DIGITAL COMMUNICATION

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Roll No. FWC22050

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Contents

Chapter 1

Two Dice

1.1 Sum of Independant Random Variables

Two dice, one blue and one grey, are thrown at the same time. The event defined by the sum of the two numbers appearing on the top of the dice can have 11 possible outcomes 2, 3, 4, 5, 6, 6, 8, 9, 10, 11 and 12. A student argues that each of these outcomes has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.

1.1.1 *The Uniform Distribution:* Let $X_i \in \{1, 2, 3, 4, 5, 6\}, i = 1, 2,$ be the random variables representing the outcome for each die. Assuming the dice to be fair, the probability mass function (pmf) is expressed as

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{6} & 1 \le n \le 6 \\ 0 & otherwise \end{cases} \tag{1.1.1.1}$$

The desired outcome is

$$X = X_1 + X_2, \tag{1.1.1.2}$$

$$\implies X \in \{1, 2, \dots, 12\} \tag{1.1.1.3}$$

The objective is to show that

$$p_X(n) \neq \frac{1}{11} \tag{1.1.1.4}$$

SOLUTION: The python code which was used to generate the results can be shared.

```
/codes/1/1.1.1.py
```

1.1.2 *Convolution:* From (1.1.1.2),

$$p_X(n) = \Pr(X_1 + X_2 = n) = \Pr(X_1 = n - X_2) \tag{1.1.2.1}$$

$$= \sum_k \Pr(X_1 = n - k | X_2 = k)\, p_{X_2}(k) \tag{1.1.2.2}$$

after unconditioning. $\because X_1$ and X_2 are independent,

$$\Pr(X_1 = n - k | X_2 = k)$$

$$= \Pr(X_1 = n - k) = p_{X_1}(n-k) \tag{1.1.2.3}$$

From (1.1.2.2) and (1.1.2.3),

$$p_X(n) = \sum_k p_{X_1}(n-k) p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \tag{1.1.2.4}$$

where $*$ denotes the convolution operation. Substituting from (1.1.1.1) in (1.1.2.4),

$$p_X(n) = \frac{1}{6} \sum_{k=1}^{6} p_{X_1}(n-k) = \frac{1}{6} \sum_{k=n-6}^{n-1} p_{X_1}(k) \tag{1.1.2.5}$$

$$\because p_{X_1}(k) = 0, \quad k \leq 1, k \geq 6. \tag{1.1.2.6}$$

From (1.1.2.5),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{1}{6}\sum_{k=1}^{n-1} p_{X_1}(k) & 1 \le n-1 \le 6 \\ \frac{1}{6}\sum_{k=n-6}^{6} p_{X_1}(k) & 1 < n-6 \le 6 \\ 0 & n > 12 \end{cases} \tag{1.1.2.7}$$

Substituting from (1.1.1.1) in (1.1.2.7),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \le n \le 7 \\ \frac{13-n}{36} & 7 < n \le 12 \\ 0 & n > 12 \end{cases} \tag{1.1.2.8}$$

satisfying (1.1.1.4).

SOLUTION: The python code which was used to generate the results can be shared.

/codes/1/1.1.2.py

1.1.3 *The Z-transform:* The Z-transform of $p_X(n)$ is defined as

$$P_X(z) = \sum_{n=-\infty}^{\infty} p_X(n) z^{-n}, \quad z \in \mathbb{C} \tag{1.1.3.1}$$

From (1.1.1.1) and (1.1.3.1),

$$P_{X_1}(z) = P_{X_2}(z) = \frac{1}{6}\sum_{n=1}^{6} z^{-n} \tag{1.1.3.2}$$

$$= \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}, \quad |z| > 1 \tag{1.1.3.3}$$

upon summing up the geometric progression.

$$\because p_X(n) = p_{X_1}(n) * p_{X_2}(n), \tag{1.1.3.4}$$

$$P_X(z) = P_{X_1}(z) P_{X_2}(z) \tag{1.1.3.5}$$

The above property follows from Fourier analysis and is fundamental to signal processing. From (1.1.3.3) and (1.1.3.5),

$$P_X(z) = \left\{ \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)} \right\}^2 \tag{1.1.3.6}$$

$$= \frac{1}{36} \frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \tag{1.1.3.7}$$

Using the fact that

$$p_X(n-k) \overset{\mathcal{H}}{\longleftrightarrow} Z P_X(z) z^{-k}, \tag{1.1.3.8}$$

$$nu(n) \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{z^{-1}}{\left(1 - z^{-1}\right)^2} \tag{1.1.3.9}$$

after some algebra, it can be shown that

$$\frac{1}{36}\left[(n-1)\, u(n-1) - 2\,(n-7)\, u(n-7) + (n-13)\, u(n-13)\right] \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{1}{36} \frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \tag{1.1.3.10}$$

where

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{1.1.3.11}$$

From (1.1.3.1), (1.1.3.7) and (1.1.3.10)

$$p_X(n) = \frac{1}{36}\left[(n-1)\, u(n-1) - 2\,(n-7)\, u(n-7) + (n-13)\, u(n-13)\right] \tag{1.1.3.12}$$

which is the same as (1.1.2.8). Note that (1.1.2.8) can be obtained from (1.1.3.10) using contour integration as well.

SOLUTION: The python code which was used to generate the results can be shared.

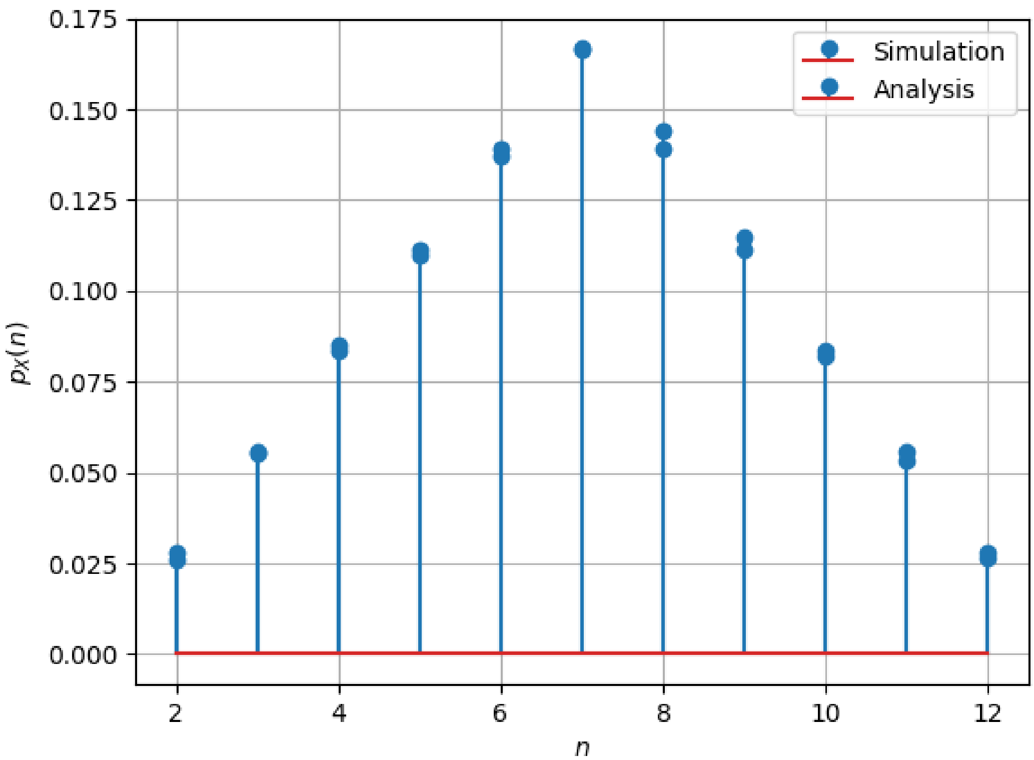

Figure 1.1.3.1: Plot of $p_X(n)$. Simulations are close to the analysis.

/codes/1/1.1.3.py

1.1.4 The experiment of rolling the dice was simulated using Python for 10000 samples. These were generated using Python libraries for uniform distribution.

The frequencies for each outcome were then used to compute the resulting pmf, which is plotted in Figure 1.1.4.1. The theoretical pmf obtained in (1.1.2.8) is plotted for comparison.

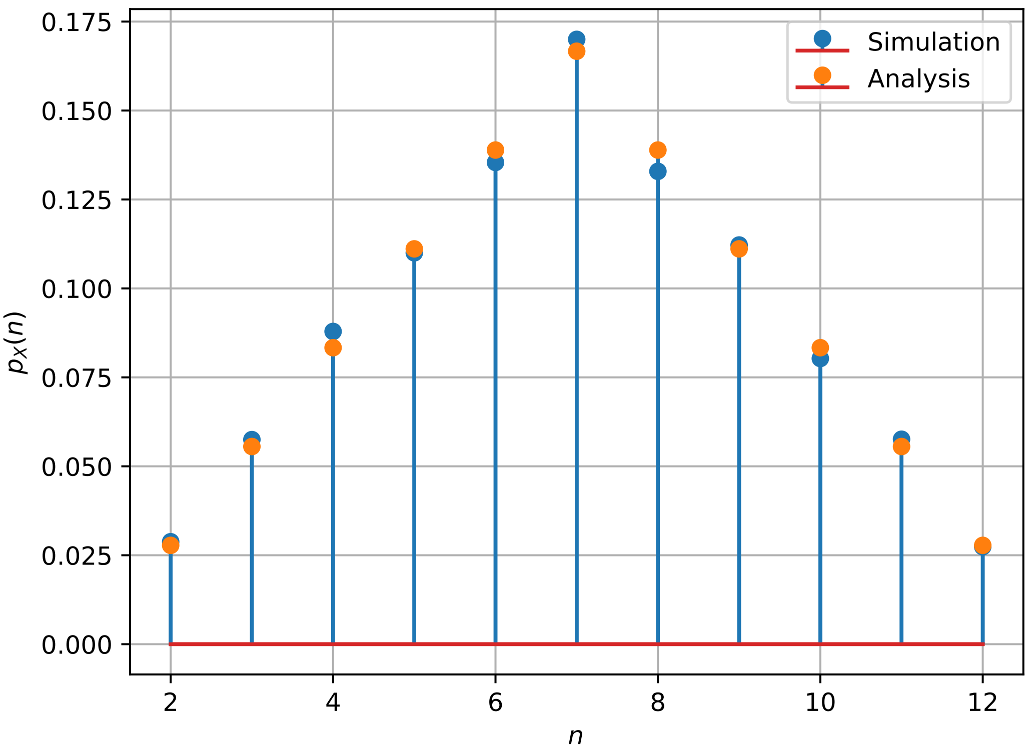

Figure 1.1.4.1: Plot of $p_X(n)$. Simulations are close to the analysis.

The python code which was used to generate the results can be shared.

/codes/1/1.1.4.py

Chapter 2

Random Numbers

2.1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

2.1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files and execute the C program.

```
/codes/coeffs.h
/codes/2/2.1.1.c
```

2.1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{2.1.2.1}$$

Solution: The following code plots Fig. 2.1.2.1

/codes/2/2.1.2.py

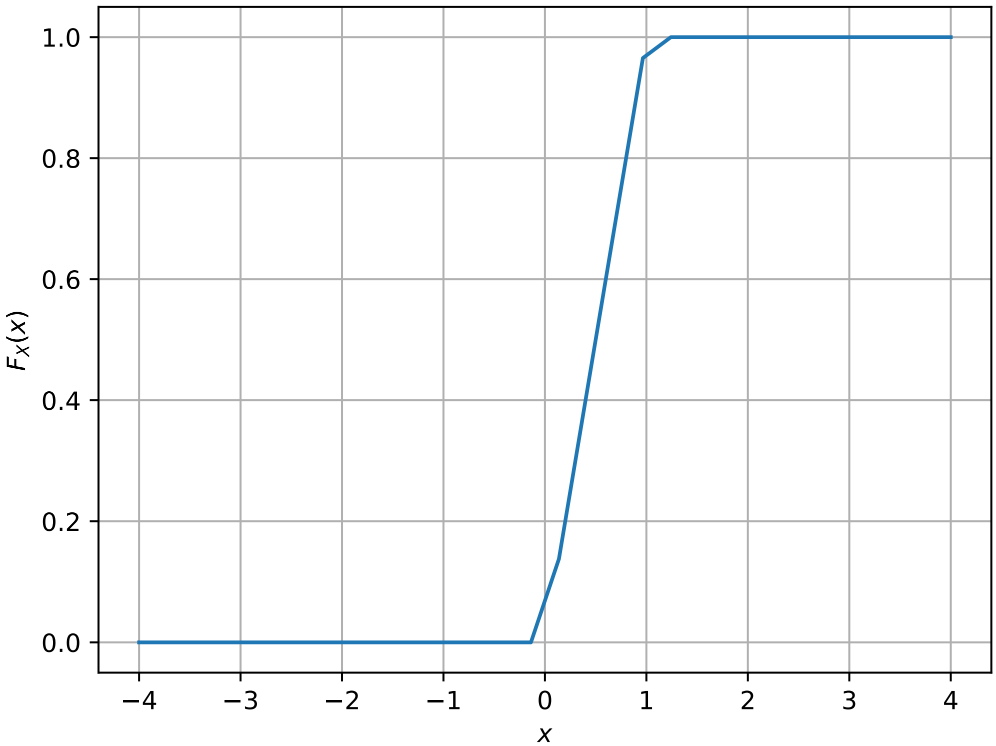

Figure 2.1.2.1: The CDF of U

2.1.3 Find a theoretical expression for $F_U(x)$.

Solution:

$$F_U(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & x \ge b \end{cases} \tag{2.1.3.1}$$

Substituting a=0 and b=1. in (2.1.3.1),

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x \ge 1 \end{cases} \tag{2.1.3.2}$$

2.1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{2.1.4.1}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{2.1.4.2}$$

Write a C program to find the mean and variance of U.

Solution:

/codes/2/2.1.4.c

/codes/2/uni.dat

2.1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{2.1.5.1}$$

Solution: : The mean μ_X and variance σ_X^2 of a random variable X,are given by

$$\mu_X = E[X] = \int_{-\infty}^{\infty} x dF_U(x) \tag{2.1.5.2}$$

$$\sigma_X^2 = E\left[X^2\right] - \mu_X^2 = \int_{-\infty}^{\infty} x^2 dF_U(x) - \mu_X^2 \tag{2.1.5.3}$$

$$\tag{2.1.5.4}$$

Substituting the CDF of U from (2.1.3.2) in (2.1.5.2) and (2.1.5.3), we get

$$
\begin{aligned}
Mean : E\left[X\right] &= \int_a^b x \cdot \frac{1}{b-a} dx \\
&= \frac{b^2-a^2}{2} \cdot \frac{1}{b-a} \\
&= \frac{(b-a)(b+a)}{2} \cdot \frac{1}{b-a} \\
&= \frac{b+a}{2}
\end{aligned}
$$

$$Here\ a = 0, b = 1$$

$$
\begin{aligned}
\mu = \mathrm{E}[X] &= \frac{1}{2} = 0.5 \\
E\left[X^2\right] &= \int_a^b x^2 \cdot \frac{1}{b-a} dx \\
&= \frac{b^3-a^3}{3} \cdot \frac{1}{b-a} \\
&= \frac{a^2+ab+b^2}{3} \\
Variance : \sigma^2 &= E\left(X^2\right) - \left[E(X)\right]^2 \\
&= \frac{(a-b)^2}{12}
\end{aligned}
$$

$$\sigma^2 = 0.834$$

theoretically calculated mean and variance are consistent with the result obtained from the problem 2.1.4

2.2 Central Limit Theorem

2.2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.2.1.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gauss.dat

Solution: Download the following files and execute the C program.

/codes/2/2.2.1.c

/codes/2/Gauss.dat

2.2.2 Load Gauss.dat in python and plot the empirical CDF of X using the samples in Gauss.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 2.2.2.1

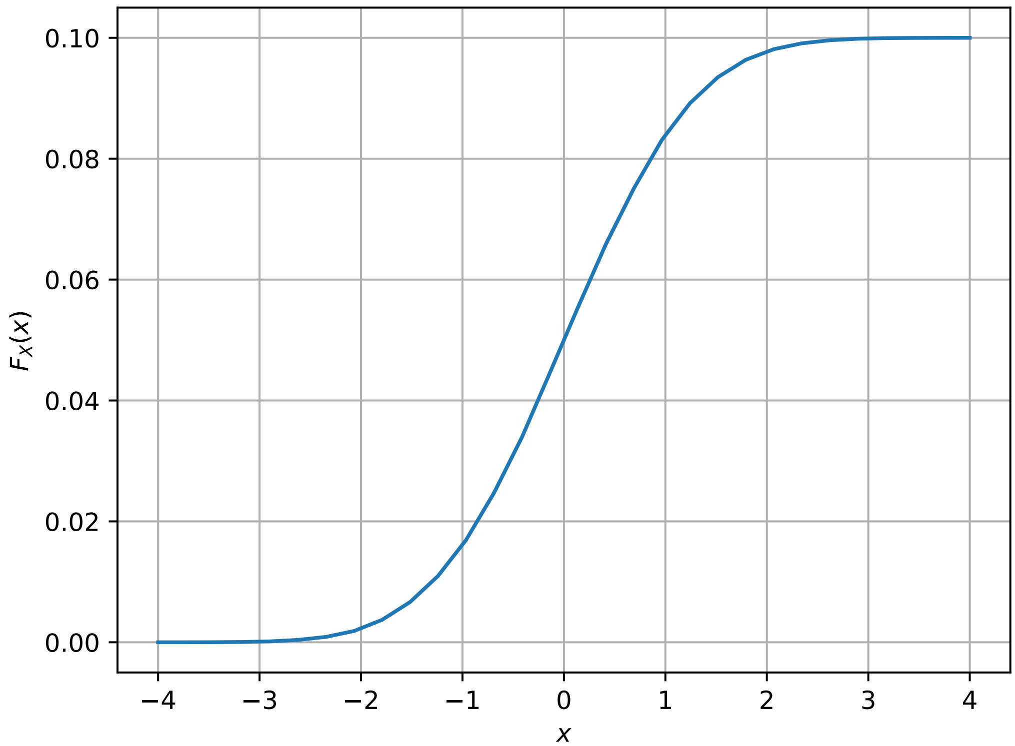

Figure 2.2.2.1: The CDF of X

Properties:

- CDF is non-decreasing function
- Right continous.
- $F_{max}(+\infty) = 1$.
- $F_{min}(-\infty) = 0$.
- $P[a \leq x \leq b] = F_x(b) - F_x(a)$

- $\frac{dF_X(x)}{dx} \geq 0$

2.2.3 Load Gaussian.dat in python and plot the empirical PDF of X using the samples in Gaussian.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2.3.1}$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 2.2.5.1 using the code below

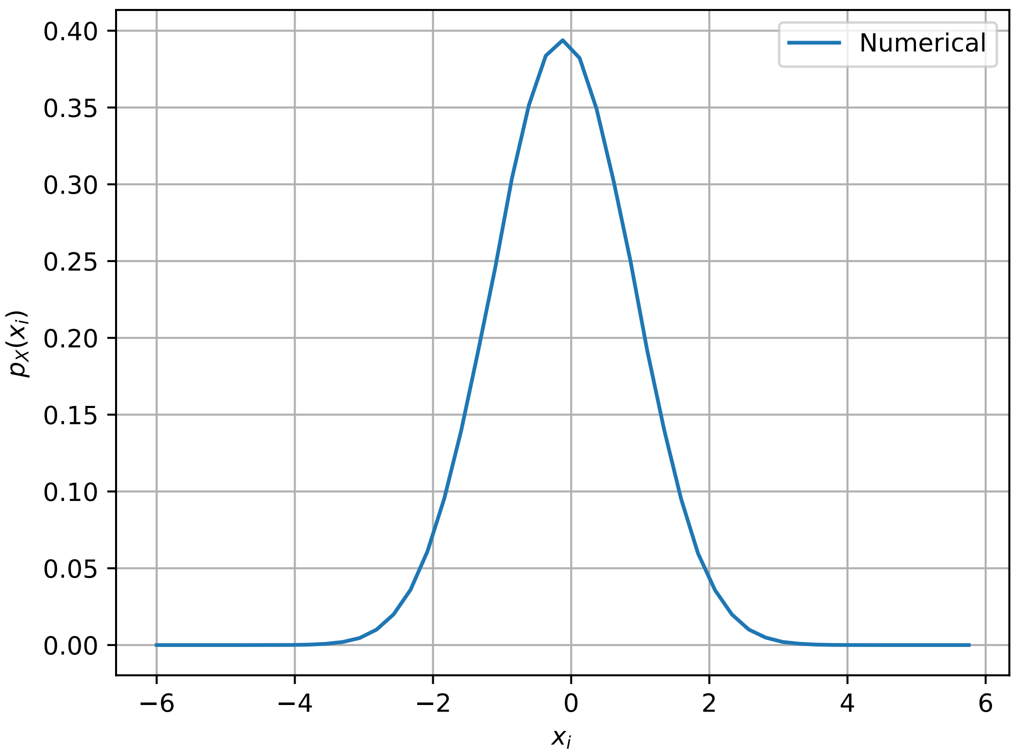

Figure 2.2.3.1: The PDF of X

/codes/2/2.2.3.py

/codes/2/Gauss.dat

Properties :

- $f_X(x) \geq 0, -\infty \leq x \leq \infty$
- $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$
- $P[x_1 \leq X \leq x_2] = \int_{x_1}^{x_2} f_X(x)\, dx$

2.2.4 Find the mean and variance of X by writing a C program.

Solution: : Download the following files and run the C program.

```
/codes/2/2.2.1.c
/codes/2/Gauss.dat
```

2.2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.2.5.1}$$

repeat the above exercise theoretically.

Solution:

$$Mean(\mu) = E(X) \tag{2.2.5.2}$$

$$E(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xe^{-\frac{x^2}{2}} dx = 0 \tag{2.2.5.3}$$

$$Variance(\sigma^2) = E\left(X\right)^2 - E^2\left(X\right) \tag{2.2.5.4}$$

$$E\left(X^2\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} dx \quad (even function) \tag{2.2.5.5}$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} x^2 e^{-\frac{x^2}{2}} dx \tag{2.2.5.6}$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \sqrt{2y} e^{-y} dy \quad \left(\frac{x^2}{2} = y\right) \tag{2.2.5.7}$$

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-y} y^{\frac{1}{2}} dy \tag{2.2.5.8}$$

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-y} y^{\left(\frac{1}{2}+1\right)-1} dy \quad \left(\Gamma(x) = \int_{0}^{\infty} z^{x-1} \cdot e^{-z}, dz\right) \tag{2.2.5.9}$$

$$= \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}+1\right) \tag{2.2.5.10}$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) \tag{2.2.5.11}$$

$$= 1 \tag{2.2.5.12}$$

where

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{2.2.5.13}$$

Thus, the variance is

$$\sigma^2 = E\left(X\right)^2 - E^2\left(X\right) = 1 \tag{2.2.5.14}$$

numerical and theoretical plots as shown in fig.2.2.5.1

/codes/2/2.2.5.py

/codes/2/Gauss.dat

2.3 From Uniform to Other

2.3.1 Generate samples of

$$V = -2\ln\left(1-U\right) \tag{2.3.1.1}$$

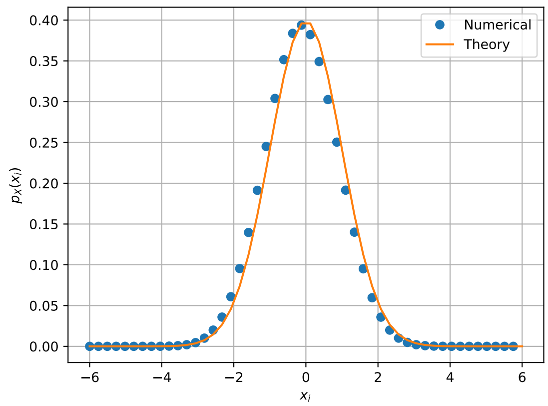

Figure 2.2.5.1: The PDF of $p\left(x\right)$

and plot its CDF.

Solution: : Code loads the samples from the v.dat file generated and plot the cdf of V

/codes/2/2.3.1.c

/codes/2/2.3.1.py

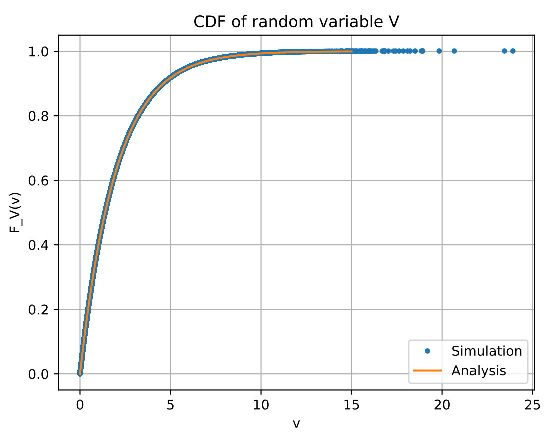

Figure 2.3.1.1: The CDF of V

2.3.2 Find a theoretical expression for $F_V(x)$.

$$F_V(x) = P(V \le x) \tag{2.3.2.1}$$
$$= P(-2\ln(1-U) \le x) \tag{2.3.2.2}$$
$$= P(\ln(1-U) \ge -\frac{x}{2}) \tag{2.3.2.3}$$
$$= P(1-U \ge e^{\frac{-x}{2}}) \tag{2.3.2.4}$$
$$= P(U \le 1-e^{\frac{-x}{2}}) \tag{2.3.2.5}$$
$$= F_U(1-e^{\frac{-x}{2}}) \tag{2.3.2.6}$$

2.4 Triangular Distribution

2.4.1 Generate

$$T = U_1 + U_2 \tag{2.4.1.1}$$

Solution: Download the following files and execute the C program.

/codes/2/2.4.1.c

2.4.2 Find the CDF of T.

Solution: Loading the samples from tri.dat in python, the CDF is plotted in Fig. 2.4.2.1

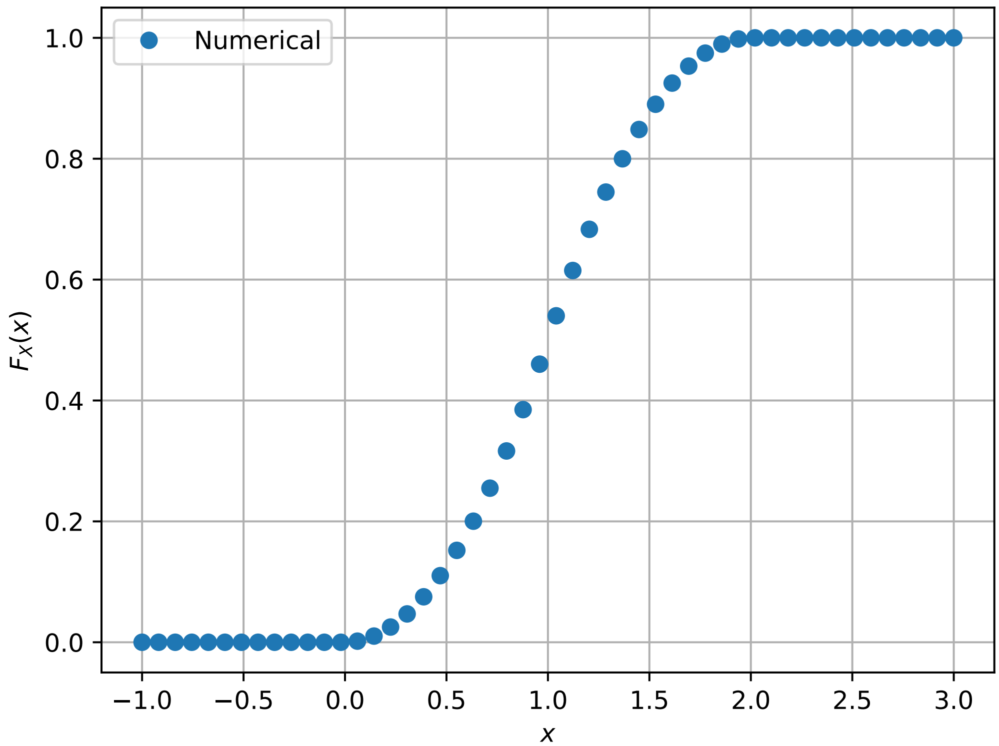

Figure 2.4.2.1: The CDF of T

2.4.3 Find the PDF of T.

Solution: The PDF of T is plotted in Fig. 2.4.3.1 using the code below **Solution:** Download the following files and execute the C program.

/codes/2/2.4.3.py

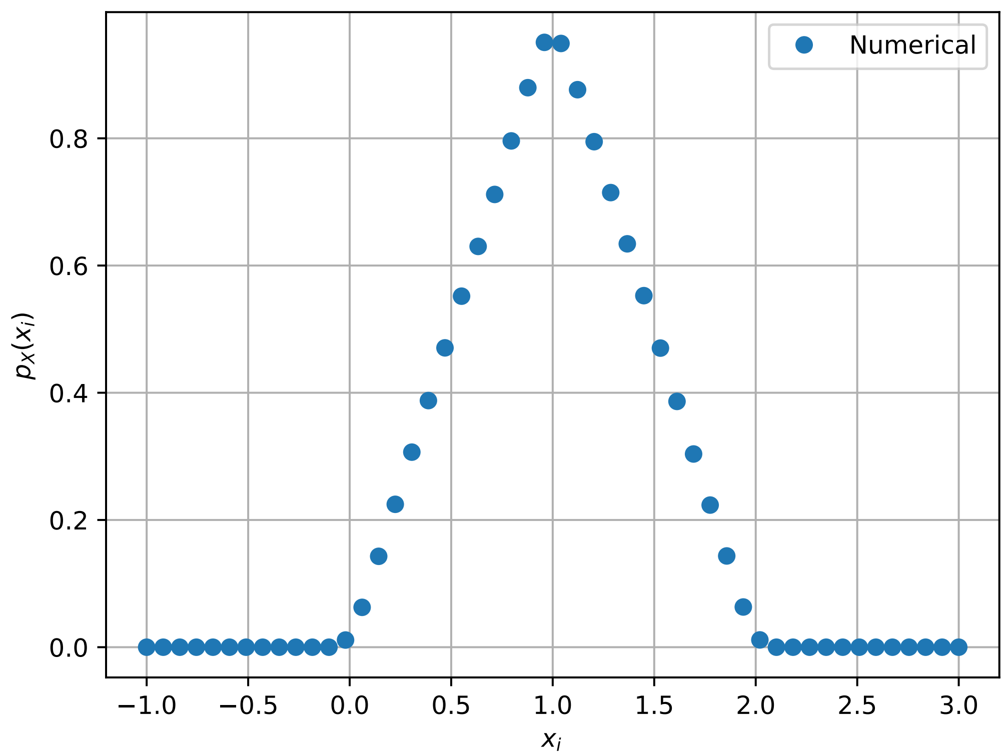

Figure 2.4.3.1: The PDF of T

2.4.4 Find the theoretical expressions for the PDF and CDF of T.

Solution: CDF ($F_T(x)$) of a triangular distribution is

$$F_T(x) = \begin{cases} 0 & x \leq a \\ \frac{(x-a)^2}{(b-a)(c-a)} & a < x \leq c \\ 1 - \frac{(b-x)^2}{(b-a)(c-a)} & c < x \leq b \\ 1 & x > b \end{cases} \tag{2.4.4.1}$$

PDF ($p_T(x)$) of a triangular distribution is

$$p_T(x) = \frac{d}{dx}F_U(x) \tag{2.4.4.2}$$

$$p_U(x) = \begin{cases} 0 & x \leq a \\ \frac{2(x-a)}{(b-a)(c-a)} & a < x \leq c \\ \frac{2(b-x)}{(b-a)(c-a)} & c < x \leq b \\ 0 & x > b \end{cases} \tag{2.4.4.3}$$

2.4.5 Verify your results through a plot.

Solution: : Compare using the Python codes provided below and the corresponding plots shown Fig. 2.4.5.1 and Fig. 2.4.5.2

```
/codes/2/2.4.4a.py
/codes/2/2.4.4b.py
```

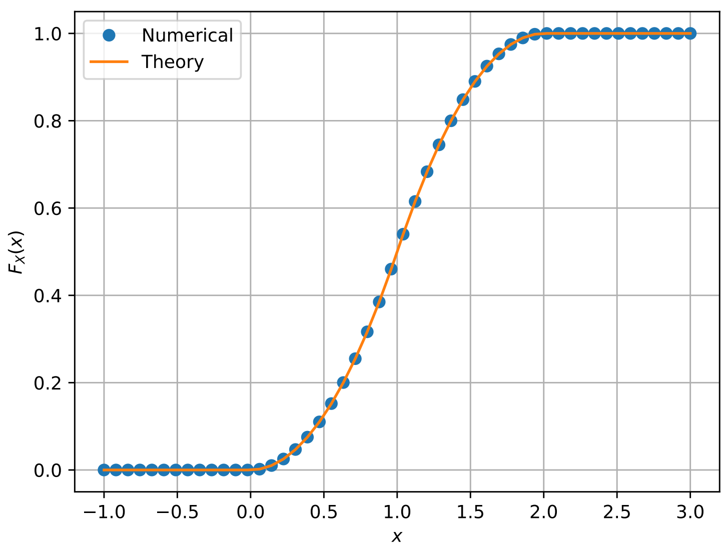

Figure 2.4.5.1: The CDF of T

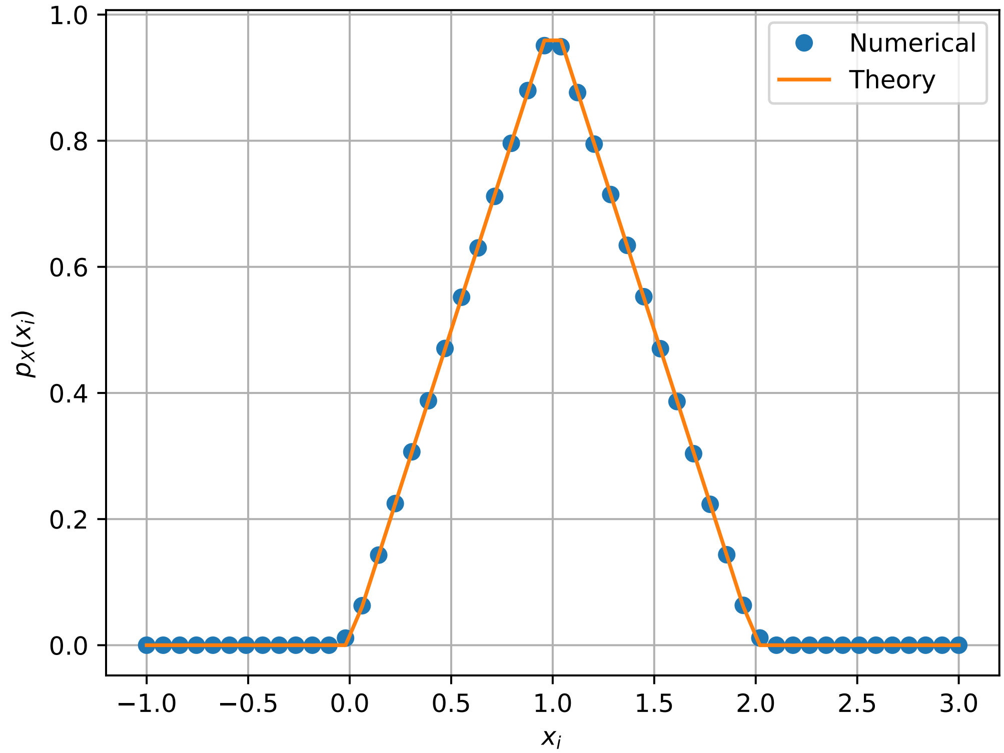

Figure 2.4.5.2: The PDF of T

Chapter 3

Maximum Likelihood Detection: BPSK

3.1 Maximum Likelihood

3.1.1 Generate equiprobable $X \in \{1, -1\}$.

Solution: X can be generated in python using the code section,

/codes/3/3.1.1.py

3.1.2 Generate

$$Y = AX + N, \tag{3.1.2.1}$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.

Solution: Y can be generated in python using the below code section,

/codes/3/3.1.2.py

3.1.3 Plot Y using a scatter plot.

Solution: The scatter plot of Y is plotted in Fig. 3.1.3.1 using the below code,

/codes/3/3.1.3.py

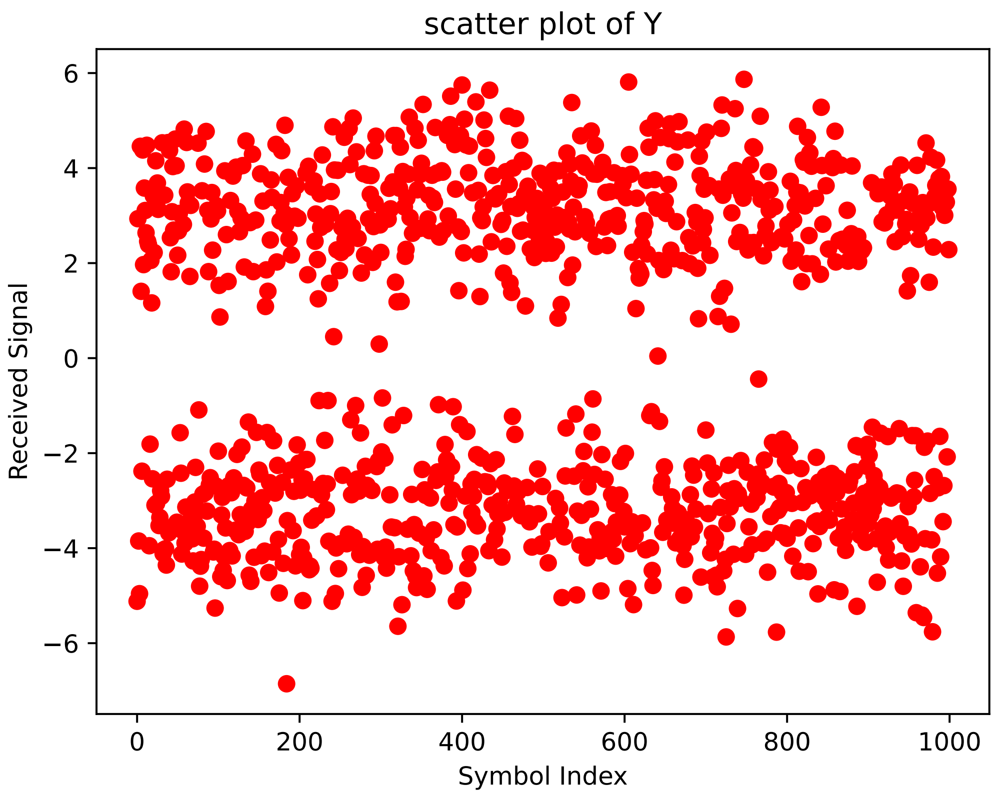

Figure 3.1.3.1: Scatter plot of Y

3.1.4 Guess how to estimate X from Y.

Solution:

$$y \underset{-1}{\overset{1}{\gtrless}} 0 \tag{3.1.4.1}$$

3.1.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{3.1.5.1}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{3.1.5.2}$$

Solution: The probability of error can be calculated using the decision rule in (3.1.4.1),

$$\begin{aligned}
P_{e|0} &= \Pr\left(\hat{X} = -1|X = 1\right) \\
&= \Pr\left(Y < 0|X = 1\right) \\
&= \Pr\left(AX + N < 0|X = 1\right) \\
&= \Pr\left(N < -AX|X = 1\right) \\
&= \Pr\left(N < -A\right) \\
&= Q(-A)
\end{aligned}$$

$$\begin{aligned}
P_{e|1} &= \Pr\left(\hat{X} = 1|X = -1\right) \\
&= \Pr\left(Y > 0|X = -1\right) \\
&= \Pr\left(AX + N > 0|X = -1\right) \\
&= \Pr\left(N > -AX|X = -1\right) \\
&= \Pr\left(N > A\right) \\
&= Q(A)
\end{aligned}$$

where,$N \sim \mathcal{N}(0,1)$

$$\therefore \Pr(N > A) = \Pr(N < -A) \tag{3.1.5.3}$$

$$P_{e|0} = P_{e|1} \tag{3.1.5.4}$$

(3.1.5.4)is known as the symmetry property of the Gaussian distribution.

3.1.6 Find P_e assuming that X has equiprobable symbols.

Solution:

$$P_e = \Pr(X = 1) P_{e|1} + \Pr(X = -1) P_{e|0} \tag{3.1.6.1}$$

given X is equiprobable

$$P_e = \frac{1}{2} P_{e|1} + \frac{1}{2} P_{e|0} \tag{3.1.6.2}$$

Substituting from (3.1.5.4)

$$P_e = \Pr(N > A) \tag{3.1.6.3}$$

Given a random varible $X \sim \mathcal{N}(0,1)$ the Q-function can be expressed as

$$Q(x) = \Pr(X > x) \tag{3.1.6.4}$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{t^2}{2}\right) dt. \tag{3.1.6.5}$$

By using the Q-function, P_e can be expressed in terms of Q-function

$$P_e = Q(A) \tag{3.1.6.6}$$

3.1.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.

Solution: in figure 3.1.7.1 both the theoretical and numerical estimations from generated samples of Y are plotted using this code

/codes/3/3.1.7.py

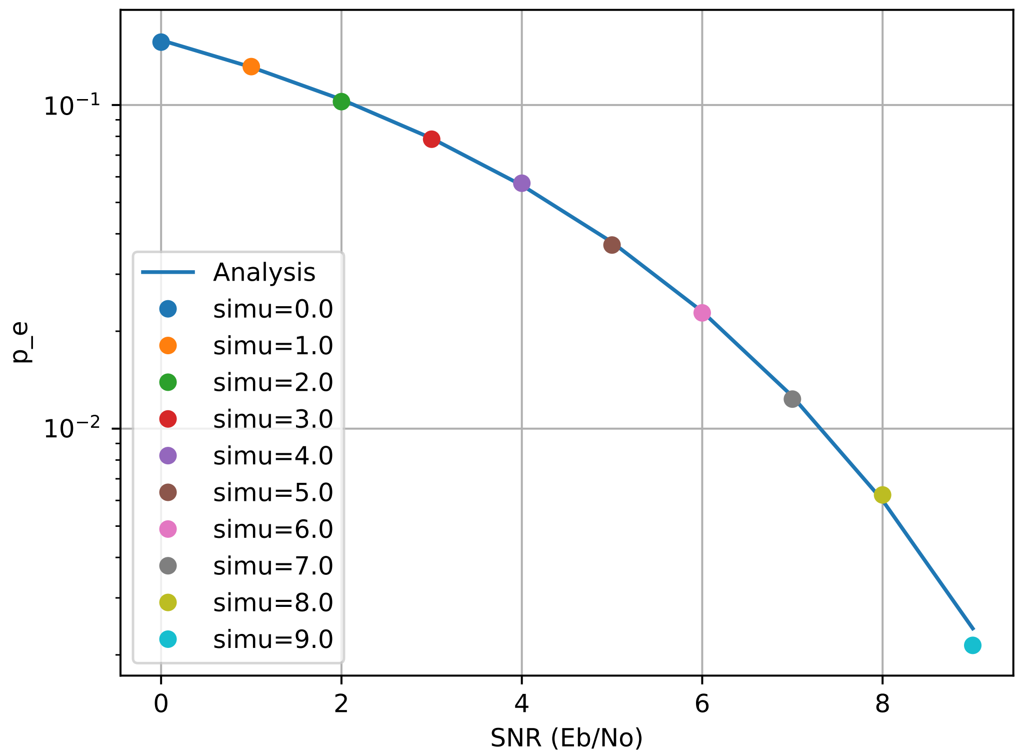

Figure 3.1.7.1: Scatter plot of Y

3.1.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.

Solution: Given the decision rule,

$$y \underset{-1}{\overset{1}{\gtrless}} \delta \tag{3.1.8.1}$$

$$\begin{aligned}
P_{e|0} &= \Pr\left(\hat{X} = -1|X = 1\right) \\
&= \Pr\left(Y < \delta|X = 1\right) \\
&= \Pr\left(AX + N < \delta|X = 1\right) \\
&= \Pr\left(A + N < \delta\right) \\
&= \Pr\left(N < -A + \delta\right) \\
&= \Pr\left(N > A - \delta\right) \\
&= Q(A - \delta)
\end{aligned}$$

$$\begin{aligned}
P_{e|1} &= \Pr\left(\hat{X} = 1|X = -1\right) \\
&= \Pr\left(Y > \delta|X = -1\right) \\
&= \Pr\left(AX + N > \delta|X = -1\right) \\
&= \Pr\left(N > A + \delta\right) \\
&= Q(A + \delta)
\end{aligned}$$

The overall P_e is the average of these two probabilities, which is given by:

$$P_e = \frac{1}{2}Q(A+\delta) + \frac{1}{2}Q(A-\delta) \tag{3.1.8.2}$$

Differentiating this equation with respect to δ and equating it to 0, we get:

$$\frac{\partial P_e}{\partial \delta} = \frac{1}{2}\frac{\partial Q(A+\delta)}{\partial \delta} - \frac{1}{2}\frac{\partial Q(A-\delta)}{\partial \delta} = 0 \tag{3.1.8.3}$$

$$\frac{\partial Q(A+\delta)}{\partial \delta} - \frac{\partial Q(A-\delta)}{\partial \delta} = 0 \tag{3.1.8.4}$$

And by the Leibniz's rule:

$$\frac{d}{dx}\int_{f(x)}^{g(x)} h(t,x)dt = h(g(x),x)g'(x) - h(f(x),x)f'(x) \tag{3.1.8.5}$$

We get

$$\frac{\frac{\partial}{\partial\delta}\int_{A+\delta}^{\infty} e^{-\frac{u^2}{2}}du - \frac{\partial}{\partial\delta}\int_{A-\delta}^{\infty} e^{-\frac{u^2}{2}}du}{\partial\delta} = 0 \tag{3.1.8.6}$$

$$\frac{e^{-\frac{(A+\delta)^2}{2}}}{\partial\delta} - \frac{e^{-\frac{(A-\delta)^2}{2}}}{\partial\delta} = 0 \tag{3.1.8.7}$$

Solving for δ we get

$$e^{-\frac{(A+\delta)^2}{2}} - e^{-\frac{(A-\delta)^2}{2}} = 0 \tag{3.1.8.8}$$

$$\frac{e^{-\frac{(A+\delta)^2}{2}}}{e^{-\frac{(A-\delta)^2}{2}}} = 1 \tag{3.1.8.9}$$

$$e^{-\frac{(A+\delta)^2-(A-\delta)^2}{2}} = 1 \tag{3.1.8.10}$$

$$e^{-2A\delta} = 1 \tag{3.1.8.11}$$

$$\ln(e^{-2A\delta}) = \ln(1) \tag{3.1.8.12}$$

$$-2A\delta = 0 \tag{3.1.8.13}$$

$$\delta = 0 \tag{3.1.8.14}$$

3.1.9 Repeat the above exercise when

$$p_X(0) = p \tag{3.1.9.1}$$

Solution:

Given X is not equiprobable, P_e is given by,

$$P_e = (1-p)P_{e|1} + pP_{e|0} \tag{3.1.9.2}$$

$$= (1-p)Q(A+\delta) + pQ(A-\delta) \tag{3.1.9.3}$$

Where $Q(x)$ is the Q-function

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{u^2}{2}} du \tag{3.1.9.4}$$

$$P_e = k((1-p)\int_{A+\delta}^\infty \exp^{-\frac{u^2}{2}} du + p\int_{A-\delta}^\infty \exp^{-\frac{u^2}{2}} du \tag{3.1.9.5}$$

where k is a constant.

we differentiate the above expression with respect to δ and equate to zero,

$$\begin{aligned} \frac{\partial P_e}{\partial \delta} &== k((1-p)\frac{\partial}{\partial \delta}\int_{A+\delta}^\infty \exp^{\left(-\frac{u^2}{2}\right)}, du - p\frac{\partial}{\partial \delta}\int_{A-\delta}^\infty \exp^{\left(-\frac{u^2}{2}\right)}, du) = 0 \\ &= k((1-p)\exp - \frac{(A+\delta)^2}{2} - p\exp - \frac{(A-\delta)^2}{2} = 0 \end{aligned}$$

Dividing both sides by $(1-p)\exp^{-\frac{(A+\delta)^2}{2}}$ we get

$$\frac{\exp\frac{(A+\delta)^2}{2}}{\exp -\frac{(A-\delta)^2}{2}} = \frac{p}{(1-p)}$$

Taking ln on both sides

$$\ln\left(\frac{\exp\left(-\frac{(A+\delta)^2}{2}\right)}{\exp\left(-\frac{(A-\delta)^2}{2}\right)}\right) = \ln\left(\frac{p}{(1-p)}\right)$$

Using the properties of ln and exp, we can simplify the above as

$$\ln\left(\exp\left(-\frac{(A+\delta)^2}{2}\right)\exp\left(\frac{(A-\delta)^2}{2}\right)\right) = \ln\left(\frac{p}{(1-p)}\right)$$

Rearranging and using the properties of exponential function

$$\ln\left(\exp\left(-\frac{(A+\delta)^2}{2}+\frac{(A-\delta)^2}{2}\right)\right)=\ln\left(\frac{p}{(1-p)}\right)$$

$$\ln\left(\exp\left(-2A\delta\right)\right)=\ln\left(\frac{p}{(1-p)}\right)$$

Solving for δ

$$-2A\delta=\ln\left(\frac{p}{(1-p)}\right)$$

$$\delta=\frac{1}{2A}\ln\left(\frac{p}{(1-p)}\right)$$

Finally taking the exponential of both sides

$$\delta=\frac{1}{2A}\log\left(\frac{1}{p}-1\right)$$

Therefore, the optimal value of δ is given by

$$\delta=\frac{1}{2A}\log\left(\frac{1}{p}-1\right)$$

Chapter 4

Transformation of Random Variables

4.1 Gaussian to Other

4.1.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{4.1.1.1}$$

Solution: The CDF and PDF of V are plotted in Fig. **??** using the below code

/codes/4/4.1.1.py

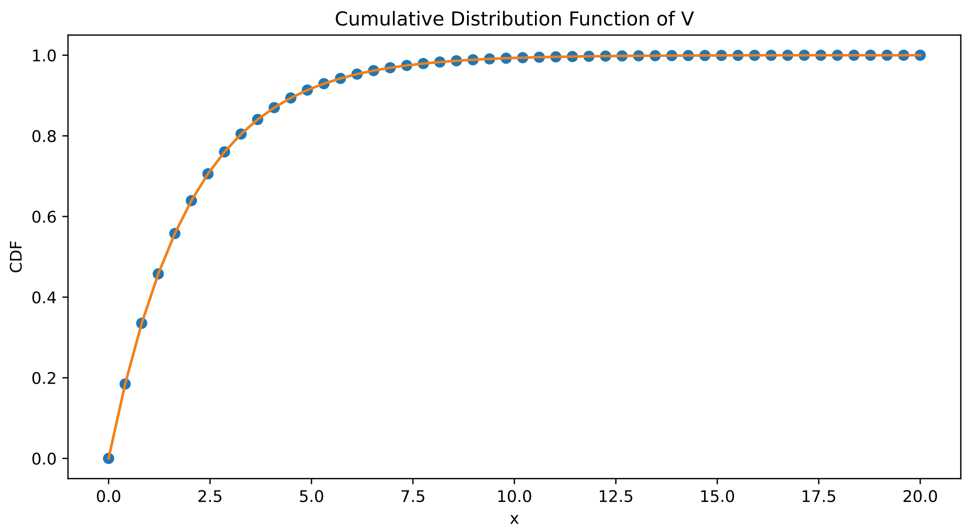

Figure 4.1.1.1: CDF of V

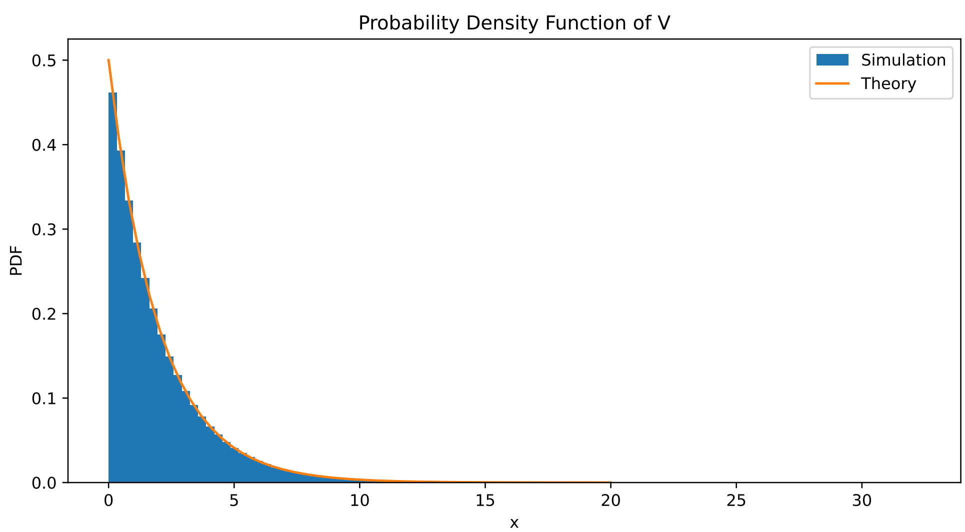

Figure 4.1.1.2: PDF of V

4.1.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{4.1.2.1}$$

find α.

Let $Z = X^2$ where $X \sim \mathcal{N}(0,1)$. Defining the CDF for Z,

$$\begin{aligned} P_Z(z) &= \Pr(Z < z) \\ &= \Pr\left(X^2 < z\right) \\ &= \Pr\left(-\sqrt{z} < X < \sqrt{z}\right) \\ &= \int_{-\sqrt{z}}^{\sqrt{z}} p_X(x), dx \end{aligned}$$

Using the derivative of the CDF, the PDF of Z is given by

$$\frac{d}{dz} P_Z(z) = p_Z(z) \tag{4.1.2.2}$$

$$= \frac{p_X(\sqrt{z}) + p_X(-\sqrt{z})}{2\sqrt{z}} \tag{4.1.2.3}$$

Substituting the standard gaussian density function $p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, we get

$$p_Z(z) = \begin{cases} \frac{1}{\sqrt{2\pi z}} e^{-\frac{z}{2}} & z \geq 0 \\ 0 & z < 0 \end{cases} \tag{4.1.2.4}$$

Now, let $V = X_1^2 + X_2^2$, where $X_1, X_2 \sim \mathcal{N}(0,1)$ are independent. The PDF of

V can be obtained by convolution of $p_{X_1^2}(x)$ and $p_{X_2^2}(x)$ is

$$\begin{aligned}
p_V(v) &= p_{X_1^2}(x_1) * p_{X_2^2}(x_2) \\
&= \frac{1}{2\pi}\int_0^v \frac{e^{-\frac{x}{2}}}{\sqrt{x}}\frac{e^{-\frac{v-x}{2}}}{\sqrt{v-x}}, dx \\
&= \frac{e^{-\frac{v}{2}}}{2\pi}\int_0^v \frac{1}{\sqrt{x(v-x)}}, dx \\
&= \frac{e^{-\frac{v}{2}}}{2\pi}[-\arcsin\left(\frac{v-2x}{v}\right)]0^v \\
&= \frac{e^{-\frac{v}{2}}}{2\pi}\pi \\
&= \frac{e^{-\frac{v}{2}}}{2} \text{ for } v \geq 0
\end{aligned}$$

The CDF of V can be obtained by integrating the PDF:

$$\begin{aligned}
F_V(v) &= \frac{1}{2}\int_0^v \exp\left(-\frac{v}{2}\right) \\
&= 1 - \exp\left(-\frac{v}{2}\right) \text{ for } v \geq 0
\end{aligned} \tag{4.1.2.5}$$

Comparing (4.1.2.5) with (4.1.2.1), $\alpha = \frac{1}{2}$

/codes/4/4.1.2.py

Code shows different values of α, and plots it along with the simulated data. The legend in the plot will show which line corresponds to which value of α.

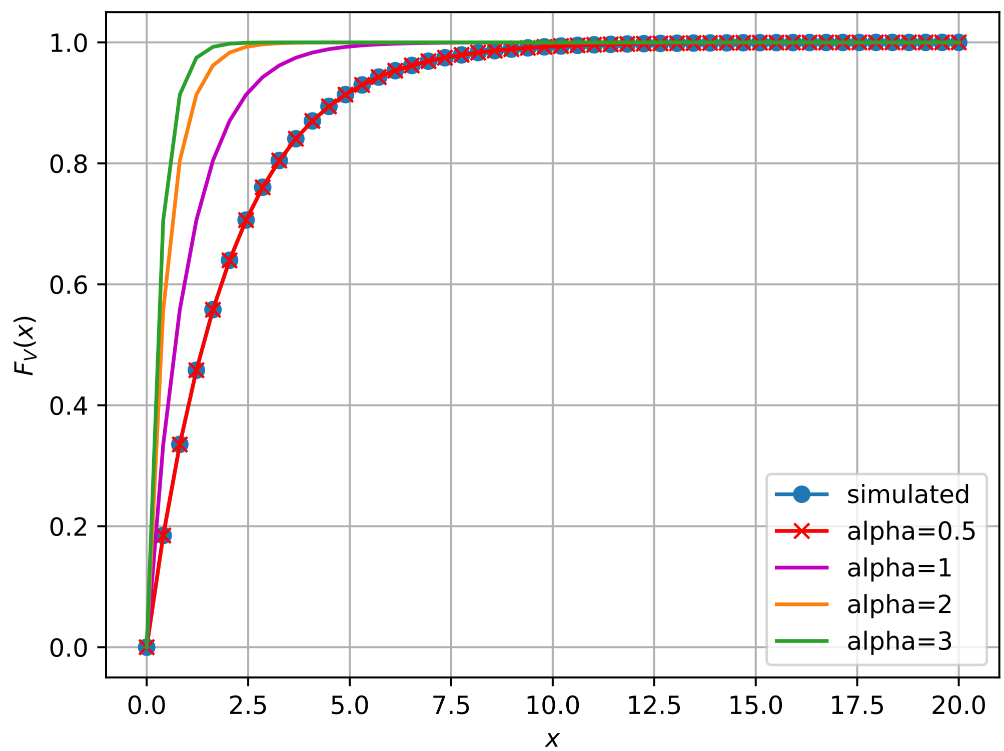

Figure 4.1.2.1: CDF of V

4.1.3 Plot the CDF and PDF of

$$A = \sqrt{V} \tag{4.1.3.1}$$

Solution: The CDF and PDF of A are plotted in Fig. 4.1.3.1 and Fig. 4.1.3.2 using the below code

/codes/4/4.1.3.py

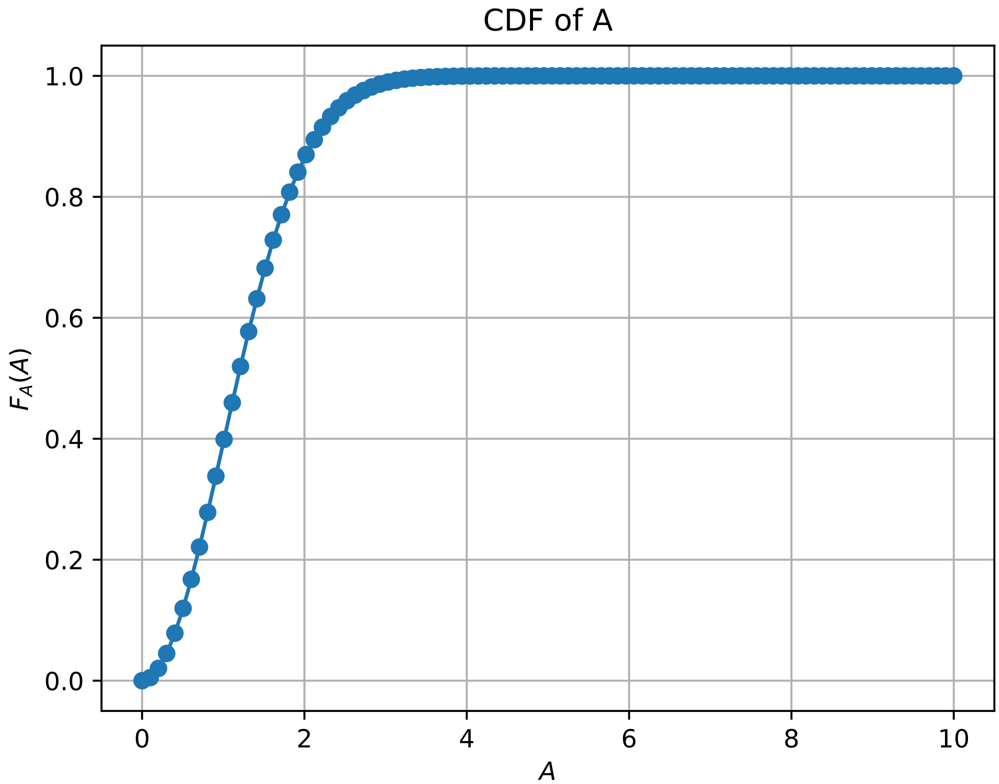

Figure 4.1.3.1: CDF of V

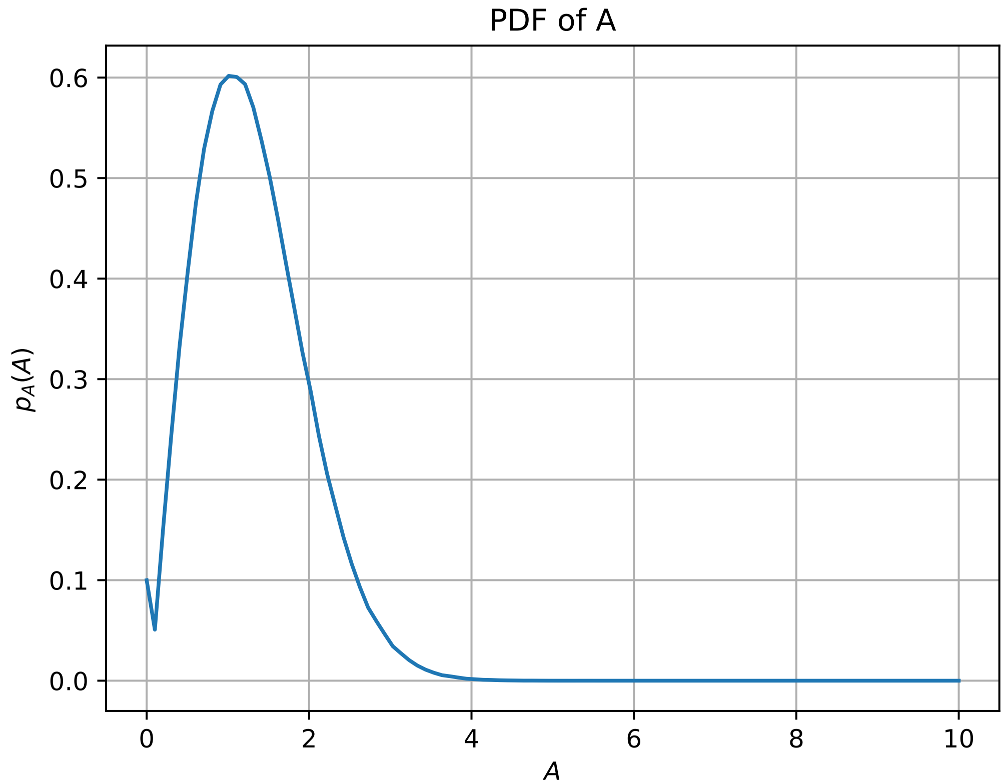

Figure 4.1.3.2: PDF of V

4.2 Conditional Probability

4.2.1 Plot

$$P_e = \Pr\left(\hat{X} = -1|X = 1\right) \tag{4.2.1.1}$$

for

$$Y = AX + N, \tag{4.2.1.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}\left(0, 1\right), X \in (-1, 1)$ for $0 \le \gamma \le 10$ dB.

Solution:

/codes/4/4.2.1.py

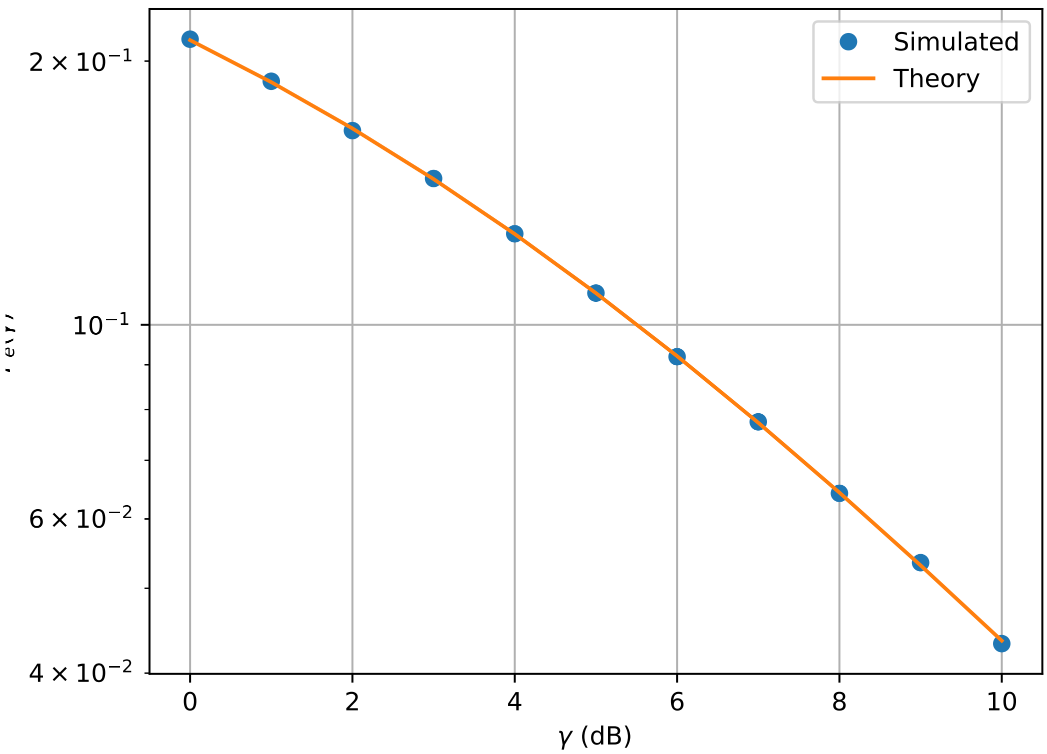

Figure 4.2.1.1: P_e versus γ

4.2.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

Solution: The probability of error P_e is given by

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{4.2.2.1}$$
$$= \Pr\left(Y < 0 | X = 1\right) \tag{4.2.2.2}$$
$$= \Pr\left(AX + N < 0 | X = 1\right) \tag{4.2.2.3}$$
$$= \Pr\left(A + N < 0\right) \tag{4.2.2.4}$$
$$= \Pr\left(A < -N\right). \tag{4.2.2.5}$$

where A is a Rayleigh random variable with PDF $f_A(x)$.

$$f_A(x) = \begin{cases} \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) & x \geq 0 \\ 0 & otherwise \end{cases} \tag{4.2.2.6}$$

Now, we need to find the CDF of A at $-N$ to calculate P_e, which is given by

$$F_A(-N) = \int_{-\infty}^{-N} f_A(x) dx \tag{4.2.2.7}$$
$$= \int_{0}^{-N} \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx. \tag{4.2.2.8}$$
$$= 1 - \exp\left(-\frac{N^2}{2\sigma^2}\right) \tag{4.2.2.9}$$

Since $F_A(-N) = 0$ for $N \geq 0$, the final expression for P_e is given by

$$P_e(N) = \begin{cases} 1 - \exp\left(-\frac{N^2}{2\sigma^2}\right) & N < 0 \\ 0 & otherwise \end{cases} \tag{4.2.2.10}$$

4.2.3 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{4.2.3.1}$$

Find $P_e = E\left[P_e(N)\right]$.

Solution: Since $N \sim \mathcal{N}(0, 1)$,

$$p_N(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{4.2.3.2}$$

And from (4.2.2.10)

$$P_e(x) = \begin{cases} 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right) & x < 0 \\ 0 & otherwise \end{cases} \tag{4.2.3.3}$$

$$P_e = E\left[P_e(N)\right] = \int_{-\infty}^{\infty} P_e(x) p_N(x)\, dx \tag{4.2.3.4}$$

For a Rayleigh Distribution with scale $= \sigma$,

$$E\left[A^2\right] = 2\sigma^2 \tag{4.2.3.5}$$

$$\gamma = 2\sigma^2 \tag{4.2.3.6}$$

$$\tag{4.2.3.7}$$

Using $P_e(N)$ from (4.2.2.10),

$$\begin{aligned} P_e &= \int_{-\infty}^{\infty} P_e(x) p_N(x)\, dx \\ &= \int_0^{\infty} \left(1 - e^{-\frac{x^2}{2\sigma^2}}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \left(1 - e^{-\frac{x^2}{\gamma}}\right) e^{-\frac{x^2}{2}}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{x^2}{2}}\, dx - \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \exp\left(-x^2\left(\frac{1}{\gamma} + \frac{1}{2}\right)\right)\, dx \\ &= \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\gamma}{2+\gamma}} \end{aligned}$$

$$P_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\gamma}{2+\gamma}} \tag{4.2.3.8}$$

4.2.4 Plot P_e in problems 4.2.1 a nd 4.2.3 on the same graph w.r.t γ. Comment.

Solution: P_e is plotted w.r.t γ in 4.2.4.1 using the code below.

/codes/4/4.2.4.py

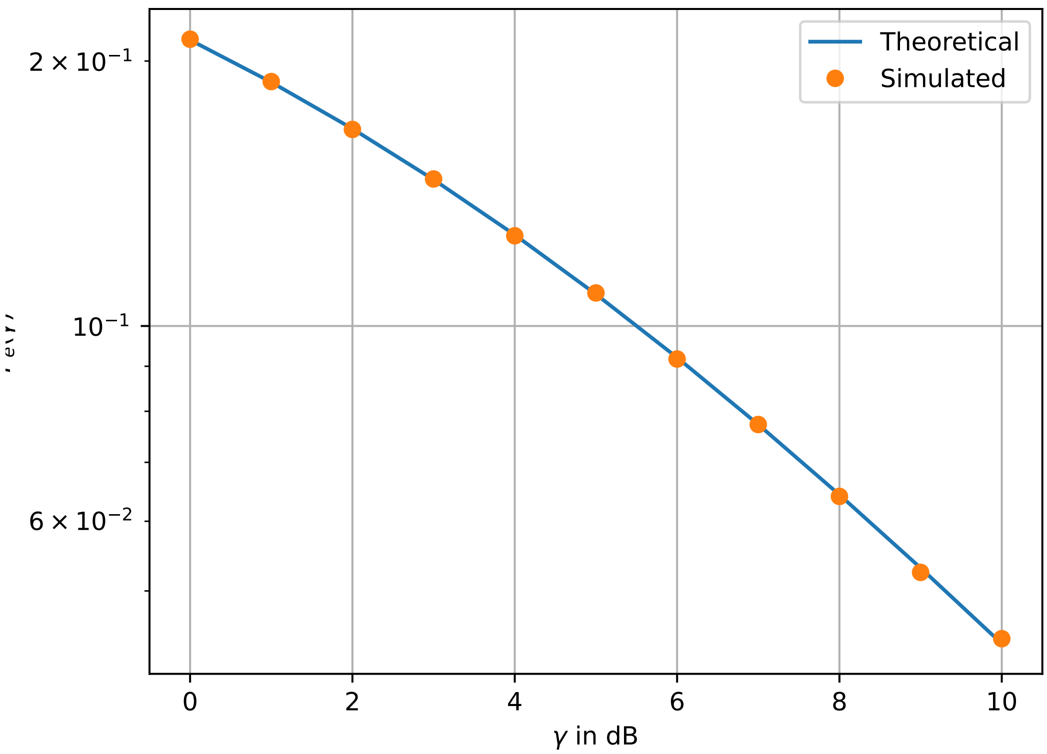

Figure 4.2.4.1: The P_e wrt γ

Chapter 5

Bivariate Random Variables: FSK

5.1 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{5.1.0.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.1.0.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \tag{5.1.0.3}$$

5.1.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{5.1.1.1}$$

on the same graph using a scatter plot.

/codes/5/5.1.1.py

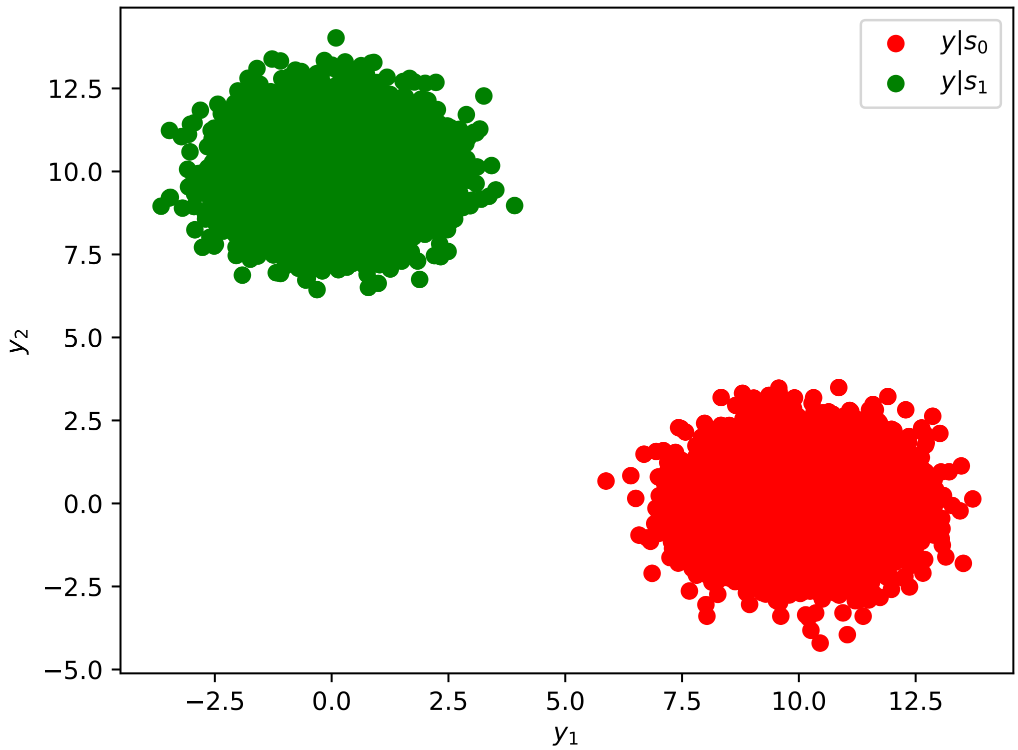

Figure 5.1.1.1: Scatter plot of $\mathbf{y}|\mathbf{s}_0$ and $\mathbf{y}|\mathbf{s}_1$

5.1.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

Solution: :

e decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$ can be expressed mathematically as follows:

Let $\mathbf{y} = \begin{bmatrix} y_1 & y_2 \end{bmatrix}$. Then the decision rule can be written as:

$$y_1 \underset{1}{\overset{0}{\gtrless}} y_2 \tag{5.1.2.1}$$

The random vector $\mathbf{y}|\mathbf{s}i$ has normally distributed components with PDF:

$$p\mathbf{y}|\mathbf{s}_i\left(\mathbf{y}\right) = \frac{1}{2\pi\sqrt{\det \mathbf{\Sigma}}} \exp\left(-\frac{1}{2}\left(\mathbf{y}-\mathbf{s}_i\right)^\top \mathbf{\Sigma}^{-1}\left(\mathbf{y}-\mathbf{s}i\right)\right) \tag{5.1.2.2}$$

where $\mathbf{\Sigma}$ is the covariance matrix. With $\mathbf{\Sigma} = \sigma\mathbf{I}$, the PDF becomes:

$$p\mathbf{y}|\mathbf{s}_i\left(\mathbf{y}\right) = \frac{1}{2\pi\sigma} \exp\left(-\frac{1}{2\sigma}\left(\mathbf{y}-\mathbf{s}_i\right)^\top \mathbf{I}\left(\mathbf{y}-\mathbf{s}_i\right)\right) \tag{5.1.2.3}$$

$$= \frac{1}{2\pi\sigma} \exp\left(-\frac{1}{2\sigma}\left(\mathbf{y}-\mathbf{s}_i\right)^\top \left(\mathbf{y}-\mathbf{s}_i\right)\right) \tag{5.1.2.4}$$

Assuming equal probability for both symbols, we can use the Maximum A Posteriori (MAP) rule to determine the optimum decision. For only two possible symbols, the optimal decision criterion is found by equating $p_{\mathbf{y}|\mathbf{s}0}$ and $p\mathbf{y}|\mathbf{s}1$

$$\begin{aligned}
\log p\mathbf{y}|\mathbf{s}0 &= \log p\mathbf{y}|\mathbf{s}_1 \\
-\frac{1}{2\sigma}\left(\mathbf{y}-\mathbf{s}_0\right)^\top\left(\mathbf{y}-\mathbf{s}_0\right) &= -\frac{1}{2\sigma}\left(\mathbf{y}-\mathbf{s}_1\right)^\top\left(\mathbf{y}-\mathbf{s}_1\right) \\
\implies \frac{1}{2\sigma}\left(\mathbf{y}-\mathbf{s}_0\right)^\top\left(\mathbf{y}-\mathbf{s}_0\right) &= \frac{1}{2\sigma}\left(\mathbf{y}-\mathbf{s}_1\right)^\top\left(\mathbf{y}-\mathbf{s}_1\right) \\
\implies \left(\mathbf{y}-\mathbf{s}_0\right)^\top\left(\mathbf{y}-\mathbf{s}_0\right) &= \left(\mathbf{y}-\mathbf{s}_1\right)^\top\left(\mathbf{y}-\mathbf{s}_1\right) \\
\implies \left(\mathbf{y}^\top-\mathbf{s}_0^\top\right)\left(\mathbf{y}-\mathbf{s}_0\right) &= \left(\mathbf{y}^\top-\mathbf{s}_1^\top\right)\left(\mathbf{y}-\mathbf{s}_1\right) \\
\implies \mathbf{y}^\top\mathbf{y}-\mathbf{y}^\top\mathbf{s}_0-\mathbf{s}_0^\top\mathbf{y}+\mathbf{s}_0^\top\mathbf{s}_0 &= \mathbf{y}^\top\mathbf{y}-\mathbf{y}^\top\mathbf{s}_1-\mathbf{s}_1^\top\mathbf{y}+\mathbf{s}_1^\top\mathbf{s}_1 \\
\implies \mathbf{s}_0^\top\mathbf{s}_0-\mathbf{s}_1^\top\mathbf{s}_1 &= 2\mathbf{y}^\top(\mathbf{s}_1-\mathbf{s}_0)-2\mathbf{s}_1^\top\mathbf{y}+2\mathbf{s}_0^\top\mathbf{y}
\end{aligned}$$

From here, one can obtain the result that $\mathbf{y} = \frac{\mathbf{s}_0+\mathbf{s}_1}{2}$ to satisfy the condition that

$$\implies \begin{pmatrix}-1\\1\end{pmatrix}^\top \mathbf{y} = 0$$

5.1.3 Plot

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{5.1.3.1}$$

with respect to the SNR from 0 to 10 dB.

/codes/5/5.1.3.py

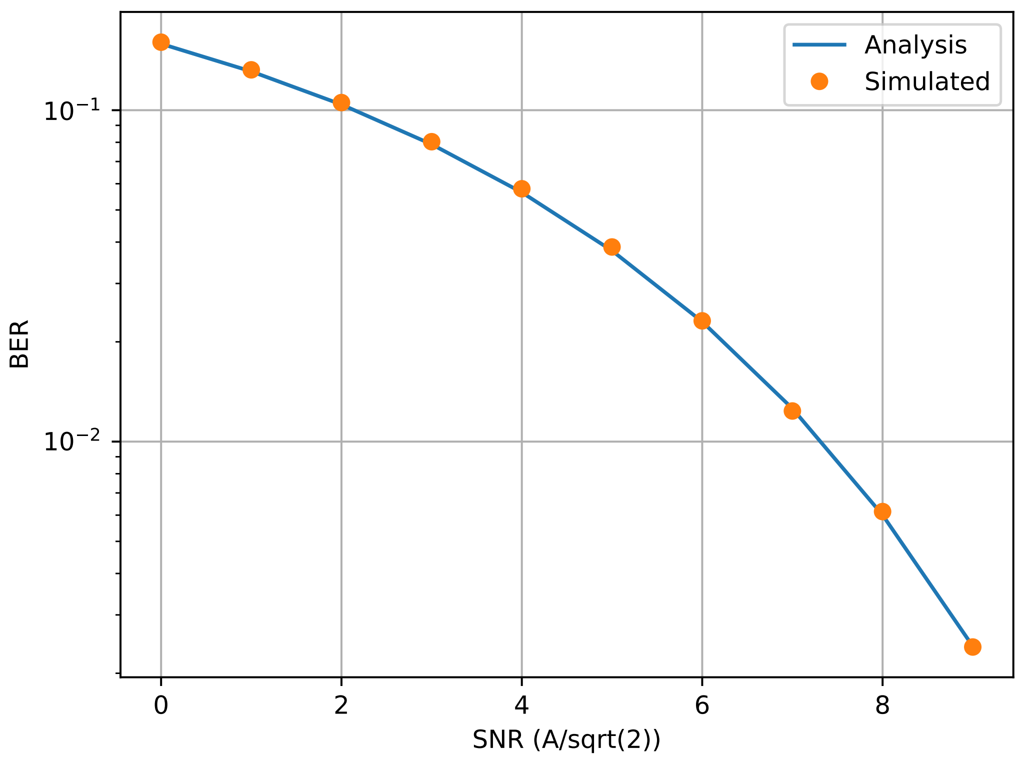

Figure 5.1.3.1: P_e versus SNR plot for FSK

5.1.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.

Solution: The Q-function defined as:

$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}} dt$

P_e, can be calculated as:

$$\begin{aligned} P_e &= \Pr(\hat{x} = s_1 | x = s_0) \\ &= \Pr(y_1 < y_2 | x = s_0) \\ &= \Pr(A + n_1 < n_2) \\ &= \Pr(n_1 - n_2 < -A) \quad (Z = n_1 - n_2) \\ &= \Pr(Z < -A) \end{aligned}$$

Using the properties of normal distributions,$n_1, n_2 \sim \mathcal{N}\left(0, \sigma^2\right), Z \sim \mathcal{N}\left(0, 2\sigma^2\right)$.

By changing the inequality sign and using the Q-function, we get:

$$\begin{aligned} P_e &= \Pr\left(Z < -A\right) \\ &= \Pr\left(Z > A\right) \\ &= Q(\frac{A}{\sqrt{2\sigma^2}}) \\ &= Q(\frac{A}{\sqrt{2}}) \quad \left(\sigma^2 = 1\right) \end{aligned}$$

Thus, P_e can be represented as the Q-function of $\frac{A}{\sqrt{2}}$

Chapter 6

Exercises

6.1 BPSK

6.1.1 The *signal constellation diagram* for BPSK is given by Fig. 6.1.1.1. The symbols s_0 and s_1 are equiprobable. $\sqrt{E_b}$ is the energy transmitted per bit. Assuming a zero mean additive white gaussian noise (AWGN) with variance $\frac{N_0}{2}$, obtain the symbols that are received

Solution: The possible received symbols are

$$y|s_0 = \sqrt{E_b} + n \tag{6.1.1.1}$$

$$y|s_1 = -\sqrt{E_b} + n \tag{6.1.1.2}$$

where the AWGN $n \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

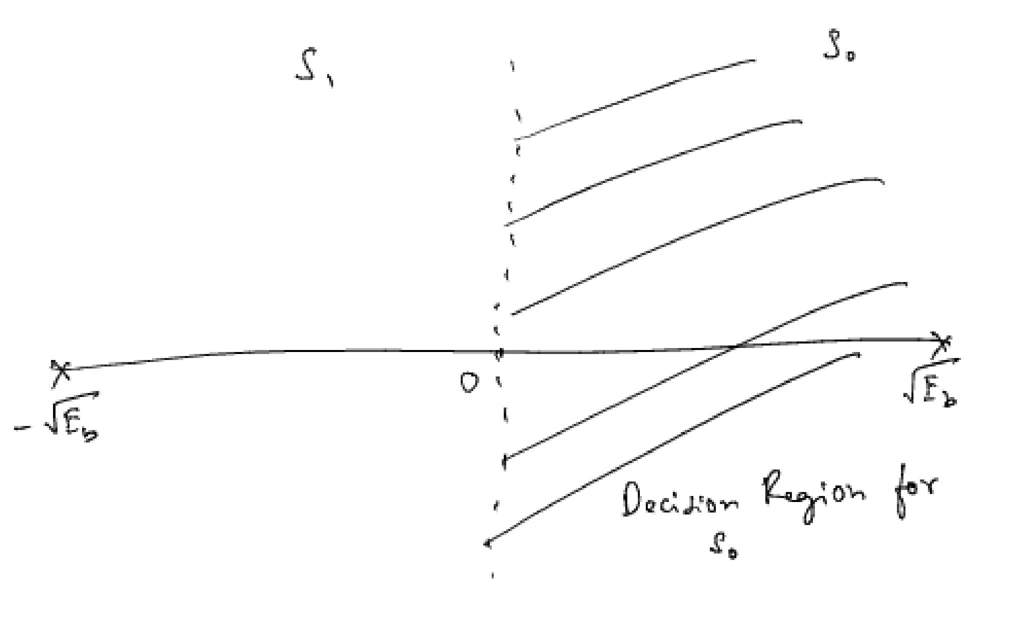

Figure 6.1.1.1

6.1.2 From Fig.6.1.1.1 obtain a decision rule for BPSK **Solution:** The decision rule is

$$y \underset{s_1}{\overset{s_0}{\gtrless}} 0 \tag{6.1.2.1}$$

6.1.3 Repeat the previous exercise using the MAP criterion.

Solution: The log-likelihood ratios for the two symbols are given by

$$\log \frac{p(y|s_0)}{p(y|s_1)} = \log \frac{\frac{1}{\sqrt{2\pi N_0/2}} \exp\left(-\frac{(y-\sqrt{E_b})^2}{2N_0/2}\right)}{\frac{1}{\sqrt{2\pi N_0/2}} \exp\left(-\frac{(y+\sqrt{E_b})^2}{2N_0/2}\right)} \tag{6.1.3.1}$$

$$= \log \exp\left(\frac{2\sqrt{E_b}y}{N_0/2}\right) - \log\left(\exp\left(\frac{2\sqrt{E_b}\sqrt{E_b}}{N_0/2}\right)\right) \tag{6.1.3.2}$$

$$= \frac{2\sqrt{E_b}y}{N_0/2} - \frac{2\sqrt{E_b}\sqrt{E_b}}{N_0/2} \tag{6.1.3.3}$$

$$= \frac{2\sqrt{E_b}(y-\sqrt{E_b})}{N_0/2} \tag{6.1.3.4}$$

Therefore, the decision rule based on the MAP criterion is

$$y \underset{s_1}{\overset{s_0}{\gtrless}} \frac{y\sqrt{E_b}}{N_0/2} \gtrless 0 \tag{6.1.3.5}$$

which is equivalent to the previous decision rule.

$$y \underset{s_1}{\overset{s_0}{\gtrless}} 0 \tag{6.1.3.6}$$

6.1.4 Using the decision rule in Problem 6.1.2, obtain an expression for the probability of error for BPSK.

Solution: Since the symbols are equiprobable, it is sufficient if the error is calculated assuming that a 0 was sent. This results in

$$P_e = \Pr\left(y < 0|s_0\right) = \Pr\left(\sqrt{E_b} + n < 0\right) \tag{6.1.4.1}$$

$$= \Pr\left(-n > \sqrt{E_b}\right) = \Pr\left(n > \sqrt{E_b}\right) \tag{6.1.4.2}$$

since n has a symmetric pdf. Let $w \sim \mathcal{N}(0,1)$. Then $n = \sqrt{\frac{N_0}{2}}w$. Substituting this in (6.1.4.2),

$$P_e = \Pr\left(\sqrt{\frac{N_0}{2}}w > \sqrt{E_b}\right) = \Pr\left(w > \sqrt{\frac{2E_b}{N_0}}\right) \tag{6.1.4.3}$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{6.1.4.4}$$

where $Q(x) \triangleq \Pr(w > x), x \geq 0$.

6.1.5 The PDF of $w \sim \mathcal{N}(0,1)$ is given by

$$p_w(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty \tag{6.1.5.1}$$

and the complementary error function is defined as

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt. \tag{6.1.5.2}$$

Show that

$$Q(x) = \frac{1}{2}\,\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right) \tag{6.1.5.3}$$

Solution:

$$Q(a) = \frac{1}{2\pi} \int_a^\infty e^{-\frac{t^2}{2}}, dt \tag{6.1.5.4}$$

$$a = \frac{a'}{\sqrt{2}}, \quad a' = \sqrt{2}a, \quad da = \frac{da'}{\sqrt{2}} \tag{6.1.5.5}$$

Using these substitutions, we can rewrite the equation for $\mathrm{erfc}(x)$ as:

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-a^2/2}, \frac{da'}{\sqrt{2}} \tag{6.1.5.6}$$

$$= \frac{2}{\sqrt{2\pi}} \int_{\sqrt{2}x}^\infty e^{-a^2/2}, da' \tag{6.1.5.7}$$

$$\implies \mathrm{erfc}(x) = 2Q(\sqrt{2}x) \tag{6.1.5.8}$$

$$\therefore Q(\sqrt{2}x) = \frac{1}{2}\,\mathrm{erfc}(x) \tag{6.1.5.9}$$

So, finally,

$$Q(x) = \frac{1}{2}\,\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right) \tag{6.1.5.10}$$

6.1.6 Verify the bit error rate (BER) plots for BPSK through simulation and analysis for 0 to 10 dB. **Solution:**

```
/codes/6/6.1.6.py
```

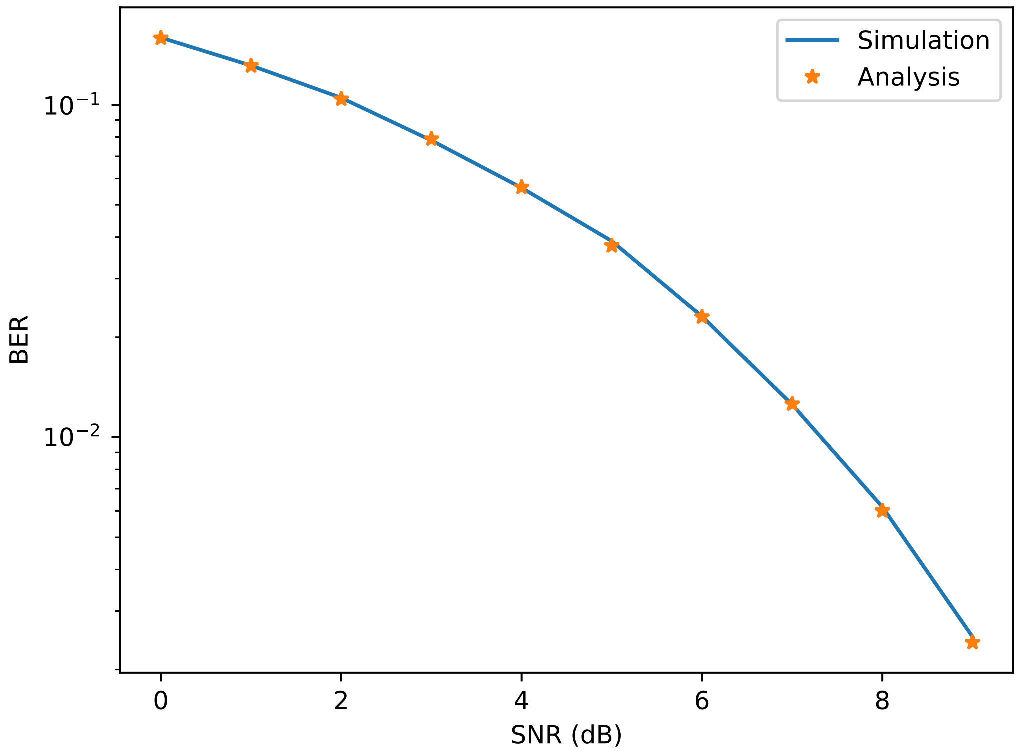

Figure 6.1.6.1

6.1.7 Show that

$$Q(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}}\, d\theta \tag{6.1.7.1}$$

Solution: The Gaussian Q-function, which is defined by

$$
\begin{aligned}
Q(x) &= \int_x^{\infty} f_u(u)\, du \\
&= \frac{1}{2\pi} \int_u^{\infty} e^{-\frac{u}{2}} \\
&= \frac{1}{2\pi} \int_z^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{u^2+v^2}{2}\right) du\, dv
\end{aligned}
$$

Transforming the Cartesian coordinates in (6.2.1.1) to polar coordinates as $u = rsin\theta, v = rcos\theta$ and $dudv = rdrd\theta$

$$
\begin{aligned}
Q(x) &= \frac{1}{2\pi} \int_0^{\pi} d\theta \int_{\frac{x}{\sin\theta}}^{\infty} \exp\left(-\frac{r^2}{2}\right) r\, dr \\
&= \frac{1}{2\pi} \int_0^{\pi} \exp\left(-\frac{x^2}{2\sin^2\theta}\right) d\theta \\
&= \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \exp\left(-\frac{x^2}{2\sin^2\theta}\right) d\theta
\end{aligned}
$$

6.2 Coherent BFSK

6.2.1 The signal constellation for binary frequency shift keying (BFSK) is given in Fig. 6.2.1.1. Obtain the equations for the received symbols.

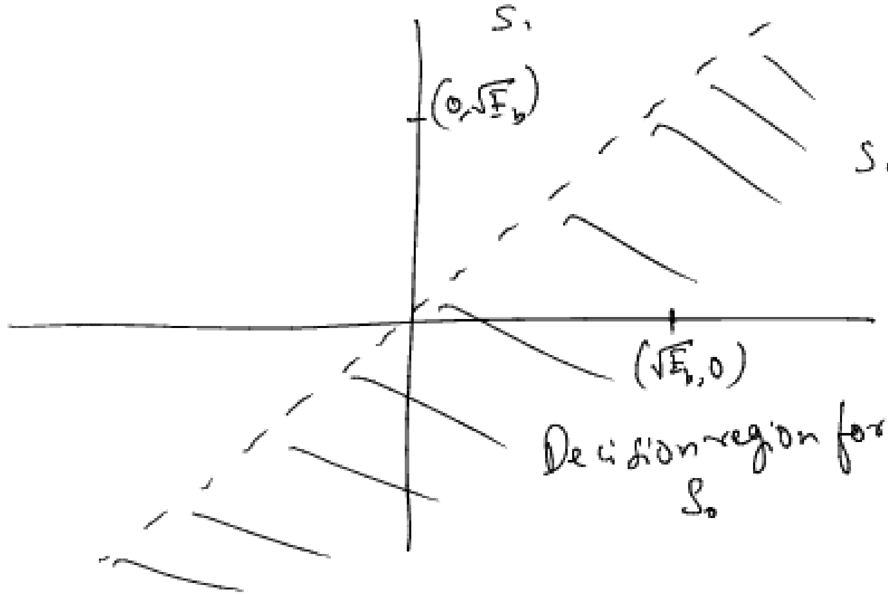

Figure 6.2.1.1

Solution: The received symbols are given by

$$\mathbf{y}|s_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{6.2.1.1}$$

and

$$\mathbf{y}|s_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{6.2.1.2}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$. and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

6.2.2 Obtain a decision rule for BFSK from Fig. 6.2.1.1.

Solution: The decision rule is

$$y_1 \underset{s_1}{\overset{s_0}{\gtrless}} y_2 \tag{6.2.2.1}$$

6.2.3 Repeat the previous exercise using the MAP criterion.

Solution: The Maximum A Posteriori (MAP) criterion for deciding the symbol s from the received signal $\mathbf{y}$ can be given as:

$$s = \arg\max_{s_0, s_1} P(s|\mathbf{y}) \tag{6.2.3.1}$$

The likelihood function can be given as:

$$P(\mathbf{y}|s_0) = \frac{1}{\sqrt{2\pi \frac{N_0}{2}}} \exp\left(-\frac{||\mathbf{y} - \mathbf{y}|s_0||^2}{2\frac{N_0}{2}}\right) \tag{6.2.3.2}$$

$$P(\mathbf{y}|s_1) = \frac{1}{\sqrt{2\pi \frac{N_0}{2}}} \exp\left(-\frac{||\mathbf{y} - \mathbf{y}|s_1||^2}{2\frac{N_0}{2}}\right) \tag{6.2.3.3}$$

where $||\cdot||$ represents the Euclidean distance.

Assuming equal prior probabilities for both symbols, i.e., $P(s_0) = P(s_1) = \frac{1}{2}$, the decision rule using the MAP criterion can be given as:

$$s = \arg\max_{s_0, s_1} \frac{1}{\sqrt{2\pi \frac{N_0}{2}}} \exp\left(-\frac{||\mathbf{y} - \mathbf{y}|s||^2}{2\frac{N_0}{2}}\right) \tag{6.2.3.4}$$

can be represented as:

$$y_1 \underset{s_1}{\overset{s_0}{\gtrless}} y_2 \tag{6.2.3.5}$$

The decision is made by comparing the values of y_1 and y_2 and choosing the symbol with the highest likelihood of being transmitted, given the received signal.