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Project 1
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1 Section 3.2 Qn.3

Find the positive minimum point of the function $f(x) = x^{-2} \cdot \tan(x)$ by computing the zeros of f' using Newton's method.

1.1 Newton's Method

Newton-Raphson iteration can be used to find the zero of a real valued function of a real variable.
For funtion $f(x)$, we can find its zero using the following iteration :

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad ; n >= 0$$

where $f(x_n)$ is the value of the function at x_n and $f'(x_n)$ is the value of the derivative of $f(x)$ at x_n.

Specific to Qn.3

For Qn.3, minimum value of $f(x)$ occurs at $f'(x) = 0$. Hence we use Newton's method to find zero of $f'(x)$ for different values of x.

$$\begin{aligned} f(x) &= x^{-2} \cdot \tan(x) \\ f'(x) &= x^{-2} \cdot \sec^2(x) - 2x^{-3} \cdot \tan(x) \\ f''(x) &= 2 \cdot x^{-2} \cdot \tan(x) \cdot \sec^2(x) - 4 \cdot x^{-3} \cdot \sec^2(x) + 6 \cdot x^{-4} \cdot \tan(x) \end{aligned}$$

And apply newton's iteration :

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)} \quad ; n >= 0$$

to find the zero of $f'(x)$ for different values of x and take the x for which $f'(x) = 0$ and $f(x) > 0$ is minimum.

x_0 was initialized from -10 to 9 with a step size of 1 and we took the resultant zero for which $f(x) > 0$ is minimum. We run Newton's method for each x_0 for a maximum of 100 iterations and use the stopping criterion : $|x_{n+1} - x_n| < \epsilon$ where we've set $\epsilon = 1e - 10$.

Our computed result is :

$$\begin{aligned} \text{zero occurs at } x &= 0.9477471335169906 \\ f(x) &= 1.549440034483616 \\ f'(x) &= 4.440892098500626e - 16 \end{aligned}$$

One of the iterations that gave us the minimum value is,

```
*********************** Using initial value x0=-3 **********************


--------------------- Beginning Newton's Iteration ---------------------


n = 0         :   x_(n) = -3.0                 ,   x_(n+1) =
    -3.6386777897469313 ,   error = |x_(n+1) - x_(n)| = 0.6386777897469313
n = 1         :   x_(n) = -3.6386777897469313 ,   x_(n+1) =
    0.741684403886512   ,   error = |x_(n+1) - x_(n)| = 4.380362193633443
n = 2         :   x_(n) = 0.741684403886512   ,   x_(n+1) =
    0.9249815041067425  ,   error = |x_(n+1) - x_(n)| = 0.1832971002202305
```

```
n = 3          :   x_(n) = 0.9249815041067425  ,   x_(n+1) = 0.9480681094491926  ,   error = |x_(n+1) - x_(n)| = 0.023086605342450106
n = 4          :   x_(n) = 0.9480681094491926  ,   x_(n+1) = 0.9477472136116464  ,   error = |x_(n+1) - x_(n)| = 0.0003208958375462423
n = 5          :   x_(n) = 0.9477472136116464  ,   x_(n+1) = 0.9477471335169952  ,   error = |x_(n+1) - x_(n)| = 8.009465113367753e-08
n = 6          :   x_(n) = 0.9477471335169952  ,   x_(n+1) = 0.9477471335169906  ,   error = |x_(n+1) - x_(n)| = 4.6629367034256575e-15






 x :  0.9477471335169906 , val :  1.549440034483616
```

1.2 Secant Method

Secant method is also used to find the zero of a real valued function of a real variable.

One of the drawbacks of Newton's method is that it involves derivative of the function whose zero is sought. To overcome this disadvantage, one of the ways is to use secant method.

For function $f(x)$, we can find its zero using the following iteration :

$$x_{n+1} = x_n - f(x_n) * \left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \right) \quad ; n >= 1$$

where $f(x_n)$ is the value of the function at x_n and $f'(x_n)$ is the value of the derivative of $f(x)$ at x_n.

Specific to Qn.3

For Qn.3, minimum value of $f(x)$ occurs at $f'(x) = 0$. Hence we use Secant method to find zero of $f'(x)$ for different values of x.

$$\begin{aligned} f(x) &= x^{-2} \cdot \tan(x) \\ f'(x) &= x^{-2} \cdot \sec^2(x) - 2x^{-3} \cdot \tan(x) \\ f''(x) &= 2 \cdot x^{-2} \cdot \tan(x) \cdot \sec^2(x) - 4 \cdot x^{-3} \cdot \sec^2(x) + 6 \cdot x^{-4} \cdot \tan(x) \end{aligned}$$

And apply the iteration :

$$x_{n+1} = x_n - f'(x_n) * \left(\frac{x_n - x_{n-1}}{f'(x_n) - f'(x_{n-1})} \right) \quad ; n >= 1$$

to find the zero of $f'(x)$ for different values of x and take the x for which $f'(x) = 0$ and $f(x) > 0$ is minimum.

x_1 was initialized from -10 to 9 with a step size of 1 and x_0 was set as $x_0 = x_1 - 1$. We took the resultant zero for which $f(x) > 0$ is minimum. We run Secant method for each (x_0, x_1) for a maximum of 100 iterations and use the stopping criterion : $|x_{n+1} - x_n| < \epsilon$ where we've set $\epsilon = 1e - 10$.

Our computed result is :

$$\begin{aligned} \text{zero occurs at } x &= 0.9477471335169904 \\ f(x) &= 1.549440034483616 \\ f'(x) &= 4.440892098500626e - 16 \end{aligned}$$

One of the iterations that gave us the minimum value is,

```
********************** Using initial value x0=2 **********************



-------------------- Beginning Secant Method --------------------


n = 1          :   x_(n-1) = 2.0                  ,   x_(n) = 1.0         ,   x_(n+1) = 0.8149646022426777   ,   error = |x_(n+1) - x_(n)| = 0.18503539775732225
n = 2          :   x_(n-1) = 1.0                  ,   x_(n) = 0.8149646022426777   ,   x_(n+1) = 0.9442098076297187   ,   error = |x_(n+1) - x_(n)| = 0.12924520538704098
n = 3          :   x_(n-1) = 0.8149646022426777   ,   x_(n) = 0.9442098076297187   ,   x_(n+1) = 0.9478916090914846   ,   error = |x_(n+1) - x_(n)| = 0.003681801461765888
n = 4          :   x_(n-1) = 0.9442098076297187   ,   x_(n) = 0.9478916090914846   ,   x_(n+1) = 0.9477467431759683   ,   error = |x_(n+1) - x_(n)| = 0.00014486591551632344
n = 5          :   x_(n-1) = 0.9478916090914846   ,   x_(n) = 0.9477467431759683   ,   x_(n+1) = 0.9477471334732425   ,   error = |x_(n+1) - x_(n)| = 3.902972741665067e-07
n = 6          :   x_(n-1) = 0.9477467431759683   ,   x_(n) = 0.9477471334732425   ,   x_(n+1) = 0.9477471335169904   ,   error = |x_(n+1) - x_(n)| = 4.3747894196144443e-11






 x :  0.9477471335169904 , val :  1.549440034483616
```

2 Section 3.2 Qn.5

The equation $2 \cdot x^4 + 24 \cdot x^3 + 61 \cdot x^2 - 16 \cdot x + 1 = 0$ has 2 roots near 0.1. Determine them by means of Newton's method. Use Horner's algorithm to compute value of the function and it's derivative.

2.1 Newton's Method

Newton-Raphson iteration can be used to find the zero of a real valued function of a real variable.
For function $f(x)$, we can find its zero using the following iteration :

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad ; n >= 0$$

where $f(x_n)$ is the value of the function at x_n and $f'(x_n)$ is the value of the derivative of $f(x)$ at x_n.

Specific to Qn.5

Here, we can use Newton's method to fund the zero of the function $f(x) = 2 \cdot x^4 + 24 \cdot x^3 + 61 \cdot x^2 - 16 \cdot x + 1$.

We consider a window around 0.1 and use various initialization within the window to find the roots of $f(x)$. For this problem, we chose a window size of 8 and initialized x_0 to 50 uniformly distributed points between -7.9 and 8.1.

We run Newton's method for each x_0 for a maximum of 100 iterations and use the stopping criterion : $|x_{n+1} - x_n| < \epsilon$ where we've set $\epsilon = 1e - 10$.

2.2 Horner's Algorithm

Horner's algorithm is to used to efficiently compute values of a polynomial. The method is also known as 'nested multiplcation' or 'synthetic division'.

The algorithm has several usecases. For example :

1. Given a complex number z_0 and polynomial p find the values of $p(z_0)$ and it's derivatives.
2. Find the deflation factors, ie., removing linear factor from polynomial like $p(z) = (z - z_0) \cdot q(z) + r$ by computing r.
3. Find coefficients of Taylor series expansion of the polynomial p around z_0 (complete Horner's algorithm)

Specific to Qn.5

Here, given x_0 and coefficients of polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + ...a_1 x + a_0$ we use Horner's algorithm to compute :

$$\alpha = p(x_0)$$
$$\beta = p'(x_0)$$

by starting with $\alpha = a_n$ and $\beta = 0$ and using the following iteration from $k = n - 1$ to 0:

$$\beta = \alpha + x_0 \times \beta$$
$$\alpha = a_k + x_0 \times \alpha$$

Hence, we can compute $f(x)$ and $f'(x)$ required for Newton's method using Horner's algorithm.

Our computed result is :

```
 ------------- Qn3_2__5 Newton's + Horner's -------------

All zero's of the polynomial :  [-8.123105625617661, 0.12310562561766183,
   -4.121320343559642, 0.12132034355964338]

f(-8.123105625617661)   = 3.637978807091713e-12
f(0.12310562561766183)  = 2.4188550468151604e-16
f(-4.121320343559642)   = 0.0
f(0.12132034355964338)  = -8.462197956249362e-17


 Zeros close to 0.1 are :
 x = 0.12310562561766183, f(0.12310562561766183) = 2.4188550468151604e-16
 x = 0.12132034355964338, f(0.12132034355964338) = -8.462197956249362e-17
```

Some of the iterations that gave us the roots are,

```
*************************** Using initial value x0=-6.593877551020409
   ***************************


-------------------- Beginning Newton's Iteration --------------------


n = 0         :   x_(n) = -6.593877551020409   ,   x_(n+1) =
   14.106976673808806   ,   error = |x_(n+1) - x_(n)| = 20.700854224829214
n = 1         :   x_(n) = 14.106976673808806   ,   x_(n+1) =
   9.989320993720831    ,   error = |x_(n+1) - x_(n)| = 4.1176556800879744
n = 2         :   x_(n) = 9.989320993720831    ,   x_(n+1) =
   6.9477073306892105   ,   error = |x_(n+1) - x_(n)| = 3.0416136630316206
n = 3         :   x_(n) = 6.9477073306892105   ,   x_(n+1) =
   4.724075163676767    ,   error = |x_(n+1) - x_(n)| = 2.2236321670124433
```

```
n = 4          :   x_(n) = 4.724075163676767   ,   x_(n+1) =
   3.1253488894905646  ,   error = |x_(n+1) - x_(n)| = 1.5987262741862027
n = 5          :   x_(n) = 3.1253488894905646  ,   x_(n+1) =
   2.005360148525364   ,   error = |x_(n+1) - x_(n)| = 1.1199887409652005
n = 6          :   x_(n) = 2.005360148525364   ,   x_(n+1) =
   1.2500755847401313  ,   error = |x_(n+1) - x_(n)| = 0.7552845637852328
n = 7          :   x_(n) = 1.2500755847401313  ,   x_(n+1) =
   0.7660778712778118  ,   error = |x_(n+1) - x_(n)| = 0.48399771346231946
n = 8          :   x_(n) = 0.7660778712778118  ,   x_(n+1) =
   0.4739665005548374  ,   error = |x_(n+1) - x_(n)| = 0.2921113707229744
n = 9          :   x_(n) = 0.4739665005548374  ,   x_(n+1) =
   0.30784717373550985 ,   error = |x_(n+1) - x_(n)| = 0.16611932681932756
n = 10         :   x_(n) = 0.30784717373550985 ,   x_(n+1) =
   0.21790709050375962 ,   error = |x_(n+1) - x_(n)| = 0.08994008323175023
n = 11         :   x_(n) = 0.21790709050375962 ,   x_(n+1) =
   0.1708529011077159  ,   error = |x_(n+1) - x_(n)| = 0.04705418939604372
n = 12         :   x_(n) = 0.1708529011077159  ,   x_(n+1) =
   0.14674832393950843 ,   error = |x_(n+1) - x_(n)| =
   0.024104577168207464
n = 13         :   x_(n) = 0.14674832393950843 ,   x_(n+1) =
   0.134549978732071 8  ,   error = |x_(n+1) - x_(n)| =
   0.012198345207436623
n = 14         :   x_(n) = 0.1345499787320718  ,   x_(n+1) =
   0.12842715294881332 ,   error = |x_(n+1) - x_(n)| =
   0.006122825783258484
n = 15         :   x_(n) = 0.12842715294881332 ,   x_(n+1) =
   0.12538747844546938 ,   error = |x_(n+1) - x_(n)| =
   0.003039674503343942
n = 16         :   x_(n) = 0.12538747844546938 ,   x_(n+1) =
   0.12392649488631077 ,   error = |x_(n+1) - x_(n)| =
   0.0014609835591586073
n = 17         :   x_(n) = 0.12392649488631077 ,   x_(n+1) =
   0.12330238625590682 ,   error = |x_(n+1) - x_(n)| =
   0.0006241086304039495
n = 18         :   x_(n) = 0.12330238625590682 ,   x_(n+1) =
   0.12312340586282922 ,   error = |x_(n+1) - x_(n)| =
   0.00017898039307760738
n = 19         :   x_(n) = 0.12312340586282922 ,   x_(n+1) =
   0.12310579934962448 ,   error = |x_(n+1) - x_(n)| = 1.7606513204732055e
   -05
n = 20         :   x_(n) = 0.12310579934962448 ,   x_(n+1) =
   0.12310562563457486 ,   error = |x_(n+1) - x_(n)| = 1.7371504962282458e
   -07
n = 21         :   x_(n) = 0.12310562563457486 ,   x_(n+1) =
   0.12310562561766183 ,   error = |x_(n+1) - x_(n)| = 1.6913026534837172e
   -11





************************** Using initial value x0=-1.2061224489795919
   **************************


-------------------- Beginning Newton's Iteration --------------------


n = 0          :   x_(n) = -1.2061224489795919 ,   x_(n+1) =
   -0.2238455622125799 ,   error = |x_(n+1) - x_(n)| = 0.982276886767012
n = 1          :   x_(n) = -0.2238455622125799 ,   x_(n+1) =
```

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    -0.03853123428303043,   error = |x_(n+1) - x_(n)| = 0.18531432792954947
n = 2          :   x_(n) = -0.03853123428303043,   x_(n+1) = 0.04429212912999636,   error = |x_(n+1) - x_(n)| = 0.0828233634130300 7
n = 3          :   x_(n) = 0.04429212912999636,   x_(n+1) = 0.08380542112658823 ,   error = |x_(n+1) - x_(n)| = 0.03951329199658859 5
n = 4          :   x_(n) = 0.08380542112658823 ,   x_(n+1) = 0.10313224240661037 ,   error = |x_(n+1) - x_(n)| = 0.01932682128002214
n = 5          :   x_(n) = 0.10313224240661037 ,   x_(n+1) = 0.1126843122373796  ,   error = |x_(n+1) - x_(n)| = 0.009552069830769228
n = 6          :   x_(n) = 0.1126843122373796  ,   x_(n+1) = 0.11741482621050053 ,   error = |x_(n+1) - x_(n)| = 0.004730513973120937
n = 7          :   x_(n) = 0.11741482621050053 ,   x_(n+1) = 0.11973279072246992 ,   error = |x_(n+1) - x_(n)| = 0.0023179645119693892
n = 8          :   x_(n) = 0.11973279072246992 ,   x_(n+1) = 0.12081266971391345 ,   error = |x_(n+1) - x_(n)| = 0.0010798789914435308
n = 9          :   x_(n) = 0.12081266971391345 ,   x_(n+1) = 0.12122837854346087 ,   error = |x_(n+1) - x_(n)| = 0.0004157088295474176
n = 10         :   x_(n) = 0.12122837854346087 ,   x_(n+1) = 0.12131605140611083 ,   error = |x_(n+1) - x_(n)| = 8.767286264996232e-05
n = 11         :   x_(n) = 0.12131605140611083 ,   x_(n+1) = 0.12132033329642408 ,   error = |x_(n+1) - x_(n)| = 4.281890313248549e-06
n = 12         :   x_(n) = 0.12132033329642408 ,   x_(n+1) = 0.12132034355958204 ,   error = |x_(n+1) - x_(n)| = 1.0263157962375757e-08
n = 13         :   x_(n) = 0.12132034355958204 ,   x_(n+1) = 0.12132034355964338 ,   error = |x_(n+1) - x_(n)| = 6.13398221105399e-14





*************************** Using initial value x0=-15.9 ***************************


--------------------- Beginning Newton's Iteration ---------------------


n = 0          :   x_(n) = -15.9                ,   x_(n+1) = -12.944147676764224 ,   error = |x_(n+1) - x_(n)| = 2.955852323235776
n = 1          :   x_(n) = -12.944147676764224 ,   x_(n+1) = -10.833691453241366 ,   error = |x_(n+1) - x_(n)| = 2.110456223522858
n = 2          :   x_(n) = -10.833691453241366 ,   x_(n+1) = -9.406043019078517  ,   error = |x_(n+1) - x_(n)| = 1.4276484341628493
n = 3          :   x_(n) = -9.406043019078517  ,   x_(n+1) = -8.557570066799421  ,   error = |x_(n+1) - x_(n)| = 0.8484729522790957
n = 4          :   x_(n) = -8.557570066799421  ,   x_(n+1) = -8.194925433500654  ,   error = |x_(n+1) - x_(n)| = 0.3626446332987676
n = 5          :   x_(n) = -8.194925433500654  ,   x_(n+1) = -8.125527666324423  ,   error = |x_(n+1) - x_(n)| = 0.06939776717623047
n = 6          :   x_(n) = -8.125527666324423  ,   x_(n+1) = -8.12310850972693 1  ,   error = |x_(n+1) - x_(n)| =
```

```
0.0024191565974920337
n = 7          :   x_(n) = -8.123108509726931   ,   x_(n+1) = -8.123105625621758   ,   error = |x_(n+1) - x_(n)| = 2.884105173350804e-06
n = 8          :   x_(n) = -8.123105625621758   ,   x_(n+1) = -8.123105625617661   ,   error = |x_(n+1) - x_(n)| = 4.0962788716569776e-12




************************** Using initial value x0=-6.104081632653061 **************************


-------------------- Beginning Newton's Iteration --------------------


n = 0          :   x_(n) = -6.104081632653061   ,   x_(n+1) = -3.076317116613531   ,   error = |x_(n+1) - x_(n)| = 3.0277645160395306
n = 1          :   x_(n) = -3.076317116613531   ,   x_(n+1) = -4.963845404932734   ,   error = |x_(n+1) - x_(n)| = 1.8875282883192028
n = 2          :   x_(n) = -4.963845404932734   ,   x_(n+1) = -4.172462205777248   ,   error = |x_(n+1) - x_(n)| = 0.7913831991554856
n = 3          :   x_(n) = -4.172462205777248   ,   x_(n+1) = -4.121870338272995   ,   error = |x_(n+1) - x_(n)| = 0.05059186750425315
n = 4          :   x_(n) = -4.121870338272995   ,   x_(n+1) = -4.121320410499956   ,   error = |x_(n+1) - x_(n)| = 0.0005499277730391938
n = 5          :   x_(n) = -4.121320410499956   ,   x_(n+1) = -4.121320343559643   ,   error = |x_(n+1) - x_(n)| = 6.694031284837365e-08
n = 6          :   x_(n) = -4.121320343559643   ,   x_(n+1) = -4.121320343559642   ,   error = |x_(n+1) - x_(n)| = 8.881784197001252e-16
```

3 Section 3.2 Qn.14b

Using Newton's method, find the roots of the nonlinear systems:

$$x + e^{-1x} + y^3 = 0$$

$$x^2 + 2xy - y^2 + \tan(x) = 0$$

3.1 Newton's Method

Newton-Raphson iteration can be used to find the zero of a real valued function of a real variable.
For funtion $f(x)$, we can find its zero using the following iteration :

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad ; n >= 0$$

where $f(x_n)$ is the value of the function at x_n and $f'(x_n)$ is the value of the derivative of $f(x)$ at x_n.

We can also use it for solving a system of non-linear equations by linearizing it and using matrices.

For example, in the case of solving 2 nonlinear equation f_1 and f_2,

$$f_1(x_1, x_2) = 0$$
$$f_2(x_1, x_2) = 0$$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$F(X) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$$

$$F'(X) = \begin{bmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} \end{bmatrix}$$

We use the iteration :

$$X_{n+1} = X_n - F'(X_n)^{-1} \cdot F(X_n)$$

Specific to Qn.14b

For Qn.14b, we use Newton's method to solve the nonlinear systems as follows :

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$F(X) = \begin{bmatrix} x_1 + e^{-x_1} + x_2^3 \\ x_1^2 + 2 \cdot x_1 \cdot x_2 - x_2^2 + \tan(x_1) \end{bmatrix}$$

$$F'(X) = \begin{bmatrix} 1 - e^{-x_1} & 3 \cdot x_2^2 \\ 2 \cdot x_1 + 2 \cdot x_2 + \sec^2 x_1 & 2 \cdot x_1 - 2 \cdot x_2 \end{bmatrix}$$

where $x_1 = x$ and $x_2 = y$ and apply newton's iteration :

$$X_{n+1} = X_n - F'(X_n)^{-1} \cdot F(X_n)$$

X was initialized as $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

We run Newton's method for a maximum of 100 iterations and use the stopping criterion :

$$\sum_{i=0}^{i=1} |x_{n+1}^{(i)} - x_n^{(i)}| < \epsilon$$

where we've set $\epsilon = 1e - 10$ and superscipt i indicates the index of the element in the column vector.

Our computed result is :

$$\text{zero occurs at } x = \begin{bmatrix} -0.71116218 \\ -1.09839831 \end{bmatrix}$$

$$f(x) = \begin{bmatrix} -2.22044605e - 16 \\ 0.00000000e + 00 \end{bmatrix}$$

Newton's iterations generated using $X_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is given below :

```


--------------------- Beginning Newton's Iteration ---------------------


n = 0          :   x_(n) = [1.0, 1.0                                   ],   x_(n+1) = [0.5209213240853966, 0.31165201305684753 ],   error = sum(|x_(n+1) - x_(n)|) = 1.1674266628577559
n = 1          :   x_(n) = [0.5209213240853966, 0.31165201305684753 ],   x_(n+1) = [0.7582343297700177, -3.949161018669593  ],   error = sum(|x_(n+1) - x_(n)|) = 4.498126037411062
n = 2          :   x_(n) = [0.7582343297700177, -3.949161018669593  ],   x_(n+1) = [-0.9656976812101966, -2.639409210831367 ],   error = sum(|x_(n+1) - x_(n)|) = 3.0336838188184405
```

```
n = 3          :   x_(n) = [-0.9656976812101966, -2.639409210831367 ],   x_(n+1) = [-0.8887252271078999, -1.8330873803997338],   error = sum(|x_(n+1) - x_(n)|) = 0.88329428453393
n = 4          :   x_(n) = [-0.8887252271078999, -1.8330873803997338],   x_(n+1) = [-0.7678648856609555, -1.357984987600598 ],   error = sum(|x_(n+1) - x_(n)|) = 0.5959627342460803
n = 5          :   x_(n) = [-0.7678648856609555, -1.357984987600598 ],   x_(n+1) = [-0.7178169022463898, -1.1456329205587816],   error = sum(|x_(n+1) - x_(n)|) = 0.26240005045638204
n = 6          :   x_(n) = [-0.7178169022463898, -1.1456329205587816],   x_(n+1) = [-0.71116696530348, -1.100308733803541  ],   error = sum(|x_(n+1) - x_(n)|) = 0.05197439247128233
n = 7          :   x_(n) = [-0.71116696530348, -1.100308733803541  ],   x_(n+1) = [-0.7111615447731828, -1.0984014409815541],   error = sum(|x_(n+1) - x_(n)|) = 0.0019124445791521838
n = 8          :   x_(n) = [-0.7111615447731828, -1.0984014409815541],   x_(n+1) = [-0.711162179287523, -1.098398306962086  ],   error = sum(|x_(n+1) - x_(n)|) = 3.768533808345964e-06
n = 9          :   x_(n) = [-0.711162179287523, -1.098398306962086  ],   x_(n+1) = [-0.711162179291851, -1.098398306954496  ],   error = sum(|x_(n+1) - x_(n)|) = 1.1917911102443668e-11


 ------------- Qn3_2__14b Newtons -------------

  zero occurs at x =  [-0.71116218 -1.09839831]

  f(x) =  [-2.22044605e-16  0.00000000e+00]
```

Note :, when we use a different initial value of $X_0 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$ we get the following result :

```


--------------------- Beginning Newton's Iteration ---------------------


n = 0          :   x_(n) = [0.1, 0.1                                   ],   x_(n+1) = [0.014660318889975002, -33.157209121251384],   error = sum(|x_(n+1) - x_(n)|) = 33.342548802361414
n = 1          :   x_(n) = [0.014660318889975002, -33.157209121251384],   x_(n+1) = [-5.6086715431112335, -22.105084495734577],   error = sum(|x_(n+1) - x_(n)|) = 16.675456487518016
n = 2          :   x_(n) = [-5.6086715431112335, -22.105084495734577],   x_(n+1) = [-5.0072128976943455, -14.807469460053127],   error = sum(|x_(n+1) - x_(n)|) = 7.899073681098337
n = 3          :   x_(n) = [-5.0072128976943455, -14.807469460053127],   x_(n+1) = [-2.8740814344651144, -9.60976211059479  ],   error = sum(|x_(n+1) - x_(n)|) = 7.330838812687569
n = 4          :   x_(n) = [-2.8740814344651144, -9.60976211059479  ],   x_(n+1) = [-2.2737279510249344, -6.4238471843861875],   error = sum(|x_(n+1) - x_(n)|) = 3.7862684096487826
n = 5          :   x_(n) = [-2.2737279510249344, -6.4238471843861875],   x_(n+1) = [-1.4712132863776082, -4.286179289718738 ],   error = sum(|x_(n+1) - x_(n)|) = 2.940182559314776
n = 6          :   x_(n) = [-1.4712132863776082, -4.286179289718738 ],   x_(n+1) = [-1.4061623790335949, -2.9058087717232746],   error = sum(|x_(n+1) - x_(n)|) = 1.4454214253394764
n = 7          :   x_(n) = [-1.4061623790335949, -2.9058087717232746],   x_(n+1) = [-1.3466203012947184, -2.035531292676814 ],   error = sum(|x_(n+1) - x_(n)|) = 0.9298195567853369
n = 8          :   x_(n) = [-1.3466203012947184, -2.035531292676814 ],   x_(n+1) = [-1.304866373717048, -1.5484121571445504 ],   error = sum(|x_(n
```

```
+1) - x_(n)|) = 0.5288730631099341
n = 9          :    x_(n) = [-1.304866373717048, -1.5484121571445504 ],    x_(n+1) = [-1.2785976125702294, -1.3536739128698807],    error = sum(|x_(n+1) - x_(n)|) = 0.22100700542148832
n = 10         :    x_(n) = [-1.2785976125702294, -1.3536739128698807],    x_(n+1) = [-1.270459688010876, -1.3193651911333484 ],    error = sum(|x_(n+1) - x_(n)|) = 0.04244664629588568
n = 11         :    x_(n) = [-1.270459688010876, -1.3193651911333484 ],    x_(n+1) = [-1.269975544587634, -1.3182427076782932 ],    error = sum(|x_(n+1) - x_(n)|) = 0.0016066268782972681
n = 12         :    x_(n) = [-1.269975544587634, -1.3182427076782932 ],    x_(n+1) = [-1.2699741889770049, -1.3182411655398052],    error = sum(|x_(n+1) - x_(n)|) = 2.8977491171033876e-06
n = 13         :    x_(n) = [-1.2699741889770049, -1.3182411655398052],    x_(n+1) = [-1.2699741889668283, -1.3182411655336301],    error = sum(|x_(n+1) - x_(n)|) = 1.635158675128423e-11


 ------------- Qn3_2__14b Newtons -------------

  zero occurs at x =  [-1.26997419 -1.31824117]

  f(x) =  [0.0000000e+00 4.4408921e-16]
```