Steiner triple systems with small circumference

Undergraduate research project

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Abstract

This document introduces the concept of *circumference* of a Steiner triple system and poses some questions worthy of study.

1 Introduction

A *Steiner triple system of order* n, briefly STS(n), is a pair $(V, \mathcal{B})$ where V is a set of n elements and $\mathcal{B}$ is a collection of three-element subsets of V called *blocks* such that for every pair of distinct elements $a, b \in V$, there is a unique block $\{a, b, c\} \in \mathcal{B}$ containing a and b. It is known that an STS(n) exists if and only if $n \equiv 1$ or $3 \pmod 6$; such values are called *admissible*.

Given an STS(n) $\mathcal{S} = (V, \mathcal{B})$ and a distinct pair $a, b \in V$, let $\{a, b, c\} \in \mathcal{B}$ be the unique block containing a and b. We define the *cycle graph* $G_{a,b}$ on vertex set $V \setminus \{a, b, c\}$ with edges given by the collection $\{\{x, y\} \mid \{a, x, y\} \in \mathcal{B} \text{ or } \{b, x, y\} \in \mathcal{B}\}$. $G_{a,b}$ is necessarily a union of cycles, and the lengths of these cycles must add up to $n - 3$.

One common problem in this area is finding perfect Steiner triple systems; an STS(n) is *perfect* if every cycle graph consists of a single cycle of length $n - 3$. We are primarily interested in the opposite problem: can we construct an STS(n) such that every cycle graph consists of a union of small cycles? For an STS(n) $\mathcal{S} = (V, \mathcal{B})$, define the *circumference* of $\mathcal{S}$ to be the length of the longest cycle across all cycle graphs $G_{a,b}$ with $a, b \in V$. We make the following conjecture.

Conjecture 1. *For every admissible n, there exists an STS(n) with circumference at most 12.*

This conjecture is easily established for $n \in \{3, 7, 9, 13, 15, 19, 21, 27, 31\}$, so the goal of this project is to find an STS(25) with circumference at most 12. While this may seem trivial for such a small value of n, it is not even known how many STS(25)s exist (there are over 10^{16} nonisomorphic STS(21)s, and there are likely many more STS(25)s).

2 Example: STS(15)s

We will look at two different STS(15)s. For the first one $\mathcal{S}_1 = (V_1, \mathcal{B}_1)$, let $V_1 = \{\infty, 0, 1, ..., 6, \overline{0}, \overline{1}, ..., \overline{6}\}$. Consider the STS(7) $(\mathbb{Z}_7, \mathcal{B}_7)$ given by the Fano plane in Figure 1, and set

$$\mathcal{B}_1 = \{\infty x \overline{x} \mid x \in \mathbb{Z}_7\} \cup \{xyz, x\overline{y}\overline{z}, \overline{x}y\overline{z}, \overline{x}\overline{y}z \mid xyz \in \mathcal{B}\}.$$

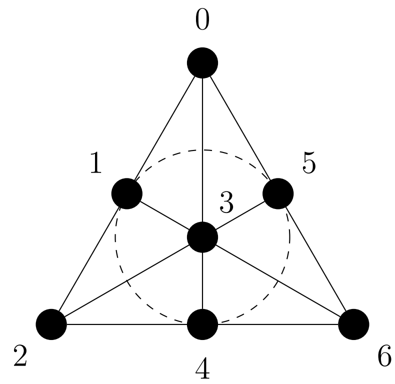

Figure 1: An STS(7) on the set $\mathbb{Z}_7 = \{0, 1, ..., 6\}$ with block set $\mathcal{B}_7$.

We will compute a few cycle graphs.

$$\begin{aligned} G_{\infty,0} &= (1\,\overline{1}\,\overline{2}\,2)\,(3\,\overline{3}\,\overline{4}\,4)\,(5\,\overline{5}\,\overline{6}\,6) \\ G_{1,6} &= (\infty\,\overline{1}\,\overline{3}\,\overline{6})\,(0\,2\,4\,3)\,(\overline{0}\,\overline{2}\,\overline{4}\,\overline{3}) \\ G_{3,\overline{3}} &= (0\,4\,\overline{0}\,\overline{4})\,(1\,6\,\overline{1}\,\overline{6})\,(2\,5\,\overline{2}\,\overline{5}) \end{aligned}$$

If we computed all of the cycle graphs, we would see that they all consist of three cycles of length 4. Therefore the circumference of $\mathcal{S}_1$ is 4.

For the second example $\mathcal{S}_2 = (V_2, \mathcal{B}_2)$, let $V_2 = \mathbb{Z}_{15}$ and set

$$\begin{aligned} \mathcal{B}_2 = \{ & \{0,1,2\}, \{0,3,4\}, \{0,5,6\}, \{0,7,8\}, \{0,9,10\}, \{0,11,12\}, \{0,13,14\}, \\ & \{3,8,10\}, \{1,8,11\}, \{1,7,9\}, \{1,4,6\}, \{1,12,14\}, \{1,10,13\}, \{1,3,5\}, \\ & \{4,9,13\}, \{2,9,12\}, \{2,8,14\}, \{2,11,13\}, \{2,4,5\}, \{2,3,6\}, \{2,7,10\}, \\ & \{5,11,14\}, \{5,7,13\}, \{3,12,13\}, \{3,9,14\}, \{3,7,11\}, \{4,7,14\}, \{4,8,12\}, \\ & \{6,7,12\}, \{6,10,14\}, \{4,10,11\}, \{5,10,12\}, \{6,8,13\}, \{5,8,9\}, \{6,9,11\} \}. \end{aligned}$$

Once again we compute a few cycle graphs.

$$\begin{aligned} G_{0,1} &= (3\,4\,6\,5)\,(7\,8\,11\,12\,14\,13\,10\,9) \\ G_{6,14} &= (0\,5\,11\,9\,3\,2\,8\,13)\,(1\,4\,7\,12) \\ G_{3,9} &= (0\,4\,13\,12\,2\,6\,11\,7\,1\,5\,8\,10) \end{aligned}$$

From this we see that some cycle graphs have cycles of length 4 and 8, but at least one cycle graph has a cycle of length 12, so the circumference of $\mathcal{S}_2$ is 12.

3 Questions to consider

1. Do any of the known STS(25)s have a circumference of at most 12?
2. Can we use a (smart) hill-climbing algorithm to build an STS(25) with small circumference?
3. Are there any patterns that we can extend to STS(n)s for $n > 25$?