SIGNAL PROCESSING

Through Practice

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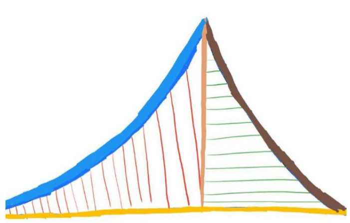

Contents

Introduction

This book introduces digital communication through probability.

Chapter 1

Two Dice

1.1. Problem

Two dice, one blue and one grey, are thrown at the same time. The event defined by the sum of the two numbers appearing on the top of the dice can have 11 possible outcomes 2, 3, 4, 5, 6, 6, 8, 9, 10, 11 and 12. A student argues that each of these outcomes has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.

1.2. Uniform Distribution: Rectangular Function

1.2.1. Let $X_i \in \{1, 2, 3, 4, 5, 6\}, i = 1, 2,$ be the random variables representing the outcome for each die. Assuming the dice to be fair, the probability mass function (pmf) is expressed as

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{6} & 1 \le n \le 6 \\ 0 & otherwise \end{cases} \tag{1.2.1.1}$$

1.3. Sum of Random Variables: Convolution

1.3.1. The desired outcome is

$$X = X_1 + X_2, \tag{1.3.1.1}$$

$$\implies X \in \{1, 2, \dots, 12\} \tag{1.3.1.2}$$

The objective is to show that

$$p_X(n) \neq \frac{1}{11} \tag{1.3.1.3}$$

1.3.2. <u>Convolution:</u> From (1.3.1.1),

$$p_X(n) = \Pr(X_1 + X_2 = n) = \Pr(X_1 = n - X_2) \tag{1.3.2.1}$$

$$= \sum_k \Pr(X_1 = n - k | X_2 = k)\, p_{X_2}(k) \tag{1.3.2.2}$$

after unconditioning. $\because$ X_1 and X_2 are independent,

$$\Pr(X_1 = n - k | X_2 = k) = \Pr(X_1 = n - k) = p_{X_1}(n - k) \tag{1.3.2.3}$$

From (1.3.2.2) and (1.3.2.3),

$$p_X(n) = \sum_k p_{X_1}(n - k) p_{X_2}(k) \triangleq p_{X_1}(n) * p_{X_2}(n) \tag{1.3.2.4}$$

where $*$ denotes the convolution operation.

1.3.3. Substituting from (1.2.1.1) in (1.3.2.4),

$$p_X(n) = \frac{1}{6}\sum_{k=1}^{6} p_{X_1}(n-k) = \frac{1}{6}\sum_{k=n-6}^{n-1} p_{X_1}(k) \tag{1.3.3.1}$$

$$\because p_{X_1}(k) = 0, \quad k \le 1, k \ge 6. \tag{1.3.3.2}$$

From (1.3.3.1),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{1}{6}\sum_{k=1}^{n-1} p_{X_1}(k) & 1 \le n-1 \le 6 \\ \frac{1}{6}\sum_{k=n-6}^{6} p_{X_1}(k) & 1 < n-6 \le 6 \\ 0 & n > 12 \end{cases} \tag{1.3.3.3}$$

1.4. The Triangular function

1.4.1. Substituting from (1.2.1.1) in (1.3.3.3),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \le n \le 7 \\ \frac{13-n}{36} & 7 < n \le 12 \\ 0 & n > 12 \end{cases} \tag{1.4.1.1}$$

satisfying (1.3.1.3).

1.5. The Z-transform

1.5.1. The Z-transform of $p_X(n)$ is defined as

$$P_X(z) = \sum_{n=-\infty}^{\infty} p_X(n) z^{-n}, \quad z \in \mathbb{C} \tag{1.5.1.1}$$

From (1.2.1.1) and (1.5.1.1),

$$P_{X_1}(z) = P_{X_2}(z) = \frac{1}{6}\sum_{n=1}^{6} z^{-n} \tag{1.5.1.2}$$

$$= \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}, \quad |z| > 1 \tag{1.5.1.3}$$

upon summing up the geometric progression.

1.5.2. Show that

$$p_X(n) = p_{X_1}(n) * p_{X_2}(n) \implies P_X(z) = P_{X_1}(z) P_{X_2}(z) \tag{1.5.2.1}$$

The above property follows from Fourier analysis and is fundamental to signal processing.

1.5.3. The unit step function is defined as

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{1.5.3.1}$$

Show that the Z transform of $u(n)$ is

$$U(z) = \frac{1}{1 - z^{-1}}, \ |z| > 1 \tag{1.5.3.2}$$

1.5.4. Show that

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2}, \ |z| > 1 \tag{1.5.4.1}$$

1.5.5. Show that

$$p_X(n-k) \overset{\mathcal{Z}}{\longleftrightarrow} P_X(z)z^{-k} \tag{1.5.5.1}$$

1.5.6. From (1.5.1.3) and (1.5.2.1),

$$P_X(z) = \left\{ \frac{z^{-1}\left(1-z^{-6}\right)}{6\left(1-z^{-1}\right)} \right\}^2 = \frac{1}{36} \frac{z^{-2}\left(1-2z^{-6}+z^{-12}\right)}{\left(1-z^{-1}\right)^2} \tag{1.5.6.1}$$

From (1.5.5.1) and (1.5.4.1), it can be shown that

$$\begin{aligned} &\frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \\ &\qquad\qquad \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{36} \frac{z^{-2}\left(1-2z^{-6}+z^{-12}\right)}{\left(1-z^{-1}\right)^2} \end{aligned} \tag{1.5.6.2}$$

1.5.7. From (1.5.1.1), (1.5.6.1) and (1.5.6.2)

$$p_X(n) = \frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \tag{1.5.7.1}$$

which is the same as (1.4.1.1). Note that (1.4.1.1) can be obtained from (1.5.6.2) using contour integration as well.

1.5.8. The experiment of rolling the dice was simulated using Python for 10000 samples. These were generated using Python libraries for uniform distribution. The frequencies for each outcome were then used to compute the resulting pmf, which is plotted in

Figure 1.5.8.1. The theoretical pmf obtained in (1.4.1.1) is plotted for comparison.

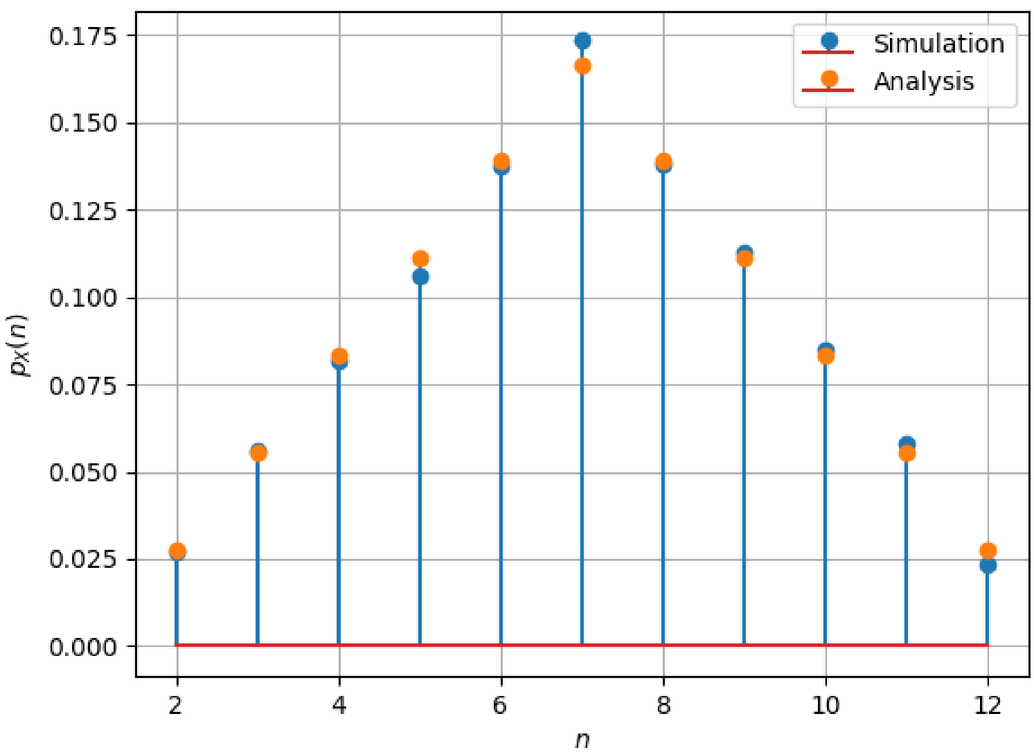

Figure 1.5.8.1: Plot of $p_X(n)$. Simulations are close to the analysis.

1.5.9. The python code is available in

/codes/sum/dice.py

Chapter 2

Pingala Series

2.1. JEE 2019

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \tag{9.1}$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \tag{9.2}$$

where α and β $(\alpha > \beta)$ are the roots of the

$$z^2 - z - 1 = 0 \tag{9.3}$$

Verify the following using a python code.

2.1.1

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{2.1.1.1}$$

2.1.2

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{2.1.2.1}$$

2.1.3

$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \tag{2.1.3.1}$$

2.1.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \tag{2.1.4.1}$$

Solution:

```
$ python3 pingala/codes/1.py
```

2.2. Pingala Series

2.2.1 The Pingala series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n), \quad x(0) = x(1) = 1, n \geq 0 \tag{2.2.1.1}$$

Generate a stem plot for $x(n)$.

Solution: The following code generates Fig. 2.2.1.1.

```
$ python3 pingala/codes/2_1.py
```

2.2.2 The one sided Z-transform of $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}, \quad z \in \mathbb{C} \tag{2.2.2.1}$$

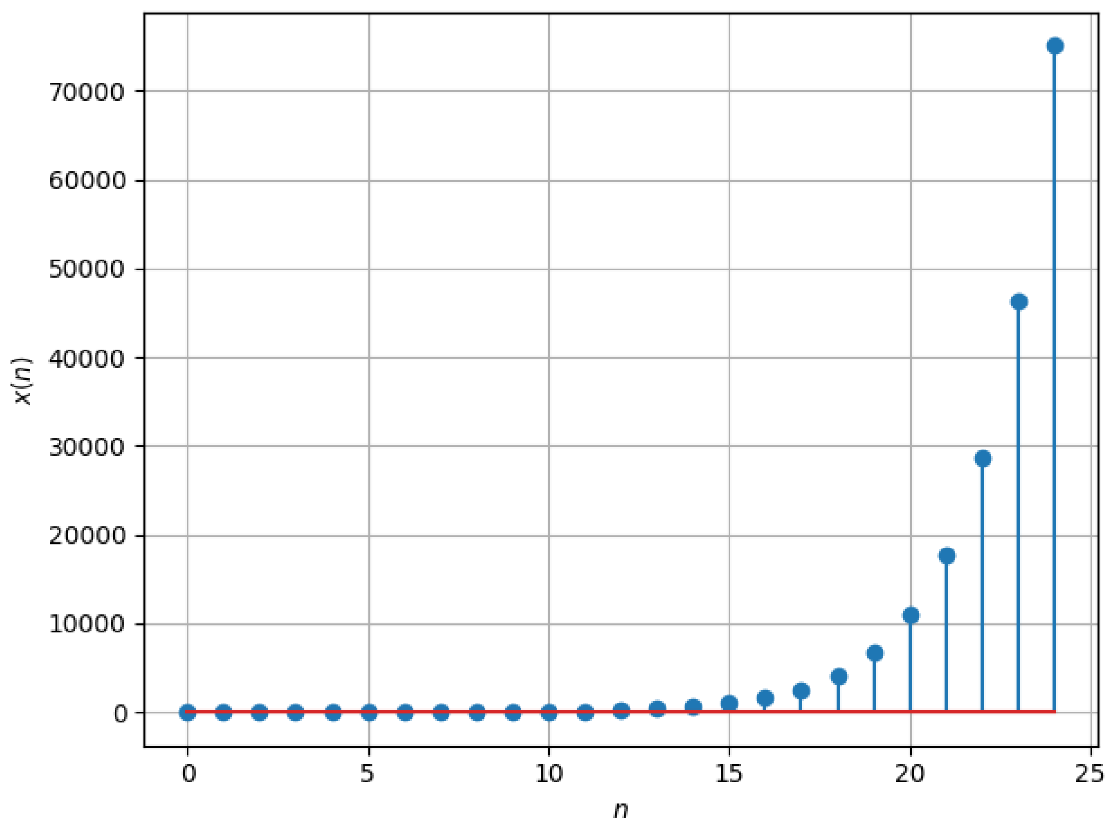

Figure 2.2.1.1: Plot of $x(n)$

Find $X^+(z)$.

Solution: Taking the one-sided Z-transform on both sides of (2.2.1.1),

$$\mathcal{Z}^+\left[x(n+2)\right] = \mathcal{Z}^+\left[x(n+1)\right] + \mathcal{Z}^+\left[x(n)\right] \tag{2.2.2.2}$$

$$\implies z^2X^+(z) - z^2x(0) - zx(1) = zX^+(z) - zx(0) + zX^+(z) \tag{2.2.2.3}$$

$$\implies \left(z^2 - z - 1\right)X^+(z) = z^2 \tag{2.2.2.4}$$

$$\implies X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} = \frac{1}{\left(1 - \alpha z^{-1}\right)\left(1 - \beta z^{-1}\right)}, \quad |z| > \alpha \tag{2.2.2.5}$$

2.2.3 Find $x(n)$.

Solution: Expanding $X^+(z)$ in (2.2.2.5) using partial fractions, we get

$$X^+(z) = \frac{1}{(\alpha - \beta)}\left[\frac{z}{1 - \alpha z^{-1}} - \frac{z}{1 - \beta z^{-1}}\right] \tag{2.2.3.1}$$

$$\implies x(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}u(n) \tag{2.2.3.2}$$

$$= a_{n+1} \tag{2.2.3.3}$$

upon comparing with (9.1).

2.3. Linear Time Invariant System

2.3.1 Sketch

$$y(n) = x\left(n-1\right) + x\left(n+1\right), \quad n \geq 0 \tag{2.3.1.1}$$

Solution: Execute

```
$ python3 pingala/codes/2_2.py
```

to obtain Fig. 2.3.1.1

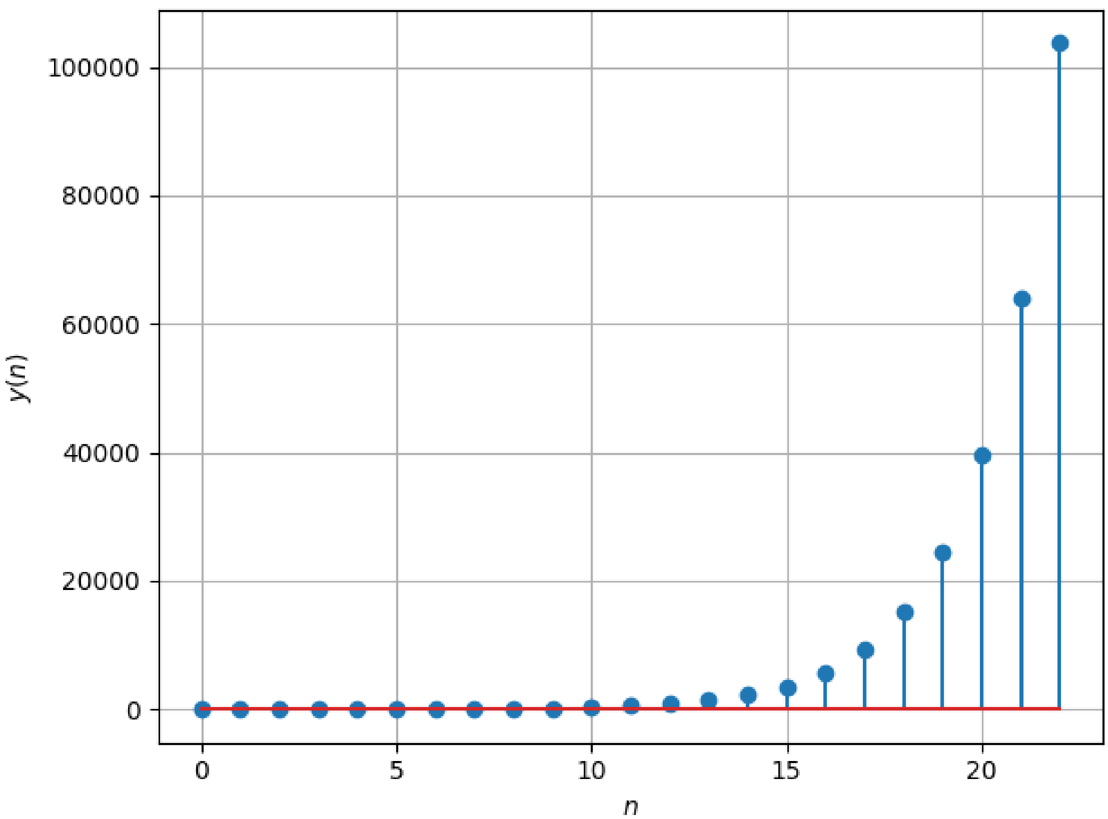

Figure 2.3.1.1: Plot of $y(n)$

2.3.2 Show that

$$x(n+1) \overset{\mathcal{Z}}{\longleftrightarrow} zX^{+}(z) - zx(0) \tag{2.3.2.1}$$

$$x(n-1) \overset{\mathcal{Z}}{\longleftrightarrow} z^{-1}X^{+}(z) + zx(-1) \tag{2.3.2.2}$$

2.3.3 Find $Y^{+}(z)$.

Solution: Taking the one-sided Z-transform on both sides of (2.3.1.1),

$$\mathcal{Z}^+ [y(n)] = \mathcal{Z}^+ [x(n+1)] + \mathcal{Z}^+ [x(n-1)] \tag{2.3.3.1}$$

$$Y^+(z) = zX^+(z) - zx(0) + z^{-1}X^+(z) + zx(-1) \tag{2.3.3.2}$$

$$= \frac{z + z^{-1}}{1 - z^{-1} - z^{-2}} - z = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}}, \quad |z| > \alpha \tag{2.3.3.3}$$

2.3.4 Show that

$$y(n) = b_{n+1}. \tag{2.3.4.1}$$

2.3.5 Find the impulse response of (2.3.1.1)

2.4. Power of the Z transform

2.4.1 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \tag{2.4.1.1}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{a_{k+1}}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} \tag{2.4.1.2}$$

$$= \frac{1}{10} X^+(10) = \frac{1}{10} \times \frac{100}{89} = \frac{10}{89} \tag{2.4.1.3}$$

Thus, (2.1.2.1) is correct.

2.4.2 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y\left(k\right)}{10^k} = \frac{1}{10} Y^{+}\left(10\right) \tag{2.4.2.1}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{b_{k+1}}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} \tag{2.4.2.2}$$

$$= \frac{1}{10} Y^{+}(z) = \frac{1}{10} \times \frac{120}{89} = \frac{12}{89} \tag{2.4.2.3}$$

Thus, (2.1.4.1) is incorrect.

2.4.3 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \tag{2.4.3.1}$$

can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \tag{2.4.3.2}$$

and find $W(z)$.

Solution: Putting $n = k + 1$ in (2.4.3.1) and using the definition of $u(n)$,

$$\alpha^n + \beta^n = \left(\alpha^{k+1} + \beta^{k+1}\right) u(k) \tag{2.4.3.3}$$

Hence, (2.4.3.1) can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \tag{2.4.3.4}$$

Therefore,

$$W(z) = Y(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{2.4.3.5}$$

Thus, by invoking (2.3.4.1), we find that (2.1.3.1) is correct

2.5. Convolution

2.5.1 Show that

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) = x(n) * u(n-1) \tag{2.5.1.1}$$

Solution: From (2.2.3.3), and noting that $x(n) = 0 \ \forall \ n < 0$,

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) = \sum_{k=-\infty}^{n-1} x(k) \tag{2.5.1.2}$$

$$= \sum_{k=-\infty}^{\infty} x(k)u(n-1-k) = x(n) * u(n-1) \tag{2.5.1.3}$$

2.5.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \tag{2.5.2.1}$$

can be expressed as

$$\left[x\left(n+1\right) - 1\right] u\left(n-1\right) \tag{2.5.2.2}$$

Solution: From (2.2.3.3),

$$a_{n+2} - 1 = [x(n+1) - 1], \quad n \geq 1 \tag{2.5.2.3}$$

and so, using the definition of $u(n)$,

$$a_{n+2} - 1 = [x(n+1) - 1]\, u(n-1) \tag{2.5.2.4}$$

2.5.3 Show that

$$[x(n+1) - 1]\, u(n-1) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1 - z^{-1} - z^{-2})(1 - z^{-1})} \tag{2.5.3.1}$$

Solution: The Z transform of the above signal can be expressed as

$$\sum_{n=1}^{\infty} x(n+1) z^{-n} - \frac{z^{-1}}{1 - z^{-1}} = \sum_{n=2}^{\infty} x(n) z^{-n+1} - \frac{z^{-1}}{1 - z^{-1}} \tag{2.5.3.2}$$

$$= z\left[X^{+}(z) - x(0) - x(1)z^{-1}\right] - \frac{z^{-1}}{1 - z^{-1}} \tag{2.5.3.3}$$

$$= \frac{z}{1 - z^{-1} - z^{-2}} - z - 1 - \frac{z^{-1}}{1 - z^{-1}} \tag{2.5.3.4}$$

$$= \frac{z}{1 - z^{-1} - z^{-2}} - \frac{z}{1 - z^{-1}} \tag{2.5.3.5}$$

$$= \frac{z^{-1}}{(1 - z^{-1} - z^{-2})(1 - z^{-1})} \tag{2.5.3.6}$$

From (2.2.3.3), we get

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1 \tag{2.5.3.7}$$

Chapter 3

Circuits and Transforms

3.1. Definitions

1. The unit step function is defined as

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{3.1.1.1}$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\, dt \tag{3.1.2.1}$$

3.2. Laplace Transform

In the circuit in Fig. 2.1, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\,\mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\,\mu C$. Use variables R_1, R_2, C_0 for the passive elements.

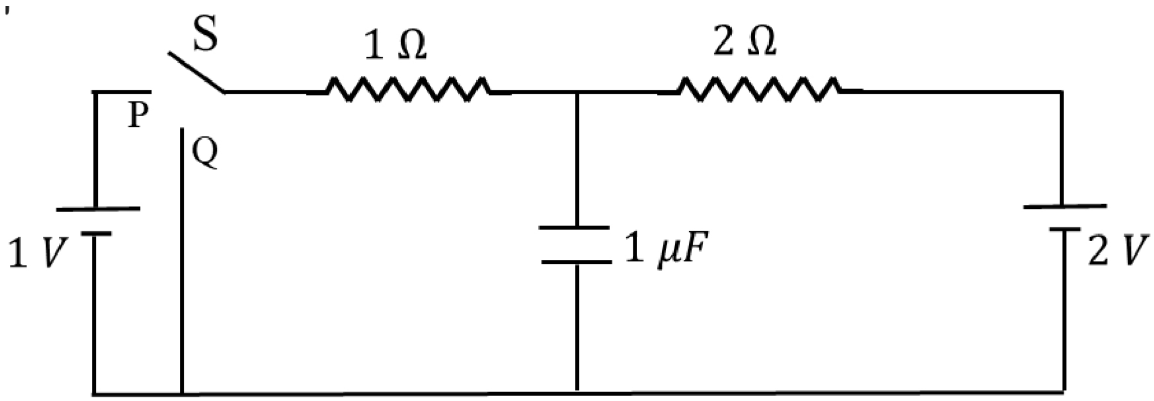

Figure 2.1:

1. Draw the circuit in position P.

   Solution: See Fig. 3.2.1.1 drawn using the code in

cktsig/figs/ckt−q1.tex

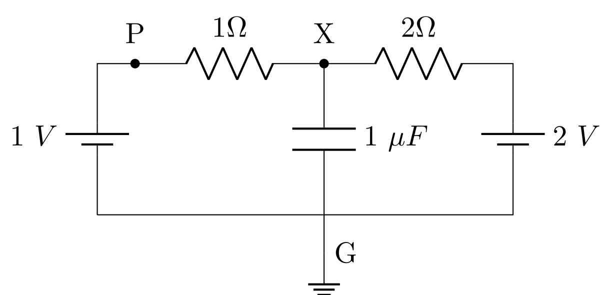

Figure 3.2.1.1:

2. Simulate the circuit in Fig. 3.2.1.1 using ngspice.

   Solution: The following ngspice script simulates the given circuit.

ngspice codes/2_7.cir

3. Find q_1 in Fig. 3.2.3.1.

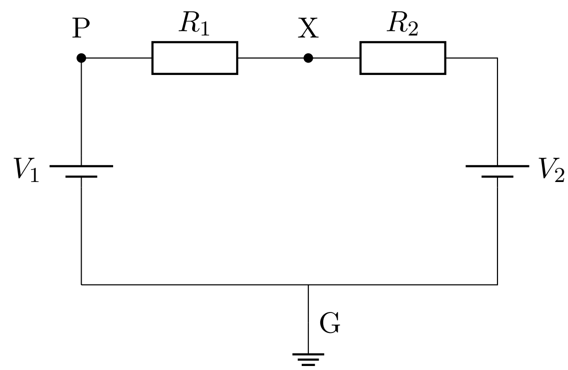

Figure 3.2.3.1:

Solution: Assuming the circuit to be grounded at G and the relative potential at point X to be V, we use KCL at X and get

$$\frac{V - V_1}{R_1} + \frac{V - V_2}{R_2} = 0 \tag{3.2.3.1}$$

$$\tag{3.2.3.2}$$

$$\implies V = \frac{V_1 R_2 + V_2 R_1}{R_1 + R_2} = \frac{4}{3}\,\mathrm{V} \tag{3.2.3.3}$$

upon substituting numerical values. Hence,

$$q_1 = CV = C\frac{(V_1 R_2 + V_2 R_1)}{R_1 + R_2} = \frac{4}{3}\,\mu\mathrm{C} \tag{3.2.3.4}$$

4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.

Solution: We have,

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} \int_0^\infty u(t)e^{-st}dt \tag{3.2.4.1}$$

$$= \int_0^0 \frac{1}{2}e^{-st}dt + \int_0^\infty e^{-st}dt \tag{3.2.4.2}$$

$$= \frac{1}{s}, \quad \Re(s) > 0 \tag{3.2.4.3}$$

5. Now consider the following resistive circuit in Fig. 3.2.5.1 transformed from Fig. 2.1 where

$$v_1(t) = u(t) \overset{\mathcal{L}}{\longleftrightarrow} V_1(s) = \frac{1}{s} \tag{3.2.5.1}$$

$$v_2(t) = 2u(t) \overset{\mathcal{L}}{\longleftrightarrow} V_2(s) = \frac{2}{s} \tag{3.2.5.2}$$

Find the voltage across the capacitor $V_{C_0}(s)$.

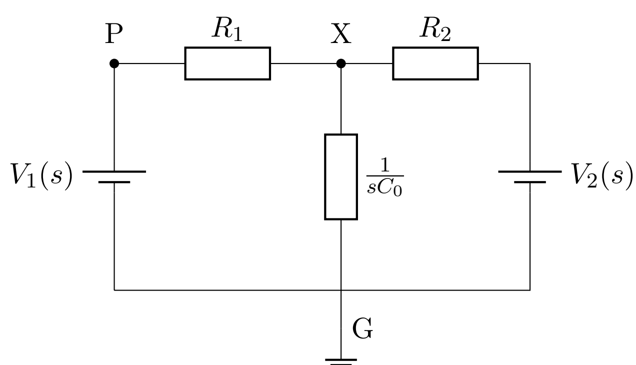

Figure 3.2.5.1:

Solution: Applying KCL at X,

$$\frac{V(s)-\frac{1}{s}}{R_1}+\frac{V(s)-\frac{2}{s}}{R_2}+sC_0V(s)=0 \tag{3.2.5.3}$$

$$\implies V(s)\left(\frac{1}{R_1}+\frac{1}{R_2}+sC_0\right)=\frac{1}{s}\left(\frac{1}{R_1}+\frac{2}{R_2}\right) \tag{3.2.5.4}$$

$$\implies V(s)=\frac{\frac{1}{R_1}+\frac{2}{R_2}}{s\left(\frac{1}{R_1}+\frac{1}{R_2}+sC_0\right)} \tag{3.2.5.5}$$

$$=\frac{\frac{1}{R_1}+\frac{2}{R_2}}{\frac{1}{R_1}+\frac{1}{R_2}}\left(\frac{1}{s}-\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s}\right) \tag{3.2.5.6}$$

6. Show that

$$e^{-at}u(t)\overset{\mathcal{L}}{\longleftrightarrow}\frac{1}{s+a},\quad a>0 \tag{3.2.6.1}$$

and find the ROC.

Solution: Note that by substituting $s := s+a$ in (3.2.4.3), and considering $a \in \mathbb{R}$,

$$e^{-at}u(t)\overset{\mathcal{L}}{\longleftrightarrow}\int_0^\infty u(t)e^{-(s+a)t}dt \tag{3.2.6.2}$$

$$=\frac{1}{s+a},\quad \Re(s)>-a \tag{3.2.6.3}$$

7. Find $v_{C_0}(t)$.

Solution: Taking the inverse Laplace transform in (3.2.5.6),

$$V(s)\overset{\mathcal{L}^{-1}}{\longleftrightarrow}\frac{2R_1+R_2}{R_1+R_2}u(t)\left(1-e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}\right) \tag{3.2.7.1}$$

from (3.2.6.1).

8. Verify your result numerically.

Solution: The following python code plots the graph in Fig. 3.2.8.1

```
python3 codes/2_6-sim.py
```

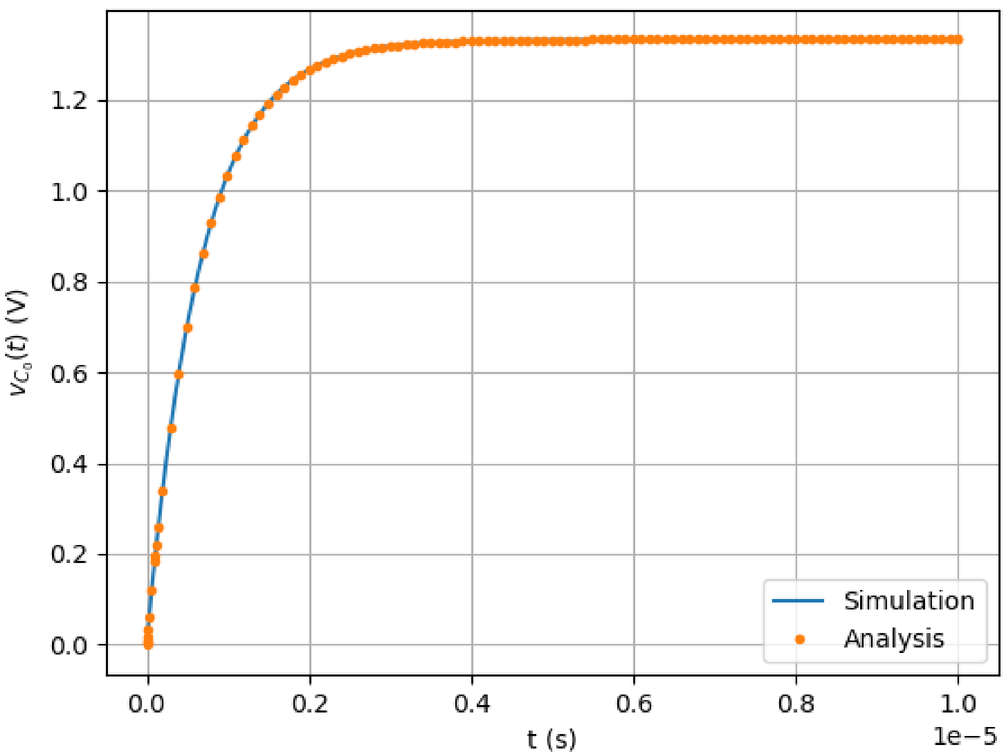

Figure 3.2.8.1: $v_{C_0}(t)$ simulation

3.3. Initial Conditions

1. Draw the equivalent circuit when the switch is at Q.

   Solution: See Fig. 3.3.1.1. below.

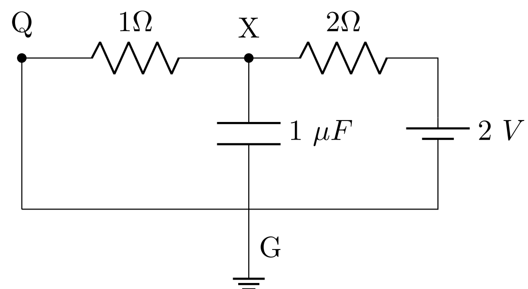

Figure 3.3.1.1:

2. Find q_2 in Fig. 3.3.2.1.

   Solution: The equivalent circuit at steady state when the switch is at Q is shown in Fig. 3.3.2.1. Since the capacitor behaves as an open circuit, we use voltage division at

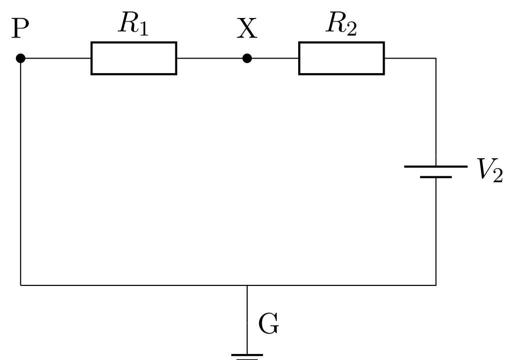

Figure 3.3.2.1:

   X to obtain

$$V = V_1 \frac{R_1}{R_1 + R_2} = \frac{2}{3}\,\mathrm{V} \tag{3.3.2.1}$$

   upon substituting numerical values. Hence,

$$q_2 = CV = \frac{2}{3}\,\mu\mathrm{C}. \tag{3.3.2.2}$$

3. Draw the equivalent s-domain resistive circuit when S is switched to position Q.

   See Fig. 3.3.3.1.

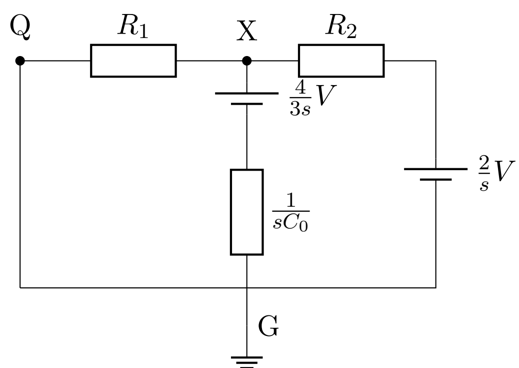

Figure 3.3.3.1:

4. $V_{C_0}(s) = ?$

   Solution: Using KCL at node X in Fig. 3.3.3.1,

$$\frac{V(s) - 0}{R_1} + \frac{V(s) - \frac{2}{s}}{R_2} + sC_0\left(V(s) - \frac{4}{3s}\right) = 0 \tag{3.3.4.1}$$

$$\implies V_{C_0}(s) = \frac{\frac{2}{sR_2} + \frac{4C_0}{3}}{\frac{1}{R_1} + \frac{2}{R_2} + sC_0} \tag{3.3.4.2}$$

5. $v_{C_0}(t) = ?$

   Solution: From (3.3.4.2), using partial fractions,

$$V_{C_0}(s) = \frac{4}{3}\left(\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right) + \frac{2}{R_2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right) \tag{3.3.5.1}$$

Taking the inverse Laplace Transform,

$$v_{C_0}(t) = \frac{4}{3}e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}u(t) + \frac{2}{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\left(1 - e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}\right)u(t) \quad (3.3.5.2)$$

from (3.2.6.1).

6. Verify your result numerically.

   Solution: Substituting numerical values,

$$v_{C_0}(t) = \frac{2}{3}\left(1 + e^{-\left(1.5\times10^6\right)t}\right)u(t) \quad (3.3.6.1)$$

   This is verified by the following code that plots Fig. 3.3.6.1 using ngspice and python.

```
python3 codes/3_4−sim.py
```

7. Find $v_{C_0}(0-), v_{C_0}(0+)$ and $v_{C_0}(\infty)$.

   Solution: From the initial conditions,

$$v_{C_0}(0-) = \frac{q_1}{C_0} = \frac{4}{3}\,\mathrm{V} \quad (3.3.7.1)$$

   Using (3.3.6.1),

$$v_{C_0}(0+) = \lim_{t\to0+} v_{C_0}(t) = \frac{4}{3}\,\mathrm{V} \quad (3.3.7.2)$$

$$v_{C_0}(\infty) = \lim_{t\to\infty} v_{C_0}(t) = \frac{2}{3}\,\mathrm{V} \quad (3.3.7.3)$$

3.4. Bilinear Transform

1. Formulate the differential equation for Fig. 3.4.1.1.

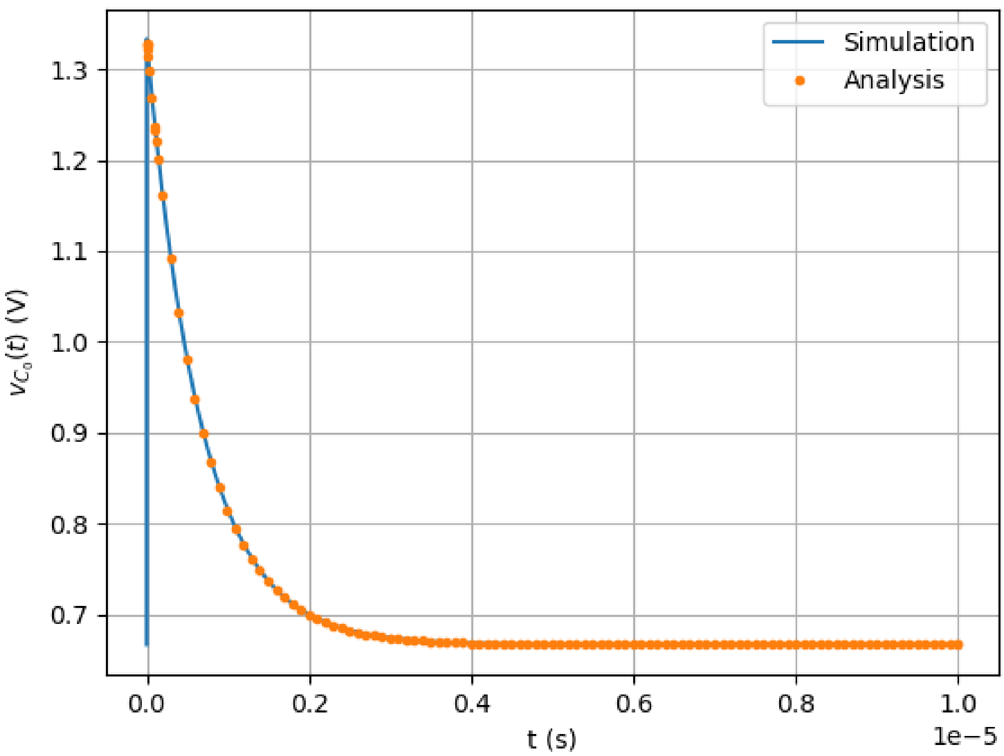

Figure 3.3.6.1: $v_{C_0}(t)$ after the switch is flipped

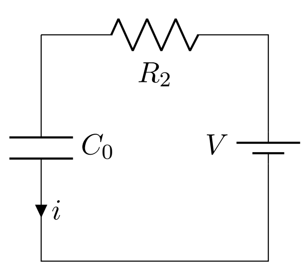

Figure 3.4.1.1:

Solution: Applying KVL on the loop,

$$V - iR_2 - \frac{1}{C_0}\int_0^t i dt = 0 \tag{3.4.1.1}$$

where $i(0) = 0$, $V_C(0) = 0$. Denote by V_C the voltage at the capacitor. Then,

$$i = C\frac{dV_C}{dt} \tag{3.4.1.2}$$

and therefore using (3.4.1.2) in (3.4.1.1), we get the differential equation

$$V - \tau\frac{dV_C}{dt} - V_C = 0 \tag{3.4.1.3}$$

where $\tau \triangleq R_2C_0$ is the time constant of the circuit.

2. Fig. 3.4.1.1 is transformed to Fig. 3.4.2.1 in the s domain. Find the system transfer function

$$H(s) = \frac{V(s)}{V_c(s)} \tag{3.4.2.1}$$

Solution: Using the voltage division formula, the voltage across the capacitor is

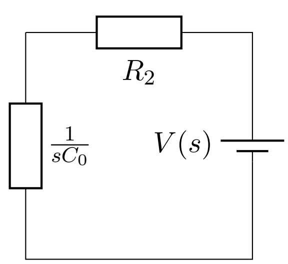

Figure 3.4.2.1:

given by

$$V_{C_0}(s) = V(s)\frac{\frac{1}{sC_0}}{\frac{1}{sC_0} + R_2} = V(s)\frac{1}{1 + sC_0R_2} \tag{3.4.2.2}$$

$$\implies H(s) = \frac{V_{C_0}(s)}{V(s)} = \frac{1}{1 + sC_0R_2} \tag{3.4.2.3}$$

3. Plot $|H(\jmath\omega|$. What kind of filter is it?

   Solution:

4. Using trapezoidal rule for integration, formulate (3.4.1.3) as a difference equation by considering

$$V_C(n) = V_C(t)|_{t=n} \tag{3.4.4.1}$$

   Solution: Integrating (3.4.1.3) between limits $n-1$ to n and applying the trapezoidal formula,

$$\frac{V_C(n) + V_C(n-1)}{2} + \tau\left(V_C(n) - V_C(n-1)\right) = \frac{V(n) + V(n-1)}{2} \tag{3.4.4.2}$$

$$\implies V_C(n)\left(\frac{1}{2} + \tau\right) + V_C(n-1)\left(\frac{1}{2} - \tau\right) = \frac{V(n) + V(n-1)}{2} \tag{3.4.4.3}$$

5. Find

$$H(z) = \frac{V_C(z)}{V(z)}. \tag{3.4.5.1}$$

   Solution: Applying the Z-transform on both sides of (3.4.4.3),

$$V_C(z)\left[(2\tau+1) - z^{-1}(2\tau-1)\right] = V(z)\left(1 + z^{-1}\right) \tag{3.4.5.2}$$

   Hence,

$$H(z) = \frac{1 + z^{-1}}{(2\tau+1) - (2\tau-1)\,z^{-1}} \tag{3.4.5.3}$$

6. Is the system defined by (3.4.4.3) stable?

Solution: From (3.4.5.3), $H(z)$ has a pole at

$$z = \frac{2\tau - 1}{2\tau + 1} \tag{3.4.6.1}$$

Since

$$\left| \frac{2\tau - 1}{2\tau + 1} \right| < 1, \tag{3.4.6.2}$$

the ROC of $H(z)$ is $|z| > 1$, which implies that (3.4.4.3) represents a stable system.

7. How can you obtain $H(z)$ from $H(s)$?

   Solution: Substituting

$$s = 2\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) \tag{3.4.7.1}$$

   in (3.4.2.3), we obtain (3.4.5.3). This is a special case of the the bilinear transformation for $T = 1$ where

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}} \tag{3.4.7.2}$$

8. Find $V_C(n)$. Verify using ngspice.

   Solution: Since $V(n) = Vu(n)$,

$$V(z) = \frac{V}{1 - z^{-1}} \tag{3.4.8.1}$$

Therefore,

$$V_C(z) = H(z)V(z) \qquad = \frac{V\left(1+z^{-1}\right)}{\left(1-z^{-1}\right)\left((2\tau+1)-(2\tau-1)\,z^{-1}\right)} \tag{3.4.8.2}$$

$$\tag{3.4.8.3}$$

Taking the inverse,

$$V_C(n) = \begin{cases} \frac{V}{2}\left[u(n)\left(1-p^n\right)+u(n-1)\left(1-p^{n-1}\right)\right] & n > 0 \\ 0 & n \le 0 \end{cases} \tag{3.4.8.4}$$

where

$$p = \frac{2\tau-1}{2\tau+1} \tag{3.4.8.5}$$

The following python code

```
cktsig/codes/1_7.py
```

plots V_C for the sampling interval $T = 10^{-7}$ in Fig. 3.4.8.1.

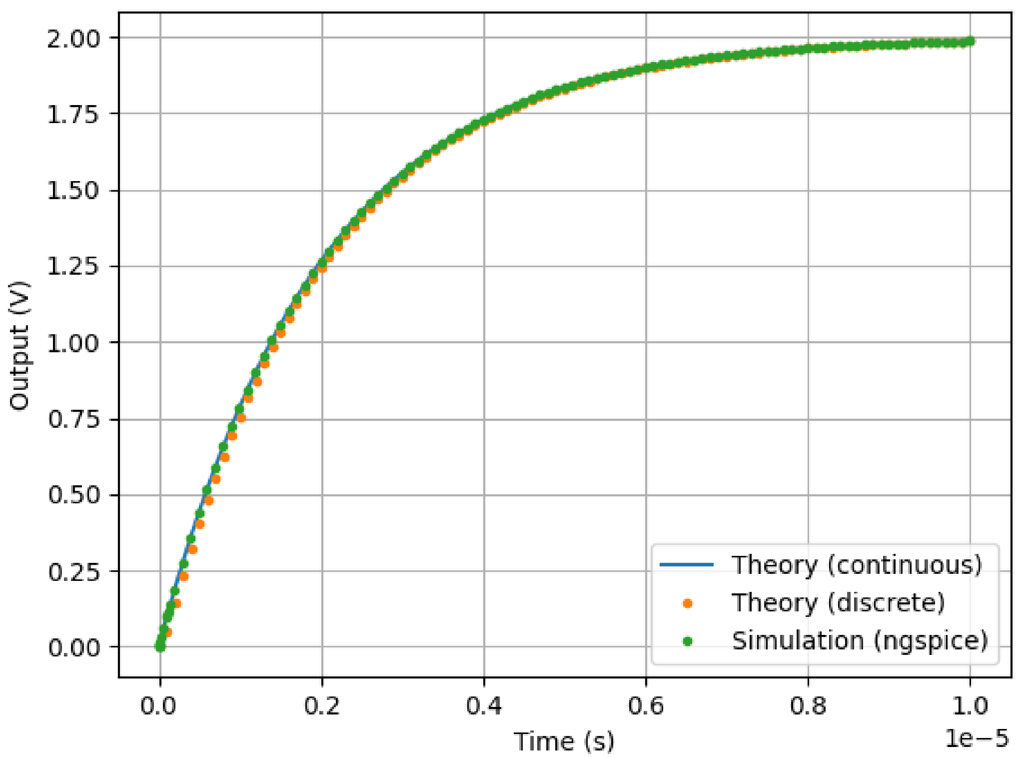

Figure 3.4.8.1: Representation of output across C.

Chapter 4

Frequency Modulation

4.1. Message

Play the message signal using

```
sudo apt install ffmpeg
ffplay /fm/codes/msg/Sound_Noise.wav
```

1. Find the sampling rate of the message.

   Solution: Executing

   ```
   python3 fm/codes/msg/sample_rate.py
   ```

   gives the sampling rate of the input signal as 44100Hz.

2. Plot the spectrum of the message signal using the builtin FFT algorithm.

   Solution:

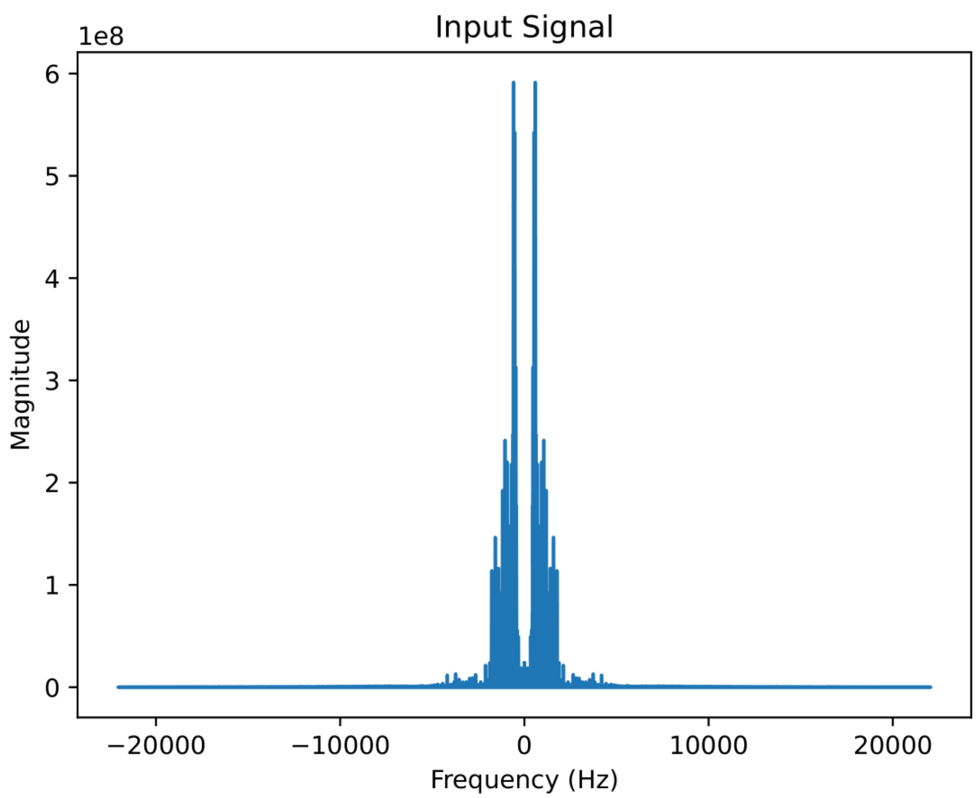

Figure 4.1.2.1: Plot of spectrum of message signal using builtin FFT algorithm.

3. Find the number of samples used to compute the FFT. The folowing code plots the spectrum in Fig. 4.1.2.1 using the builtin FFT algorithm of the python library 'Numpy.'

```
/fm/codes/msg/msg_spec_amp.py
```

4. What does the following command do?

```
f_i = np.fft.fftfreq(len(audio_data), d=1/sample_rate)
```

5. Plot the spectrum of the message signal by writing your own FFT algorithm.

   Solution:

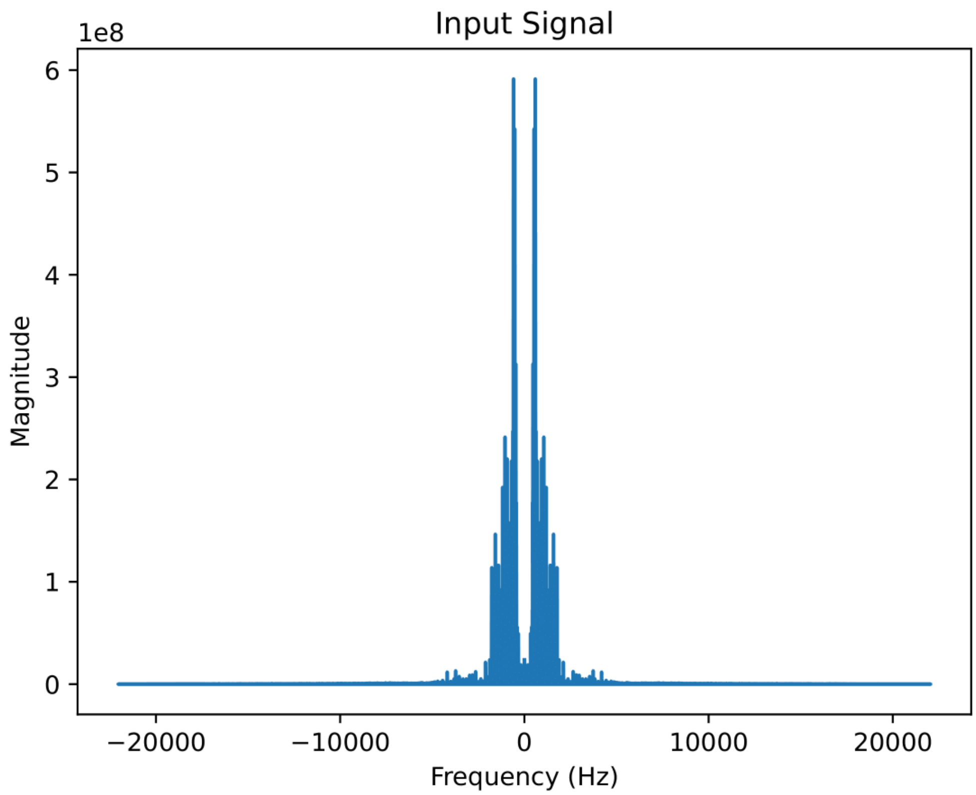

Figure 4.1.5.1: Plot of spectrum of message signal using own FFT algorithm.

   The folowing code plots the spectrum in Fig. 4.1.5.1 using the DFT defined in (A.1.1.1).

```
python3 fm/codes/msg/FFTmsg.py
```

6. Compute and plot the PSD of the message signal using (A.3.1.1).

7. Find the bandwidth of the message signal.

   Solution:

4.2. Transmitter

1. The modulated signal is given by

$$s(t) = \cos(2\pi f_c t + \phi(t)) \tag{4.2.1.1}$$

where

$$\phi(t) = 2\pi k_f \int_0^t m(\tau)\, d\tau \tag{4.2.1.2}$$

List the various parameters in a table.

Solution:

2. Obtain a difference equation for computing $\phi(t)$. Suggest a sampling rate.

3. Plot the spectrum of the transmitted signal.

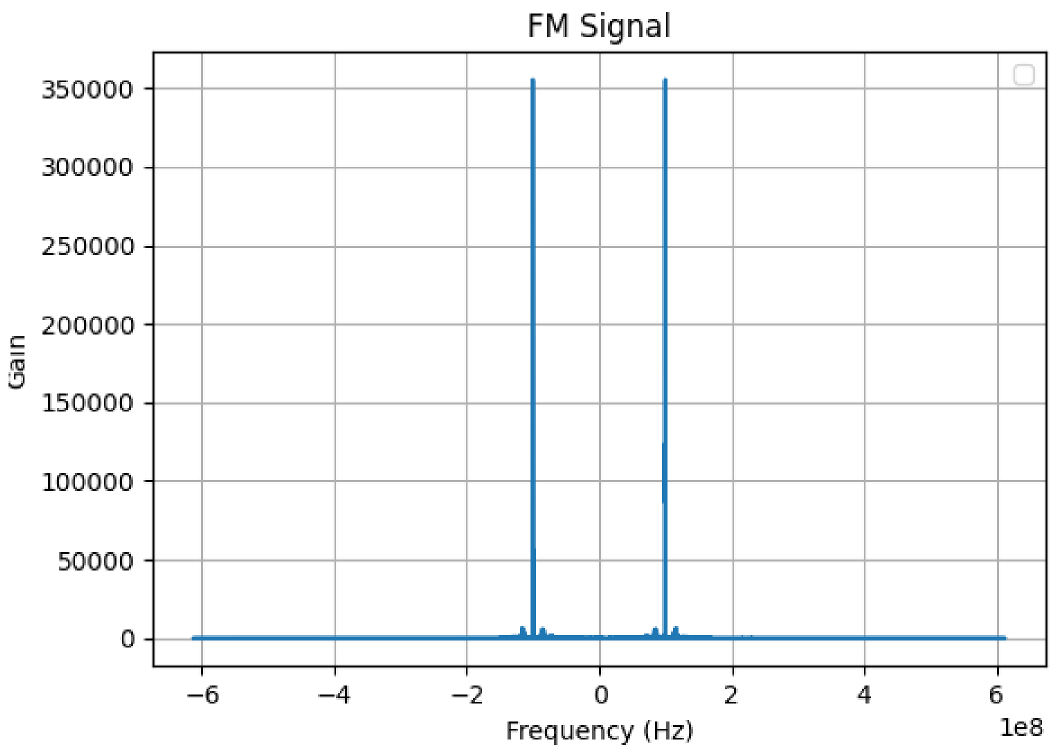

Figure 4.2.3.1: Plot of spectrum of transmitted signal.

4. Compute the bandwidth of the transmitted signal.

   Solution: Calculate the bandwidth of the transmitted signal by computing the Fourier Transform:

$$S_k = \sum_{n=0}^{N-1} s(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{4.2.4.1}$$

   In code for computing the FFT of s(n)

$$S_k = \texttt{fft}(s(n), N) \tag{4.2.4.2}$$

   Where S_k is the frequency representation of the signal, s(n) is the transmitted signal.

we need to calculate its power spectral density. This can be done using the equation (4.2.4.3)

Calculating the Power Spectral Density:

$$PSD(s) = |S_k|^2 \tag{4.2.4.3}$$

Identify the frequency range with significant power using a mask function.code is given below

```
/codes/mod.py
```

Appendix A

Discrete Fourier Transform

A.1. FFT

1. The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{A.1.1.1}$$

2. Let

$$W_N = e^{-j2\pi/N} \tag{A.1.2.1}$$

Then the N-point DFT matrix is defined as

$$\mathbf{F}_N = [W_N^{mn}], \quad 0 \le m, n \le N-1 \tag{A.1.2.2}$$

where W_N^{mn} are the elements of $\mathbf{F}_N$.

A.2. FFT

1. Let

$$\mathbf{I}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^2 & \mathbf{e}_4^3 & \mathbf{e}_4^4 \end{pmatrix} \tag{A.2.1.1}$$

be the 4×4 identity matrix. Then the 4 point <u>DFT permutation matrix</u> is defined as

$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{A.2.1.2}$$

2. The 4 point <u>DFT diagonal matrix</u> is defined as

$$\mathbf{D}_4 = diag \begin{pmatrix} W_8^0 & W_8^1 & W_8^2 & W_8^3 \end{pmatrix} \tag{A.2.2.1}$$

3. Show that

$$W_N^2 = W_{N/2} \tag{A.2.3.1}$$

4. Show that

$$\mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{A.2.4.1}$$

5. Show that

$$\mathbf{F}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N \tag{A.2.5.1}$$

6. Find

$$\mathbf{P}_4 \mathbf{x} \tag{A.2.6.1}$$

7. Show that

$$\mathbf{X} = \mathbf{F}_N \mathbf{x} \tag{A.2.7.1}$$

where $\mathbf{x}, \mathbf{X}$ are the vector representations of $x(n), X(k)$ respectively.

8. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{A.2.8.1}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{A.2.8.2}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{A.2.8.3}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{A.2.8.4}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{A.2.8.5}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{A.2.8.6}$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{A.2.8.7}$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{A.2.8.8}$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{A.2.8.9}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{A.2.8.10}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{A.2.8.11}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{A.2.8.12}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{A.2.8.13}$$

9. For

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{A.2.9.1}$$

compte the DFT using (A.2.7.1)

10. Repeat the above exercise using the FFT after zero padding $\mathbf{x}$.

11. Write a C program to compute the 8-point FFT.

A.3. Power Spectral Density(PSD)

1. Power spectral density (PSD) is a measure of the power distribution over frequency components of a signal.The PSD of (A.1.1.1)is given by

$$X(k) = |X(k)|^2 \tag{A.3.1.1}$$

Appendix B

Filter Design

B.1. Introduction

We are supposed to design the equivalent FIR and IIR filter realizations for filter number 114. This is a bandpass filter whose specifications are available below.

B.2. Filter Specifications

The sampling rate for the filter has been specified as $F_s = 48$ kHz. If the un-normalized discrete-time (natural) frequency is F, the corresponding normalized digital filter (angular) frequency is given by $\omega = 2\pi\left(\frac{F}{F_s}\right)$.

B.2.1. The Digital Filter

1. Tolerances: The passband (δ_1) and stopband (δ_2) tolerances are given to be equal, so we let $\delta_1 = \delta_2 = \delta = 0.15$.

2. Passband: The passband of filter number j, j going from 109 to 135 is from $\{3 + 0.6(\text{j-109})\}$kHz to $\{3 + 0.6(\text{j-107})\}$kHz. Since our filter number is 114, substituting

$j = 114$ gives the passband range for our bandpass filter as 6 kHz - 7.2 kHz. Hence, the un-normalized discrete time filter passband frequencies are $F_{p1} = 7.2$ kHz and $F_{p2} = 6.0$ kHz. The corresponding normalized digital filter passband frequencies are $\omega_{p1} = 2\pi\frac{F_{p1}}{F_s} = 0.3\pi$ and $\omega_{p2} = 2\pi\frac{F_{p2}}{F_s} = 0.25\pi$ kHz. The centre frequency is then given by $\omega_c = \frac{\omega_{p1}+\omega_{p2}}{2} = 0.275\pi$.

3. Stopband: The transition band for bandpass filters is $\Delta F = 0.3$ kHz on either side of the passband. Hence, the un-normalized stopband frequencies are $F_{s1} = 7.2+0.3 = 7.5$ kHz and $F_{s2} = 6.0 - 0.3 = 5.7$ kHz. The corresponding normalized frequencies are $\omega_{s1} = 0.3125\pi$ and $\omega_{s2} = 0.2375\pi$.

B.2.2. The Analog filter

In the bilinear transform, the analog filter frequency (Ω) is related to the corresponding digital filter frequency (ω) as $\Omega = \tan\frac{\omega}{2}$. Using this relation, we obtain the analog passband and stopband frequencies as $\Omega_{p1} = 0.5095$, $\Omega_{p2} = 0.4142$ and $\Omega_{s1} = 0.5345$, $\Omega_{s2} = 0.3914$ respectively.

B.3. The IIR Filter Design

Filter Type: We are supposed to design filters whose stopband is monotonic and passband equiripple. Hence, we use the Chebyschev approximation to design our bandpass IIR filter.

B.3.1. The Analog Filter

1. Low Pass Filter Specifications: If $H_{a,BP}(j\Omega)$ be the desired analog band pass filter, with the specifications provided in Section 2.2, and $H_{a,LP}(j\Omega_L)$ be the equivalent low pass filter, then

$$\Omega_L = \frac{\Omega^2 - \Omega_0^2}{B\Omega} \tag{1.1}$$

where $\Omega_0 = \sqrt{\Omega_{p1}\Omega_{p2}} = 0.4594$ and $B = \Omega_{p1} - \Omega_{p2} = 0.0953$. The low pass filter has the passband edge at $\Omega_{Lp} = 1$ and stopband edges at $\Omega_{Ls_1} = 1.4653$ and $\Omega_{Ls_2} = -1.5511$. We choose the stopband edge of the analog low pass filter as $\Omega_{Ls} = \min(|\Omega_{Ls_1}|, |\Omega_{Ls_2}|) = 1.4653$.

2. The Low Pass Chebyschev Filter Paramters: The magnitude squared of the Chebyschev low pass filter is given by

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\Omega_L/\Omega_{Lp})} \tag{2.1}$$

where $c_N(x) = \cosh(N \cosh^{-1} x)$ and the integer N, which is the order of the filter, and ϵ are design paramters. Since $\Omega_{Lp} = 1$, (2.1) may be rewritten as

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\Omega_L)} \tag{2.2}$$

Also, the design paramters have the following constraints

$$\begin{aligned} \frac{\sqrt{D_2}}{c_N(\Omega_{Ls})} \le \epsilon \le \sqrt{D_1}, \\ N \ge \left\lceil \frac{\cosh^{-1}\sqrt{D_2/D_1}}{\cosh^{-1}\Omega_{Ls}} \right\rceil, \end{aligned} \tag{2.3}$$

where $D_1 = \frac{1}{(1-\delta)^2} - 1$ and $D_2 = \frac{1}{\delta^2} - 1$. After appropriate substitutions, we obtain

$N \geq 4$ and $0.3184 \leq \epsilon \leq 0.6197$. In Figure 1, we plot $|H(j\Omega)|$ for a range of values of ϵ, for $N = 4$. We find that for larger values of ϵ, $|H(j\Omega)|$ decreases in the transition band. We choose $\epsilon = 0.4$ for our IIR filter design.

3. <u>The Low Pass Chebyschev Filter:</u> Thus, we obtain

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + 0.16c_4^2(\Omega_L)} \tag{3.1}$$

where

$$c_4(x) = 8x^4 + 8x^2 + 1. \tag{3.2}$$

The poles of the frequency response in (2.1) lying in the left half plane are in general obtained as $r_1 \cos\phi_k + jr_2 \sin\phi_k$, where

$$\begin{gathered} \phi_k = \frac{\pi}{2} + \frac{(2k+1)\pi}{2N}, k = 0, 1, \ldots, N-1 \\ r_1 = \frac{\beta^2 - 1}{2\beta}, r_2 = \frac{\beta^2 + 1}{2\beta}, \beta = \left[\frac{\sqrt{1+\epsilon^2}+1}{\epsilon}\right]^{\frac{1}{N}} \end{gathered} \tag{3.3}$$

Thus, for N even, the low-pass stable Chebyschev filter, with a gain G has the form

$$H_{a,LP}(s_L) = \frac{G_{LP}}{\prod_{k=1}^{\frac{N}{2}-1}(s_L^2 - 2r_1\cos\phi_k s_L + r_1^2\cos^2\phi_k + r_2^2\sin^2\phi_k)} \tag{3.4}$$

Substituting $N = 4$, $\epsilon = 0.5$ and $H_{a,LP}(j) = \frac{1}{\sqrt{1+\epsilon^2}}$, from (3.3) and (3.4), we obtain

$$H_{a,LP}(s_L) = \frac{0.3125}{s_L^4 + 1.1068s_L^3 + 1.6125s_L^2 + 0.9140s_L + 0.3366} \tag{3.5}$$

In Fig. 4.2 we plot $|H(j\Omega)|$ using (3.1) and (3.5), thereby verifying that our low-pass Chebyschev filter design meets the specifications.

4. The Band Pass Chebyschev Filter: The analog bandpass filter is obtained from (3.5) by substituting $s_L = \frac{s^2+\Omega_0^2}{Bs}$. Hence

$$H_{a,BP}(s) = G_{BP} H_{a,LP}(s_L)\Big|_{s_L=\frac{s^2+\Omega_0^2}{Bs}}, \tag{4.1}$$

where G_{BP} is the gain of the bandpass filter. After appropriate substitutions, and evaluating the gain such that $H_{a,BP}(j\Omega_{p1}) = 1$, we obtain

$$H_{a,BP}(s) = \frac{2.7776 \times 10^{-5} s^4}{s^8 + 0.1055s^7 + 0.8589s^6 + 0.0676s^5 + 0.2735s^4 + 0.0143s^3 + 0.0383s^2 + 0.001s + 0.002} \tag{4.2}$$

In Fig 4.3, we plot $|H_{a,BP}(j\Omega)|$ as a function of Ω for both positve as

well as negative frequencies. We find that the passband and stopband frequencies in the figure match well with those obtained analytically through the bilinear transformation.

B.3.2. The Digital Filter

From the bilinear transformation, we obtain the digital bandpass filter from the corresponding analog filter as

$$H_{d,BP}(z) = G H_{a,BP}(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \tag{4.3}$$

where G is the gain of the digital filter. From (4.2) and (4.3), we obtain

$$H_{d,BP}(z) = G\frac{N(z)}{D(z)} \tag{4.4}$$

where $G = 2.7776 \times 10^{-5}$,

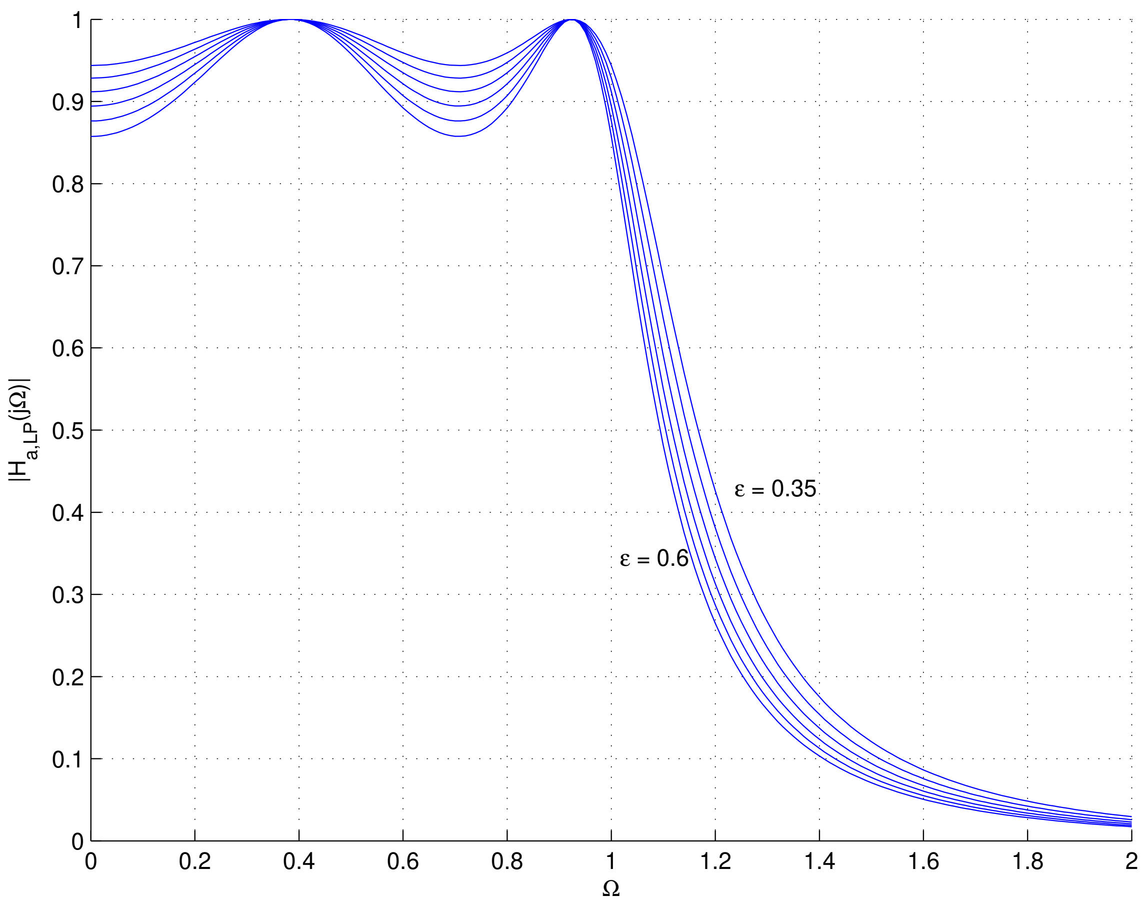

Figure 4.1: The Analog Low-Pass Frequency Response for $0.35 \leq \epsilon \leq 0.6$

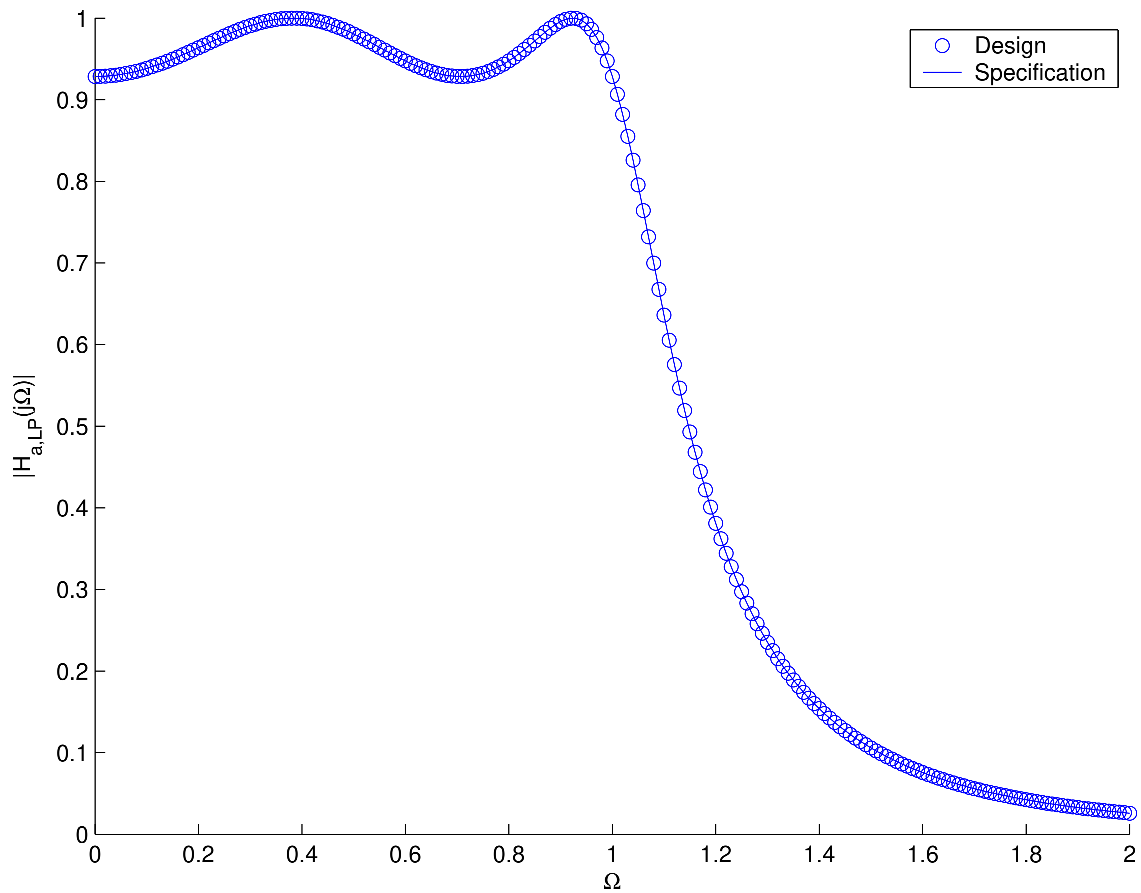

Figure 4.2: The magnitude response plots from the specifications in Equation 3.1 and the design in Equation 3.5

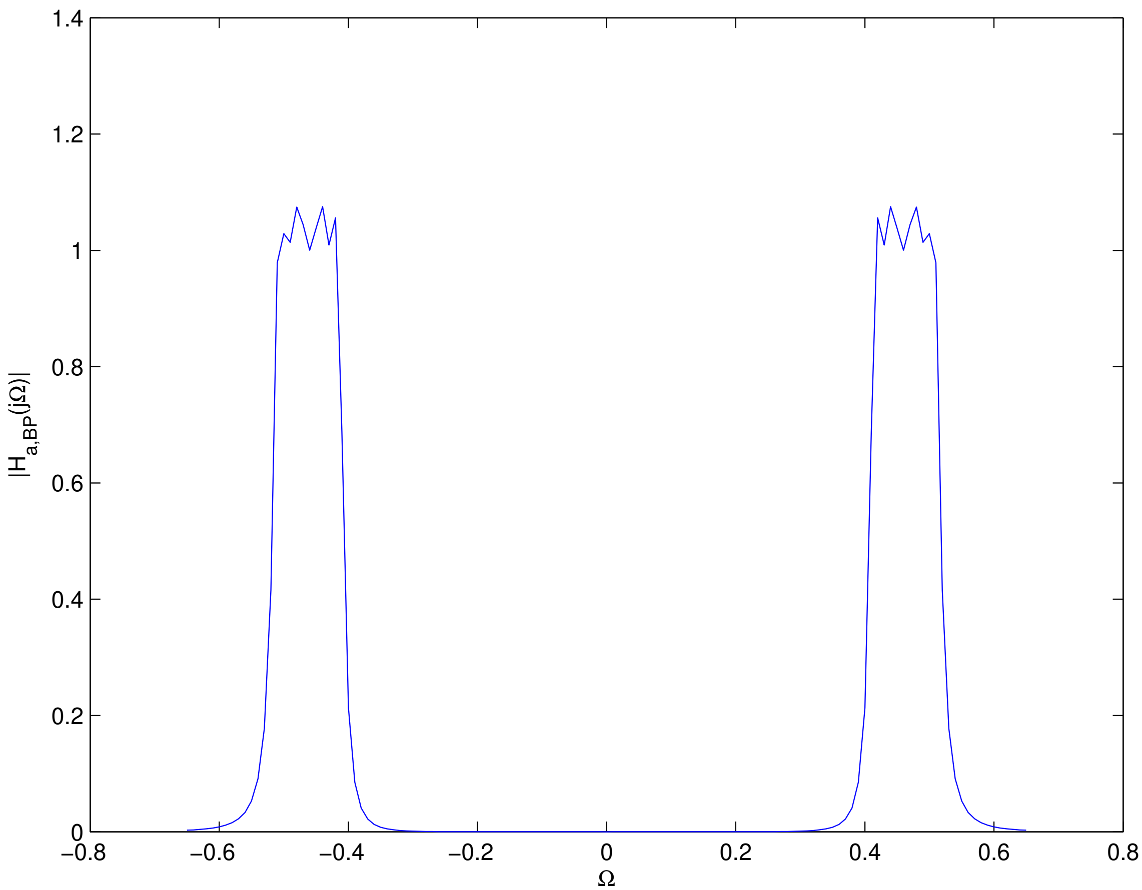

Figure 4.3: The analog bandpass magnitude response plot from Equation 4.2

$$N(z) = 1 - 4z^{-2} + 6z^{-4} - 4z^{-6} + z^{-8} \tag{4.5}$$

and

$$D(z) = 2.3609 - 12.0002z^{-1} + 31.8772z^{-2} - 53.7495z^{-3} + 62.8086z^{-4}$$
$$-51.4634z^{-5} + 29.2231z^{-6} - 10.5329z^{-7} + 1.9842z^{-8} \tag{4.6}$$

The plot of $|H_{d,BP}(z)|$ with respect to the normalized angular freqency (normalizing factor π) is available in Fig. 4.4. Again we

find that the passband and stopband frequencies meet the specifications well enough.

B.4. The FIR Filter

We design the FIR filter by first obtaining the (non-causal) lowpass equivalent using the Kaiser window and then converting it to a causal bandpass filter.

B.4.1. The Equivalent Lowpass Filter

The lowpass filter has a passband frequency ω_l and transition band $\Delta\omega = 2\pi\frac{\Delta F}{F_s} = 0.0125\pi$. The stopband tolerance is δ.

1. The passband frequency ω_l is defined as $\omega_l = \frac{\omega_{p1}-\omega_{p2}}{2}$. Substituting the values of ω_{p1} and ω_{p2} from section 2.1, we obtain $\omega_l = 0.025\pi$.

2. The impulse response $h_{lp}(n)$ of the desired lowpass filter with cutoff frequency ω_l is

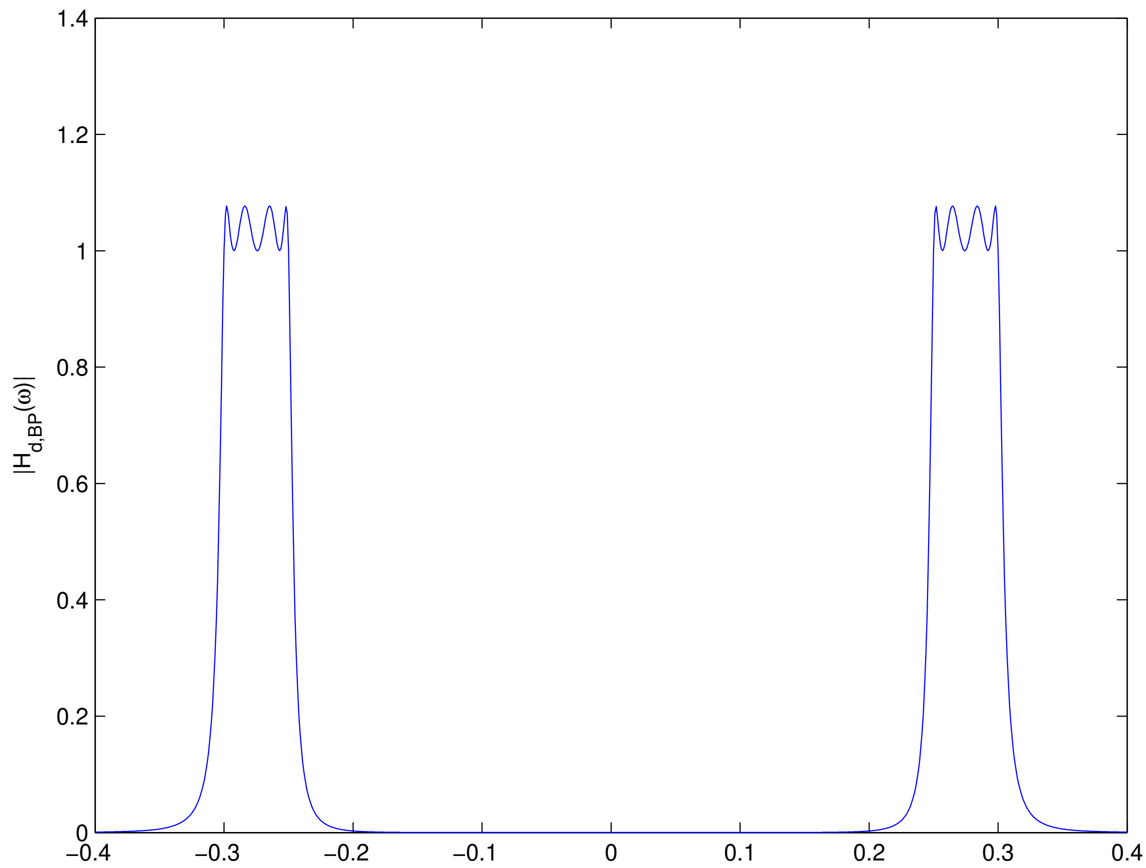

Figure 4.4: The magnitude response of the bandpass digital filter designed to meet the given specifications

given by

$$h_l(n) = \frac{\sin(n\omega_l)}{n\pi} w(n), \tag{2.1}$$

where $w(n)$ is the Kaiser window obtained from the design specifications.

B.4.2. The Kaiser Window

The Kaiser window is defined as

$$\begin{aligned} w(n) &= \frac{I_0\left[\beta N\sqrt{1-\left(\frac{n}{N}\right)^2}\right]}{I_0(\beta N)}, \qquad -N \le n \le N, \qquad \beta > 0 \\ &= 0 \qquad\qquad\qquad\qquad\qquad \text{otherwise}, \end{aligned} \tag{2.2}$$

where $I_0(x)$ is the modified Bessel function of the first kind of order zero in x and β and N are the window shaping factors. In the following, we find β and N using the design parameters in section 2.1.

1. N is chosen according to
$$N \ge \frac{A-8}{4.57\Delta\omega}, \tag{1.1}$$
where $A = -20\log_{10}\delta$. Substituting the appropriate values from the design specifications, we obtain $A = 16.4782$ and $N \ge 48$.

2. β is chosen according to

$$\beta N = \begin{cases} 0.1102(A-8.7) & A > 50 \\ 0.5849(A-21)^{0.4} + 0.07886(A-21) & 21 \le A \le 50 \\ 0 & A < 21 \end{cases} \tag{2.1}$$

In our design, we have $A = 16.4782 < 21$. Hence, from (2.1) we obtain $\beta = 0$.

3. We choose $N = 100$, to ensure the desired low pass filter response. Substituting in (2.2) gives us the rectangular window

$$\begin{aligned} w(n) &= 1, -100 \leq n \leq 100 \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{3.1}$$

From (2.1) and (3.1), we obtain the desired lowpass filter impulse response

$$\begin{aligned} h_{lp}(n) &= \frac{\sin(\frac{n\pi}{40})}{n\pi} \qquad -100 \leq n \leq 100 \\ &= 0, \qquad \text{otherwise} \end{aligned} \tag{3.2}$$

The magnitude response of the filter in (3.2) is shown in Figure 5.

B.4.3. The FIR Bandpass Filter

The centre of the passband of the desired bandpass filter was found to be $\omega_c = 0.275\pi$ in Section 2.1. The impulse response of the desired bandpass filter is obtained from the impulse response of the corresponding lowpass filter as

$$h_{bp}(n) = 2h_{lp}(n)cos(n\omega_c) \tag{3.3}$$

Thus, from (3.2), we obtain

$$\begin{aligned} h_{bp}(n) &= \frac{2\sin(\frac{n\pi}{40})\cos(\frac{11n\pi}{40})}{n\pi} \qquad -100 \leq n \leq 100 \\ &= 0, \qquad \text{otherwise} \end{aligned} \tag{3.4}$$

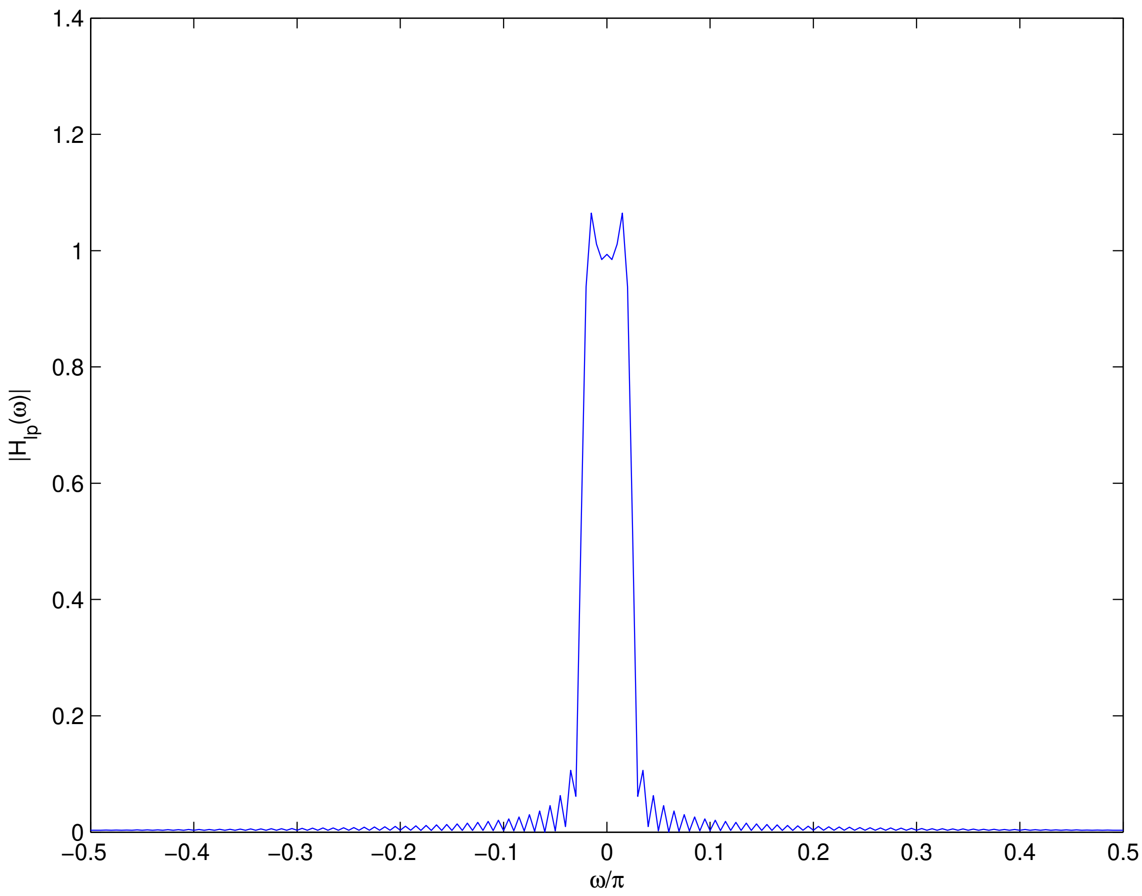

Figure 3.1: The magnitude response of the FIR lowpass digital filter designed to meet the given specifications

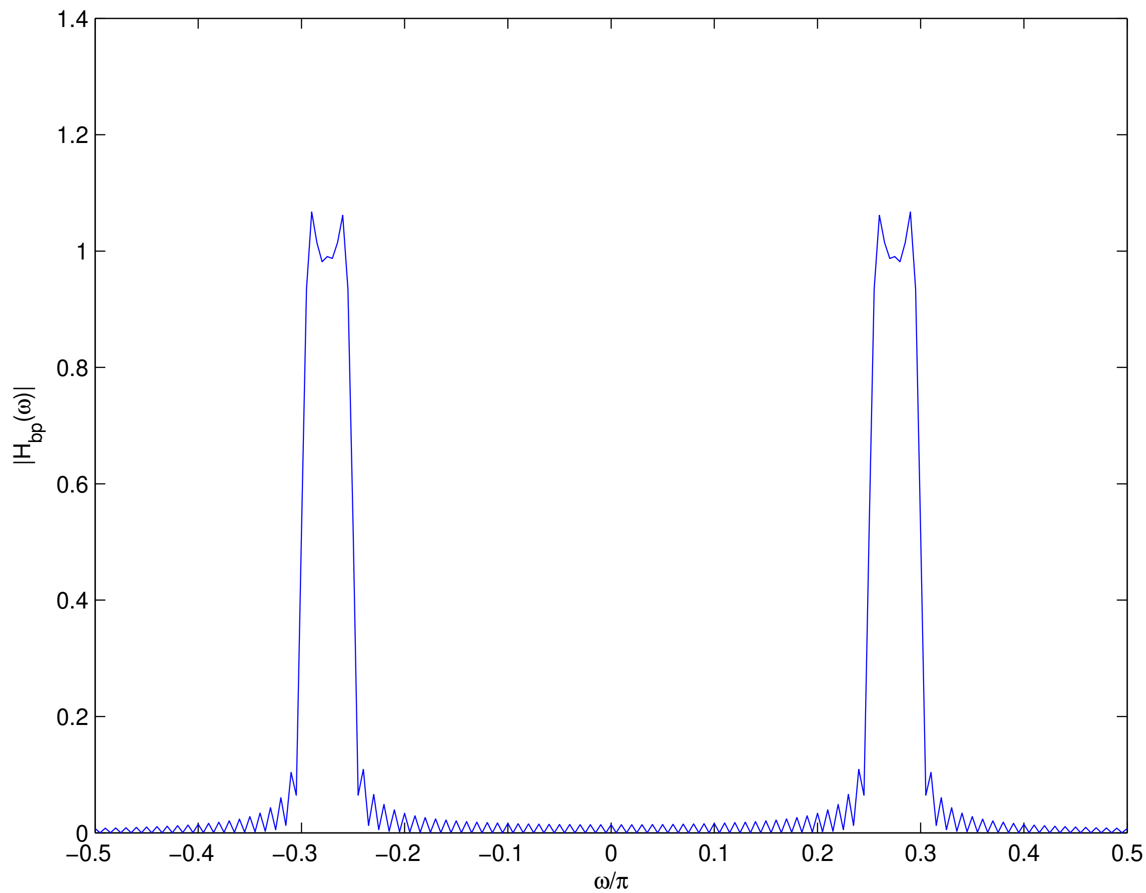

Figure 3.2: The magnitude response of the FIR bandpass digital filter designed to meet the given specifications

The magnitude response of the FIR bandpass filter designed to meet the given specifications is plotted in Fig. 3.2.