Signal Processing in High School

G. V. V. Sharma
Dept. of Electrical Engg.,
IIT Hyderabad.

March 16, 2023

NCERT

Two dice, one blue and one grey, are thrown at the same time. The event defined by the sum of the two numbers appearing on the top of the dice can have 11 possible outcomes 2, 3, 4, 5, 6, 6, 8, 9, 10, 11 and 12. A student argues that each of these outcomes has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.

Uniform Distribution: Rectangular Function

Let $X_i \in \{1, 2, 3, 4, 5, 6\}$, $i = 1, 2$, be the random variables representing the outcome for each die. Assuming the dice to be fair, the probability mass function (pmf) is expressed as

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{6} & 1 \leq n \leq 6 \\ 0 & otherwise \end{cases} \tag{2.1}$$

Sum of Random Variables: Convolution

2.1. The desired outcome is

$$X = X_1 + X_2, \tag{2.2}$$

$$\implies X \in \{1, 2, \ldots, 12\} \tag{2.3}$$

The objective is to show that

$$p_X(n) \neq \frac{1}{11} \tag{2.4}$$

2.2. *Convolution:* From (2.2),

$$p_X(n) = \Pr(X_1 + X_2 = n) = \Pr(X_1 = n - X_2) \tag{2.5}$$

$$= \sum_k \Pr(X_1 = n - k | X_2 = k)\, p_{X_2}(k) \tag{2.6}$$

after unconditioning. $\because$ X_1 and X_2 are independent,

$$\Pr(X_1 = n - k | X_2 = k) = \Pr(X_1 = n - k) = p_{X_1}(n - k)$$

$$\implies p_X(n) = \sum_k p_{X_1}(n - k) p_{X_2}(k) \triangleq p_{X_1}(n) * p_{X_2}(n) \tag{2.7}$$

Convolution Cont.

$$p_X(n) = \frac{1}{6}\sum_{k=1}^{6} p_{X_1}(n-k) = \frac{1}{6}\sum_{k=n-6}^{n-1} p_{X_1}(k) \tag{2.8}$$

$$\because p_{X_1}(k) = 0, \quad k \leq 1, k \geq 6. \tag{2.9}$$

From (2.8),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{1}{6}\sum_{k=1}^{n-1} p_{X_1}(k) & 1 \leq n-1 \leq 6 \\ \frac{1}{6}\sum_{k=n-6}^{6} p_{X_1}(k) & 1 < n-6 \leq 6 \\ 0 & n > 12 \end{cases} \tag{2.10}$$

Triangular Distribution: Student is wrong

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \leq n \leq 7 \\ \frac{13-n}{36} & 7 < n \leq 12 \\ 0 & n > 12 \end{cases} \tag{2.11}$$

Z Transform

The Z-transform of $p_X(n)$ is defined as

$$P_X(z) = \sum_{n=-\infty}^{\infty} p_X(n) z^{-n}, \quad z \in \mathbb{C} \tag{2.12}$$

From (2.1) and (2.12),

$$P_{X_1}(z) = P_{X_2}(z) = \frac{1}{6} \sum_{n=1}^{6} z^{-n} \tag{2.13}$$

$$= \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}, \quad |z| > 1 \tag{2.14}$$

upon summing up the geometric progression.

Convolution vs Multiplication

$$p_X(n) = p_{X_1}(n) * p_{X_2}(n) \implies P_X(z) = P_{X_1}(z) P_{X_2}(z) \tag{2.15}$$

Thus,

$$P_X(z) = \left\{ \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)} \right\}^2 = \frac{1}{36} \frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \tag{2.16}$$

Inverse Z transform

The Z transform of $u(n)$ is

$$u(n) \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-z^{-1}}, \; |z|>1 \tag{2.17}$$

and

$$nu(n) \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2}, \; |z|>1 \tag{2.18}$$

$$\begin{aligned} \frac{1}{36}\left[(n-1)\,u(n-1)-2\,(n-7)\,u(n-7)+(n-13)\,u(n-13)\right] \\ \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{36}\frac{z^{-2}\left(1-2z^{-6}+z^{-12}\right)}{(1-z^{-1})^2} \end{aligned} \tag{2.19}$$

Problem Statement

Let

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \tag{3.1}$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \tag{3.2}$$

where α and β ($\alpha > \beta$) are the roots of the

$$z^2 - z - 1 = 0 \tag{3.3}$$

JEE 2019

Which of the following options is correct?

3.1

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{3.4}$$

3.2

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{3.5}$$

3.3

$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \tag{3.6}$$

3.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \tag{3.7}$$

Difference Equation

The *Pingala* series is generated using the difference equation

$$x(n+2) = x\left(n+1\right) + x\left(n\right), \quad x(0) = x(1) = 1, n \geq 0 \tag{3.8}$$

One Sided Z transform

The *one sided* Z-transform of $x(n)$ is defined as

$$X^{+}(z) = \sum_{n=0}^{\infty} x(n) z^{-n}, \quad z \in \mathbb{C} \tag{3.9}$$

Taking the one-sided Z-transform on both sides of (3.8),

$$\mathcal{Z}^{+}\left[x(n+2)\right] = \mathcal{Z}^{+}\left[x(n+1)\right] + \mathcal{Z}^{+}\left[x(n)\right] \tag{3.10}$$

$$\implies z^2 X^{+}(z) - z^2 x(0) - zx(1) = zX^{+}(z) - zx(0) + zX^{+}(z) \tag{3.11}$$

$$\implies \left(z^2 - z - 1\right) X^{+}(z) = z^2 \tag{3.12}$$

$$\implies X^{+}(z) = \frac{1}{1 - z^{-1} - z^{-2}} = \frac{1}{\left(1 - \alpha z^{-1}\right)\left(1 - \beta z^{-1}\right)}, \quad |z| > \alpha \tag{3.13}$$

Inverse

Expanding $X^+(z)$ in (3.13) using partial fractions, we get

$$X^+(z) = \frac{1}{(\alpha - \beta)} \left[\frac{z}{1 - \alpha z^{-1}} - \frac{z}{1 - \beta z^{-1}} \right] \tag{3.14}$$

$$\implies x(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} u(n) \tag{3.15}$$

$$= a_{n+1} \tag{3.16}$$

upon comparing with (3.1).

Linear Time Invariant System

Let

$$y(n) = x\,(n-1) + x\,(n+1)\,, \quad n \geq 0 \tag{3.17}$$

Taking the one-sided Z-transform on both sides of (3.17),

$$\mathcal{Z}^{+}\left[y(n)\right] = \mathcal{Z}^{+}\left[x(n+1)\right] + \mathcal{Z}^{+}\left[x(n-1)\right] \tag{3.18}$$

$$Y^{+}(z) = zX^{+}(z) - zx(0) + z^{-1}X^{+}(z) + zx(-1) \tag{3.19}$$

$$= \frac{z+z^{-1}}{1-z^{-1}-z^{-2}} - z = \frac{1+2z^{-1}}{1-z^{-1}-z^{-2}}, \quad |z| > \alpha \tag{3.20}$$

Power of the Z transform

$$\alpha^n + \beta^n, \quad n \geq 1 \tag{3.21}$$

can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \tag{3.22}$$

Therefore,

$$W(z) = Y(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{3.23}$$

Power of the Z transform Cont.

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{a_{k+1}}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} \tag{3.24}$$

$$= \frac{1}{10} X^+(10) = \frac{1}{10} \times \frac{100}{89} = \frac{10}{89} \tag{3.25}$$

Thus, (3.5) is correct.

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{b_{k+1}}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} \tag{3.26}$$

$$= \frac{1}{10} Y^+(z) = \frac{1}{10} \times \frac{120}{89} = \frac{12}{89} \tag{3.27}$$

Thus, (3.7) is incorrect.

Convolution

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) = \sum_{k=-\infty}^{n-1} x(k) \tag{3.28}$$

$$= \sum_{k=-\infty}^{\infty} x(k)u(n-1-k) = x(n) * u(n-1) \tag{3.29}$$

$$a_{n+2} - 1, \quad n \geq 1 \tag{3.30}$$

can be expressed as

$$[x(n+1) - 1]\, u(n-1) \tag{3.31}$$

Convolution Cont.

The Z transform of the above signal can be expressed as

$$\sum_{n=1}^{\infty} x(n+1)z^{-n} - \frac{z^{-1}}{1-z^{-1}} = \sum_{n=2}^{\infty} x(n)z^{-n+1} - \frac{z^{-1}}{1-z^{-1}} \tag{3.32}$$

$$= z\left[X^{+}(z) - x(0) - x(1)z^{-1}\right] - \frac{z^{-1}}{1-z^{-1}} \tag{3.33}$$

$$= \frac{z}{1-z^{-1}-z^{-2}} - z - 1 - \frac{z^{-1}}{1-z^{-1}} \tag{3.34}$$

$$= \frac{z}{1-z^{-1}-z^{-2}} - \frac{z}{1-z^{-1}} \tag{3.35}$$

$$= \frac{z^{-1}}{\left(1-z^{-1}-z^{-2}\right)\left(1-z^{-1}\right)} \tag{3.36}$$

From (3.16), we get

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1 \tag{3.37}$$