☆ SSTreatment $= \sum_{j=1}^{v} \frac{V_j^2}{r_j} - \frac{G^2}{n}$ (unadjusted)

$(V^T R^{-1} V)$

SSBlock (adjusted) $= \rho^T \hat{\beta}$ (adjusted)

☆ Here again SSTreatment, SSBlock and corresponding SSError are orthogonal, ∴ we can construct F statistic.

☆ $$F = \frac{SSBlock\ (adjusted)\ / b-1}{Error\ \ n-b-v+1} \overset{H_0}{\sim} F_{b-1,\, n-b-v+1}$$

☆ SSTreatment unadjusted, SSBlock unadjusted are not orthogonal.

Interblock Analysis

☆ block totals contain some information on treatment effects

→ How to retrieve that information

→ How to use that information in data analysis.

Assumption → block effects are random (mixed effect model) (If there are large no of blocks, take a sample of blocks).

$v = 9$, $b = 2$

9 treatments, 2 blocks

Block-1	y_{14}	y_{16},	y_{17},	y_{19}
Block-2	y_{23}	y_{25}	y_{26}	y_{27}

Block totals
$$\begin{cases} B_1 = y_{14} + y_{16} + y_{17} + y_{19} \\ B_2 = y_{23} + y_{25} + y_{26} + y_{27} \end{cases}$$

(y_{ij} → jth treatment, ith block)

$y_{ij} = \mu + \overset{*}{\beta_i} + \tau_j + e_{ij}$

$\overset{\uparrow}{\text{random}}$

$\overset{*}{\beta_i} \sim N(0, \sigma_b^2)$

β_i's are independent of e_{ij}.

$$y_{14} = \mu + B_1^* + \tau_4 + e_{14}$$
$$y_{16} = \mu + B_1^* + \tau_6 + e_{16}$$
$$y_{17} = \mu + B_1^* + \tau_7 + e_{17}$$
$$y_{19} = \mu + B_1^* + \tau_9 + e_{19}$$

$$y_{23} = \mu + B_2^* + \tau_3 + e_{23}$$
$$y_{25} = \mu + B_2^* + \tau_5 + e_{25}$$
$$y_{26} = \mu + B_2^* + \tau_6 + e_{26}$$
$$y_{27} = \mu + B_2^* + \tau_7 + e_{27}$$

$$B_1 - B_2 = 4(B_1^* - B_2^*) + (\tau_4 + \tau_9) - (\tau_3 + \tau_5) + (e_{14} + e_{16} + e_{17} + e_{19}) - (e_{23} + e_{25} + e_{26} + e_{27})$$

$$E(B_1 - B_2) = (\tau_4 + \tau_9) - (\tau_3 + \tau_5)$$

→ n_{ij} = # j^{th} treatment occurring in i^{th} block.

→ Assumptions

① $n_{ij} = 0$ or 1

② $k_1 = k_2 = k_3 = \cdots = k_b = k$ (block size)

$$B_i = \sum_{j=1}^{v} n_{ij} y_{ij} = \sum_{j=1}^{v} n_{ij}(\mu + B_i^* + \tau_j + e_{ij})$$

$$= k\mu + \sum_{j=1}^{v} n_{ij}\tau_j + \underbrace{\sum_{j=1}^{v} n_{ij} e_{ij} + kB_i^*}_{f_i \to \text{random}}$$

$E(f_i) = 0$

$var(f_i) = k^2\sigma_b^2 + k\sigma^2 = \sigma_f^2$

$$B_i = k\mu + \sum_{j=1}^{v} n_{ij}\tau_j + f_i$$

estimate μ, τ_j's by LS method (least square).

$$S = \left(B_i - k\mu - \sum_{j=1}^{v} n_{ij}\tau_j\right)^2$$

$$\frac{\partial S}{\partial \mu} = 0, \quad \frac{\partial S}{\partial \tau_j} = 0 \quad \forall j = 1 \cdots v$$

$$\sum_{j=1}^{v} r_j \tau_j = 0 \qquad \tau = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_v \end{pmatrix} \qquad B = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_b \end{pmatrix}$$

$$k^2 b \hat{\mu} + k E_{v1}' R \hat{\tau} = kG \qquad (N' = N^T)$$

$$k R E_{v1}' \hat{\mu} + N'N \hat{\tau} = N'B$$

$$\hat{\mu} = \frac{G}{bk} \qquad (bk = n)$$

$$\tilde{\tau} = (N'N)^{-1} \left(N'B - \frac{k R E_{v1} G}{bk} \right)$$

$$= (N'N)^{-1} N'B - \frac{G E_{v1}}{bk}$$

$$R E_{v1} = N'_{v \times b} E_{b1} \quad , \quad k \times E_{b1} = N E_{v1}$$

$$= (N'N)^{-1} \left(N'B - \frac{kG}{bk} N' E_{b1} \right)$$

$$= (N'N)^{-1} \left(N'B - \frac{G}{bk} N'N E_{v1} \right)$$

$$= (N'N)^{-1} N'B - \frac{G E_{v1}}{bk}$$

combine both $\tilde{\tau}, \hat{\tau}$:

☆ $\tilde{\tau} \rightarrow$ inter block, $\hat{\tau} \rightarrow$ intra block

$$\tilde{\tau} = (N'N)^{-1} N'B - \frac{G E_{v1}}{bk} \qquad \hat{\tau} \Rightarrow Q = C\hat{\tau}$$

☆ for parameter θ there are estimators ϕ_1, ϕ_2

$$E(\phi_1) = \theta \qquad E(\phi_2) = \theta$$

$$E(a_1 \phi_1 + a_2 \phi_2) = \theta \quad \Rightarrow \quad a_1 + a_2 = 1$$

$$\rightarrow \phi^* = a_1 \phi_1 + a_2 \phi_2 \quad \Leftrightarrow \quad \phi^* = \frac{a_1}{a_1 + a_2} \phi_1 + \frac{a_2}{a_1 + a_2} \phi_2$$

minimize $var(\phi^*)$ subject to $a_1 + a_2 = 1$

$$\min_{a_1, a_2} Var(\phi^*)$$

Solution $a_1 \propto \frac{1}{var(\phi_1)}$ $\quad a_2 \propto \frac{1}{var(\phi_2)}$

☆ Let $\tau = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix}$ $\quad l'\tau \rightarrow$ linear contrasts of τ

$l = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$ $\quad \sum_{j=1}^{v} l_j \tau_j$

☆ $l'\tau$ is a linear contrasts of τ if $\sum_{j=1}^{v} l_j = 0$

$\left(= \sum_{j=1}^{v} l_j \tau_j\right)$

☆ $l'\tau = \sum_{j=1}^{v} l_j \tau_j \quad \left(\sum_{j=1}^{v} l_j = 0\right)$

estimator of $l'\tau$ is (Intra block) $= l'\hat{\tau} = L_1$

(Inter block) $= l'\tilde{\tau} = L_2$

$var(L_1) = \sigma^2 \underset{1\times v}{l'} \underset{v\times v}{C^-} \underset{v\times 1}{l}$

C^- = generalized inverse of C

$var(L_2) = \sigma_f^2 \, l'(N'N)^{-1}_{N}$

$var(f_i)$

$= k^2 \sigma_\beta^2 + k\sigma^2$

$= \sigma_f^2$

$$\frac{\theta_1}{\theta_2} = \frac{Var(L_2)}{Var(L_1)}$$

→ the above estimators are unbiased and uncorrelated

☆ pooled estimator of $l'\tau$ is

$$\frac{\theta_1}{\theta_1+\theta_2} L_1 + \frac{\theta_2}{\theta_1+\theta_2} L_2 \quad \text{where } \frac{\theta_1}{\theta_2} = \frac{Var(L_2)}{Var(L_1)}$$

☆ $$\hat{\sigma}^2 = \frac{SSError(t)}{n-b-v+1}$$

SST = SSTreatment (adj.) + SSBlock (unadj.) + SSError(t)

☆ $$\hat{\sigma}_\beta^2 = \frac{1}{n-v}\left(SS\,Block\,(adj) - \frac{b-1}{n-v-b+1} SSError(t) \right)$$

(SSBlock (adj) $= P'\hat{\beta}$)

☆ Let $\tau = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix}$ $l = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$ $l'\tau \rightarrow$ linear contrasts of τ

$\sum_{j=1}^{v} l_j \tau_j$

☆ $l'\tau$ is a linear contrasts of τ if $\sum_{j=1}^{v} l_j = 0$

$\left(= \sum_{j=1}^{v} l_j \tau_j \right)$

☆ $l'\tau = \sum_{j=1}^{v} l_j \tau_j \quad \left(\sum_{j=1}^{v} l_j = 0 \right)$

estimator of $l'\tau$ is (Intra block) $= l'\hat{\tau} = L_1$

(Inter block) $= l'\tilde{\tau} = L_2$

$var(L_1) = \sigma^2 \, l'_{1\times v} \, C^{-}_{v\times v} \, l_{v\times 1}$

C^{-} = generalized inverse of C

$var(L_2) = \sigma_f^2 \, l'(N'N)^{-1} N' l$

$var(f_i)$

$= k^2 \sigma_\beta^2 + k\sigma^2$

$= \sigma_f^2$

$$\frac{\theta_1}{\theta_2} = \frac{var(L_2)}{var(L_1)}$$

$\rightarrow$ the above estimators are unbiased and uncorrelated

☆ pooled estimator of $l'\tau$ is

$$\frac{\theta_1}{\theta_1+\theta_2} L_1 + \frac{\theta_2}{\theta_1+\theta_2} L_2 \quad \text{where} \quad \frac{\theta_1}{\theta_2} = \frac{var(L_2)}{var(L_1)}$$

☆ $$\hat{\sigma}^2 = \frac{SS\,Error(t)}{n-b-v+1}$$

$SST = SS\,Treatment\,(adj.) + SS\,Block\,(unadj.) + SS\,Error(t)$

☆ $$\hat{\sigma}_\beta^2 = \frac{1}{n-v}\left(SS\,Block\,(adj.) - \frac{b-1}{n-v-b+1} SS\,Error(t) \right)$$

$(SS\,Block\,(adj.) = P'\hat{\beta})$

~~Bla~~ Balanced ~~In~~complete Block Design

★ A special incomplete block design which follows few rules

→ b blocks each of size k

→ every treatment repeated r times

→ $n_{ij} = 1$ or 0

→ every pair of treatment occurs together in λ of the b blocks.

ex: $b = 10$, $v = 6$, $k = 3$, $r = 5$, $\lambda = 2$

B_1 - $T_1 T_2 T_5$	B_6 - $T_2 T_3 T_4$	(T_1, T_2) in B_1, B_2
B_2 - $T_1 T_2 T_6$	B_7 - $T_2 T_3 T_5$	(T_5, T_6) in B_9, B_{10}
B_3 - $T_1 T_3 T_4$	B_8 - $T_2 T_4 T_6$	$\lambda = 2$
B_4 - $T_1 T_3 T_6$	B_9 - $T_3 T_5 T_6$	
B_5 - $T_1 T_4 T_5$	B_{10} - $T_4 T_5 T_6$	

★ $D(b, k, v, r, \lambda)$

it satisfies
→ $bk = rv$
→ $\lambda(v-1) = r(k-1)$
→ $b \geq v$

★ → b blocks of size k. In each block pairs of eu's is ${}^kC_2 = \frac{(k-1)k}{2}$

→ Total numbers of pairs eus where in a pair both eu's come from the same block $= \frac{b \times k(k-1)}{2}$

$$\frac{\lambda \times v(v-1)}{\cancel{2}} = \frac{bk(k-1)}{\cancel{2}}$$

$$\lambda v(v-1) = rv(k-1)$$

$$\lambda(v-1) = r(k-1)$$

* $(N'N)$ is non singular $\text{rank}(N'N) = v$

$$\text{rank}(N_{b \times v}) = \text{rank}(N'N) = v$$
$$\text{rank}(N) \le b$$
$$v \le b$$

__Balanced Incomplete block design: (BIBD)__

Intrablock analysis $\quad y_{ij} = \mu + \beta_i + \tau_j + e_{ij}$

$$Q = C\tau \qquad Q = \begin{pmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_v \end{pmatrix}$$

$$Q_j = V_j - \sum_{i=1}^{b} \frac{n_{ij} B_i}{k_i}$$

under BIBD

$$Q_j = V_j - \frac{1}{k} \sum_{i=1}^{b} n_{ij} B_i$$

$$T_j = \sum_{i=1}^{b} n_{ij} B_i = \sum_{i(j)} B_i$$

$\sum_{i(j)}$ → denotes the sum of block totals where jth treatment is present.