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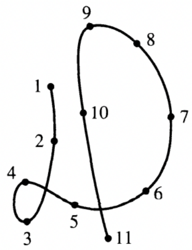
1 Section 6.4 Qn.7

Draw a script letter, such as the one shown in Figure 6.7. Then reproduce it with the aid of cubic splines and a plotter. Proceed as follows: Select a modest number of points on the curve, say n = 11. Label these t = 1, 2,... , n. For each point, obtain the corresponding x- and y-coordinates. Then fit $x = S_x(t)$ and $y = S_y(t)$, using cubic spline interpolating functions S_x and S_y. This will produce a parametric representation of the original curve. Compute a large number of values of $S_x(t)$ and $S_y(t)$ to give to the plotter. To learn more about how spline curves are used in designing typefaces, the reader should consult Knuth [1979].

FIGURE 6.7
Script letter from 11 knots



1.1 Cubic Spline Interpolation

Spline function consists of polynomial pieces on subintervals joined together with certain continuity conditions. $n+1$ points $t_0, t_1, ..., t_n$ are called knots. A spline function of degree k having knots $t_0, t_1, ..., t_n$ is a function S such that :

1. On each interval $[t_{i-1}, t_i$, S is a polynomial of degree $\leq k$.
2. S has a continuous $(k-1)^{st}$ derivative on $[t_0, t_n]$.

In a cubic spline function S, each polynomial will of degree ≤ 3. Given data (t_i, y_i) for $0 \leq i \leq n$, we can compute the cubic spline interpolation given by,

$$S(x) = \begin{cases} S_0(x) & x \in [t_0, t_1] \\ S_1(x) & x \in [t_1, t_2] \\ \vdots & \vdots \\ S_{n-1}(x) & x \in [t_{n-1}, t_n] \end{cases}$$

by solving the system,

$$\begin{bmatrix} u_1 & h_1 & & & & \\ h_1 & u_2 & h_2 & & & \\ & h_2 & u_3 & h_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & h_{n-3} & u_{n-2} & h_{n-2} \\ & & & & h_{n-2} & u_{n-1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_{n-2} \\ z_{n-1} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-2} \\ v_{n-1} \end{bmatrix}$$

where

$$h_i = t_{i+1} - t_i$$

$$u_i = 2(h_i + h_{i-1})$$

$$b_i = \frac{6}{h_i}(y_{i+1} - y_i)$$

$$v_i = b_i - b_{i-1}$$

This can be solved using the algorithm,

input n, (t_i), (y_i)
for $i = 0$ **to** $n-1$ **do**
  $h_i \leftarrow t_{i+1} - t_i$
  $b_i \leftarrow 6(y_{i+1} - y_i)/h_i$
end do
$u_1 \leftarrow 2(h_0 + h_1)$
$v_1 \leftarrow b_1 - b_0$
for $i = 2$ **to** $n-1$ **do**
  $u_i \leftarrow 2(h_i + h_{i-1}) - h_{i-1}^2/u_{i-1}$
  $v_i \leftarrow b_i - b_{i-1} - h_{i-1}v_{i-1}/u_{i-1}$
end do
$z_n \leftarrow 0$
for $i = n-1$ **to** 1 **step** -1 **do**
  $z_i \leftarrow (v_i - h_i z_{i+1})/u_i$
end do
$z_0 \leftarrow 0$
output (z_i)

after which we can compute $S_i(x)$ using,

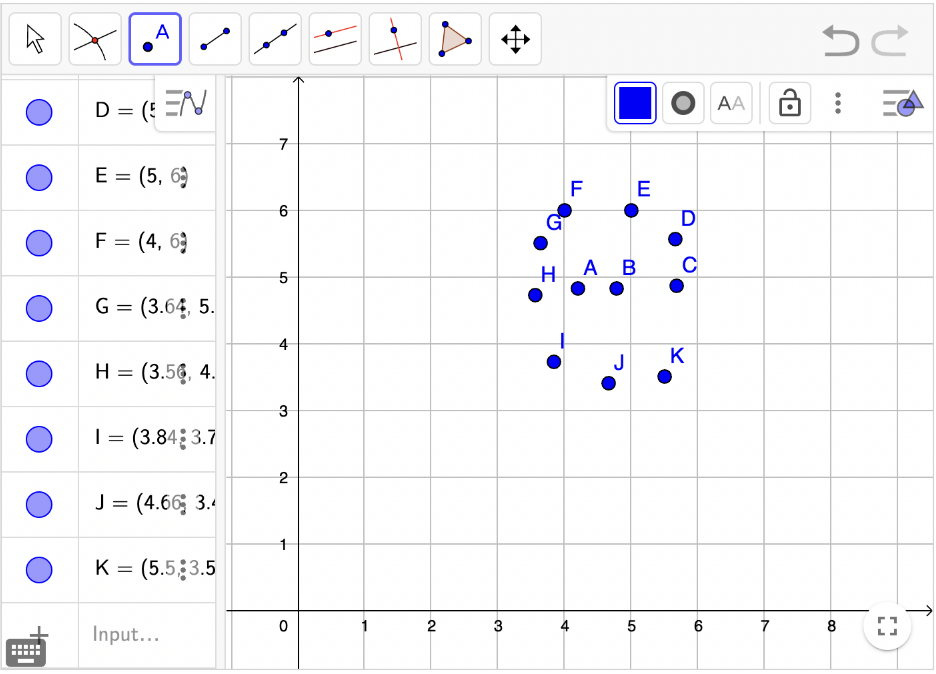
$$S_i(x) = y_i + (x - t_i)\Big[C_i + (x - t_i)\big[B_i + (x - t_i)A_i\big]\Big]$$

where

$$\begin{aligned} A_i &= \frac{1}{6h_i}(z_{i+1} - z_i) \\ B_i &= \frac{z_i}{2} \\ C_i &= -\frac{h_i}{6}z_{i+1} - \frac{h_i}{3}z_i + \frac{1}{h_i}(y_{i+1} - y_i) \end{aligned}$$

Specific to Qn.7

Samples coordinates (x_i, y_i) were obtained using geogebra.



Using the method described above, we generated samples from $S_x(t)$ and $S_y(t)$ (parametric representation of the original curve).

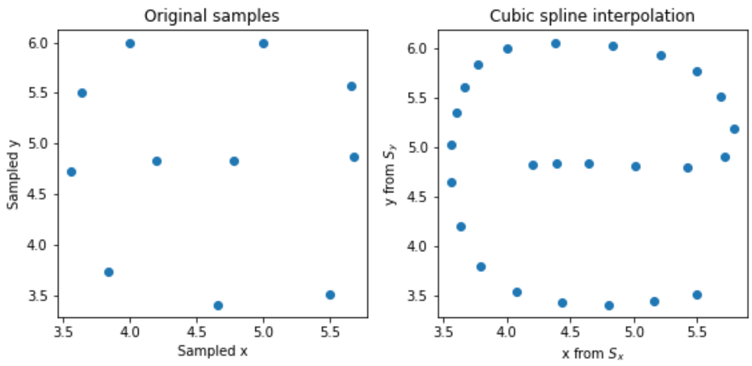


Abbildung 1: Using 25 values of $S_x(t)$ and $S_y(t)$

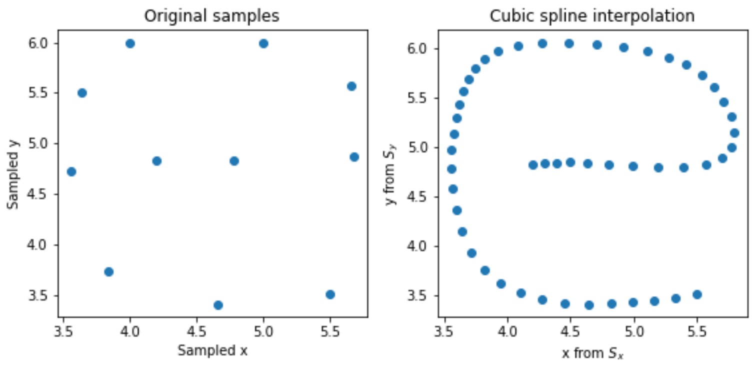


Abbildung 2: Using 50 values of $S_x(t)$ and $S_y(t)$

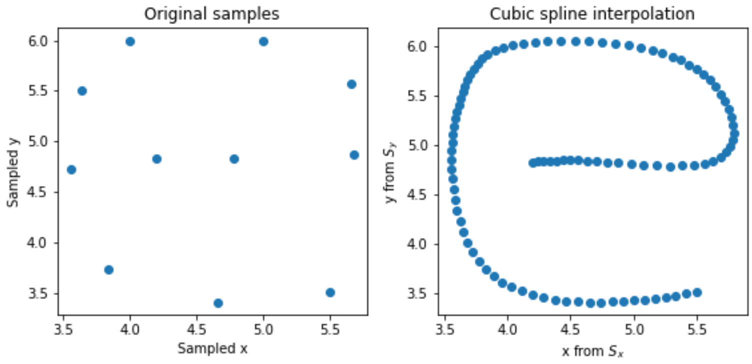


Abbildung 3: Using 100 values of $S_x(t)$ and $S_y(t)$

2 Section 6.4 Qn.8

Interpret the results of the following numerical experiment and draw some conclusions.

(a) Define p to be the polynomial of degree 20 that interpolates the function $f(x) = (1 + 6x^2)^{-1}$ at 21 equally spaced nodes in the interval $[|1, 1]$. Include the endpoints as nodes. Print a table of $f(x), p(x)$ and $f(x)|p(x)$ at 41 equally spaced points on the interval.

(b) Repeat the experiment using the Chebyshev nodes given by

$$x_i = cos[(i - 1)\pi/20] \qquad (1 < i < 21)$$

(c) With 21 equally spaced knots, repeat the experiment using a cubic interpolating spline

2.1 Polynomial Interpolation : Using Lagrange Form

Given data (x_i, y_i) for $0 \leq i \leq n$, there is one and only one interpolation polynomial of degree $\leq n$ associated with the data if x_i are unique. There are different ways of expressing the interpolation polynomial. The Lagrange form of the interpolation polynomial is given by,

$$p(x) = \sum_{k=0}^{n} y_k \cdot l_k(x)$$

where

$$l_k(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} \qquad 0 \leq i \leq n$$

Specific to Qn.8

Interpolated data from the three methods :

Tabelle 1: Polynomial Interpolation with equally spaced nodes

	f(x)	p(x)	f(x)-p(x)
0	0.14285714285714300	0.14285714285714800	-5.32907051820075E-15
1	0.1558846453624320	-0.17832013255646900	0.3342047779189010
2	0.1706484641638230	0.17064846416382700	-4.55191440096314E-15
3	0.18744142455482700	0.21509981222146900	-0.027658387666642400
4	0.20661157024793400	0.20661157024793800	-4.30211422042248E-15
5	0.22857142857142900	0.22454987513417800	0.0040215534372501900
6	0.2538071065989850	0.2538071065989890	-4.27435864480685E-15
7	0.28288543140028300	0.28374814088352700	-0.000862709483244406
8	0.3164556962025320	0.31645569620253600	-4.32986979603811E-15
9	0.35523978685612800	0.3549897790936030	0.0002500077625247620
10	0.4	0.40000000000000400	-4.44089209850063E-15
11	0.4514672686230250	0.4515595128036700	-9.22441806447405E-05
12	0.5102040816326530	0.5102040816326580	-4.55191440096314E-15
13	0.5763688760806920	0.5763278165374390	4.10595432527305E-05
14	0.6493506493506500	0.6493506493506540	-4.2188474935756E-15
15	0.7272727272727270	0.7272932866688490	-2.05593961221107E-05

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Tabelle 1: Polynomial Interpolation with equally spaced nodes (Continued)

	f(x)	p(x)	f(x)-p(x)
16	0.806451612903226	0.8064516129032300	-3.5527136788005E-15
17	0.8810572687224670	0.881047106765446	1.0161957021193E-05
18	0.9433962264150940	0.9433962264150970	-2.33146835171283E-15
19	0.9852216748768470	0.985224803920944	-3.12904409660586E-06
20	1.0	1.0000000000000000	-2.22044604925031E-16
21	0.9852216748768470	0.9852248039209420	-3.12904409449644E-06
22	0.9433962264150940	0.9433962264150930	1.77635683940025E-15
23	0.8810572687224670	0.8810471067654400	1.01619570267442E-05
24	0.8064516129032260	0.8064516129032220	3.10862446895044E-15
25	0.7272727272727270	0.7272932866688420	-2.05593961151163E-05
26	0.6493506493506490	0.6493506493506450	3.99680288865056E-15
27	0.5763688760806920	0.5763278165374310	4.10595432607241E-05
28	0.5102040816326530	0.5102040816326490	4.2188474935756E-15
29	0.45146726862302500	0.45155951280366100	-9.22441806360808E-05
30	0.4	0.3999999999999960	4.32986979603811E-15
31	0.35523978685612800	0.35498977909359500	0.00025000776253258900
32	0.3164556962025320	0.31645569620252700	4.27435864480685E-15
33	0.2828854314002830	0.28374814088352000	-0.000862709483237023
34	0.25380710659898500	0.2538071065989800	4.2188474935756E-15
35	0.22857142857142900	0.22454987513417900	0.004021553437249390
36	0.20661157024793400	0.20661157024793000	4.24660306919122E-15
37	0.1874414245548270	0.21509981222152900	-0.027658387666702400
38	0.17064846416382200	0.17064846416381800	4.46864767411626E-15
39	0.15588464536243200	-0.17832013255535900	0.3342047779177910
40	0.14285714285714300	0.1428571428571380	5.24580379135387E-15

Tabelle 2: Polynomial Interpolation with Chebyshev nodes

	f(x)	p(x)	f(x)-p(x)
0	0.14285714285714300	0.14285714285716200	-1.92346139016308E-14
1	0.1558846453624320	0.1558796363073600	5.00905507189176E-06
2	0.1706484641638230	0.17060650048173000	4.19636820921665E-05
3	0.18744142455482700	0.18757450572406900	-0.0001330811692422660
4	0.20661157024793400	0.20656431165746500	4.72585904686129E-05
5	0.22857142857142900	0.2283997540424080	0.00017167452902097600
6	0.2538071065989850	0.25384728108005900	-4.01744810746285E-05

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Tabelle 2: Polynomial Interpolation with Chebyshev nodes (Continued)

	f(x)	p(x)	f(x)-p(x)
7	0.28288543140028300	0.283108099554567	-0.00022266815428406700
8	0.3164556962025320	0.31652807961325	-7.23834107184129E-05
9	0.35523978685612800	0.35503303308817800	0.00020675376794959700
10	0.4	0.3997609464322750	0.00023905356772552200
11	0.4514672686230250	0.4514931038143060	-2.58351912814092E-05
12	0.5102040816326530	0.5104813085025360	-0.0002772268698827410
13	0.5763688760806920	0.5765987179829180	-0.00022984190222674600
14	0.6493506493506500	0.6492949112755870	5.57380750628722E-05
15	0.7272727272727270	0.7270083412934760	0.0002643859792518240
16	0.806451612903226	0.8062566095410580	0.00019500336216815600
17	0.8810572687224670	0.8810843102493840	-2.70415269166824E-05
18	0.9433962264150940	0.9435320431203240	-0.00013581670522944100
19	0.9852216748768470	0.9852876712088110	-6.59963319639134E-05
20	1.0	0.9999999999999990	7.7715611723761E-16
21	0.9852216748768470	0.9852876712088130	-6.59963319661339E-05
22	0.9433962264150940	0.943532043120326	-0.0001358167052317730
23	0.8810572687224670	0.8810843102493840	-2.70415269167934E-05
24	0.8064516129032260	0.8062566095410550	0.00019500336217015400
25	0.7272727272727270	0.7270083412934730	0.00026438597925460000
26	0.6493506493506490	0.6492949112755860	5.57380750638714E-05
27	0.5763688760806920	0.5765987179829200	-0.00022984190222830000
28	0.5102040816326530	0.5104813085025380	-0.0002772268698846280
29	0.45146726862302500	0.45149310381430600	-2.58351912810761E-05
30	0.4	0.39976094643227200	0.0002390535677278540
31	0.35523978685612800	0.3550330330881760	0.0002067537679522610
32	0.3164556962025320	0.3165280796132510	-7.2383410718968E-05
33	0.2828854314002830	0.2831080995545700	-0.00022266815428706500
34	0.25380710659898500	0.2538472810800600	-4.01744810754612E-05
35	0.22857142857142900	0.2283997540424040	0.00017167452902436300
36	0.20661157024793400	0.2065643116574640	4.72585904697509E-05
37	0.1874414245548270	0.1875745057240740	-0.00013308116924729000
38	0.17064846416382200	0.17060650048172900	4.19636820938318E-05
39	0.15588464536243200	0.15587963630736100	5.0090550711146E-06
40	0.14285714285714300	0.1428571428571240	1.9151347174784E-14

Tabelle 3: Cubic spline Interpolation with equally spaced nodes

	f(x)	Sy(x)	f(x)-Sy(x)
0	0.14285714285714300	0.14285714285714300	0.0
1	0.1558846453624320	0.1561525125508040	-0.00026786718837179500
2	0.1706484641638230	0.1706484641638230	0.0
3	0.18744142455482700	0.18736647079337900	7.49537614479767E-05
4	0.20661157024793400	0.20661157024793400	0.0
5	0.22857142857142900	0.22858723439149700	-1.58058200688538E-05
6	0.2538071065989850	0.2538071065989850	0.0
7	0.28288543140028300	0.2828763076213170	9.12377896572503E-06
8	0.3164556962025320	0.3164556962025320	0.0
9	0.35523978685612800	0.3552395394585850	2.47397542563199E-07
10	0.4	0.4	0.0
11	0.4514672686230250	0.4514770596555750	-9.79103254983293E-06
12	0.5102040816326530	0.5102040816326530	0.0
13	0.5763688760806920	0.5764147498071410	-4.58737264495968E-05
14	0.6493506493506500	0.6493506493506500	0.0
15	0.7272727272727270	0.7273902443057580	-0.00011751703303097800
16	0.806451612903226	0.806451612903226	0.0
17	0.8810572687224670	0.881155815455686	-9.85467332188517E-05
18	0.9433962264150940	0.9433962264150940	0.0
19	0.9852216748768470	0.9849678630805	0.0002538117963474250
20	1.0	1.0	0.0
21	0.9852216748768470	0.9849678630805000	0.00025381179634709200
22	0.9433962264150940	0.9433962264150940	0.0
23	0.8810572687224670	0.8811558154556860	-9.85467332188517E-05
24	0.8064516129032260	0.8064516129032260	0.0
25	0.7272727272727270	0.7273902443057580	-0.00011751703303086700
26	0.6493506493506490	0.6493506493506490	0.0
27	0.5763688760806920	0.5764147498071410	-4.58737264495968E-05
28	0.5102040816326530	0.5102040816326530	0.0
29	0.45146726862302500	0.45147705965557400	-9.79103254977742E-06
30	0.4	0.4	0.0
31	0.35523978685612800	0.35523953945858500	2.47397542507688E-07
32	0.3164556962025320	0.3164556962025320	0.0
33	0.2828854314002830	0.2828763076213170	9.12377896572503E-06
34	0.25380710659898500	0.25380710659898500	0.0
35	0.22857142857142900	0.22858723439149700	-1.58058200687983E-05

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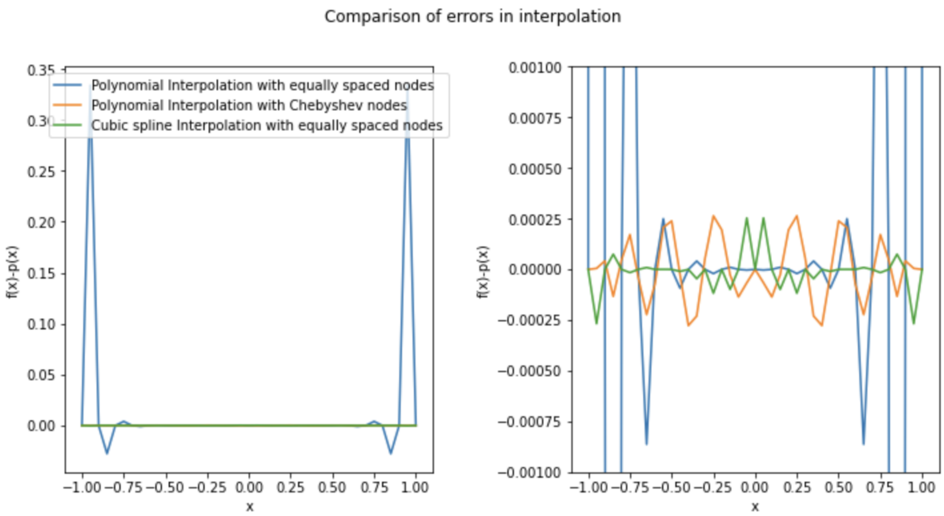
Tabelle 3: Cubic spline Interpolation with equally spaced nodes (Continued)

	f(x)	Sy(x)	f(x)-Sy(x)
36	0.20661157024793400	0.20661157024793400	0.0
37	0.1874414245548270	0.18736647079337900	7.4953761447949E-05
38	0.17064846416382200	0.17064846416382200	0.0
39	0.15588464536243200	0.15615251255080400	-0.0002678671883717680
40	0.14285714285714300	0.14285714285714300	0.0

Aggregating results



	Polynomial Interpolation with equally spaced nodes	Polynomial Interpolation with Chebyshev nodes	Cubic spline Interpolation with equally spaced nodes
Max	0.3342047779189009	0.00026438597925459995	0.0002538117963474251
Min	-0.027658387666702366	-0.0002772268698846281	-0.00026786718837179535
Mean	0.015116611260860295	-1.573236101873795e-07	-1.0598282896892778e-05
Std_dev	0.07251251050940774	0.00015180945290261418	8.986271219774465e-05

As we can see from above, Polynomial Interpolation with Chebyshev nodes and Cubic spline Interpolation with equally spaced nodes give the lowest interpolation errors with Polynomial Interpolation with Chebyshev nodes slightly better, but not significantly better.