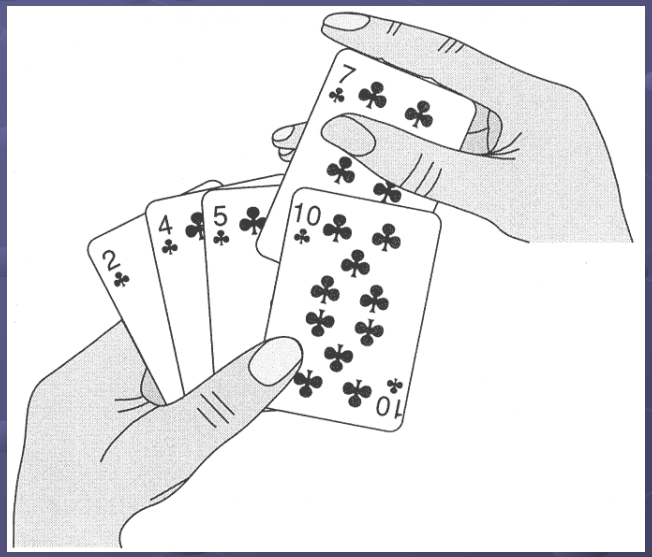
Getting Started

Insertion Sort

- **Sorting problem**
  - **Input: A sequence of n numbers $\langle a_1, a_2, \ldots, a_n \rangle$.**
  - **Output: A permutation (reordering) $\langle a'_1, a'_2, \ldots, a'_n \rangle$ of the input sequence such that $a'_1 \le a'_2 \le \ldots \le a'_n$.**

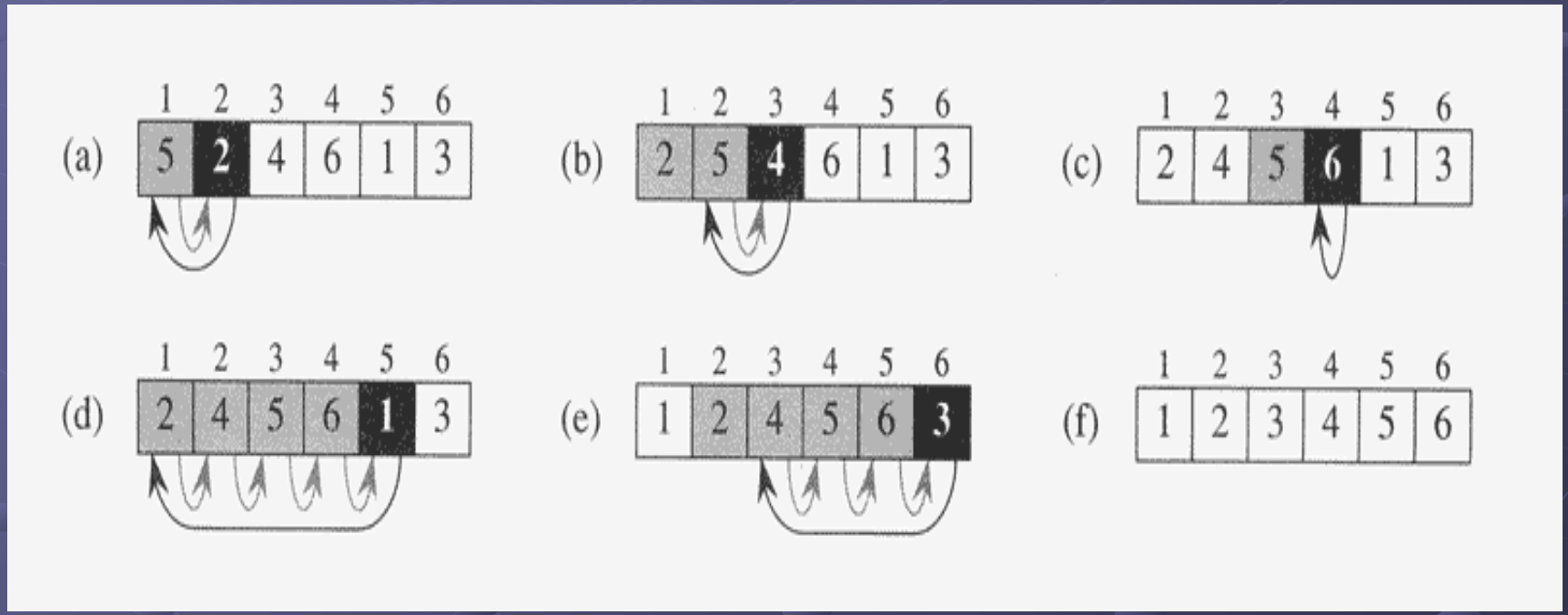


- ***Keys*: The numbers that we wish to sort.**

Pseudo Code

```
Insertion-Sort(A)
1 for j  = 2 to A.length
2     key = A[j]
3     // Insert A[j] into the sorted sequence A[1 .. j-1]
4     i  = j - 1
5     while i > 0 and A[i] > key
6           A[i+1] = A[i]
7           i  = i - 1
8     A[i +1] = key
```

Insertion Sort



Loop Invariant and the Correctness

- **Sorted in place**
  - **The numbers are rearranged within the array *A*, with at most a constant number of them stored outside the array at any time.**
- ***Loop Invariant***
  - **At the start of each iteration of the for loop of lines 1-8, the subarray A[1..j-1] consists of the elements originally in A[1..j-1], but in sorted order.**
- **Loop invariant is used to help us understand why an algorithm is correct.**

Loop Invariant and the Correctness

- We must show three things about a loop invariant:
  - ***Initialization***: It is true prior to the first iteration of the loop.
  - ***Maintenance***: If it is true before an iteration of the loop, it remains true before the next iteration.
  - ***Termination***: When the loop terminates, the invariant gives us a useful property that helps show that the algorithm is correct.

Analyzing Algorithms

- Analyzing algorithms
  - Predicting the resources that the algorithm requires.
- *Resources*
  - Memory, communication, bandwidth, logic gate, time.
- Assumption
  - One processor, random-access machine (RAM) model
- The best notion for *input size* depends on the problem being studied.
  - *Number of items in the input* : Sorting.
  - *Total number of bits* : Multiplying two integers.

Analysis of Insertion Sort

- ***Two numbers to represent the size*** : Graph algorithms.

☞ The ***running time*** of an algorithm

- The number of primitive operations or "steps" executed.

☞ Time to execute each line of the pseudo code: c_i.

☞ The number of times to execute the while loop test: t_j.

☞ Even for inputs of a given size, an algorithm's running time may depend on which input of that size is given.

- The ***best case*** *v.s.* the ***worst case*** running time.

Analysis of Insertion Sort

INSERTION-SORT(A)	*cost*	*times*
1 **for** $j = 2$ **to** $A.length$	c_1	n
2 $key = A[j]$	c_2	$n-1$
3 // Insert $A[j]$ into the sorted sequence $A[1..j-1]$.	0	$n-1$
4 $i = j-1$	c_4	$n-1$
5 **while** $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$
6 $A[i+1] = A[i]$	c_6	$\sum_{j=2}^{n} (t_j - 1)$
7 $i = i-1$	c_7	$\sum_{j=2}^{n} (t_j - 1)$
8 $A[i+1] = key$	c_8	$n-1$

Analysis of Insertion Sort

- **Running time**

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j +$$

$$c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1)$$

- **The best case running time: Sorted, $t_j = 1$**

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5(n-1) + c_8(n-1)$$

$$= (c_1 + c_2 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8)$$

Analysis of Insertion Sort

- **The worst case: Reversed ordered, $t_j = j$.**

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5\left(\frac{n(n+1)}{2} - 1\right) +$$

$$c_6\left(\frac{n(n-1)}{2}\right) + c_7\left(\frac{n(n-1)}{2}\right) + c_8(n-1)$$

$$= \left(\frac{c_5 + c_6 + c_7}{2}\right)n^2 - \left(c_1 + c_2 + c_4 + \frac{c_5 - c_6 - c_7}{2} + c_8\right)n$$

$$-(c_2 + c_4 + c_5 + c_8)$$

$$T(n) = an^2 + bn + c$$

Analyzing Algorithms

- We usually concentrate on the worst case. Why?
  - The worst case running time of an algorithm is an upper bound on the running for any input.
  - For some algorithms, the worst case occurs fairly often.
  - The "average case" is often roughly as bad as the worst case.
- *Order of growth* (*rate of growth*)
  - A simplifying abstraction
  - Only consider the leading term of a formula.
  - Also ignores the leading term's constant coefficient

$$T(n) = an^2 + bn + c = \quad (n^2)$$

Designing Algorithms

- Many ways to design algorithms
  - *Incremental approach*: insertion sort.
  - *Divide-and-Conquer*: merge sort.
- *Divide-and-Conquer approach*
  - Paradigm
    - *Divide*: Subdivide the problem into smaller subproblems.
    - *Conquer*: If subproblem size small enough, solve it straightforward.
    - *Combine*: Integrate solutions of subproblems into the solution of the original problem.
  - Recursive structure

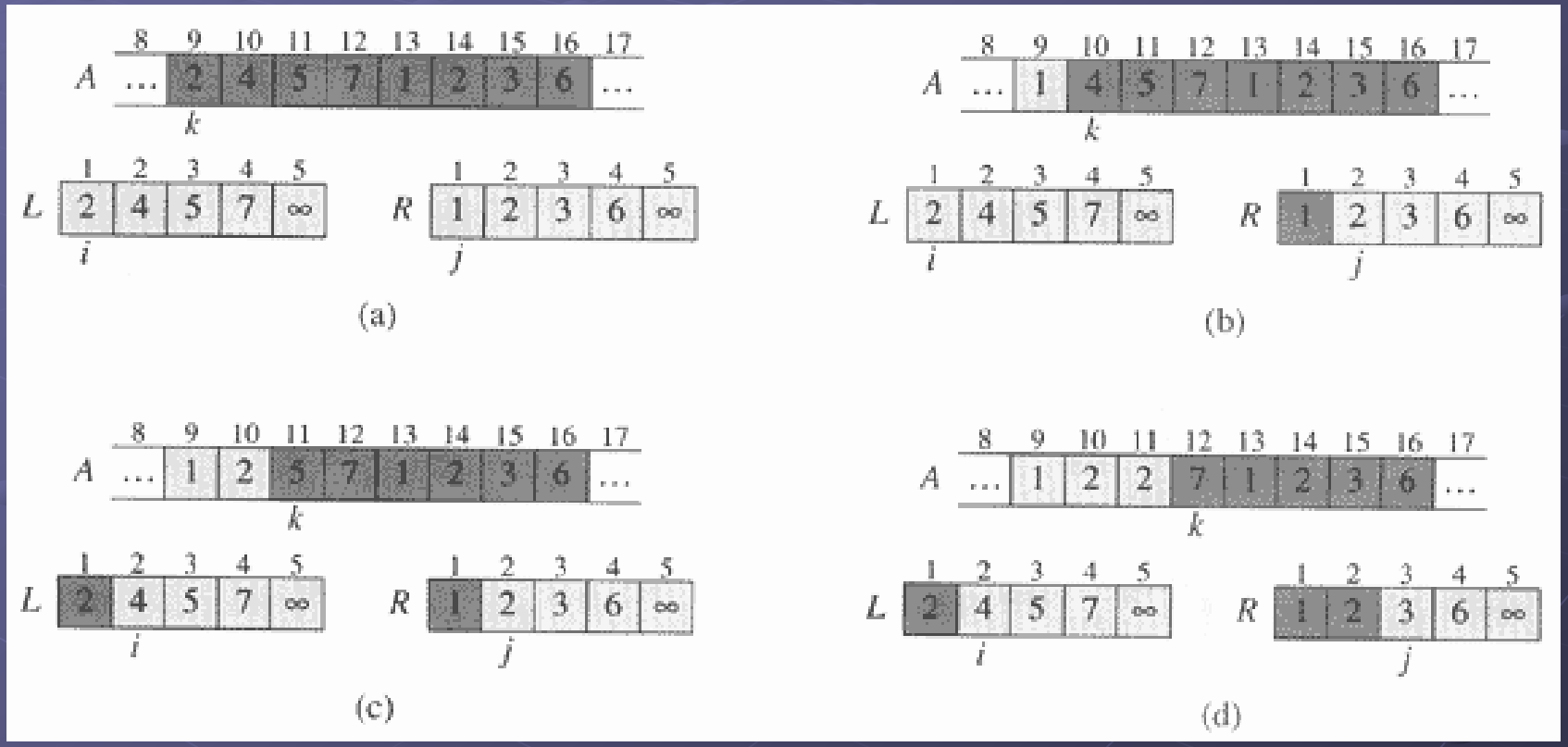
Merge Sort

- Follow the divide-and-conquer paradigm:
  - Divide: Divide the *n*-element sequence to be sorted into two subsequences of *n*/2 elements each.
  - Conquer: Sort the two subsequences recursively using merge sort.
  - Combine: Merge the two sorted subsequences to produce a sorted sequence.
- Key operation
  - The merging of two sorted sequences.
- *Sentinel card*

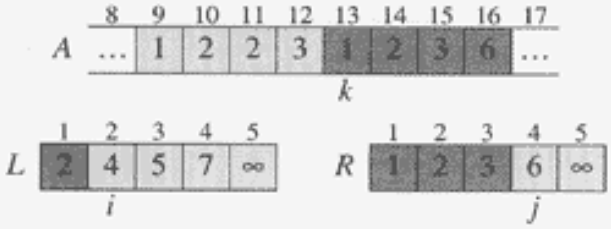
Merge

```
MERGE(A, p, q, r)
 1  n1 = q − p + 1
 2  n2 = r − q
 3  let L[1 .. n1 + 1] and R[1 .. n2 + 1] be new arrays
 4  for i = 1 to n1
 5      L[i] = A[p + i − 1]
 6  for j = 1 to n2
 7      R[j] = A[q + j]
 8  L[n1 + 1] = ∞
 9  R[n2 + 1] = ∞
10  i = 1
11  j = 1
12  for k = p to r
13      if L[i] ≤ R[j]
14          A[k] = L[i]
15          i = i + 1
16      else A[k] = R[j]
17          j = j + 1
```

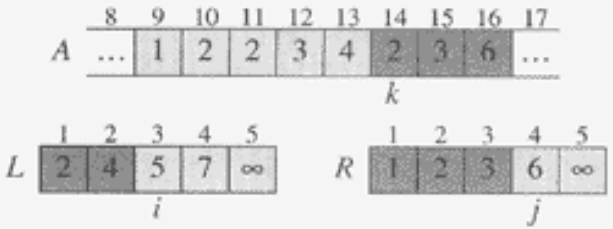
Merge


A … 2 4 5 7 1 2 3 6 …
k
L 2 4 5 7 ∞
i
R 1 2 3 6 ∞
j
(a)
A … 1 4 5 7 1 2 3 6 …
k
L 2 4 5 7 ∞
i
R 1 2 3 6 ∞
j
(b)
A … 1 2 5 7 1 2 3 6 …
k
L 2 4 5 7 ∞
i
R 1 2 3 6 ∞
j
(c)
A … 1 2 2 7 1 2 3 6 …
k
L 2 4 5 7 ∞
i
R 1 2 3 6 ∞
j
(d)

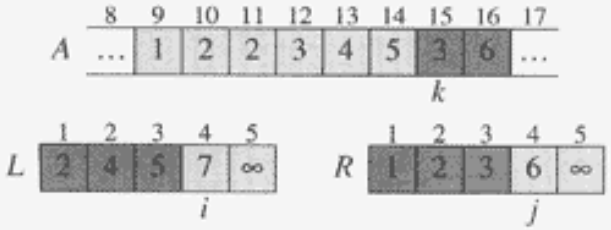
Merge


8 9 10 11 12 13 14 15 16 17
A … 1 2 2 3 1 2 3 6 …
k
1 2 3 4 5
L 2 4 5 7 ∞
i
1 2 3 4 5
R 1 2 3 6 ∞
j

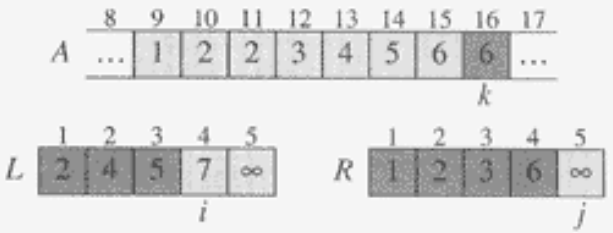
(e)


8 9 10 11 12 13 14 15 16 17
A … 1 2 2 3 4 2 3 6 …
k
1 2 3 4 5
L 2 4 5 7 ∞
i
1 2 3 4 5
R 1 2 3 6 ∞
j

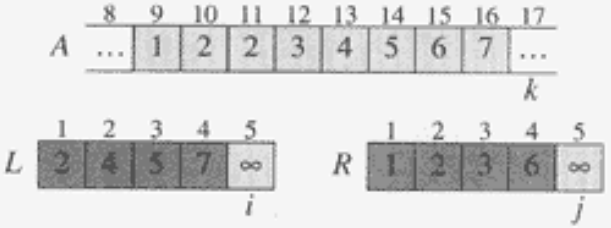
(f)


8 9 10 11 12 13 14 15 16 17
A … 1 2 2 3 4 5 3 6 …
k
1 2 3 4 5
L 2 4 5 7 ∞
i
1 2 3 4 5
R 1 2 3 6 ∞
j

(g)


8 9 10 11 12 13 14 15 16 17
A … 1 2 2 3 4 5 6 6 …
k
1 2 3 4 5
L 2 4 5 7 ∞
i
1 2 3 4 5
R 1 2 3 6 ∞
j

(h)


8 9 10 11 12 13 14 15 16 17
A … 1 2 2 3 4 5 6 7 …
k
1 2 3 4 5
L 2 4 5 7 ∞
i
1 2 3 4 5
R 1 2 3 6 ∞
j

(i)

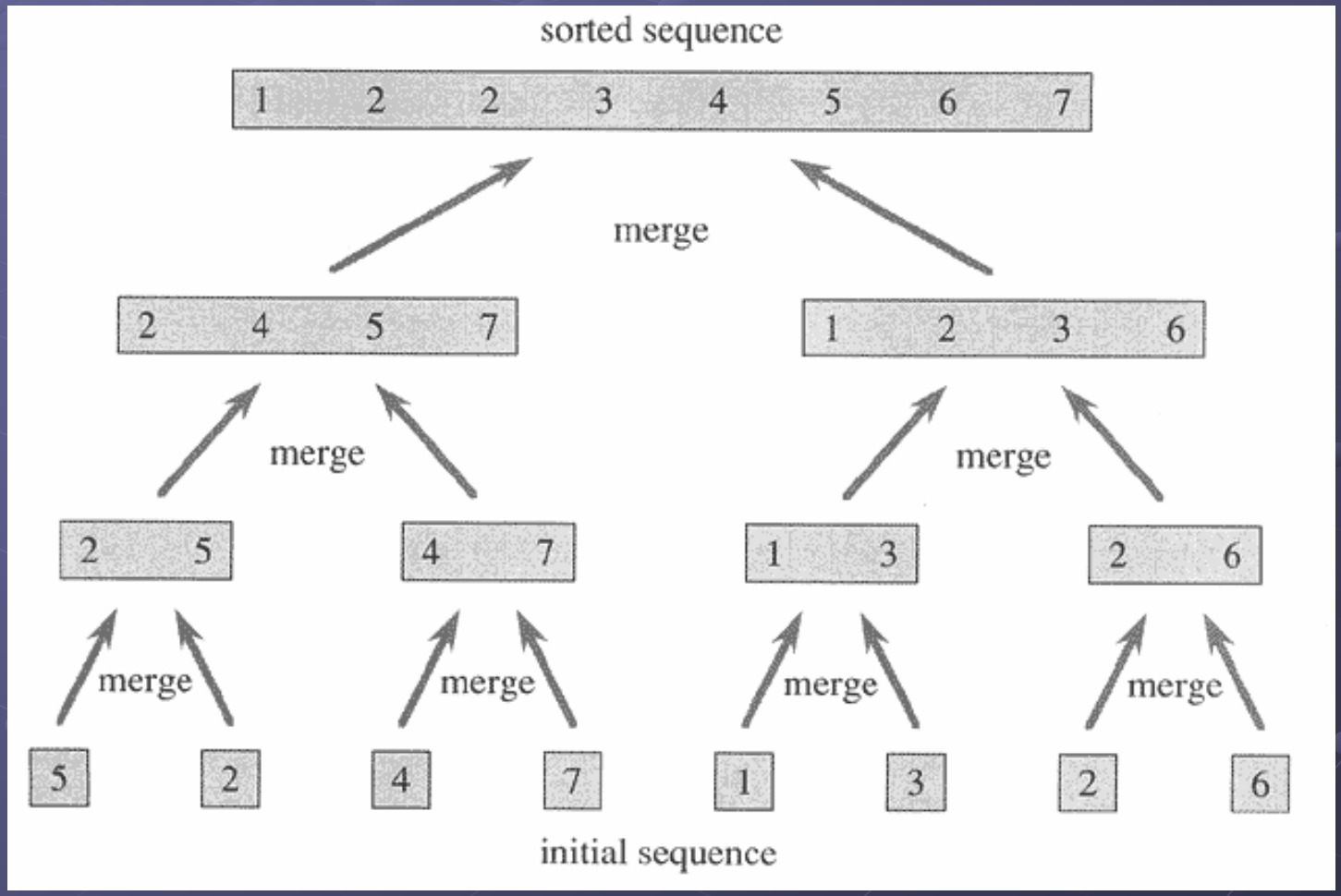
Merge

- Loop invariant
  - At the start of each iteration of the for loop of lines 12-17, the subarray $A[p \,..\, k\text{-}1]$ contains the $k - p$ smallest elements of $L[1 \,..\, n_1\text{+}1]$ and $R[1 \,..\, n_2\text{+}1]$, in sorted order. Moreover, $L[i]$ and $R[j]$ are the smallest elements of their arrays that have not been copied back into A.
  - Initialization
  - Maintenance
  - Termination

Merge Sort

```
MERGE-SORT(A, p, r)
1   if p < r
2       q = ⌊(p + r)/2⌋
3       MERGE-SORT(A, p, q)
4       MERGE-SORT(A, q + 1, r)
5       MERGE(A, p, q, r)
```

Merge Sort


sorted sequence
1 2 2 3 4 5 6 7
merge
2 4 5 7
1 2 3 6
merge
merge
2 5
4 7
1 3
2 6
merge
merge
merge
merge
5
2
4
7
1
3
2
6
initial sequence

Analyzing Divide-and-Conquer Algorithms

- Recurrence equation (or recurrence)
  - When an algorithm contains a recursive call to itself, its running time can often be described by recurrence.

$$T(n) = \begin{cases} \Theta(1) & if\ n \le c \\ aT(n/b) + D(n) + C(n) & otherwise \end{cases}$$

Analysis of Merge Sort

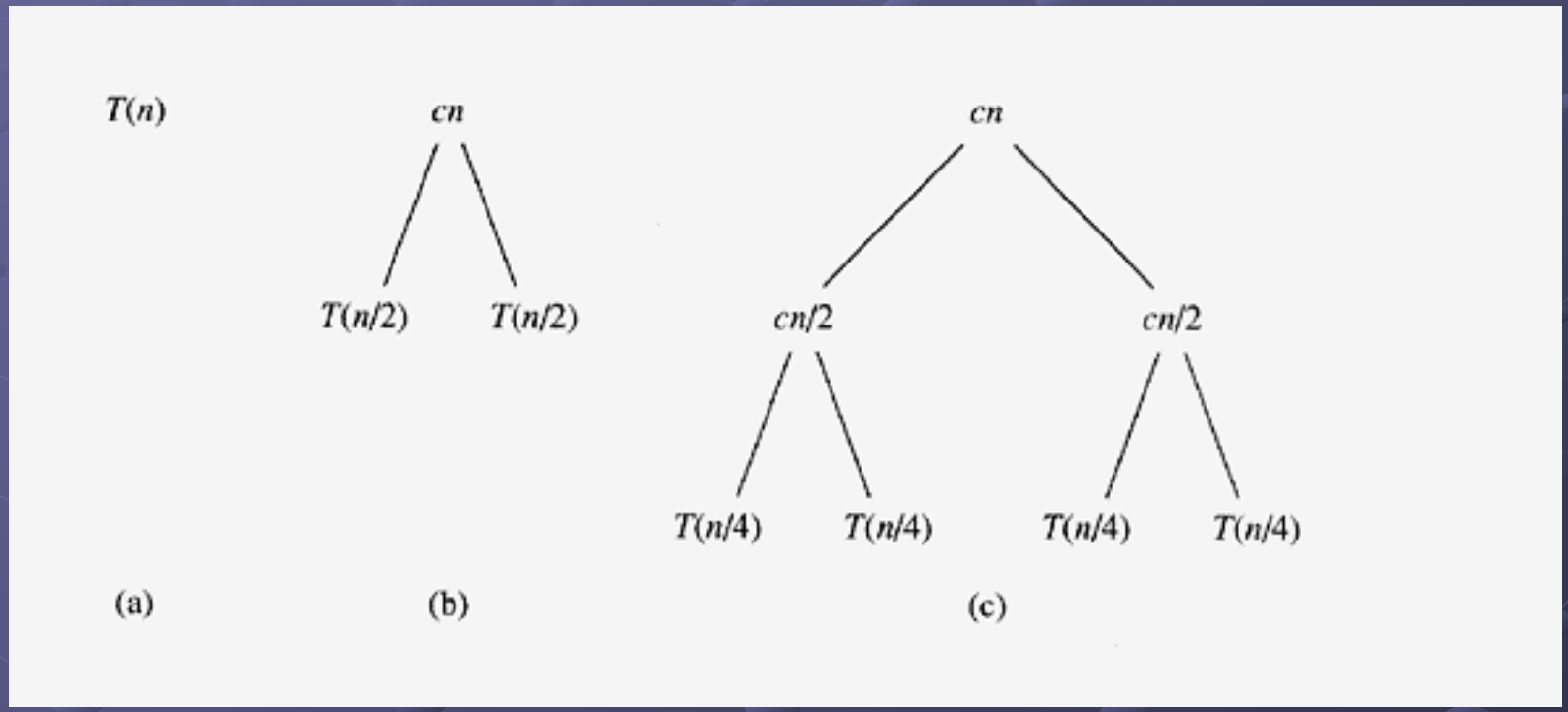
- **Divide: Constant time $D(n) = \Theta(1)$.**
- **Conquer: Recursively solve two sub problems $2T(n/2)$.**
- **Combine: Merge into an n-element array $C(n) = \Theta(n)$.**

$$T(n) = \begin{cases} \Theta(1) & if\ n = 1, \\ 2T(n/2) + \Theta(n) & if\ n > 1. \end{cases}$$

$$T(n) = \begin{cases} c & if\ n = 1, \\ 2T(n/2) + cn & if\ n > 1. \end{cases}$$

$$T(n) = \Theta(n\ lg\ n)$$

Analysis of Merge Sort



Analysis of Merge Sort