Ganna Fagerberg / Statistical methods

REPORT

The detailed R-codes for the problems are given in the attachment to this report, HA2_StatMethods_G.Fagerberg_codes.

<u>Problem 1.</u> Consider a linear regression model:

$$y_i = \beta_0 + \beta_1 x_i^2 + \varepsilon_i, \text{ where } \varepsilon_i \sim N(0, \sigma^2).$$

The design region is $\chi = [0,6]$ and a proposed design is given by

$$\xi_1 = \begin{Bmatrix} 0 & 3 & 6 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{Bmatrix},$$

that is the design weights, w_i, for the three design points are equal ($w_i = \frac{1}{3}$). The number of unknown parameters in the model, p, is $p = 2$.

A) Derive an expression for the standardized matrix for the least square's estimators β_0 and β_1. Calculate the standardized information matrix for the design ξ_1.

<u>1A.1. Derivation of the standardized information matrix</u>

The LS estimator for β, $\hat{\beta}$, is given by **(1)**: $\hat{\beta} = (X^T X) X^T Y$

The variance-covariance matrix is given by **(2)**: $cov(\hat{\beta}) = \sigma^2 (X^T X)^{-1}$

The Fisher's information matrix is the inverse of $cov(\hat{\beta})$, **(3)**:

$$I(\xi) = \frac{1}{\sigma^2} X^T X$$

The standardized information matrix, $M(\xi)$, which gives average information per observation, is given by **(4)**:

$$M(\xi) = \frac{I(\xi)}{N} = \frac{1}{N\sigma^2} X^T X = \frac{1}{N\sigma^2} \sum_{i=1}^{N} x_i x_i^T = \frac{1}{\sigma^2} \sum_{i=1}^{n} \frac{n_i}{N} x_i x_i^T = \frac{1}{\sigma^2} \sum_{i=1}^{n} w_i x_i x_i^T = \frac{1}{\sigma^2} \sum_{i=1}^{n} w_i M(x_i),$$

where $M(x_i)$ is the information at design point x_i.

$$M(x_i) = f(x) f(x)^T.$$

For $y_i = \beta_0 + \beta_1 x_i^2 + \varepsilon$,

$$f(x) = \begin{bmatrix} \frac{dy_i}{d\beta_0} \\ \frac{dy_i}{d\beta_1} \end{bmatrix} = \begin{bmatrix} 1 \\ x_i^2 \end{bmatrix}.$$

Then, the information at design point x_i is given by **(5)**:

$$M(x_i) = \begin{bmatrix} 1 \\ x_i^2 \end{bmatrix} \begin{bmatrix} 1 & x_i^2 \end{bmatrix} = \begin{bmatrix} 1 & x_i^2 \\ x_i^2 & x_i^4 \end{bmatrix}.$$

Substituting (5) into (4), we get **(6)**:

$$M(\xi) = \frac{1}{\sigma^2}\sum_{i=1}^{n} w_i\, f(x)f(x)^T \propto \sum_{i=1}^{n} w_i \begin{bmatrix} 1 & x_i^2 \\ x_i^2 & x_i^4 \end{bmatrix}$$

1A.2. Computation of the standardized information matrix

To compute the standardized information matrix for the design ξ_1, I use the expression **(6)**:

$$M(\xi_1) \propto \frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 & 9 \\ 9 & 81 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 & 36 \\ 36 & 1296 \end{bmatrix} \propto \begin{bmatrix} 1 & 15 \\ 15 & 459 \end{bmatrix}$$

B) Derive an expression for the standardized predictor variance of the design ξ_1. Plot this function over the design region and comment.

1B.1. *Derive an expression for the standardized predictor variance of the design ξ_1*

The standardized predictor variance of the design matrix at the design point x_0 is given by **(7)**:

$$s(x_0, \xi_1) = \frac{N}{\sigma^2} var(\hat{y}(x_0))$$

The variance of the prediction, $var(\hat{y}(x_0))$, is in turn equal to **(8)**:

$$var\big(\hat{y}(x_0)\big) = f(x_0)^T cov(\beta) f(x_0).$$

Since the variance-covariance matrix, $cov(\beta)$, is the inverse of the information matrix, $I(\xi)$, it can be expressed in terms of the standardized information matrix, $M(\xi)$, as **(9)**:

$$cov(\beta) = I(\xi)^{-1} = \left(\frac{N}{N} I(\xi)\right)^{-1} = \big(NM(\xi)\big)^{-1} = \frac{1}{N}\big(M(\xi)\big)^{-1}.$$

Substituting expressions (8) and (9) into expression (7), we get expression **(10)**:

$$s(x_0, \xi_1) = \frac{N}{\sigma^2} f(x_0)^T \frac{1}{N}\big(M(\xi)\big)^{-1} f(x_0) = \frac{1}{\sigma^2} f(x_0)^T \big(M(\xi)\big)^{-1} f(x_0).$$

Substituting expression (6) into expression (10), we get an expression for the standardized predictor variance of ξ_1, **(11)**:

$$s(x_0, \xi_1) = \frac{1}{\sigma^2} f(x_0)^T \left(\frac{1}{\sigma^2}\sum_{i=1}^{n} w_i\, f(x)f(x)^T\right)^{-1} f(x_0) =$$

$$= f(x_0)^T \big(M(\xi_1)\big)^{-1} f(x_0)$$

1B.2. *Plot this function over the design region and comment.*

The standardized predictor variance, s($\boldsymbol{x_0}, \boldsymbol{\xi_1}$)

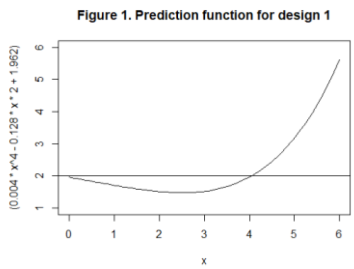
The inverse of the standardized information matrix $M(\xi_1)^{-1}$ is:

$$M(\xi_1)^{-1} \approx \begin{bmatrix} 1{,}962 & -0{,}064 \\ -0{,}064 & 0{,}004 \end{bmatrix}$$

The function for the standardized predictor variance

$$f(x_0, \xi_1) = \begin{bmatrix} 1 & x_0^2 \end{bmatrix} \begin{bmatrix} 1{,}962 & -0{,}064 \\ -0{,}064 & 0{,}004 \end{bmatrix} \begin{bmatrix} 1 \\ x_0^2 \end{bmatrix}$$
$$= 0{,}004x_0^4 - 0{,}128x_0^2 + 1.962$$

The plot of the function, s($\boldsymbol{x_0}, \boldsymbol{\xi_1}$)



Comment on the plot. We use the (standardized) predicted variance as a means to make as efficient predictions as possible over the design region. The goal is to find a design with the smallest possible standardized variance of the predicted response. The design that achieves this is called G-optimal. A G-optimal design has one important feature: the standardized variance for a continuous G-optimal design is always less than or equal to the number of parameters in the model. The standardized prediction variance is also equal at the ends of the design points. Since the variance is not equal at the ends of the design points, e.g., at $x = 0$ and $x = 6$, and $\mathbf{s}(\boldsymbol{x_0}, \boldsymbol{\xi_1}) > \mathbf{2}$, i.e. larger than the number of the parameters in the given model (see Figure 1), the design ξ_1 is not G-optimal (Berger *et al*., 2009, p.42).

C) Use the plot in b) to check whether the design ξ_1. is D-optimal or not. If not, obtain a D-optimal design and verify its optimality via a corresponding plot of the standardized predictor variance.

1C.1. Use the plot in b) to check whether the design ξ_1. is D-optimal or not.

It has been proven that D- and G-optimal designs are the same when the model's errors all have constant variance. In the current case, the errors are constant, $\varepsilon_i \sim N(0, \sigma^2)$. Therefore, we can draw the conclusion that the design ξ_1 is not D-optimal, since it is not G-optimal (Berger *et al*., 2009, p.42).

1C.2. Obtain a D- optimal design and verify its optimality via a corresponding plot of the standardized predictor variance

The universally optimal design in this case is the design that assigns equal weights at the end points of the design (Berger *et al*., 2009, p. 36). That is the design, ξ_2, given below

$$\xi_2 = \begin{Bmatrix} 0 & 3 & 6 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{Bmatrix},$$

is optimal.

To doublecheck that the design ξ_2 is indeed optimal, I derive its standardized prediction function and plot it.

The standardized predictor variance, s(x_0, ξ_2)

The standardized information matrix, $M(\xi_2)$:

$$M(\xi_2) \propto \begin{bmatrix} 1 & 18 \\ 18 & 648 \end{bmatrix}$$

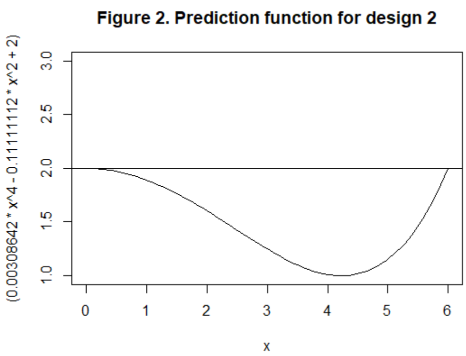
The inverse of the standardized information matrix $M(\xi_2)^{-1}$ is:

$$M(\xi_2)^{-1} \approx \begin{bmatrix} 2 & -0.056 \\ -0.056 & 0.003 \end{bmatrix}$$

The function for the standardized predictor variance

$$f(x_0, \xi_2) = [1 \quad x_0^2] \begin{bmatrix} 2 & -0.056 \\ -0.056 & 0.003 \end{bmatrix} \begin{bmatrix} 1 \\ x_0^2 \end{bmatrix} = 0.003x_0^4 - 0.112x_0^2 + 2$$

The plot of the function, s(x_0, ξ_2)



The chosen design, $\xi_{2,}$ is indeed G- and D-optimal, since the standardized prediction variance is equal at the end points of the design, i.e., at x=0 and x=6, and is equal to the number of the parameters in the model, i.e. $s(x_0 = 0, \xi_2) = s(x_0 = 6, \xi_2) = 2$. So, $\xi_{2,} = \xi_{opt}$.

D) Calculate the relative D-efficiency of ξ_1.

The relative efficiency based on the D-criterion is given by **(12)**:

$$RE_D = \left(\sqrt{\frac{\det\left(M^{-1}(\xi_{opt})\right)}{\det\left(M^{-1}(\xi)\right)}} \right)^{\frac{1}{p}},$$

Where det stands for the determinant of the inverse standardized information matrix, and p denotes the number of parameters in the model we intend to estimate. Since, in our model we are estimating two parameters, β_0 and β_1, expression (12) simplifies to

$$RE_D = \sqrt{\frac{\det\left(M^{-1}(\xi_{opt})\right)}{\det\left(M^{-1}(\xi_1)\right)}} \approx 0.85$$

It follows that the sample size of the design ξ_1 is to be multiplied by a factor of ca. $\frac{1}{RE_D} \approx 1.18$ for it to obtain the same efficiency as the optimal design, ξ_{opt}, or equivalently, the sample size has to be enlarged by 18%.

E) Obtain A-optimal design by modifying R-code provided in lecture 2.

The A-optimal design minimizes the sum, or average, of the variances of the parameter estimates. This is equivalent of minimizing the trace of the variance–covariance matrix, $cov(\hat{\beta})$:

$$\min\left(tr\left(M(\xi)\right)^{-1}\right).$$

The detailed codes are given in the attachment. The suggested A-optimal design is given below.

Design points, x	Weights	A-criterion
0	0.97	1.05
6	0.03	1.05

This design is very different from the D-optimal design. However, if the variances of the parameters are very different in magnitude, the A-optimal design is known to be misleading. This seems to be the case in the current example.

Problem 2

2A. Use the data given in the table to estimate the parameters in the logistic model

The data describes the classical dose-response experiment where beetles were exposed to gaseous carbon disulphide at different concentrations. In total, we have information on three variables:

x_i: = the dose in units of $\log_{10} \mathrm{CS_2}\ \mathrm{mgl}^{-1}$

r_i: = number of beetles exposed to each concentration of dose

y_i: = number of killed beetles at each x_i.

To estimate the three parameters of the logistic model

$$p(x) = \frac{exp(\beta_0+\beta_1 x+\beta_2 x^2)}{1+exp(\beta_0+\beta_1 x+\beta_2 x^2)},$$

that is β_0, β_1 and β_2, I use the *glm* function in R with a binomial link. The estimated parameters are given below in Table 1.

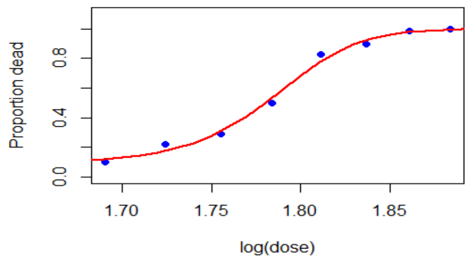
Table 1. The estimated parameters of the given logistic model

Parameters	Estimate	Standard error	Z value	P-value
β_0	431.11	180.65	2.386	0.01702
β_1	-520.62	204.52	-2.546	0.01091
β_2	156.41	57.86	2.703	0.00687

All the parameters obtained are significant (see the p-values, Table 1). However, all the estimated parameters are very large. Since the estimated parameters represent odds ratios, it might not be rational to interpret them in a standard way, i.e., in terms of increase in one unit in the dose. A more sensible approach might be to consider instead, e.g., a 0.01 increase in the dose.

Also, judging from Figure 2.1, the model seems to fit well.

Figure 2.1. The fitted line for $p(x)$



<u>2B. Obtain the locally D-optimal design assuming the parameter estimates from a) are the true values. (Modify R-code given in lecture-2 to find D-optimal design)</u>.

In contrast to linear regression models, e.g., as given in question 1, the D-optimal design for a logistic regression will depend on the initial guess or estimation of a model's parameters. The optimal design might therefore differ for different values of the initial parameters. If we assume that the estimates obtained in part (A) are true parameter values, then the resulting D-optimal design, obtained by modifying the codes in lecture 2, is as given below:

Table 2. The suggested D-optimal design for the logistic model

Design points, x	Weights	D-criterion
1.6907	0.3333	23.9919
1.7704	0.3333	23.9919
1.8350	0.3333	23.9919

The design obtained by means of the IOACD package is similar: the suggested design points are 1.6907, 1.7707, and 1.8345 with the corresponding weights of 0.3307, 0.3337, and 0.3356.

<u>2C. How robust is the locally D-optimal design from b)? Do a small sensitivity analysis where you evaluate the D-efficiency for some alternative parameter values</u>.

I have conducted a small sensitivity analysis for some alternative parameter values to the estimated $\boldsymbol{\beta} = (431.11, -520.62, 156.41)$. Since I do not dispose of any other prior information about the plausible values for the parameters in question except for the information I obtained from estimating the logistic model for given data, the alternative values that I chose all lie in the range of one-two standard deviations from the initially estimated values. The results are given in Table 3 below. When the alternatives values lie within one standard deviation from the initially estimated values, the design resembles that obtained in Table 2 (see two first lines of Table 3), though the D-criterion is larger. Otherwise the design stray quote considerably from the one given in Table 2 (check the two last rows of Table 3), both in terms of design points, x, and the assigned weight, w. Moreover, the returned value fir the D-criterion is infinity.

This signified that the locally D-optimal design is indeed sensitive to the choice of starting values. In the absence of reliable preliminary point estimates, other methods for finding optimal designs, e.g. the minimax criterion, are to be preferred.

Table 3. Sensitivity analysis for alternative parameter values

Parameters: $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2)$	Design points, $\boldsymbol{x}$	Weights, $\boldsymbol{w}$	D-criterion
$(400, -450, 130)$	$(1.6907, 1.7607, 1.8570)$	$(0.3333, 0.3333, 0.3333))$	51.01
$(300, -400, 100)$	$(1.6907, 1.7163, 1.8107)$	$(0.3333, 0.3333, 0.3333)$	298.39
$(250, -200, 100)$	$(1.6967, 1.7625, 1.7837, 1.8259, 1.8214\ )$	$(0.3280, 0.3025, 0.0238, 0.1130, 0.2326)$	Inf
$(500, -600, 250)$	$(1.6967, 1.7625, 1.7837, 1.8259, 1.8413\ )$	$(0.3280, 0.3025, 0.0238, 0.1130, 0.2325)$	Inf

<u>2D. Describe two strategies (other than local optimality) to deal with the parameter dependency issue for the logistic model</u>

The optimal design of the logistic model directly depends on the unknown parameters of the model. To achieve an optimal design, a researcher first has to preliminary estimate the parameters of the model. She may obtain information about the unknown parameters either by relying on the expert knowledge, or from pilot studies, etc. But in any case, the design achieved in such a way will depend on the chosen initial values of the parameters. Should she choose another set of parameters, she may obtain (vastly) different results. Therefore, a design that depend on the unknown parameters of the model is referred to as locally optimal.

The objective is to find a design that is not overly dependent on the unknown parameters of the model and to which researchers may resort when there is not sufficient (preliminary) information about the parameters in the logistic model.

Strategy 1. One viable and simple alternative to local design is a design which assigns equal weight to the design points in the experiment. It has been shown that this design provides a similar estimation of the variances of the slope coefficients in the model (Berger *et al*., 2009, p.119).

Strategy 2. A more efficient design than that of the equal weights, though not necessarily optimal, is the so-called maximin design. The idea of this design is to minimize the worst- performance of the experiment over the range of user-selected region for the parameters. This means that a range (or, an interval) of possible values for the parameters has to be specified in advance. If the true values of the parameters are in the prespecified range for the parameters, this design is said to be robust, in contrast to the locally optimal design which is dependent on the preliminary point estimation of the unknown parameters. Let us denote by R the design region for all the parameters in the given logistic model, so that $R = (\boldsymbol{\beta_0}, \boldsymbol{\beta_1}, \boldsymbol{\beta_2})$. Let us also assume that the intervals $\boldsymbol{\beta_0}, \boldsymbol{\beta_1}, \boldsymbol{\beta_2}$ are chosen so that they comprise all plausible values of the parameters. To find minimax D-optimal design, e.g., we have to minimize the following

$$\max\{-\log(\det(M(\xi;\ \boldsymbol{\beta_0}, \boldsymbol{\beta_1}, \boldsymbol{\beta_2})))\},$$

where $M(\xi;\ \boldsymbol{\beta_0}, \boldsymbol{\beta_1}, \boldsymbol{\beta_2})$ is the information matrix (Berger *et al.,* 2000).

2E. Obtain the locally E-optimal design assuming the parameter estimates from a) are the true values. (Modify R-code given in lecture-2 to find E-optimal design).

The E-optimal criterion minimizes the length of largest axis of a confidence ellipsoid. This is equivalent to minimizing the maximum eigenvalue of $cov(\hat{\beta})$:

$$\min\left(eigen(M(\xi))^{-1}\right).$$

The detailed codes are given in the attachment. The suggested E-optimal design is given in Table 3 below.

Table 4. The suggested E-optimal design for the logistic model

Design points, x	Weights	E-criterion
1.6907	0.2596	22034658
1.7672	0.3105	22034658
1.8563	0.4299	22034658

This design differs from the D-optimal design given in Table 2, especially in the *Weights* column. However, the E-optimal design is not often used in practice as it does not use all the information on the parameters in the model (lecture 1).

The design obtained by means of the IOACD package somewhat differs: the suggested design points are 1.6907, 1.7647, and 1.8289 with the corresponding weights of 0.3198, 0.2995, and 0.3807.

Problem 3

3A. The Emax model and dose–response relationships

The Emax model I given by

$$y_i = E_0 + \frac{E_{max}x_i}{ED_{50}+x_i},$$

where the parameterE_0 is the placebo response, E_{max} is the maximal effect of the drug for a hypothetically infinite dose and ED_{50} is the dose with half of the maximal effect.

To obtain the locally D-optimal design, I use the IOACD package with the initial values of the three parameters, E_0, E_{max} and ED_{50}, $(E_0, E_{max}, ED_{50}) = (22, 11.2, 70)$, with the design region given by $X = [0, 100]$. The suggested design is given in Table 4. It follows that the D-optimal design assigns equal weights both at the extreme end points of the design, i.e., at $x = 0$ and $x = 100$, and also at the point $x = 29$.

Table 5. The suggested D-optimal design for the EMAX model

Design points, $\approx x$	Weights	D-criterion
0	0.3333	12.91
29	0.3333	12.91
100	0.3333	12.91

3B. The 3PL model

We are given the three-parameter logistic model, $\beta = (a, b, c)$,

$$p(\theta) = c + \frac{1-c}{1+\exp\left(-a(\theta-b)\right)},$$

where θ stands for the ability of a test-taker.

3B.1. Derive an expression for the standardized information matrix for the item with item parameter β.

We need to find the expression of *the standardized information matrix, **(13)***

$$M(\xi) = \sum_{i=1}^{n} w_i V(\theta_i)\,\eta(\theta)\eta(\theta)^T,$$

where $V(\theta_i)$ stands for the variance of θ_i, and $V(\theta_i) = p(\theta_i)(1-\theta_i)$.

To derive the expression, I follow the steps below:

1. The probability of success, i.e., the probability of correctly answering to item, is

$$p(\theta) = c + \frac{1-c}{1+\exp\left(-a(\theta-b)\right)} = \frac{1+c*exp(-a(\theta-b))}{1+exp(-a(\theta-b))}.$$

2. The probability of failure, i.e., the probability of incorrectly answering to item, is

$$1-p(\theta) = 1 - \frac{1+c*\exp(-a(\theta-b))}{1+\exp(-a(\theta-b))} = \frac{(1-c)\exp(-a(\theta-b))}{1+exp(-a(\theta-b))}.$$

3. The odds of correctly answering the question is

$$\frac{p(\theta)}{1-p(\theta)} = \frac{1+c*\exp(-a(\theta-b))}{1+\exp(-a(\theta-b))} * \frac{1+\exp(-a(\theta-b))}{(1-c)\exp(-a(\theta-b))} = \frac{1+c*\exp(-a(\theta-b))}{(1-c)\exp(-a(\theta-b))}.$$

4. The natural logarithm of the odds is

$$\log\left(\frac{p(\theta)}{1-p(\theta)}\right) = \log\left(1+c*\exp\left(-a(\theta-b)\right)\right) - \log(1-c) + a\left((\theta-b)\right) = \eta(\theta).$$

5. Let's find $\frac{d\eta}{d\beta} = \begin{bmatrix} \frac{d\eta}{da} \\ \frac{d\eta}{db} \\ \frac{d\eta}{dc} \end{bmatrix}$:

- $\frac{d\eta}{da} = \frac{(1+c*\exp(-a(\theta-b)))'}{1+c*\exp(-a(\theta-b))} - 0 + (\theta-b) = \frac{c*\exp(-a(\theta-b))*(-(\theta-b)))}{1+c*\exp(-a(\theta-b))} + (\theta-b) = (\theta-b)\left(1-\frac{c*\exp(-a(\theta-b))}{1+c*\exp(-a(\theta-b))}\right) = \frac{\theta-b}{1+c*\exp(-a(\theta-b))}$

- $\frac{d\eta}{db} = \frac{(1+c*\exp(-a(\theta-b)))\prime}{1+c*\exp(-a(\theta-b))} - 0 - a = \frac{c*\exp(-a(\theta-b))*a}{1+c*\exp(-a(\theta-b))} - a = a\left(\frac{c*\exp(-a(\theta-b))}{1+c*\exp(-a(\theta-b))} - 1\right) = -\frac{a}{1+c*\exp(-a(\theta-b))}$
- $\frac{d\eta}{dc} = \frac{(1+c*\exp(-a(\theta-b)))\prime}{1+c*\exp(-a(\theta-b))} - \frac{(1-c)'}{1-c} + 0 = \frac{\exp(-a(\theta-b))}{1+c*\exp(-a(\theta-b))} - \frac{-1}{1-c} = \frac{1+\exp(-a(\theta-b))}{(1-c)(1+c*\exp(-a(\theta-b)))}$

6. Let's find $\eta(\theta)\eta(\theta)^T$, (14):

$$\eta(\theta)\eta(\theta)^T = \begin{bmatrix} \frac{d\eta}{da} \\ \frac{d\eta}{db} \\ \frac{d\eta}{dc} \end{bmatrix} \begin{bmatrix} \frac{d\eta}{da} & \frac{d\eta}{db} & \frac{d\eta}{dc} \end{bmatrix} =$$

$$= \frac{1}{1+c*\exp(-a(\theta-b))} \begin{bmatrix} \theta-b \\ -a \\ \frac{1+\exp(-a(\theta-b))}{1-c} \end{bmatrix} \begin{bmatrix} \theta-b & -a & \frac{1+\exp(-a(\theta-b))}{1-c} \end{bmatrix} =$$

$$= \frac{1}{1+c*\exp(-a(\theta-b))} \begin{bmatrix} (\theta-b)^2 & -a(\theta-b) & \frac{(\theta-b)(1+\exp(-a(\theta-b)))}{1-c} \\ \theta-b & a^2 & \frac{-a(1+\exp(-a(\theta-b)))}{1-c} \\ \frac{(\theta-b)(1+\exp(-a(\theta-b)))}{1-c} & \frac{-a(1+\exp(-a(\theta-b)))}{1-c} & \frac{(1+\exp(-a(\theta-b)))^2}{(1-c)^2} \end{bmatrix}$$

7. Substitute (14) into (13) to get an expression for the standardized information matrix in question:

$$M(\xi) = \sum_{i=1}^{n} w_i p(\theta_i)(1-p(\theta_i))\,\eta(\theta)\eta(\theta)^T.$$

3B.2. Find the D-optimal design assuming $a = 0.5$, $b = 1$, $c = 0.05$ are the true values for item parameters. *The design region is* $X = [-3, 3]$. *Modify R-code given in lecture-2 to find D-optimal design.*

The suggested D-optimal design obtained by modifying the codes in lecture 2 is given in Table 5 below.

Table 6. The suggested D-optimal design for the given 3PL model

Design points, $\approx \boldsymbol{x}$	Weights	Variance
-3	0.3333	4.8418
-0.35	0.3333	4.8418
3	0.3333	4.8418

The design is similar to the one obtained by means of the IAOCD package: the suggested design points are -3, 0.27, and 3 with the corresponding weights of 0.3381, 0.3283, and 0.3335.

Reference list:

Berger *et al.*, 2009, John Wiley & Sons; An Introduction to Optimal Designs for Social and Biomedical Research

Berger *et al.*, 2000, Psychometrika, vol. 65, no. 3, p. 377-390; Minimax D-optimal designs for item response theory models

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