02  -1.53e-14   2.00e+02]

Orientation relative to the base of the manipulator R = [[  6.12e-17   6.12e-17   1.00e+00]
 [  6.12e-17   1.00e+00  -6.12e-17]
 [ -1.00e+00   6.12e-17   6.12e-17]]
```

Figure 3: T_0^n

The calculated T_{base}^{elbow} is given as follows :

```
[Gangas-Air:Downloads gangameghanath$ python fw_kinematics.py

Choose 1 or 2 :

         1. T {0} to {n}

         2. T {i} to {j}

         3. T {k-1} to {k}
2

Enter value of theta as : theta1, theta2, etc
0,0,0,0,90,-90

Enter value of 'a' as : a1, a2, etc
0,100,150,100,0,0

Enter value of 'd' as : d1, d2, etc
200,0,0,0,0,250

Enter value of alpha as : alpha1, alpha2, etc
-90,0,0,90,90,0

Enter the value of i (0 to n-1):
0

Enter the value of j (1 to n) :
2

The transformation matrix is given as T = [[  1.00e+00   0.00e+00   0.00e+00   1.00e+02]
 [  0.00e+00   6.12e-17   1.00e+00   0.00e+00]
 [  0.00e+00  -1.00e+00   6.12e-17   2.00e+02]
 [  0.00e+00   0.00e+00   0.00e+00   1.00e+00]]

The position coordinate is given as P = [ 100.    0.  200.]

Orientation relative to the base of the manipulator R = [[  1.00e+00   0.00e+00   0.00e+00]
 [  0.00e+00   6.12e-17   1.00e+00]
 [  0.00e+00  -1.00e+00   6.12e-17]] _
```

Figure 4: T_0^2

The calculated T_{base}^{elbow} is given as follows :

```
[Gangas-Air:Downloads gangameghanath$ python fw_kinematics.py

 Choose 1 or 2 :

         1. T {0} to {n}

         2. T {i} to {j}

         3. T {k-1} to {k}
3

 Enter the value of theta_k (degrees) :
30

 Enter the value of 'a_k' :
100

 Enter the value of 'd_k' :
0

 Enter the value of alpha_k (degrees) :
0

 The transformation matrix is given as T = [[  0.87  -0.5    0.    86.6 ]
  [  0.5    0.87  -0.    50.  ]
  [  0.     0.     1.     0.  ]
  [  0.     0.     0.     1.  ]]

 The position coordinate is given as P = [ 86.6  50.    0. ]

 Orientation relative to the base of the manipulator R = [[ 0.87 -0.5   0.  ]
  [ 0.5   0.87 -0.  ]
  [ 0.    0.    1.  ]]
```

Figure 5: T_{k-1}^{k}

For arbitrary θ value [30, 90, -45, 25]:

```
[Gangas-Air:Downloads gangameghanath$ python fw_kinematics.py

Choose 1 or 2 :

         1. T {0} to {n}

         2. T {i} to {j}

         3. T {k-1} to {k}
1

Enter value of theta as : theta1, theta2, etc
30, 90, -45, 25

Enter value of 'a' as : a1, a2, etc
0,100,150,100,0,0

Enter value of 'd' as : d1, d2, etc
200,0,0,0,0,250

Enter value of alpha as : alpha1, alpha2, etc
-90,0,0,90,90,0

The transformation matrix is given as T = [[  2.96e-01  -5.00e-01   8.14e-01   1.21e+02]
 [  1.71e-01   8.66e-01   4.70e-01   7.01e+01]
 [ -9.40e-01   4.03e-17   3.42e-01  -1.00e+02]
 [  0.00e+00   0.00e+00   0.00e+00   1.00e+00]]

The position coordinate is given as P = [ 121.48   70.13 -100.04]

Orientation relative to the base of the manipulator R = [[  2.96e-01  -5.00e-01   8.14e-01]
 [  1.71e-01   8.66e-01   4.70e-01]
 [ -9.40e-01   4.03e-17   3.42e-01]] _
```

Figure 6: T_0^n for arbitrary theta vector