DIVIDE AND CONQUER PARADIGM

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THE PARADIGM I

Solves a problem **recursively** applying at each level of the recursion

- **DIVIDE** the problem into smaller sub-problems via recursive calls.
- **CONQUER** combining solutions of sub-problems into one for the original problem.

Also consider a **base case** when the sub-problems become small enough.

THE PARADIGM II

MERGE SORT: RETAKEN

MERGE-SORT(A, p, r)

```
if p < r then
    q = ⌊(p + r)/2⌋
    MERGE-SORT(A, p, q)
    MERGE-SORT(A, q, r)
    MERGE(A, p, q, r)
end if
```

MERGE(A, p, q, r)

```
B = 1^st part of array.
C = 2^nd part of array.
i = 1, j = 1
for k = 1 to n do
    if B[i] < C[j] then
        A[k] = B[i++]
    else
        A[k] = C[j++]
    end if
end for
```

THE PARADIGM III

INSIGHTS

- Sub-problems sizes can be any! e.g. 1/2, 1/3, etc.
- Base case is often too naive.
- Generally, in the third step relies the good performance.
- Trying to make third step simple is the best way to tackle the problem.

RECURRENCES I

A RECURRENCE

- provide a natural way to describe the running time of divide-and-conquer algorithms.
- describes $T(n)$ in terms of the running time of recursive calls.
- consider a base case for which

$$T(n) \leq a$$

- consider the larger imputs as

$$T(n) \leq f(n)$$

Recurrences II

Computing Recurrences: Merge Sort

- Base case: sort two numbers. Swap positions in the worst case.

$$T(1) = 1$$

- Larger inputs: two sub-problems of size $n/2$ plus one call to merge with size n, i.e.

$$T(n) = 2T(n/2) + \Theta(n)$$

RECURRENCES III

leading to

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 2T(n/2) + \Theta(n) & \text{if } n > 1 \end{cases}$$

help us to get the running time

$$T(n) = \Theta(n \log_2 n)$$

and for the properties of asymptotic notation $T(n) = O(n \log n)$

EXAMPLE: COUNTING INVERSIONS I

MOTIVATION

- List of ranked objects $A = [a_1, a_2, a_3, ..., a_n]$ and a reference list B.
- We want to figure out how similar these lists are.
- Application Collaborative Filtering.

Let B be the sorted list of A, then what we seek is the number of inversions in list A, i.e. the number of pairs (i, j) such that

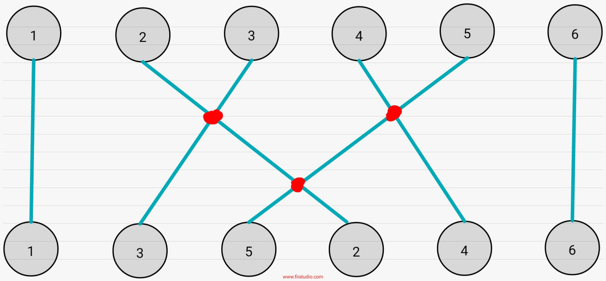
$$i < j \text{ and } A[i] > A[j]$$

EXAMPLE: COUNTING INVERSIONS II

EXAMPLE: $A = [1, 3, 5, 2, 4, 6]$

Inversions:

$(3, 2), (5, 2), (5, 4)$



In general the upper bound of number of inversions is $n(n-1)/2$.

EXAMPLE: COUNTING INVERSIONS III

A brute force algorithm

BRUTE-FORCE(A, n)

```
count = 0
for i = 1, ..., n do
    for j = i + 1, ..., n do
        if A[i] > A[j] then
            count = count + 1
        end if
    end for
end for
return count
```

It is correct but

$$T(n) = O(n^2)$$

We can do better!

Example: Counting Inversions IV

The key idea is to use divide and conquer. Let be (i, j) an inversion with $i < j$

1. Let's call it a left inversion if $i, j \leq n/2$.
2. Let's call it a right inversion if $i, j > n/2$.
3. Let's call it a split inversion if $i \leq n/2 < j$

compute 1 and 2 recursively and implement other routine for 3.

EXAMPLE: COUNTING INVERSIONS V

COUNT-INVERSIONS(A, n)

if $n = 1$ **then**

  return 0

end if

x=COUNT-INVERSIONS(first half of A, $n/2$)

y=COUNT-INVERSIONS(second half of$A$, $n/2$)

z=COUNT-SPLIT-INVERSIONS$(A, n/2)$

return $x + y + z$

The goal is to implement efficiently COUNT-SPLIT-INVERSIONS for good performance.

EXAMPLE: COUNTING INVERSIONS VI

The key idea to implement COUNT-SPLIT-INVERSIONS is to consider sorting, more precisely the merge sub-routine of merge-sort

- $B =$ left part of array.
- $C =$ second part of array.
- $A =$ output array.

MERGE(A, p, q, r)

```
i = 1, j = 1
for k = 1 to n do
    if B[i] < C[j] then
        A[k] = B[i++]
    else
        A[k] = C[j++]
    end if
end for
```

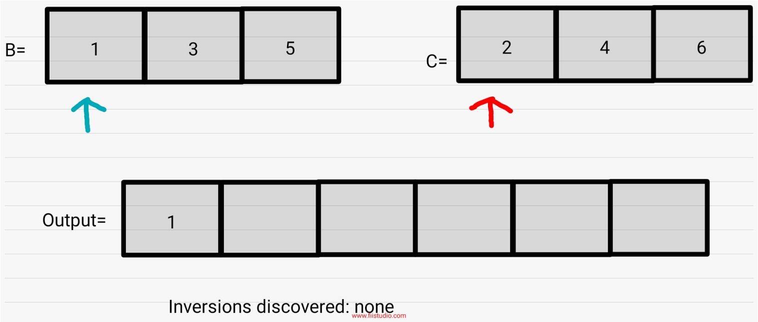
Example: Counting Inversions VII

Now suppose A has no split inversions, then following properties holds

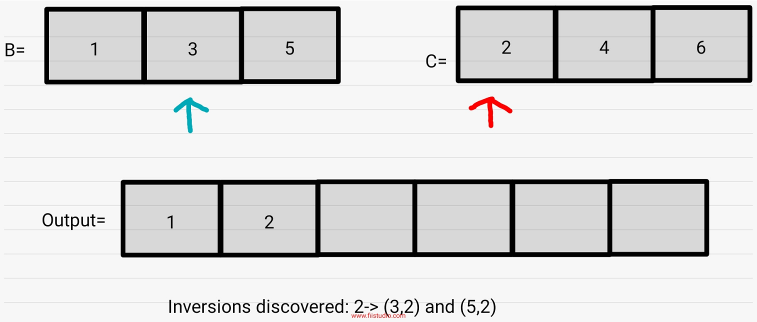
- All elements in B are less than the elements in C.
- All elements in B are copied back to A before the elements in C.

The second half has the clue to discover the inversions!

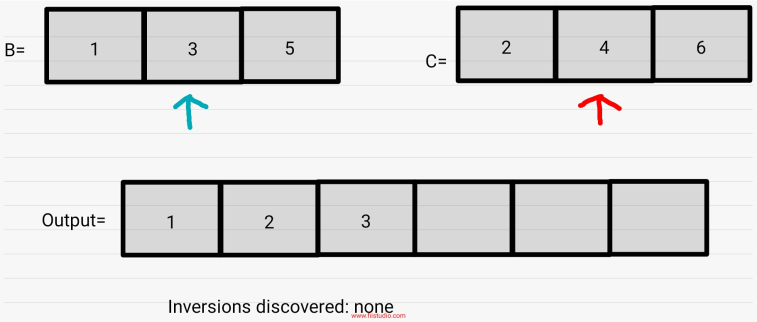
EXAMPLE: COUNTING INVERSIONS VIII



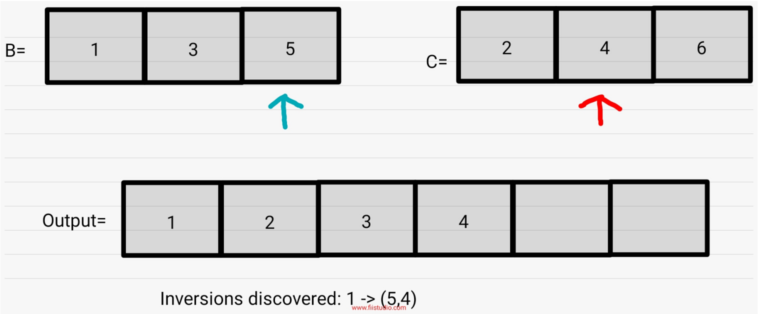
Example: Counting Inversions IX



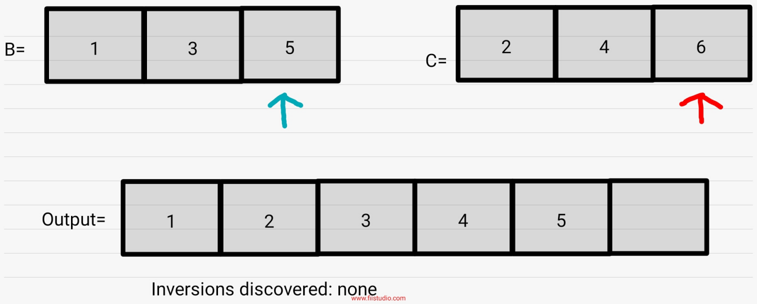
EXAMPLE: COUNTING INVERSIONS X



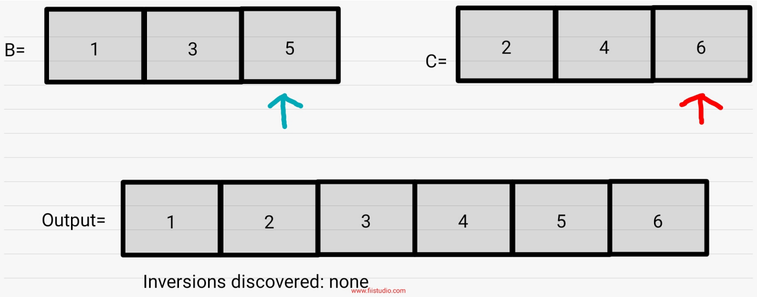
EXAMPLE: COUNTING INVERSIONS XI



EXAMPLE: COUNTING INVERSIONS XII

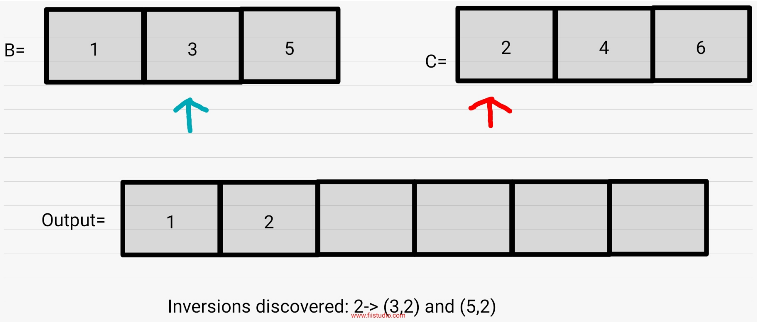


Example: Counting Inversions XIII



EXAMPLE: COUNTING INVERSIONS XIV

The split inversions involving an element $y_i \in C$ is the number of remaining positions from i in B when it is copied to the output array D.



EXAMPLE: COUNTING INVERSIONS XV

THE ALGORITHM

COUNT-INVERSIONS(A, n)

if $n = 1$ **then**
    return 0
end if
B, x=COUNT-INVERSIONS(first half of A, $n/2$)
C, y=COUNT-INVERSIONS(second half of$A$, $n/2$)
D, z=COUNT-SPLIT-INVERSIONS($B, C, n/2$)
return $x + y + z, D$

EXAMPLE: COUNTING INVERSIONS XVI

COUNT-SPLIT-INVERSIONS(B, C, n)

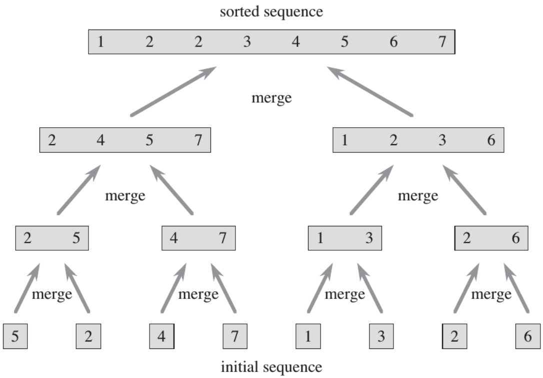
```
i = 1, j = 1, inv = 0
for k = 1 to n do
    if B[i] < C[j] then
        D[k] = B[i++]
    else
        inv += n/2 - i
        D[k] = C[j++]
    end if
end for
return inv, D
```

The recurrence is

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ 2T(n/2) + O(n) & \text{if } n > 1 \end{cases}$$

What is the running time?

SOLVING RECURRENCES : RECURSION TREE



At each level $j = 0, 1, ..., log_2(n)$, there are 2^j subproblems of size $n/2^j$.

SOLVING RECURRENCES : THE MASTER METHOD I

- Also known as the Master Theorem.
- Black box for solving recurrences.
- Take as input few parameters to get the solution.
- Assumption all sub-problems have the same size.
- We will only consider the method that yields upper bounds, i.e. $O()$.

SOLVING RECURRENCES : THE MASTER METHOD II

RECURRENCE FORMAT

- Base case: $T(n) \leq \alpha$ for sufficiently small n.
- For larger n

$$T(n) \leq aT(n/b) + O(n^d)$$

where

a is the number of recursive calls (≥ 1).
b is the factor by which the input size shrinks (> 1).
d exponent in summing time of combining step (≥ 0).

and $a, b, d \perp n$

Solving Recurrences : The Master Method III

The Method

$$T(n) = \begin{cases} O(n^d \log n) & \text{if } a = b^d \text{ (1)} \\ O(n^d) & \text{if } a < b^d \text{ (2)} \\ O(n^{\log_b a}) & \text{if } a > b^d \text{ (3)} \end{cases}$$

SOLVING RECURRENCES : THE MASTER METHOD IV

COUNTING INVERSIONS SOLUTION

The parameters

- Two recursive calls to COUNT-INVERSIONS: $a = 2$.
- Divide the array into two sub-problems: $b = 2$.
- COUNT-SPLIT-INVERSIONS runs in linear time $d = 1$.
- $a = b^d? : 2 = 2^1$.

Falls into case # 1

$$T(n) = O(n \log n)$$

SOLVING RECURRENCES : THE MASTER METHOD V

MORE: BINARY SEARCH

BINARY-SEARCH(A, a, i, p, r)

if $p < r$ **then**
  $q = \lfloor (p + r)/2 \rfloor$
  if $A[q] == a$ **then**
    $i = q$
  end if
  if $A[q] > a$ **then**
    BINARY-SEARCH$(A, a, i, p, q - 1)$
  else
    BINARY-SEARCH$(A, a, i, q + 1, r)$
  end if
end if

The parameters

- $a = 1$.
- $b = 2$.
- $d = 0$, no combining step.

Falls into case # 1

$$T(n) = O(\log n)$$

SOLVING RECURRENCES : THE MASTER METHOD VI

MORE: STRASSEN ALGORITHM FOR MATRICES MULTIPLICATION

The recurrence is

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ 7T(n/2) + O(n^2) & \text{if } n > 1 \end{cases}$$

The parameters: $a = 7$, $b = 2$, $d = 2$. $a > b^d$, i.e. $7 > 4$, falls into case # 3

$$T(n) = O(n^{log_2 7}) = O(n^{2.81})$$

and beats the naive $O(n^3)$.