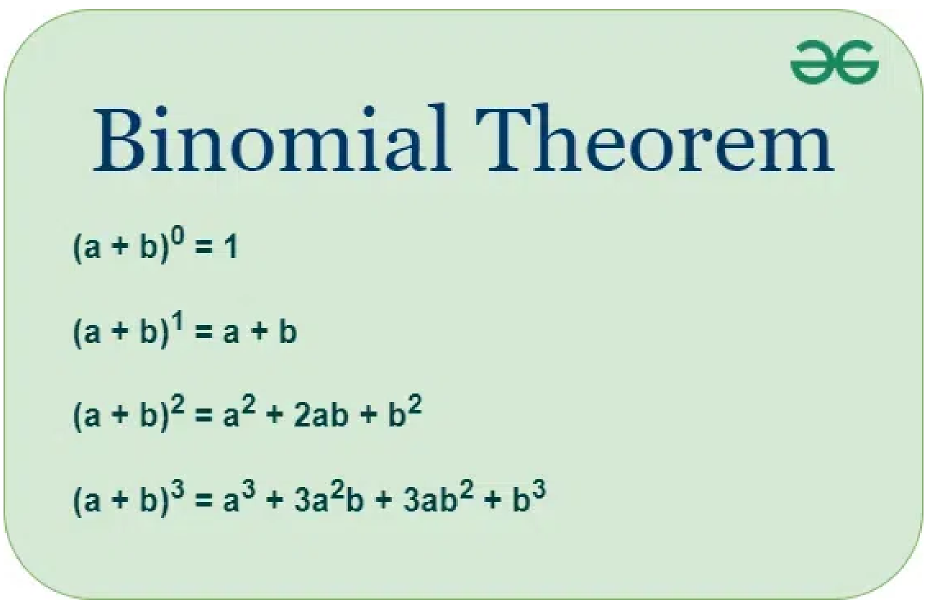
Binomial Theorem

Last Updated : 02 Jan, 2025

Binomial theorem is a fundamental principle in algebra that describes the algebraic expansion of powers of a binomial. According to this theorem, the expression $(a + b)^n$ where a and b are any numbers and n is a non-negative integer. It can be expanded into the sum of terms involving powers of a and b.

Binomial theorem is used to find the expansion of **two terms** hence it is called the Binomial Theorem.



Binomial Expansion

Binomial theorem is used to solve binomial expressions simply. This theorem was first used somewhere around 400 BC by Euclid, a famous Greek mathematician.

It gives an expression to calculate the expansion of algebraic expression

Binomial Theorem Statement

Binomial theorem for the expansion of $(a+b)^n$ is stated as,

$$(a + b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1}\, b^1 + {}^nC_2\, a^{n-2}\, b^2 + ... + {}^nC_r\, a^{n-r}\, b^r + ... + {}^nC_n\, a^0 b^n$$

where $n > 0$ and the ${}^nC_k$ is the binomial coefficient.

Example: Find the expansion of $(x + 5)^6$ using the binomial theorem.

Solution:

We know that here,

$$(a + b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1}\, b^1 + {}^nC_2\, a^{n-2}\, b^2 + ... + {}^nC_r\, a^{n-r}\, b^r + ... + {}^nC_n\, a^0 b^n$$

Thus, $(x + 5)^6 = {}^6C_0\, x^6 5^0 + {}^6C_1\, x^{6-1} 5^1 + {}^6C_2\, x^{6-2} 5^2 + {}^6C_3\, x^{6-3} 5^3 + {}^6C_4\, x^{6-4} 5^4 + {}^6C_5\, x^{6-5} 5^5 + {}^6C_6\, x^{6-6} 5^6$

$= {}^6C_0\, x^6 + {}^6C_1\, x^5 5 + {}^6C_2\, x^4 5^2 + {}^6C_3\, x^3 5^3 + {}^6C_4\, x^2 5^4 + {}^6C_5\, x^1 5^5 + {}^6C_6\, x^0 5^6$

$= x^6 + 30x^5 + 375x^4 + 2500x^3 + 9375x^2 + 18750x + 15625$

Binomial Expansion Formula

terms of the $(a+b)^n$. The binomial expansion of $(a + b)^n$ can easily be represented with the summation formula.

Binomial Theorem Formula for the expansion of $(a + b)^n$ is,

$$(a + b)^n = \sum^n{}_r {}^nC_r \, a^{n-r} \, b^r$$

where,

- **n** is a positive integer,
- **a, b** are real numbers, and **$0 < r \leq n$**

We can easily find the expansion of the various identities such as $(x+y)^7$, $(x+9)^{11}$, and others using the Binomial Theorem Formula. We can also find the expansion of $(ax + by)^n$ using the Binomial Theorem Formula,

Expansion formula for $(ax + by)^n$ is,

$$(ax + by)^n = \sum^n{}_r {}^nC_r \, (ax)^{n-r} \, (by)^r$$

where **$0< r \leq n$.**

Also using combination formula we know that,

Binomial Expansion Formula

$${}^nC_r = n! / [r! \, (n - r)!]$$

Binomial Theorem Proof

We can easily proof Binomial Theorem Expansion using the concept of Principle of Mathematical Induction

Let's take x, a, n ∈ N, and then the Binomial Theorem says that,

$$\mathbf{(x+y)^n = {}^nC_0 \, x^n y^0 + {}^nC_1 \, x^{n-1} y^1 + {}^nC_2 \, x^{n-2} \, y^2 + \ldots + {}^nC_{n-1} \, x^1 y^{n-1} + {}^nC_n \, x^0 y^n}$$

$(x+y)^1 = x+y$ which is true.

For n = 2,

$(x+y)^2 = (x+y)(x+y)$
$\Rightarrow (x+y)^2 = x^2 + xy + xy + y^2$ (using distributive property of multiplication over addition)
$\Rightarrow (x+y)^2 = x^2 + 2xy + y^2$

which is also true.

Thus, theorem is true for n = 1 and n = 2.

Let's take a positive integer k.

$(x+y)^k = {}^kC_0\, x^k y^0 + {}^kC_1\, x^{k-1}y^1 + {}^kC_2\, x^{k-2}\, y^2 + \dots + {}^kC_{k-1}\, x^1 y^{k-1} + {}^kC_k\, x^0 y^k$

Now consider the expansion for n = k + 1

$(x+y)^{k+1} = (x+y)(x+y)^k$

$\Rightarrow (x+y)^{k+1} = (x+y)(x^k + {}^kC_1\, x^{k-1}y^1 + {}^kC_2\, x^{k-2}\, y^2 + \dots + {}^kC_r\, x^{k-r}y^r + \dots + y^k)$

$\Rightarrow (x+y)^{k+1} = x^{k+1} + (1 + {}^kC_1)x^k y + ({}^kC_1 + {}^kC_2)\, x^{k-1}y^2 + \dots + ({}^kC_{r-1} + {}^kC_r)\, x^{k-r+1}y^r + \dots + ({}^kC_{k-1} + 1)\, xy^k + yk^{+1}$

$\Rightarrow (x+y)^{k+1} = x^{k+1} + {}^{k+1}C_1 x^k y + {}^{k+1}C_2\, x^{k-1}y^2 + \dots + {}^{k+1}C_r\, x^{k-r+1}y^r + \dots + {}^{k+1}C_k\, xy^k + y^{k+1}$

[As we know, ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$]

This result is true for n = k+1.

Thus, by concept of mathematical induction, the result is true for all positive integers 'n'. Proved.

Binomial Expansion

Binomial Theorem is used to expand the algebraic identity $(x + y)^n$. Hence it is also called the binomial expansion. The binomial expansion of $(x + y)^n$ with the help of the binomial theorem is,

(x+y)n = nC0 xny0 + nC1 xn-1y1 + nC2 xn-2 y2 + … + nCn-1 x1yn-1 + nCnx0yn

Solution:

$(x+y)^2 = {}^2C_0\, x^2y^0 + {}^2C_1\, x^{2-1}y^1 + {}^2C_2\, x^{2-2}\, y^2$
$\Rightarrow (x+y)^2 = x^2 + 2xy + y^2$
$And\ (x+y)^3 = {}^3C_0\, x^3y^0 + {}^3C_1\, x^{3-1}y^1 + {}^3C_2\, x^{3-2}\, y^2 + {}^3C_3\, x^{3-3}\, y^3$
$\Rightarrow (x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
$\Rightarrow (x+y)^3 = x^3 + 3xy(x+y) + y^3$

Properties of Binomial Theorem

There are various properties related to the binomial theorem, some of those properties are as follows:

- **Number of Terms:** In the binomial Expansion of $(x + y)^n$ using the binomial theorem, there are n+1 terms and coefficients.
- First and last terms in the binomial expansion of $(x + y)^n$ are x^n and y^n, respectively.
- **General Term:** In the binomial expansion of $(x + y)^n$ the general term can be represented as T(r + 1) and is given by $T(r + 1) = {}^nC_r \times x^{(n-r)} \times y^r$.
- **Pascal's Triangle:** The binomial coefficients in the expansion are arranged in Pascal's triangle which is the pattern of number where each number is the sum of the two numbers above it.
- **Specific Values:** When n is a non-negative integer, the expansion simplifies for specific values of n:
    - $(a + b)^0 = 1$
    - $(a + b)^1 = a + b$
    - $(a + b)^2 = a^2 + 2ab + b^2$
    - $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
    - and so on

- If n is even, then in the expansion of $(x + y)^n$, the middle term is ((n/2) + 1). If n is odd, then the middle terms are ((n + 1)/2) and ((n + 3)/2) in the expansion of $(x + y)^n$.

Pascal's Triangle Binomial Expansion

The number associated with the terms of the binomial expansion is called the coefficient of the binomial expansion. These variables can easily be found using Pascal's Triangle or by using combination formulas.

Binomial Theorem Coefficients

After examining the coefficient of the various terms in the expansion of algebraic identities using the binomial theorem, we have observed a trend in the coefficient of the terms of the expansion. The trend is that the coefficient of the terms in the expansion of the binomial terms is directly similar to the rows of the Pascal triangle.

For example, the coefficient of each term in the expansion of the $(x+y)^4$ is directly equivalent to the terms in the 4th row of the Pascal Triangle.

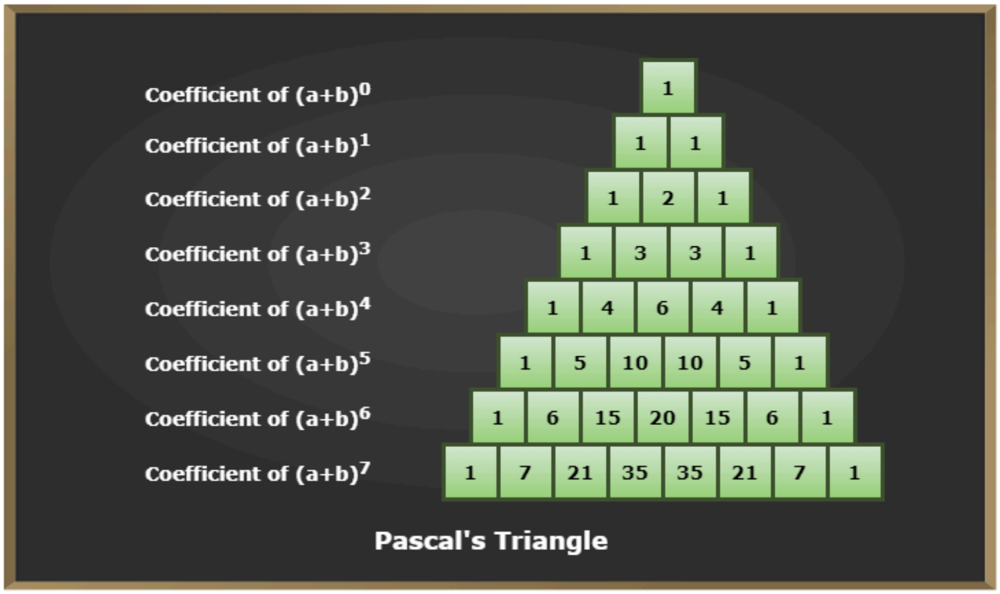
Now, let's learn about Pascal Triangle in detail.

Pascal Triangle

Pascal Triangle is an alternative method of the calculation of coefficients that come in binomial expansions, using a diagram rather than algebraic methods.

In the diagram shown below, it is noticed that each number in the triangle is the sum of the two directly above it. This pattern continues indefinitely to obtain coefficients of any index of the binomial expression.

When we observe the pattern of the coefficients of the expansion $(a + b)^n$, **Pascal's triangle** for the pattern of the coefficients of the expansion (a +



Learn More: Pascal's Triangle

Thus, from the above diagram, the expansion of small powers of n (e.g. n=0, 1, 2, 3, 4, 5, 6, 7) can be calculated as:

- $(a + b)^0 = 1$
- $(a + b)^1 = a + b$
- $(a + b)^2 = a^2 + 2ab + b^2$
- $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
- $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
- $(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$
- $(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$
- $(a + b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$

In this way, the expansion from $(a + b)^0$ to $(a + b)^7$ is obtained by the application of Pascal's Triangle. But finding $(a + b)^{15}$ is really a long process using Pascal's triangle. So, here we use combinations formulas.

Combinations

The combination formula used for choosing **r** objects out of **n** total objects is widely used in the binomial expansion and it is defined as,

> ***^nC_r = n! / [r! (n - r)!]***

Also, some common combination formulas used in the Binomial Expansion are,

- $^nC_n = {}^nC_0 = 1$
- $^nC_1 = {}^nC_{n-1} = n$
- $^nC_r = {}^nC_{r-1}$

Some properties of combination which are widely used in simplifying binomial expansion are,

- $C_1 + C_2 + C_3 + C_4 + ...C_n = 2^n$
- $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$
- $C_0 - C_1 + C_2 - C_3 + C_4 - C_5 + ... = 0$
- $C_1 + 2C_2 + 3C_3 + 4C_4 + ...nC_n = n2^{n-1}$
- $C_1 - 2C_2 + 3C_3 - 4C_4 + ...(-1)^n nC_n = 0$
- $C_1^2 + C_2^2 + C_3^2 + C_4^2 + ...C_n^2 = (2n)! / (n!)^2$

Learn more: Permutations and Combinations

Binomial Expansion for Negative Exponent

Binomial theorem is also used for finding the expansion of the identities which have negative exponents. The coefficients terms in the negative expansion are similar in magnitude to the terms in the corresponding positive exponent.

Some of the simplified expansions of the negative exponents which are widely used are,

- $(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + ...$

These expansions are easily proved using binomial expansion and replacing (+) with (-)

Important Terms of Binomial Theorem

Various terms related to Binomial Expansion that is used widely include,

General Term of Expansion

The general term of the binomial expansion signifies a term that produces all the terms of the binomial expansion by simply replacing the value of one component of the term.

For example in the binomial expansion of $(x+y)^n$ the general term is,

> $T_{r+1} = {}^nC_r\, x^{n-r} y^r$
> *Where,* $0 \leq r \leq n$

Substituting the values of r in the above term we get all the terms of the expansion. We observe that r takes n+1 values which is also true as there are n+1 terms in the expansion of $(x+y)^n$

Middle Term of Expansion

We know that the total number of terms in the expansion of $(x + y)^n$ is n + 1. And so the middle term in the binomial expansion of the $(x+y)^n$ depends on the value of n. The value of n can be either even or odd which decides the number/numbers and values of the middle term.

- If n is even then we have odd (n+1) terms in the expansion of $(x+y)^n$. Thus, there is one middle term, in this case, and the middle term is $(n/2 + 1)^{th}$ term.
- If n is odd then we have even (n+1) terms in the expansion of $(x+y)^n$. Thus, there are two middle terms, in this case, and the middle terms are

Identifying a Particular Term in Expansion

Any particular term can easily be identified in the expansion of $(x+y)^n$. We follow two steps to get a particular term in the expansion of $(x+y)^n$.

For example, if we have to find the term p^{th} term in the expansion of $(x+y)^n$.

Step 1: Find the general term in the expansion of $(x+y)^n$. The general term is,

$$T_{r+1} = {}^nC_r\, x^{n-r} y^r$$

Step 2: Then substitute the value of 'r' for the value of the term which we have to find. In this case, r = p and then simplify to get the p^{th} term.

Example: Find the 6^{th} term in the expansion of $(3x + 4)^8$

Solution:

The general term in the expansion of $(3x + 4)^8$ is,
$T_{r+1} = {}^8C_r(3x)^{8-r}(4)^r$
For 6^{th} term r = 5
$T_6 = T_{5+1} = {}^8C_5(3x)^{8-5}(4)^5$
Simplifying the above term we get our answer,
$T_6 = {}^8C_5(3x)^{8-5}(4)^5$
$= (8.7.6/3.2.1)(27)(1024)x^3$
$= 1593648x^3$

Term Independent of X in Expansion

The term independent of 'x' can easily be identified in the expansion of $(x+y)^n$. We follow two steps to get a particular term in the expansion of $(x+y)^n$

Step 1: Find the general term in the expansion of $(x+y)^n$. the general term is,

$$T_{r+1} = {}^nC_r\, x^{n-r} y^r$$

Step 2: Then substitute the value of 'r' as r = n to get rid of the x term.

Example: Find the term independent of x in the expansion of $(2x + 1)^8$

Solution:

The general term in the expansion of $(2x + 1)^8$ is,
$T_{r+1} = {}^8C_r(2x)^{8-r}(1)^r$
For the term independent of x
$8-r = 0$
$r = 8$
$T_9 = T_{8+1} = {}^8C_8(2x)^{8-8}(1)^8$
Simplifying the above term we get our answer,
$T_9 = {}^8C_8(2x)^{8-8}(1)^8$
$= (1)(1)(1)$
$= 1$

Numerically Greatest Term of Expansion

We can easily find the numerically greatest term in the expansion of $(1+x)^n$. The formula to find the greatest term in the expansion of $(1+x)^n$ is,

$$[(n+1)|x|]/(1+|x|)$$

While using this formula we have to make sure that the expansion is in $(1+x)^n$ form only, and |x| gives only the numerical value.

Applications of Binomial Theorem

- Finding Remainder in the division of very large numbers.
- Finding Last Digits of a Number
- Checking the Divisibility

Finding Remainder using Binomial Theorem

This can easily be understood with the help of the following example.

Example: Find the remainder when 2^{97} is divided by 15

Solution:

$(2^{97} / 15) = [2(2^4)^{24} / 15)]$

$= [2(15 + 1)^{24} / 15]$

As each term of the expansion of $(1 + 15)^{24}$ contains 15 except the first term which is only 1.

Thus, Remainder when 2^{97} is divided by 15 is 1.

Finding the Last Digits of a Number

How to find the last digit of an expansion can be understood using the example,

Example: Find the last digit of $(7)^{10}$

Solution:

$(7)^{10} = (49)^5 = (50-1)^5$

$(50-1)^5 = {}^5C_0\ (50)^5 - {}^5C_1\ (50)^4 + 5C_2\ (50)3 - {}^5C_3\ (50)^2 + {}^5C_4\ (50) - {}^5C_5$

$= {}^5C_0\ (50)^5 - {}^5C_1\ (50)^4 + 5C_2\ (50)3 - {}^5C_3\ (50)^2 + 5(50) - 1$

$= {}^5C_0\ (50)^5 - {}^5C_1\ (50)^4 + 5C_2\ (50)3 - {}^5C_3\ (50)^2 + 249$

A multiple of 50 + 249 = 50K + 249

Thus, the last digit is 9.

Binomial Theorem for any index, including non-integer and negative indices, generalizes the familiar binomial expansion that applies to positive integer exponents.

This generalized form involves the use of binomial coefficients that are defined for any real number index using the concept of factorial functions extended to the gamma function for non-integer values.

Multinomial Theorem

We know that the binomial theorem expansion of $(x + a)^n$ is,

> $(x+a)^n = {}^n\sum_r {}^nC_r\, x^{n-r} a^r$
> *Where* $n \in N$

We can generalize this result to get the expansion of,

$(x_1 + x_2 + \dots + x_k)^n = \sum_{(r1 + r2 + \dots + rk = n)} [n! / r_1!r_2!\dots r_k!]\, x_1^{r1}\, x_2^{r2} \dots x_k r^k$

The general term in the above expansion is

$[n! / r_1!r_2!\dots r_k!]\, x_1^{r1}\, x_2^{r2} \dots x_k r^k$

The number of terms in the above expansion is equal to the number of non-negative integral solutions of the equation $r_1 + r_2 + \dots + r_k = n$

Each solution of this equation gives a term in the above multinomial expansion. The total number of solutions can be given by ${}^{n+k-1}C_{k-1}$.

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Binomial Distribution and Binomial Coefficients

The binomial distribution models the number of successes in a fixed number of independent and identical Bernoulli trials (experiments with two possible outcomes: success or failure). It's characterized by two parameters: nn (the number of trials) and pp (the probability of success in each trial).

Hypergeometric Distribution

The hypergeometric distribution models the probability of obtaining a specific number of successes in a sample of a fixed size drawn without replacement from a finite population containing a known number of successes and failures.

Binomial Theorem Class 11

The Binomial Theorem is a fundamental theorem in algebra that describes the algebraic expansion of powers of a binomial. For Class 11 students, it is an essential topic covered under the curriculum and plays a significant role in understanding higher mathematics.

Also Check:

- *Binomial Theorem Class 11 NCERT Solutions*
- *Binomial Theorem Class 11 Notes*

Problems on Binomial Theorem

Example 1: Expand the binomial expression $(2x + 3y)^2$.

Solution:

$(2x + 3y)^2$

$= 4x^2 + 12xy + 9y^2$

Solution:

Put 1 - x = y

Then,

$(1 - x + x^2)^4 = (y + x^2)^4$

$= {}^4C_0 y^4 (x^2)^0 + {}^4C_1 y^3 (x^2)^1 + {}^4C_2 y^2 (x^2)^2 + {}^4C_3 y (x^2)^3 + {}^4C_4 (x^2)^4$

$= y^4 + 4y^3x^2 + 6y^2x^4 + 4yx^6 + x^8$

$= (1 - x)^4 + 4(1 - x)^3x^2 + 6(1 - x)^2x^4 + 4(1 - x)x^6 + x^8$

$= 1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8$

Example 3: Find the 4th term from the end in the expansion of $((x^3/2) - (2/x^2))^8$.

Solution:

Since r^{th} term from the end in the expansion of $(a + b)^n$ is

$(n - r + 2)^{th}$ term from the beginning.

Therefore, 4^{th} term from the end is 8 - 4 + 2,

i.e., 6th term from the beginning, which is given by

$T_6 = {}^8C_5 (x^3/2)^3 (-2/x^2)^5$

$= {}^8C_3 (x^9/8)(-32/x^{10})$

$= -224/x$

Example 4: Find the middle term (terms) in the expansion of $((p/x) + (x/p))^9$

Solution:

Since the power of binomial is odd. Therefore, we have two middle terms which are 5th and 6th terms. These are given by

$T_5 = {}^9C_4 (p/x)^5 (x/p)^4$

$= {}^9C_4 (p/x)$

$= {}^9C_5(x/p)$

$= 126(x/p)$

Example 5: Find the 3^{th} term in the expansion of $(2x + 1)^4$

Solution:

The general term in the expansion of $(2x + 1)^4$ is,

$T_{r+1} = {}^4C_r(2x)^{4-r}(1)^r$

For 3^{th} term r = 2

$T_3 = T_{2+1} = {}^4C_2(2x)^{4-2}(1)^2$

Simplifying the above term we get our answer,

$T_3 = {}^4C_2(2x)^{4-2}(1)^2$

$= (6)(4)x^2$

$= 24x^2$

Example 6: Find the term independent of x in the expansion of $(3x + 2)^8$

Solution:

The general term in the expansion of $(2x + 1)^8$ is,

$T_{r+1} = {}^8C_r(3x)^{8-r}(2)^r$

For the term independent of x

$8-r = 0$

$r = 8$

$T_9 = T_{8+1} = {}^8C_8(3x)^{8-8}(2)^8$

Simplifying the above term we get our answer,

$T_9 = {}^8C_8(3x)^{8-8}(2)^8$

$= (1)(1)(256)$

$= 256$

We know that,

$(x + y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y^1 + {}^nC_2\, x^{n-2}\, y^2 + \dots + {}^nC_{n-1}\, x^1 y^{n-1} + {}^nC_n\, x^0 y^n$

$(x^3 + 1)^3 = {}^3C_0\, (x^3)^3 1^0 + {}^3C_1\, (x^3)^2 1^1 + {}^3C_2\, (x^3)^1 1^2 + {}^3C_3\, (x^3)^0 1^3$

$(x^3 + 1)^3 = (1)x^9 + (3)x^6 + (3)x^3 + (1)$

$(x^3 + 1)^3 = x^9 + 3x^6 + 3x^3 + 1$

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Pascal's Triangle

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