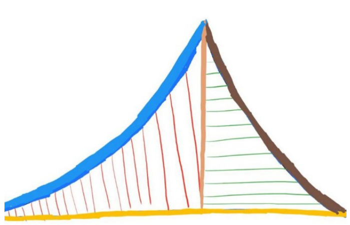
SIGNAL PROCESSING

Through Practice

G. V. V. Sharma



Contents

Introduction

This book introduces digital communication through probability.

Chapter 1

Two Dice

1.1. Problem

Two dice, one blue and one grey, are thrown at the same time. The event defined by the sum of the two numbers appearing on the top of the dice can have 11 possible outcomes 2, 3, 4, 5, 6, 6, 8, 9, 10, 11 and 12. A student argues that each of these outcomes has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.

1.2. Uniform Distribution: Rectangular Function

1.2.1. Let $X_i \in \{1, 2, 3, 4, 5, 6\}, i = 1, 2,$ be the random variables representing the outcome for each die. Assuming the dice to be fair, the probability mass function (pmf) is expressed as

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{6} & 1 \le n \le 6 \\ 0 & otherwise \end{cases} \quad (1.2.1.1)$$

1.3. Sum of Random Variables: Convolution

1.3.1. The desired outcome is

$$X = X_1 + X_2, \tag{1.3.1.1}$$

$$\implies X \in \{1, 2, \dots, 12\} \tag{1.3.1.2}$$

The objective is to show that

$$p_X(n) \neq \frac{1}{11} \tag{1.3.1.3}$$

1.3.2. Convolution: From (1.3.1.1),

$$p_X(n) = \Pr(X_1 + X_2 = n) = \Pr(X_1 = n - X_2) \tag{1.3.2.1}$$

$$= \sum_k \Pr(X_1 = n - k | X_2 = k) \, p_{X_2}(k) \tag{1.3.2.2}$$

after unconditioning. $\because$ X_1 and X_2 are independent,

$$\Pr(X_1 = n - k | X_2 = k) = \Pr(X_1 = n - k) = p_{X_1}(n - k) \tag{1.3.2.3}$$

From (1.3.2.2) and (1.3.2.3),

$$p_X(n) = \sum_k p_{X_1}(n - k) p_{X_2}(k) \triangleq p_{X_1}(n) * p_{X_2}(n) \tag{1.3.2.4}$$

where $*$ denotes the convolution operation.

1.3.3. Substituting from (1.2.1.1) in (1.3.2.4),

$$p_X(n) = \frac{1}{6}\sum_{k=1}^{6} p_{X_1}(n-k) = \frac{1}{6}\sum_{k=n-6}^{n-1} p_{X_1}(k) \tag{1.3.3.1}$$

$$\because p_{X_1}(k) = 0, \quad k \leq 1, k \geq 6. \tag{1.3.3.2}$$

From (1.3.3.1),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{1}{6}\sum_{k=1}^{n-1} p_{X_1}(k) & 1 \leq n-1 \leq 6 \\ \frac{1}{6}\sum_{k=n-6}^{6} p_{X_1}(k) & 1 < n-6 \leq 6 \\ 0 & n > 12 \end{cases} \tag{1.3.3.3}$$

1.4. The Triangular function

1.4.1. Substituting from (1.2.1.1) in (1.3.3.3),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \leq n \leq 7 \\ \frac{13-n}{36} & 7 < n \leq 12 \\ 0 & n > 12 \end{cases} \tag{1.4.1.1}$$

satisfying (1.3.1.3).

1.5. The Z-transform

1.5.1. The Z-transform of $p_X(n)$ is defined as

$$P_X(z) = \sum_{n=-\infty}^{\infty} p_X(n) z^{-n}, \quad z \in \mathbb{C} \tag{1.5.1.1}$$

From (1.2.1.1) and (1.5.1.1),

$$P_{X_1}(z) = P_{X_2}(z) = \frac{1}{6} \sum_{n=1}^{6} z^{-n} \tag{1.5.1.2}$$

$$= \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}, \quad |z| > 1 \tag{1.5.1.3}$$

upon summing up the geometric progression.

1.5.2. Show that

$$p_X(n) = p_{X_1}(n) * p_{X_2}(n) \implies P_X(z) = P_{X_1}(z) P_{X_2}(z) \tag{1.5.2.1}$$

The above property follows from Fourier analysis and is fundamental to signal processing.

1.5.3. The unit step function is defined as

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{1.5.3.1}$$

Show that the Z transform of $u(n)$ is

$$U(z) = \frac{1}{1 - z^{-1}}, \, |z| > 1 \tag{1.5.3.2}$$

1.5.4. Show that

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{\left(1-z^{-1}\right)^2}, \; |z| > 1 \tag{1.5.4.1}$$

1.5.5. Show that

$$p_X(n-k) \overset{\mathcal{Z}}{\longleftrightarrow} P_X(z)z^{-k} \tag{1.5.5.1}$$

1.5.6. From (1.5.1.3) and (1.5.2.1),

$$P_X(z) = \left\{\frac{z^{-1}\left(1-z^{-6}\right)}{6\left(1-z^{-1}\right)}\right\}^2 = \frac{1}{36}\frac{z^{-2}\left(1-2z^{-6}+z^{-12}\right)}{\left(1-z^{-1}\right)^2} \tag{1.5.6.1}$$

From (1.5.5.1) and (1.5.4.1), it can be shown that

$$\frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{36}\frac{z^{-2}\left(1-2z^{-6}+z^{-12}\right)}{\left(1-z^{-1}\right)^2} \tag{1.5.6.2}$$

1.5.7. From (1.5.1.1), (1.5.6.1) and (1.5.6.2)

$$p_X(n) = \frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \tag{1.5.7.1}$$

which is the same as (1.4.1.1). Note that (1.4.1.1) can be obtained from (1.5.6.2) using contour integration as well.

1.5.8. The experiment of rolling the dice was simulated using Python for 10000 samples. These were generated using Python libraries for uniform distribution. The frequencies for each outcome were then used to compute the resulting pmf, which is plotted in

Figure 1.5.8.1. The theoretical pmf obtained in (1.4.1.1) is plotted for comparison.

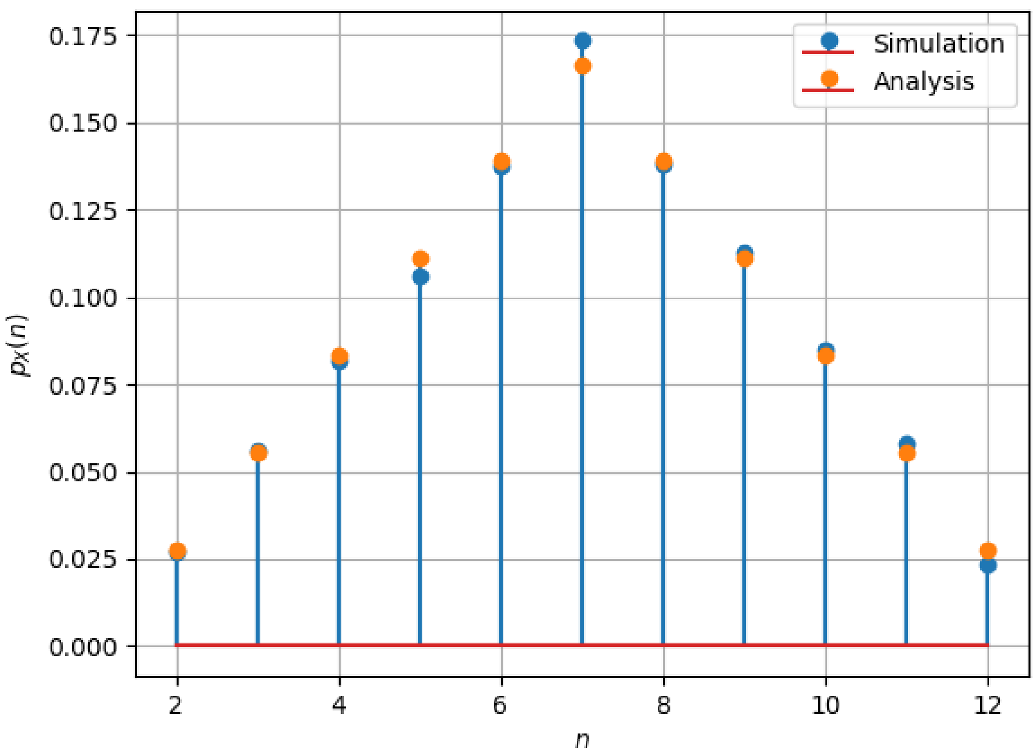


Figure 1.5.8.1: Plot of $p_X(n)$. Simulations are close to the analysis.

1.5.9. The python code is available in

```
/codes/sum/dice.py
```

Chapter 2

Pingala Series

2.1. JEE 2019

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \tag{9.1}$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \tag{9.2}$$

where α and β $(\alpha > \beta)$ are the roots of the

$$z^2 - z - 1 = 0 \tag{9.3}$$

Verify the following using a python code.

2.1.1

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{2.1.1.1}$$

2.1.2

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{2.1.2.1}$$

2.1.3

$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \tag{2.1.3.1}$$

2.1.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \tag{2.1.4.1}$$

Solution:

```
$ python3 pingala/codes/1.py
```

2.2. Pingala Series

2.2.1 The Pingala series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n), \quad x(0) = x(1) = 1, n \geq 0 \tag{2.2.1.1}$$

Generate a stem plot for $x(n)$.

Solution: The following code generates Fig. 2.2.1.1.

```
$ python3 pingala/codes/2_1.py
```

2.2.2 The one sided Z-transform of $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}, \quad z \in \mathbb{C} \tag{2.2.2.1}$$

Find $X^+(z)$.

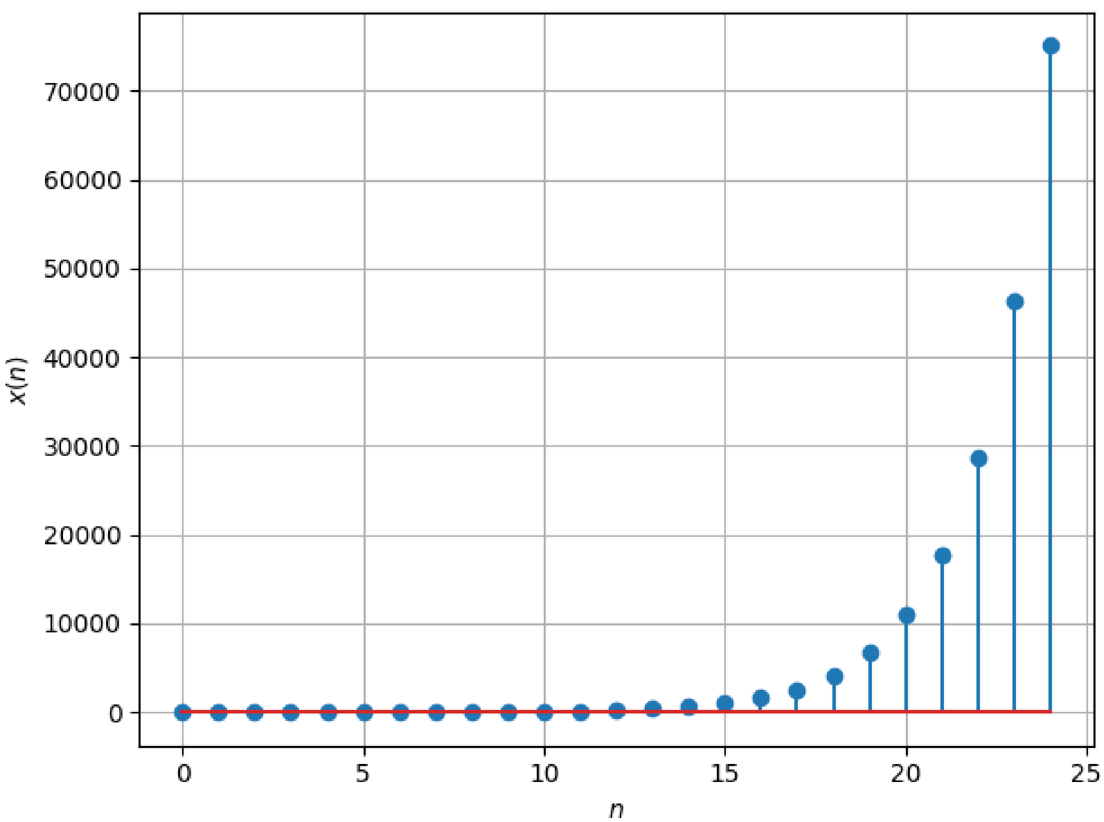


Figure 2.2.1.1: Plot of $x(n)$

Solution: Taking the one-sided Z-transform on both sides of (2.2.1.1),

$$\mathcal{Z}^+\left[x(n+2)\right] = \mathcal{Z}^+\left[x(n+1)\right] + \mathcal{Z}^+\left[x(n)\right] \tag{2.2.2.2}$$

$$\implies z^2X^+(z) - z^2x(0) - zx(1) = zX^+(z) - zx(0) + zX^+(z) \tag{2.2.2.3}$$

$$\implies \left(z^2 - z - 1\right) X^+(z) = z^2 \tag{2.2.2.4}$$

$$\implies X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} = \frac{1}{\left(1 - \alpha z^{-1}\right)\left(1 - \beta z^{-1}\right)}, \quad |z| > \alpha \tag{2.2.2.5}$$

2.2.3 Find $x(n)$.

Solution: Expanding $X^+(z)$ in (2.2.2.5) using partial fractions, we get

$$X^+(z) = \frac{1}{(\alpha - \beta)}\left[\frac{z}{1 - \alpha z^{-1}} - \frac{z}{1 - \beta z^{-1}}\right] \tag{2.2.3.1}$$

$$\implies x(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}u(n) \tag{2.2.3.2}$$

$$= a_{n+1} \tag{2.2.3.3}$$

upon comparing with (9.1).

2.3. Linear Time Invariant System

2.3.1 Sketch

$$y(n) = x\left(n-1\right) + x\left(n+1\right), \quad n \geq 0 \tag{2.3.1.1}$$

Solution: Execute

```
$ python3 pingala/codes/2_2.py
```

to obtain Fig. 2.3.1.1

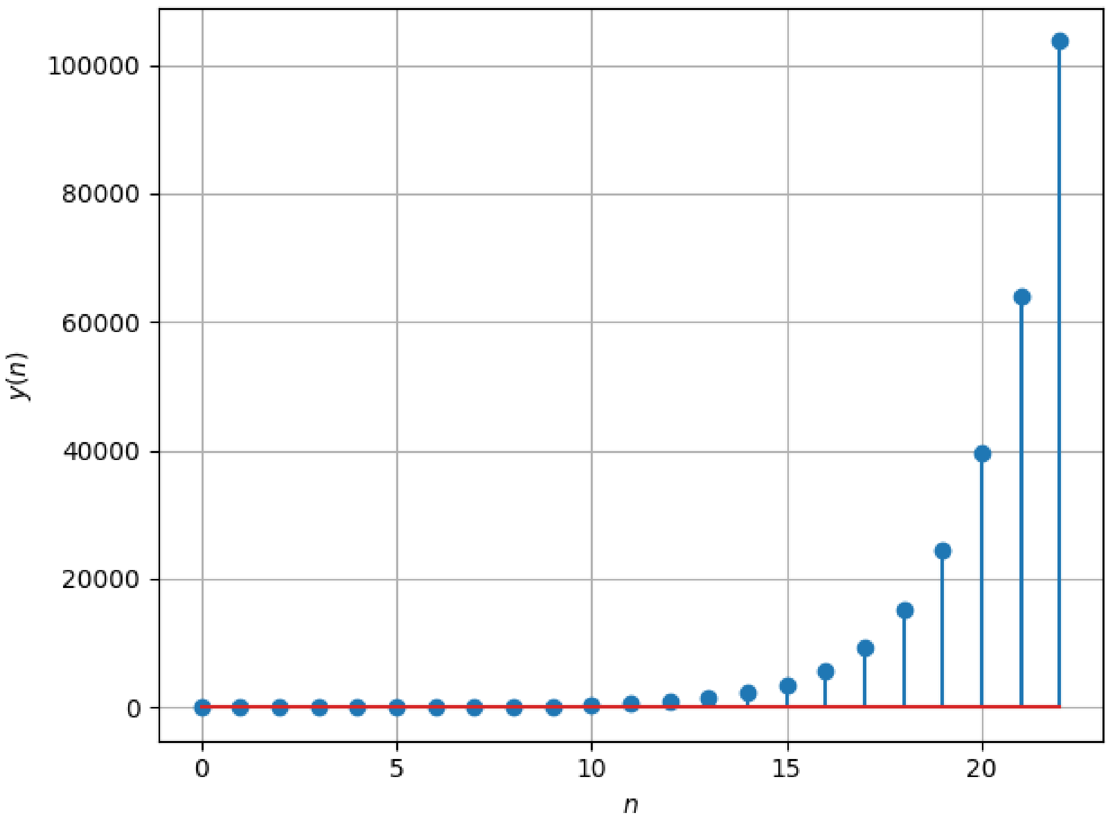


Figure 2.3.1.1: Plot of $y(n)$

2.3.2 Show that

$$x(n+1) \overset{\mathcal{Z}}{\longleftrightarrow} zX^{+}(z) - zx(0) \tag{2.3.2.1}$$

$$x(n-1) \overset{\mathcal{Z}}{\longleftrightarrow} z^{-1}X^{+}(z) + zx(-1) \tag{2.3.2.2}$$

2.3.3 Find $Y^{+}(z)$.

Solution: Taking the one-sided Z-transform on both sides of (2.3.1.1),

$$\mathcal{Z}^+\left[y(n)\right] = \mathcal{Z}^+\left[x(n+1)\right] + \mathcal{Z}^+\left[x(n-1)\right] \tag{2.3.3.1}$$

$$Y^+(z) = zX^+(z) - zx(0) + z^{-1}X^+(z) + zx(-1) \tag{2.3.3.2}$$

$$= \frac{z + z^{-1}}{1 - z^{-1} - z^{-2}} - z = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}}, \quad |z| > \alpha \tag{2.3.3.3}$$

2.3.4 Show that

$$y(n) = b_{n+1}. \tag{2.3.4.1}$$

2.3.5 Find the impulse response of (2.3.1.1)

2.4. Power of the Z transform

2.4.1 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10}\sum_{k=0}^{\infty} \frac{x\left(k\right)}{10^k} = \frac{1}{10}X^+\left(10\right) \tag{2.4.1.1}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10}\sum_{k=0}^{\infty} \frac{a_{k+1}}{10^k} = \frac{1}{10}\sum_{k=0}^{\infty} \frac{x(k)}{10^k} \tag{2.4.1.2}$$

$$= \frac{1}{10}X^+(10) = \frac{1}{10} \times \frac{100}{89} = \frac{10}{89} \tag{2.4.1.3}$$

Thus, (2.1.2.1) is correct.

2.4.2 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y\left(k\right)}{10^k} = \frac{1}{10} Y^{+}\left(10\right) \tag{2.4.2.1}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{b_{k+1}}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} \tag{2.4.2.2}$$

$$= \frac{1}{10} Y^{+}(z) = \frac{1}{10} \times \frac{120}{89} = \frac{12}{89} \tag{2.4.2.3}$$

Thus, (2.1.4.1) is incorrect.

2.4.3 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \tag{2.4.3.1}$$

can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \tag{2.4.3.2}$$

and find $W(z)$.

Solution: Putting $n = k + 1$ in (2.4.3.1) and using the definition of $u(n)$,

$$\alpha^n + \beta^n = \left(\alpha^{k+1} + \beta^{k+1}\right) u(k) \tag{2.4.3.3}$$

Hence, (2.4.3.1) can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \tag{2.4.3.4}$$

Therefore,

$$W(z) = Y(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{2.4.3.5}$$

Thus, by invoking (2.3.4.1), we find that (2.1.3.1) is correct

2.5. Convolution

2.5.1 Show that

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) = x(n) * u(n-1) \tag{2.5.1.1}$$

Solution: From (2.2.3.3), and noting that $x(n) = 0 \ \forall \ n < 0$,

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) = \sum_{k=-\infty}^{n-1} x(k) \tag{2.5.1.2}$$

$$= \sum_{k=-\infty}^{\infty} x(k)u(n-1-k) = x(n) * u(n-1) \tag{2.5.1.3}$$

2.5.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \tag{2.5.2.1}$$

can be expressed as

$$\left[x\left(n+1\right) - 1\right] u\left(n-1\right) \tag{2.5.2.2}$$

Solution: From (2.2.3.3),

$$a_{n+2} - 1 = [x(n+1) - 1], \quad n \geq 1 \tag{2.5.2.3}$$

and so, using the definition of $u(n)$,

$$a_{n+2} - 1 = [x(n+1) - 1]\, u(n-1) \tag{2.5.2.4}$$

2.5.3 Show that

$$[x(n+1) - 1]\, u(n-1) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1 - z^{-1} - z^{-2})(1 - z^{-1})} \tag{2.5.3.1}$$

Solution: The Z transform of the above signal can be expressed as

$$\begin{aligned}
\sum_{n=1}^{\infty} x(n+1) z^{-n} - \frac{z^{-1}}{1 - z^{-1}} &= \sum_{n=2}^{\infty} x(n) z^{-n+1} - \frac{z^{-1}}{1 - z^{-1}} && (2.5.3.2)\\
&= z\left[X^{+}(z) - x(0) - x(1) z^{-1}\right] - \frac{z^{-1}}{1 - z^{-1}} && (2.5.3.3)\\
&= \frac{z}{1 - z^{-1} - z^{-2}} - z - 1 - \frac{z^{-1}}{1 - z^{-1}} && (2.5.3.4)\\
&= \frac{z}{1 - z^{-1} - z^{-2}} - \frac{z}{1 - z^{-1}} && (2.5.3.5)\\
&= \frac{z^{-1}}{(1 - z^{-1} - z^{-2})(1 - z^{-1})} && (2.5.3.6)
\end{aligned}$$

From (2.2.3.3), we get

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1 \tag{2.5.3.7}$$

Chapter 3

Circuits and Transforms

3.1. Definitions

3.1.1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{3.1.1.1}$$

3.1.2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\, dt \tag{3.1.2.1}$$

3.2. Laplace Transform

3.2.1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\, \mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\, \mu C$.

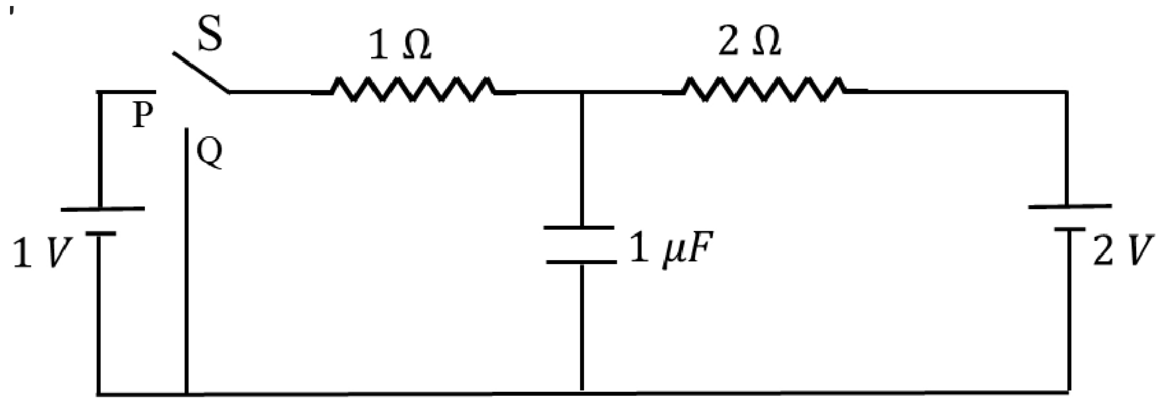


Figure 3.2.1.1:

3.2.2. Draw the circuit using latex-tikz.

3.2.3. Find q_1.

3.2.4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.

3.2.5. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad a > 0 \tag{3.2.5.1}$$

and find the ROC.

3.2.6. Now consider the following resistive circuit transformed from Fig. 3.2.1.1 where

$$u(t) \xleftrightarrow{\mathcal{L}} V_1(s) \tag{3.2.6.1}$$

$$2u(t) \xleftrightarrow{\mathcal{L}} V_2(s) \tag{3.2.6.2}$$

Find the voltage across the capacitor $V_{C_0}(s)$.

3.2.7. Find $v_{C_0}(t)$. Plot using python.

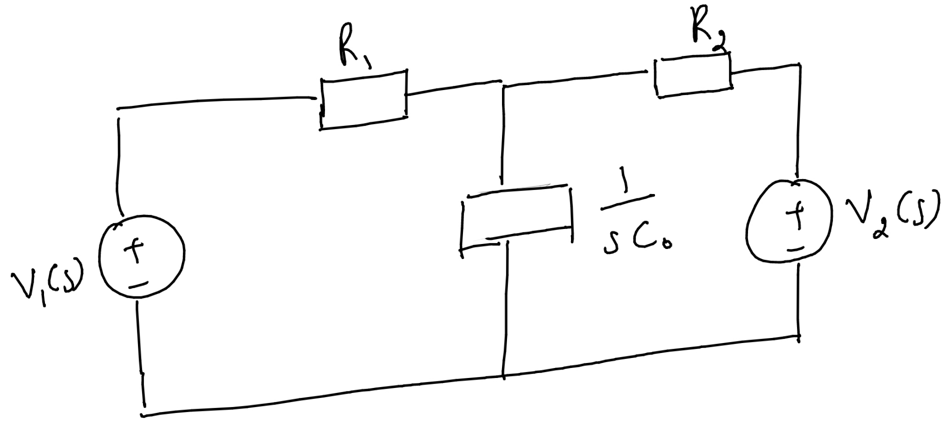


Figure 3.2.6.1:

3.2.8. Verify your result using ngspice.

3.2.9. Obtain Fig. 3.2.6.1 using the equivalent differential equation.

3.3. Initial Conditions

3.3.1. Find q_2 in Fig. 3.2.1.1.

3.3.2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz.

3.3.3. $V_{C_0}(s) = ?$

3.3.4. $v_{C_0}(t) = ?$ Plot using python.

3.3.5. Verify your result using ngspice.

3.3.6. Find $v_{C_0}(0-), v_{C_0}(0+)$ and $v_{C_0}(\infty)$.

3.3.7. Obtain the Fig. in problem 3.3.2 using the equivalent differential equation.

3.4. Bilinear Transform

3.4.1. In Fig. 3.2.1.1, consider the case when S is switched to Q right in the beginning. Formulate the differential equation.

3.4.2. Find $H(s)$ considering the ouput voltage at the capacitor.

3.4.3. Plot $H(s)$. What kind of filter is it?

3.4.4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \tag{3.4.4.1}$$

3.4.5. Find $H(z)$.

3.4.6. How can you obtain $H(z)$ from $H(s)$?

Chapter 4

Filter

4.1. Software Installation

Run the following commands

```
sudo apt-get update
sudo apt-get install libffi-dev libsndfile1 python3-scipy python3-numpy
sudo pip install cffi pysoundfile
```

4.2. Digital Filter

4.2.1 Download the sound file from

```
wget https://raw.githubusercontent.com/gadepall/
EE1310/master/filter/codes/Sound_Noise.wav
```

4.2.2 You will find a spectrogram at `https://academo.org/demos/spectrum-analyzer`. Upload the sound file that you downloaded in Problem 4.2.1 in the spectrogram and play. Observe the spectrogram. What do you find?

Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

4.2.3 Write the python code for removal of out of band noise and execute the code.

Solution:

```
codes/Cancel_noise.py
```

4.2.4 The output of the python script in Problem 4.2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 4.2.2. What do you observe?

Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

4.3. Difference Equation

4.3.1 Let

$$x(n) = \left\{\underset{\uparrow}{1}, 2, 3, 4, 2, 1\right\} \tag{4.3.1.1}$$

Sketch $x(n)$.

4.3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{4.3.2.1}$$

Sketch $y(n)$.

Solution: The following code yields Fig. 4.3.2.1.

```
wget https://github.com/gadepall/EE1310/raw/master/filter/codes/xnyn.py
```

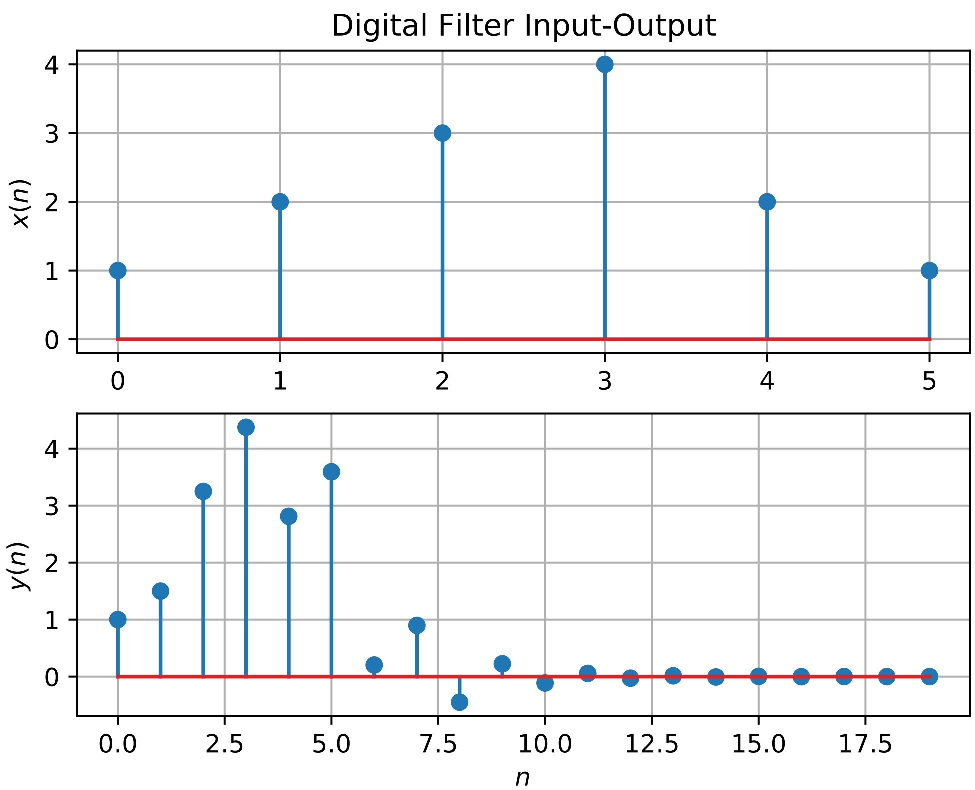


Figure 4.3.2.1:

4.3.3 Repeat the above exercise using a C code.

4.4. Z-transform

4.4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{4.4.1.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.4.1.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.4.1.3}$$

Solution: From (4.4.1.1),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4.1.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.4.1.5}$$

resulting in (4.4.1.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.4.1.6}$$

4.4.2 Obtain $X(z)$ for $x(n)$ defined in problem 4.3.1.

4.4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.4.3.1}$$

from (4.3.2.1) assuming that the Z-transform is a linear operation.

Solution: Applying (4.4.1.6) in (4.3.2.1),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.4.3.2}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{4.4.3.3}$$

4.4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.4.4.1}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.4.4.2}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.4.4.3}$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.4.4.4}$$

and from (4.4.4.2),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.4.4.5}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.4.4.6}$$

using the fomula for the sum of an infinite geometric progression.

4.4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.4.5.1}$$

4.4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.4.6.1}$$

Plot $|H(e^{\jmath\omega})|$. Is it periodic? If so, find the period. $H(e^{\jmath\omega})$ is known as the Discret Time Fourier Transform (DTFT) of $h(n)$.

Solution: The following code plots Fig. 4.4.6.1.

```
wget https://raw.githubusercontent.com/gadepall/EE1310/master/filter/codes/dtft
```

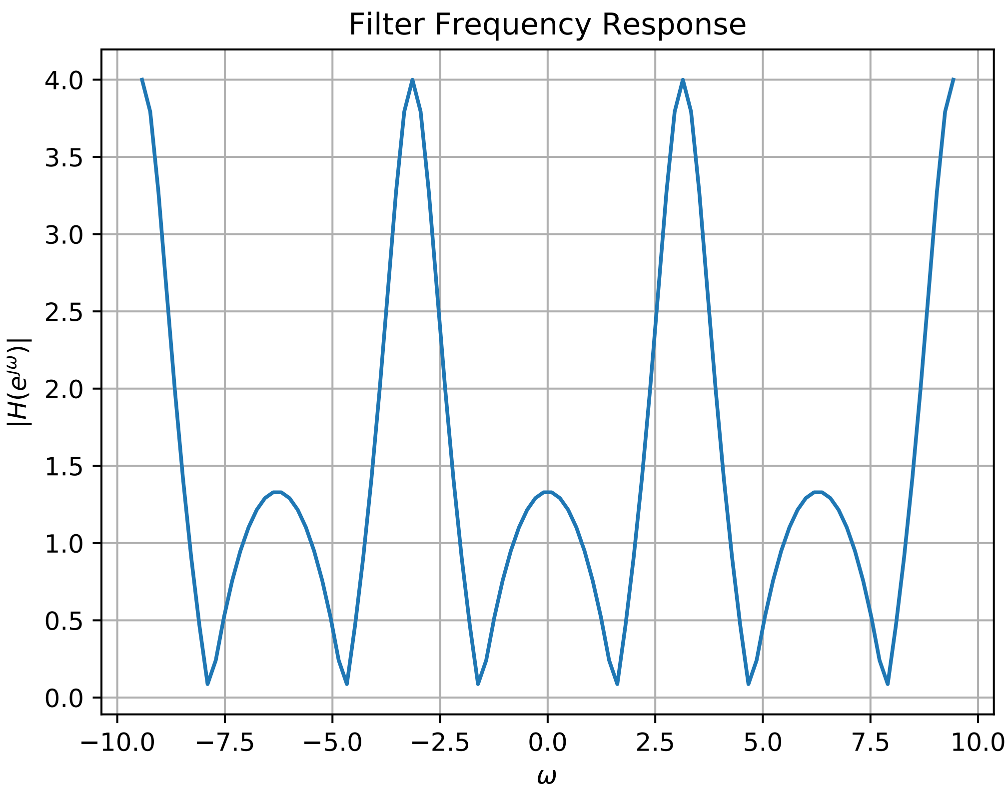


Figure 4.4.6.1: $|H(e^{\jmath\omega})|$

4.4.7 Express $h(n)$ in terms of $H(e^{\jmath\omega})$.

4.5. Impulse Response

4.5.1 Using long division, find

$$h(n), \quad n < 5 \tag{4.5.1.1}$$

for H(z) in (4.4.3.3).

4.5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{4.5.2.1}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the impulse response of the system defined by (4.3.2.1).

Solution: From (4.4.3.3),

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{4.5.2.2}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{4.5.2.3}$$

using (4.4.5.1) and (4.4.1.6).

4.5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.

Solution: The following code plots Fig. 4.5.3.1.

```
wget https://raw.githubusercontent.com/gadepall/EE1310/master/filte
```

4.5.4 Convergent? Justify using the ratio test.

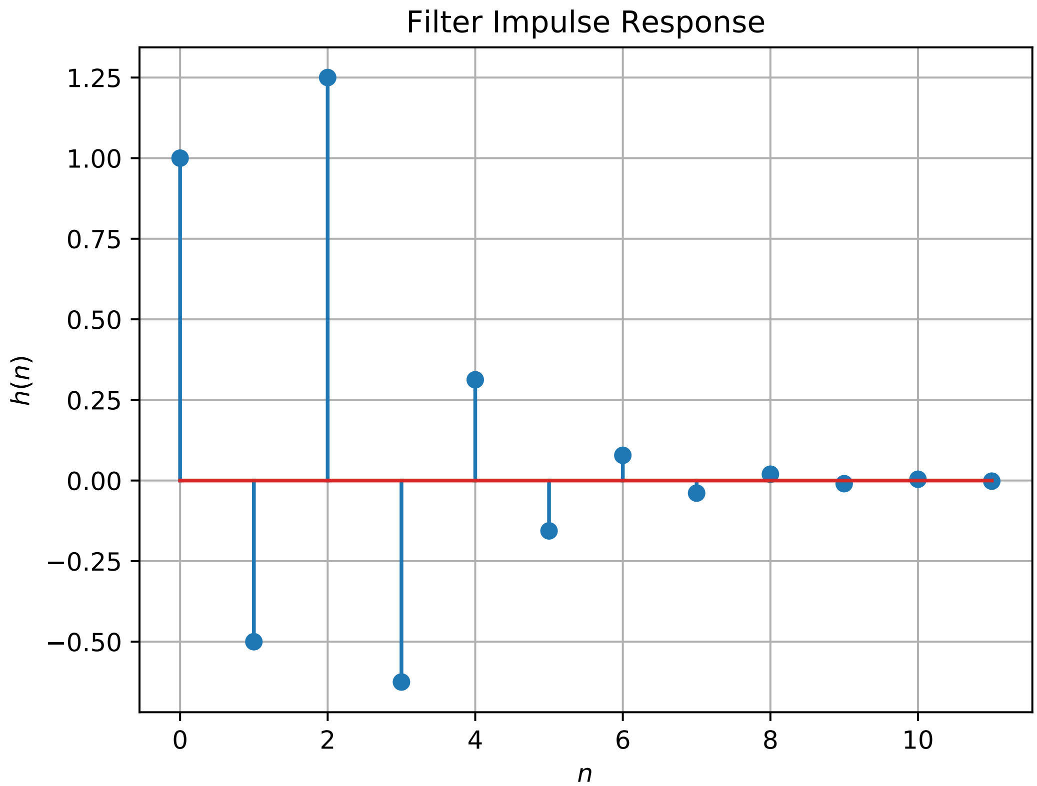


Figure 4.5.3.1: $h(n)$ as the inverse of $H(z)$

4.5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{4.5.5.1}$$

Is the system defined by (4.3.2.1) stable for the impulse response in (4.5.2.1)?

4.5.6 Verify the above result using a python code.

4.5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{4.5.7.1}$$

This is the definition of $h(n)$.

Solution: The following code plots Fig. 4.5.7.1. Note that this is the same as Fig. 4.5.3.1.

```
wget https://raw.githubusercontent.com/gadepall/EE1310/master/filter
```

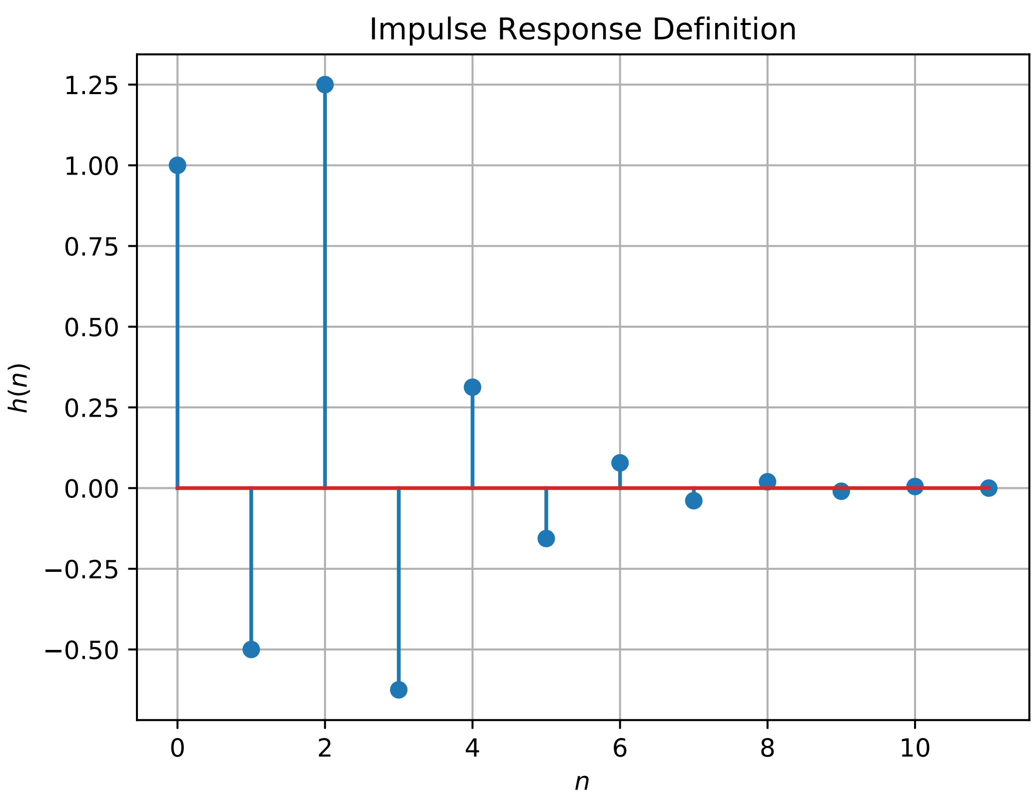


Figure 4.5.7.1: $h(n)$ from the definition

4.5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \tag{4.5.8.1}$$

Comment. The operation in (4.5.8.1) is known as convolution.

Solution: The following code plots Fig. 4.5.8.1. Note that this is the same as $y(n)$ in Fig. 4.3.2.1.

```
wget https://raw.githubusercontent.com/gadepall/EE1310/master/filter/codes/ynco
```

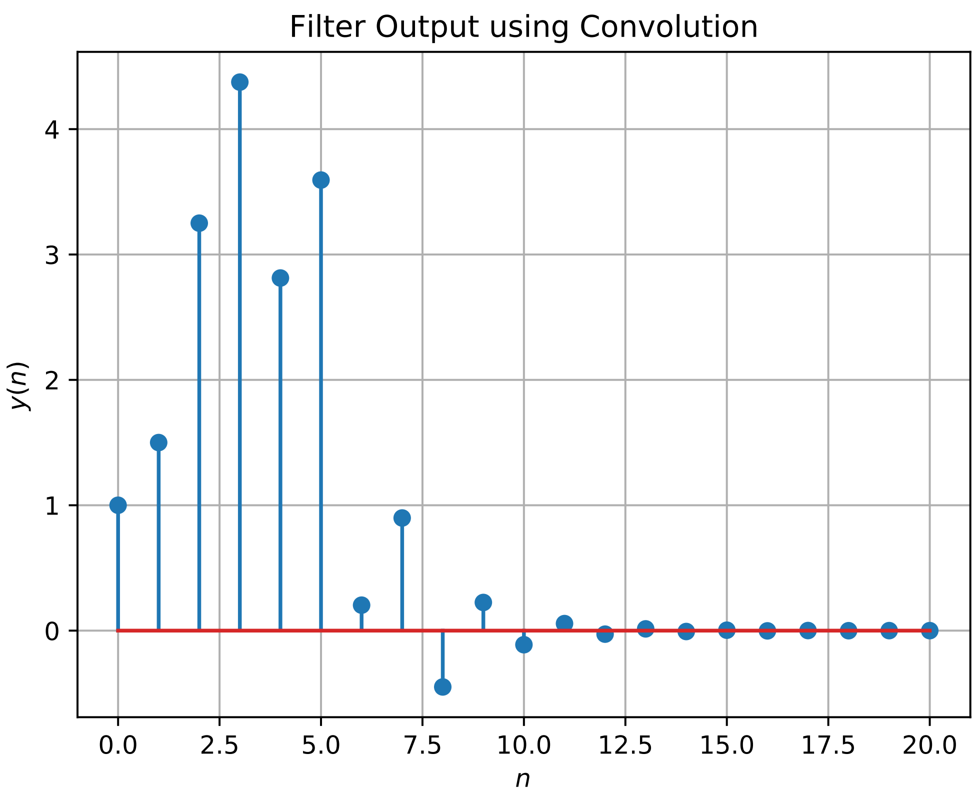


Figure 4.5.8.1: $y(n)$ from the definition of convolution

4.5.9 Express the above convolution using a Teoplitz matrix.

4.5.10 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{4.5.10.1}$$

4.6. DFT

4.6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{4.6.1.1}$$

and $H(k)$ using $h(n)$.

4.6.2 Compute

$$Y(k) = X(k)H(k) \tag{4.6.2.1}$$

4.6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k) \cdot e^{\jmath 2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \tag{4.6.3.1}$$

Solution: The following code plots Fig. 4.5.8.1. Note that this is the same as $y(n)$ in Fig. 4.3.2.1.

```
wget https://raw.githubusercontent.com/gadepall/EE1310/master/filter
```

4.6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.

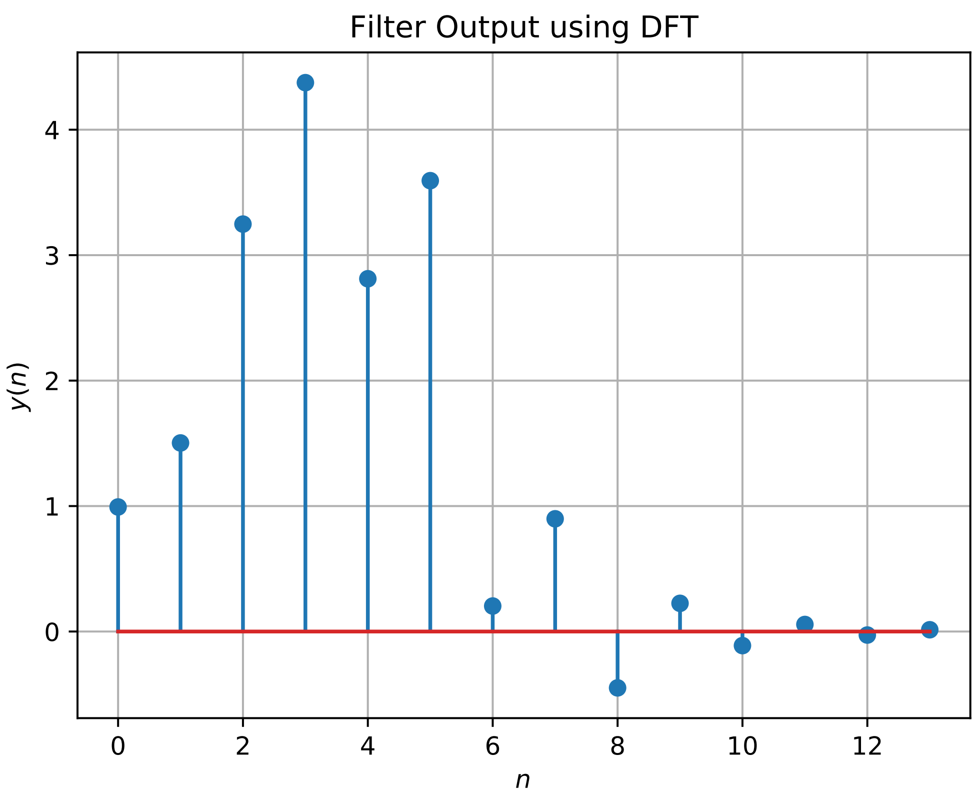


Figure 4.6.3.1: $y(n)$ from the DFT

4.7. FFT

1. The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{4.7.1}$$

2. Let

$$W_N = e^{-j2\pi/N} \tag{4.7.1}$$

Then the N-point DFT matrix is defined as

$$\mathbf{F}_N = \left[W_N^{mn}\right], \quad 0 \leq m, n \leq N-1 \tag{4.7.2}$$

where W_N^{mn} are the elements of $\mathbf{F}_N$.

3. Let

$$\mathbf{I}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^2 & \mathbf{e}_4^3 & \mathbf{e}_4^4 \end{pmatrix} \tag{4.7.1}$$

be the 4×4 identity matrix. Then the 4 point DFT permutation matrix is defined as

$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{4.7.2}$$

4. The 4 point DFT diagonal matrix is defined as

$$\mathbf{D}_4 = diag \begin{pmatrix} W_8^0 & W_8^1 & W_8^2 & W_8^3 \end{pmatrix} \tag{4.7.1}$$

5. Show that

$$W_N^2 = W_{N/2} \tag{4.7.1}$$

6. Show that

$$\mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{4.7.1}$$

7. Show that

$$\mathbf{F}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N \tag{4.7.1}$$

8. Find

$$\mathbf{P}_4\mathbf{x} \tag{4.7.1}$$

9. Show that

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \tag{4.7.1}$$

where $\mathbf{x}, \mathbf{X}$ are the vector representations of $x(n), X(k)$ respectively.

10. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{4.7.1}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{4.7.2}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{4.7.3}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{4.7.4}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{4.7.5}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{4.7.6}$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{4.7.7}$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{4.7.8}$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{4.7.9}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{4.7.10}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{4.7.11}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{4.7.12}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{4.7.13}$$

11. For

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{4.7.1}$$

compte the DFT using (4.7.1)

12. Repeat the above exercise using the FFT after zero padding $\mathbf{x}$.

13. Write a C program to compute the 8-point FFT.

4.8. Exercises

Answer the following questions by looking at the python code in Problem 4.2.3.

4.8.1 The command

```
output_signal = signal.lfilter(b, a, input_signal)
```

in Problem 4.2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a\left(m\right) y\left(n-m\right) = \sum_{k=0}^{N} b\left(k\right) x\left(n-k\right) \tag{4.8.1}$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.

4.8.2 Repeat all the exercises in the previous sections for the above a and b.

4.8.3 What is the sampling frequency of the input signal?

Solution: Sampling frequency(fs)=44.1kHZ.

4.8.4 What is type, order and cutoff-frequency of the above butterworth filter

Solution: The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.

4.8.5 Modifying the code with different input parameters and to get the best possible output.

Appendix A

Filter Design

A.1. Introduction

We are supposed to design the equivalent FIR and IIR filter realizations for filter number 114. This is a bandpass filter whose specifications are available below.

A.2. Filter Specifications

The sampling rate for the filter has been specified as $F_s = 48$ kHz. If the un-normalized discrete-time (natural) frequency is F, the corresponding normalized digital filter (angular) frequency is given by $\omega = 2\pi\left(\frac{F}{F_s}\right)$.

A.2.1. The Digital Filter

1. Tolerances: The passband (δ_1) and stopband (δ_2) tolerances are given to be equal, so we let $\delta_1 = \delta_2 = \delta = 0.15$.

2. Passband: The passband of filter number j, j going from 109 to 135 is from $\{3 + 0.6(\text{j-109})\}$kHz to $\{3 + 0.6(\text{j-107})\}$kHz. Since our filter number is 114, substituting

$j = 114$ gives the passband range for our bandpass filter as 6 kHz - 7.2 kHz. Hence, the un-normalized discrete time filter passband frequencies are $F_{p1} = 7.2$ kHz and $F_{p2} = 6.0$ kHz. The corresponding normalized digital filter passband frequencies are $\omega_{p1} = 2\pi\frac{F_{p1}}{F_s} = 0.3\pi$ and $\omega_{p2} = 2\pi\frac{F_{p2}}{F_s} = 0.25\pi$ kHz. The centre frequency is then given by $\omega_c = \frac{\omega_{p1}+\omega_{p2}}{2} = 0.275\pi$.

3. Stopband: The transition band for bandpass filters is $\Delta F = 0.3$ kHz on either side of the passband. Hence, the un-normalized stopband frequencies are $F_{s1} = 7.2+0.3 = 7.5$ kHz and $F_{s2} = 6.0 - 0.3 = 5.7$ kHz. The corresponding normalized frequencies are $\omega_{s1} = 0.3125\pi$ and $\omega_{s2} = 0.2375\pi$.

A.2.2. The Analog filter

In the bilinear transform, the analog filter frequency (Ω) is related to the corresponding digital filter frequency (ω) as $\Omega = \tan\frac{\omega}{2}$. Using this relation, we obtain the analog passband and stopband frequencies as $\Omega_{p1} = 0.5095$, $\Omega_{p2} = 0.4142$ and $\Omega_{s1} = 0.5345$, $\Omega_{s2} = 0.3914$ respectively.

A.3. The IIR Filter Design

Filter Type: We are supposed to design filters whose stopband is monotonic and passband equiripple. Hence, we use the Chebyschev approximation to design our bandpass IIR filter.

A.3.1. The Analog Filter

1. Low Pass Filter Specifications: If $H_{a,BP}(j\Omega)$ be the desired analog band pass filter, with the specifications provided in Section 2.2, and $H_{a,LP}(j\Omega_L)$ be the equivalent low pass filter, then

$$\Omega_L = \frac{\Omega^2 - \Omega_0^2}{B\Omega} \tag{A.3.1}$$

where $\Omega_0 = \sqrt{\Omega_{p1}\Omega_{p2}} = 0.4594$ and $B = \Omega_{p1} - \Omega_{p2} = 0.0953$. The low pass filter has the passband edge at $\Omega_{Lp} = 1$ and stopband edges at $\Omega_{Ls_1} = 1.4653$ and $\Omega_{Ls_2} = -1.5511$. We choose the stopband edge of the analog low pass filter as $\Omega_{Ls} = \min(|\Omega_{Ls_1}|, |\Omega_{Ls_2}|) = 1.4653$.

2. The Low Pass Chebyschev Filter Paramters: The magnitude squared of the Chebyschev low pass filter is given by

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\Omega_L/\Omega_{Lp})} \tag{A.3.1}$$

where $c_N(x) = \cosh(N \cosh^{-1} x)$ and the integer N, which is the order of the filter, and ϵ are design paramters. Since $\Omega_{Lp} = 1$, (A.3.1) may be rewritten as

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\Omega_L)} \tag{A.3.2}$$

Also, the design paramters have the following constraints

$$\begin{aligned} \frac{\sqrt{D_2}}{c_N(\Omega_{Ls})} &\leq \epsilon \leq \sqrt{D_1}, \\ N &\geq \left\lceil \frac{\cosh^{-1}\sqrt{D_2/D_1}}{\cosh^{-1}\Omega_{Ls}} \right\rceil, \end{aligned} \tag{A.3.3}$$

where $D_1 = \frac{1}{(1-\delta)^2} - 1$ and $D_2 = \frac{1}{\delta^2} - 1$. After appropriate substitutions, we obtain

$N \geq 4$ and $0.3184 \leq \epsilon \leq 0.6197$. In Figure 1, we plot $|H(j\Omega)|$ for a range of values of ϵ, for $N = 4$. We find that for larger values of ϵ, $|H(j\Omega)|$ decreases in the transition band. We choose $\epsilon = 0.4$ for our IIR filter design.

3. The Low Pass Chebyschev Filter: Thus, we obtain

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + 0.16c_4^2(\Omega_L)} \tag{A.3.1}$$

where

$$c_4(x) = 8x^4 + 8x^2 + 1. \tag{A.3.2}$$

The poles of the frequency response in (A.3.1) lying in the left half plane are in general obtained as $r_1 \cos\phi_k + jr_2 \sin\phi_k$, where

$$\begin{aligned} \phi_k &= \frac{\pi}{2} + \frac{(2k+1)\pi}{2N}, k = 0, 1, \ldots, N-1 \\ r_1 = \frac{\beta^2 - 1}{2\beta}, r_2 &= \frac{\beta^2 + 1}{2\beta}, \beta = \left[\frac{\sqrt{1+\epsilon^2} + 1}{\epsilon}\right]^{\frac{1}{N}} \end{aligned} \tag{A.3.3}$$

Thus, for N even, the low-pass stable Chebyschev filter, with a gain G has the form

$$H_{a,LP}(s_L) = \frac{G_{LP}}{\prod_{k=1}^{\frac{N}{2}-1}(s_L^2 - 2r_1\cos\phi_k s_L + r_1^2\cos^2\phi_k + r_2^2\sin^2\phi_k)} \tag{A.3.4}$$

Substituting $N = 4$, $\epsilon = 0.5$ and $H_{a,LP}(j) = \frac{1}{\sqrt{1+\epsilon^2}}$, from (A.3.3) and (A.3.4), we obtain

$$H_{a,LP}(s_L) = \frac{0.3125}{s_L^4 + 1.1068s_L^3 + 1.6125s_L^2 + 0.9140s_L + 0.3366} \tag{A.3.5}$$

In Fig. 4.2 we plot $|H(j\Omega)|$ using (A.3.1) and (A.3.5), thereby verifying that our low-pass Chebyschev filter design meets the specifications.

4. The Band Pass Chebyschev Filter: The analog bandpass filter is obtained from (A.3.5) by substituting $s_L = \frac{s^2+\Omega_0^2}{Bs}$. Hence

$$H_{a,BP}(s) = G_{BP} H_{a,LP}(s_L)\Big|_{s_L=\frac{s^2+\Omega_0^2}{Bs}}, \tag{A.3.1}$$

where G_{BP} is the gain of the bandpass filter. After appropriate substitutions, and evaluating the gain such that $H_{a,BP}(j\Omega_{p1}) = 1$, we obtain

$$H_{a,BP}(s) = \frac{2.7776 \times 10^{-5} s^4}{s^8 + 0.1055s^7 + 0.8589s^6 + 0.0676s^5 + 0.2735s^4 + 0.0143s^3 + 0.0383s^2 + 0.001s + 0.002} \tag{A.3.2}$$

In Fig 4.3, we plot $|H_{a,BP}(j\Omega)|$ as a function of Ω for both positve as

well as negative frequencies. We find that the passband and stopband frequencies in the figure match well with those obtained analytically through the bilinear transformation.

A.3.2. The Digital Filter

From the bilinear transformation, we obtain the digital bandpass filter from the corresponding analog filter as

$$H_{d,BP}(z) = G H_{a,BP}(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \tag{A.3.3}$$

where G is the gain of the digital filter. From (A.3.2) and (A.3.3), we obtain

$$H_{d,BP}(z) = G\frac{N(z)}{D(z)} \tag{A.3.4}$$

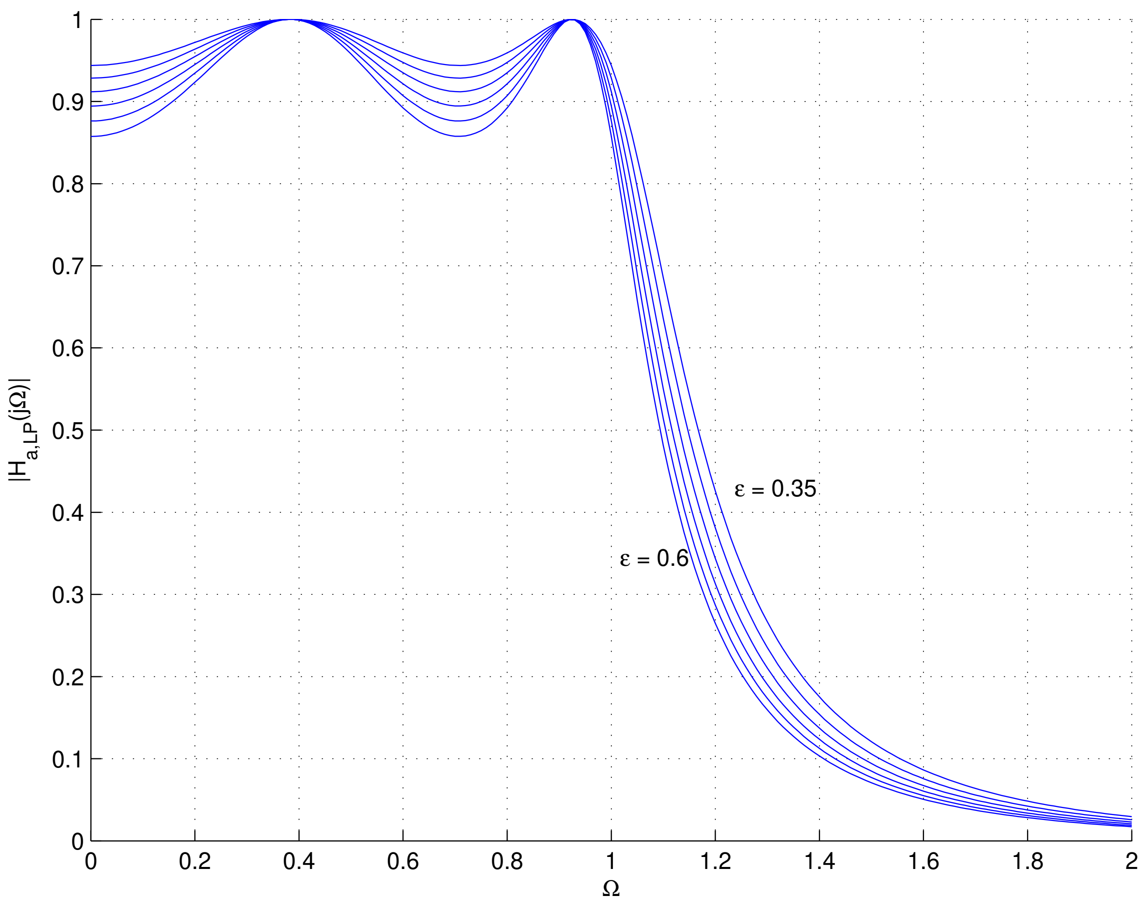


Figure 4.1: The Analog Low-Pass Frequency Response for $0.35 \leq \epsilon \leq 0.6$

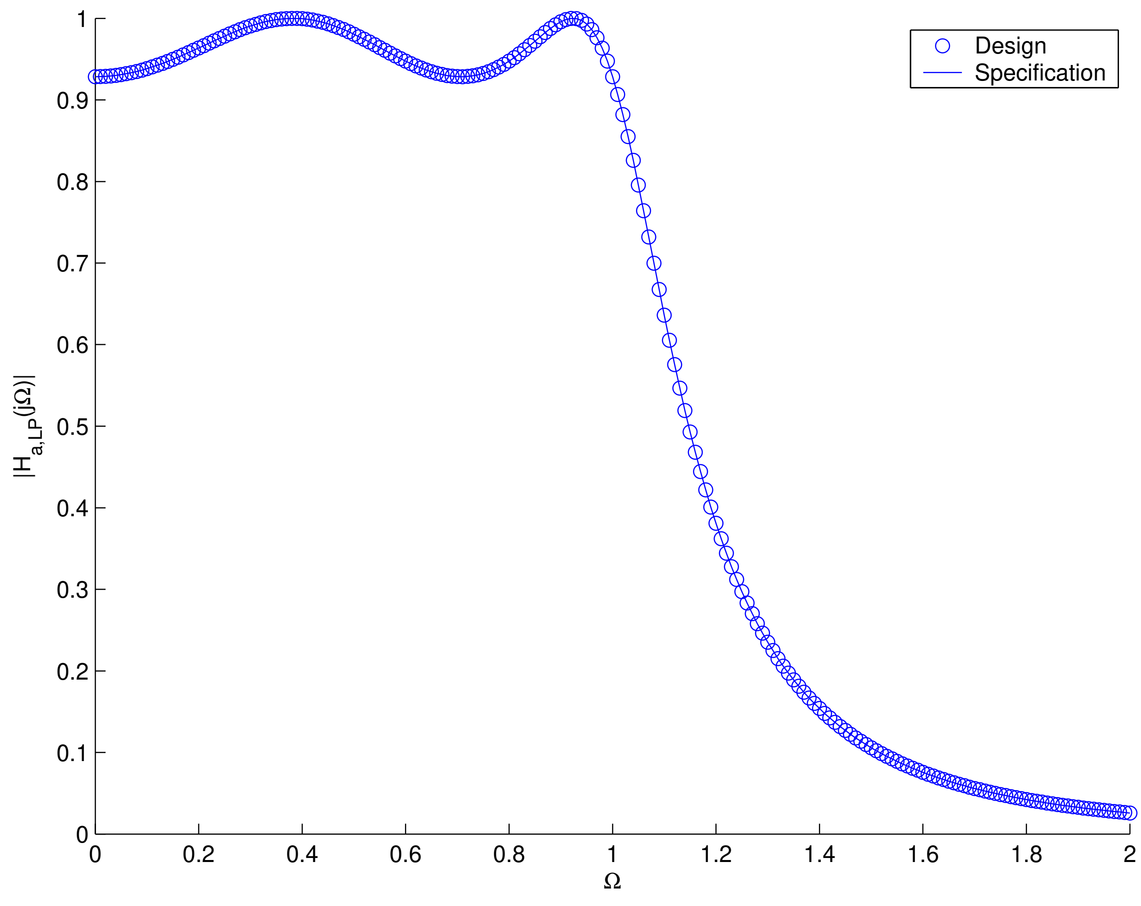


Figure 4.2: The magnitude response plots from the specifications in Equation A.3.1 and the design in Equation A.3.5

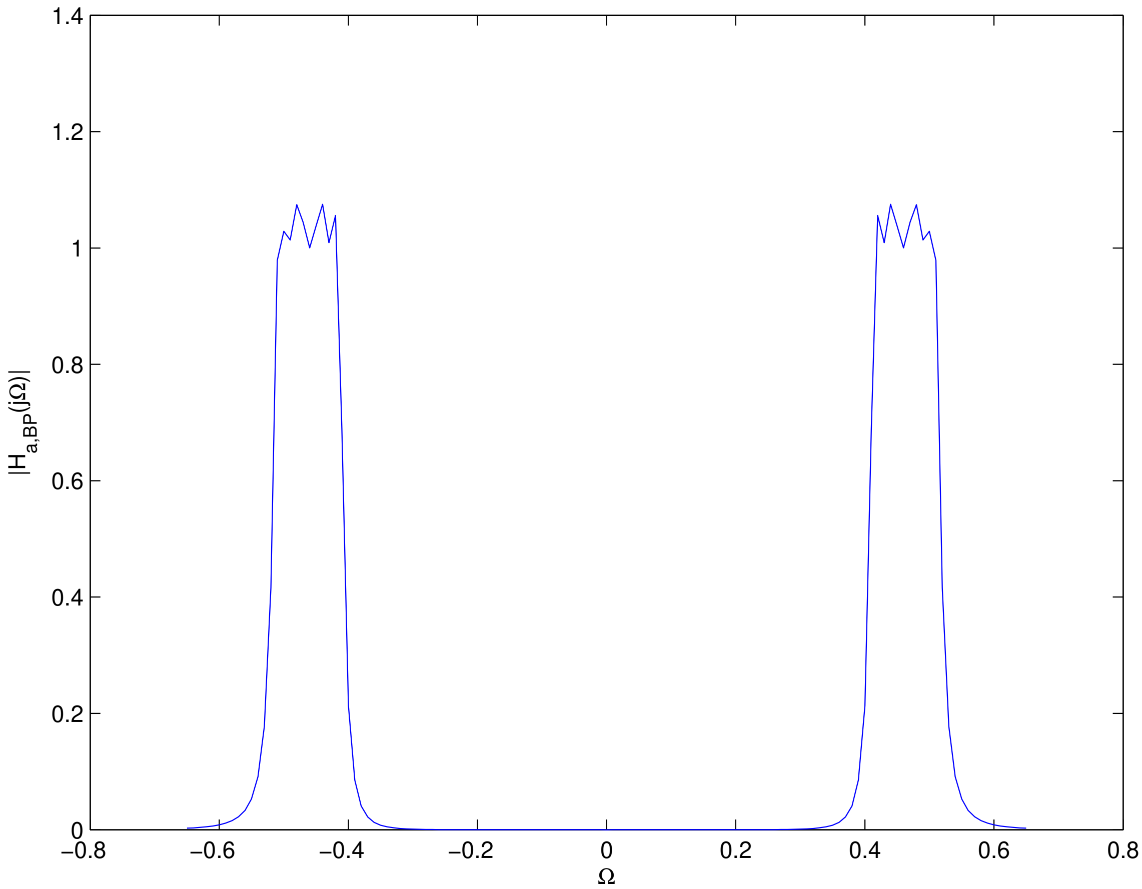


Figure 4.3: The analog bandpass magnitude response plot from Equation A.3.2

where $G = 2.7776 \times 10^{-5}$,

$$N(z) = 1 - 4z^{-2} + 6z^{-4} - 4z^{-6} + z^{-8} \tag{A.3.5}$$

and

$$\begin{aligned} D(z) = 2.3609 - 12.0002z^{-1} + 31.8772z^{-2} - 53.7495z^{-3} + 62.8086z^{-4} \\ -51.4634z^{-5} + 29.2231z^{-6} - 10.5329z^{-7} + 1.9842z^{-8} \end{aligned} \tag{A.3.6}$$

The plot of $|H_{d,BP}(z)|$ with respect to the normalized angular freqency (normalizing factor π) is available in Fig. 4.4. Again we

find that the passband and stopband frequencies meet the specifications well enough.

A.4. The FIR Filter

We design the FIR filter by first obtaining the (non-causal) lowpass equivalent using the Kaiser window and then converting it to a causal bandpass filter.

A.4.1. The Equivalent Lowpass Filter

The lowpass filter has a passband frequency ω_l and transition band $\Delta\omega = 2\pi\frac{\Delta F}{F_s} = 0.0125\pi$. The stopband tolerance is δ.

1. The passband frequency ω_l is defined as $\omega_l = \frac{\omega_{p1}-\omega_{p2}}{2}$. Substituting the values of ω_{p1} and ω_{p2} from section 2.1, we obtain $\omega_l = 0.025\pi$.

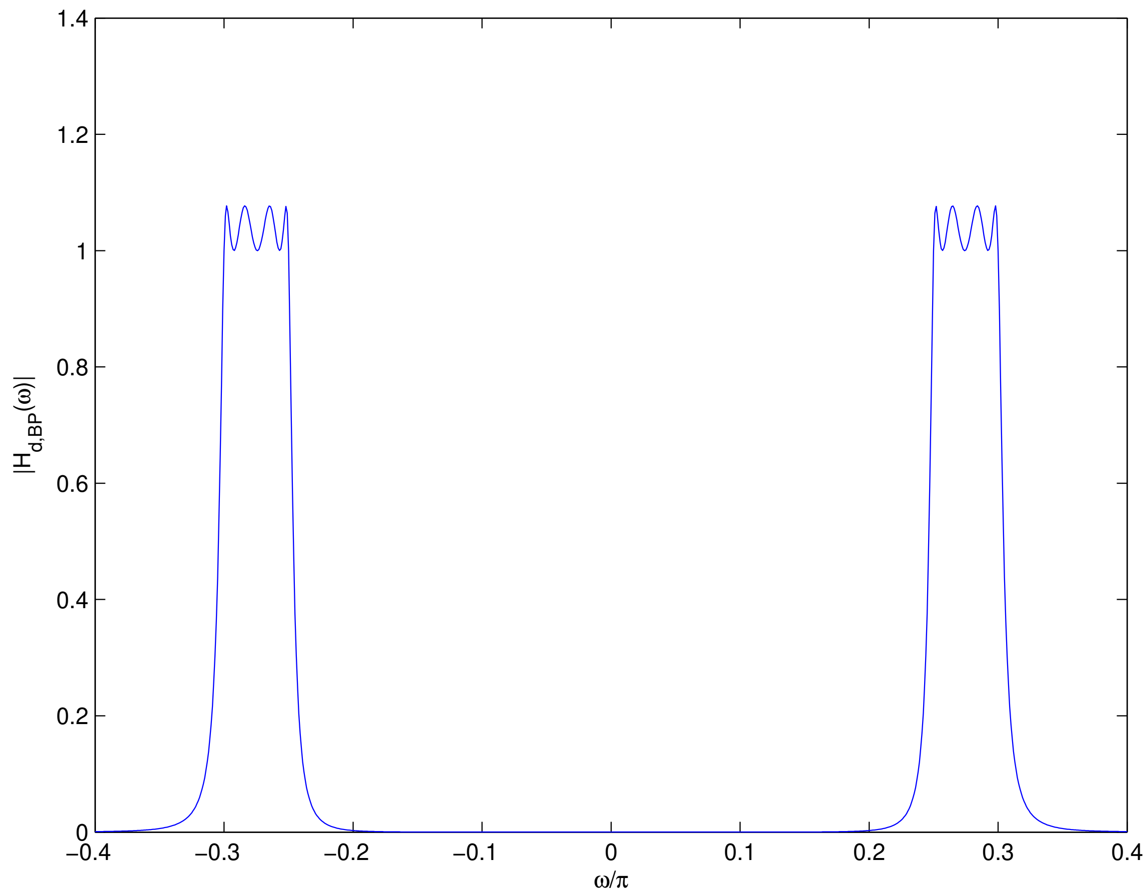


Figure 4.4: The magnitude response of the bandpass digital filter designed to meet the given specifications

2. The impulse response $h_{lp}(n)$ of the desired lowpass filter with cutoff frequency ω_l is given by

$$h_l(n) = \frac{\sin(n\omega_l)}{n\pi} w(n), \tag{A.4.1}$$

where $w(n)$ is the Kaiser window obtained from the design specifications.

A.4.2. The Kaiser Window

The Kaiser window is defined as

$$\begin{aligned} w(n) &= \frac{I_0\left[\beta N\sqrt{1-\left(\frac{n}{N}\right)^2}\right]}{I_0(\beta N)}, \qquad -N \leq n \leq N, \qquad \beta > 0 \\ &= 0 \qquad\qquad\qquad\qquad\qquad \text{otherwise}, \end{aligned} \tag{A.4.2}$$

where $I_0(x)$ is the modified Bessel function of the first kind of order zero in x and β and N are the window shaping factors. In the following, we find β and N using the design parameters in section 2.1.

1. N is chosen according to

$$N \geq \frac{A-8}{4.57\Delta\omega}, \tag{A.4.1}$$

where $A = -20\log_{10}\delta$. Substituting the appropriate values from the design specifications, we obtain $A = 16.4782$ and $N \geq 48$.

2. β is chosen according to

$$\beta N = \begin{cases} 0.1102(A-8.7) & A > 50 \\ 0.5849(A-21)^{0.4} + 0.07886(A-21) & 21 \leq A \leq 50 \\ 0 & A < 21 \end{cases} \tag{A.4.1}$$

In our design, we have $A = 16.4782 < 21$. Hence, from (A.4.1) we obtain $\beta = 0$.

3. We choose $N = 100$, to ensure the desired low pass filter response. Substituting in (A.4.2) gives us the rectangular window

$$\begin{aligned} w(n) &= 1, -100 \leq n \leq 100 \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{A.4.1}$$

From (A.4.1) and (A.4.1), we obtain the desired lowpass filter impulse response

$$\begin{aligned} h_{lp}(n) &= \frac{\sin(\frac{n\pi}{40})}{n\pi} \qquad -100 \leq n \leq 100 \\ &= 0, \qquad \text{otherwise} \end{aligned} \tag{A.4.2}$$

The magnitude response of the filter in (A.4.2) is shown in Figure 5.

A.4.3. The FIR Bandpass Filter

The centre of the passband of the desired bandpass filter was found to be $\omega_c = 0.275\pi$ in Section 2.1. The impulse response of the desired bandpass filter is obtained from the impulse response of the corresponding lowpass filter as

$$h_{bp}(n) = 2h_{lp}(n)cos(n\omega_c) \tag{A.4.3}$$

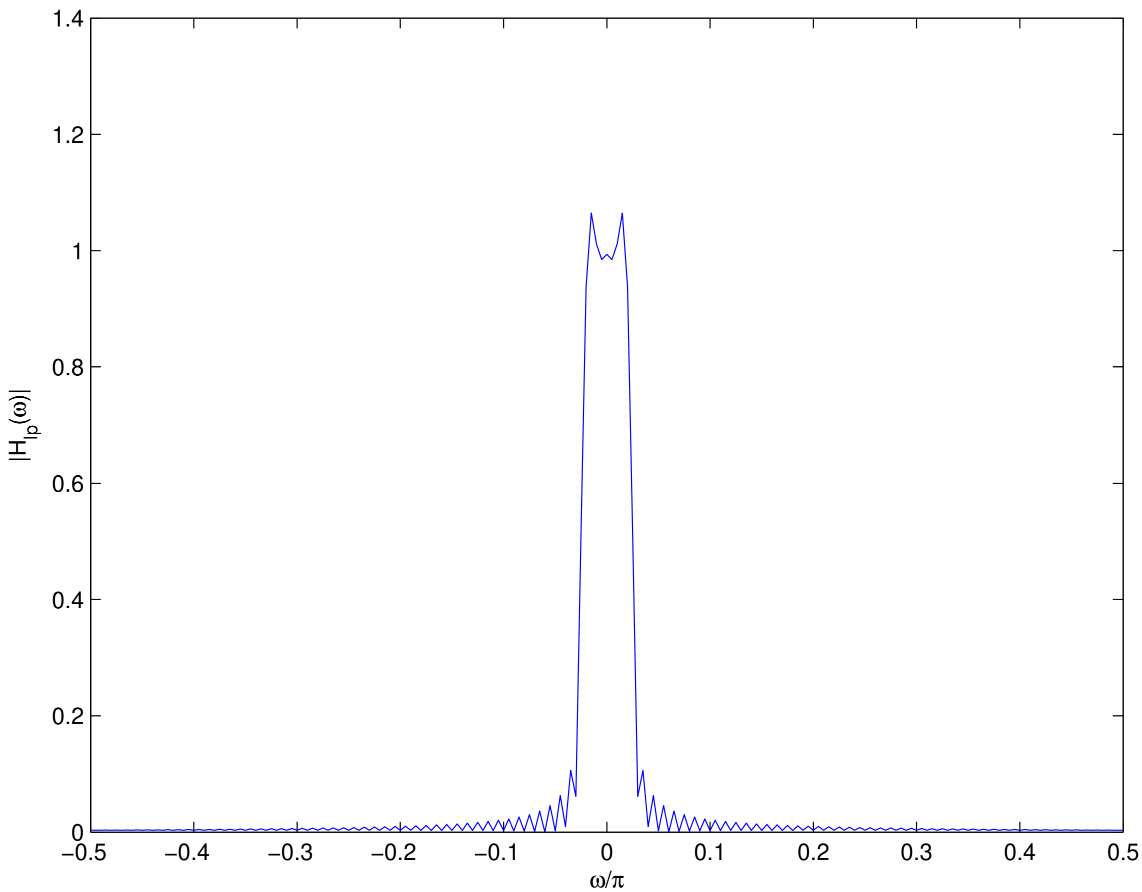


Figure 3.1: The magnitude response of the FIR lowpass digital filter designed to meet the given specifications

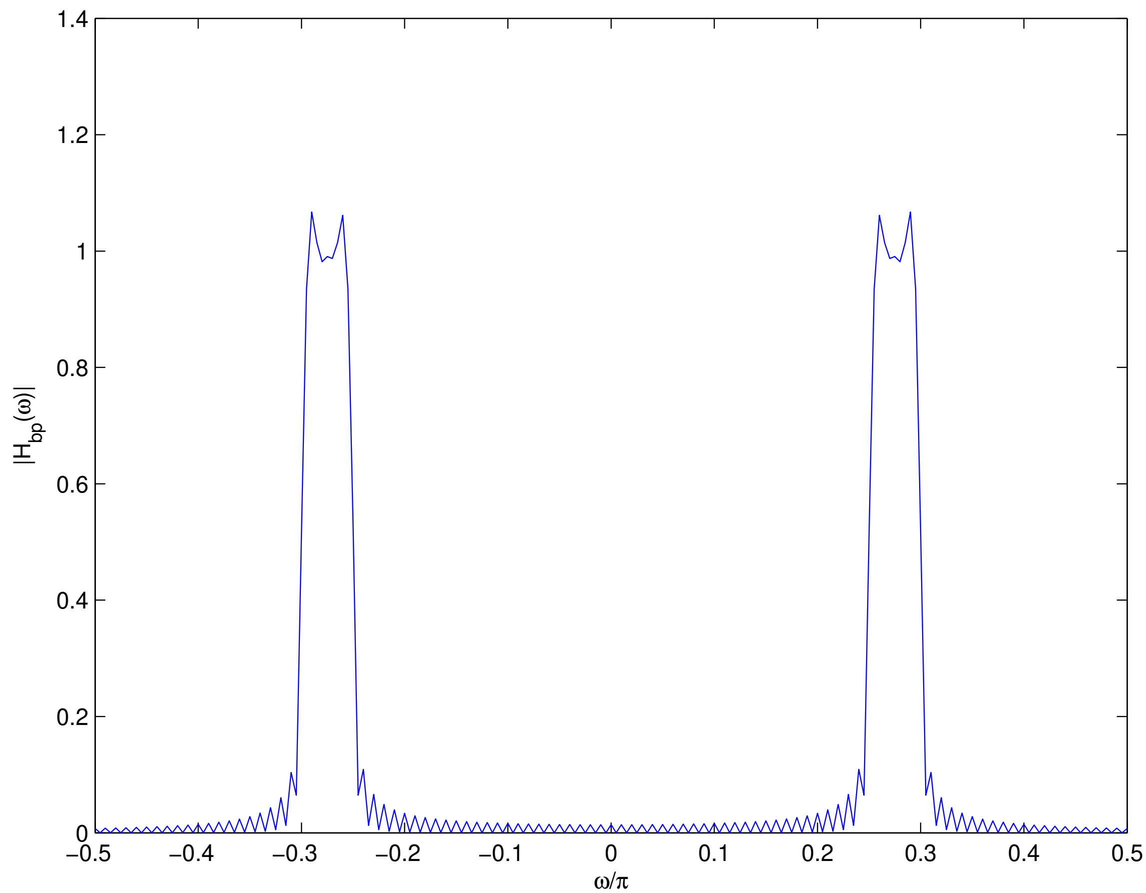


Figure 3.2: The magnitude response of the FIR bandpass digital filter designed to meet the given specifications

Thus, from (A.4.2), we obtain

$$\begin{aligned} h_{bp}(n) &= \frac{2\sin(\frac{n\pi}{40})\cos(\frac{11n\pi}{40})}{n\pi} \qquad -100 \le n \le 100 \\ &= 0, \qquad\qquad\qquad\qquad \text{otherwise} \end{aligned} \tag{A.4.4}$$

The magnitude response of the FIR bandpass filter designed to meet the given specifications is plotted in Fig. 3.2.