

# Importance sampling

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## 1 Importance sampling

Let  $X_1, X_2, \dots$  be i.i.d. uniform variables on  $(-1.9, 2)$  and define

$$S_n = 30 + \sum_{k=1}^n X_k$$

Consider the ruin probability

$$p(n) = P(\exists k \leq n : S_k \leq 0)$$

We use monte carlo integration to estimate  $p(100)$ . The key insight is that

$$p(n) = \mathbb{E} \left[ I_{(\exists k \leq n : S_k \leq 0)} \right]$$

As a result we have by the strong law of large numbers

$$p(100) \approx \frac{1}{N} \sum_{i=1}^N I_{((\exists k \leq 100 : S_k \leq 0), i)}$$

Where  $N$  is the number of realizations of the experiment.

Define:

$$\varphi(\theta) = \int_{-1.9}^2 \exp(\theta z) dz = \frac{\exp(2\theta) - \exp(-1.9\theta)}{\theta}$$

Now let

$$g_{\theta, n}(x) = \frac{1}{\varphi(\theta)^n} \exp \left( \theta \sum_{k=1}^n x_k \right)$$

for  $x \in (-1.9, 2)^n$ . Now we easily see that

$$g_{\theta, n} = \prod_{i=1}^n (g_{\theta, 1})_i$$

With  $i = 1, \dots, n$  different realizations of  $g_{\theta, 1}$ . So we can manage to sample from  $g_{\theta, 1}$ .

This we do by the quantile method. Firstly, the distribution function is

$$G_{\theta, 1}(x) = \frac{1}{\varphi(\theta)} \int_{-1.9}^x \exp(\theta t) dt = \frac{1}{\varphi(\theta)\theta} [\exp(\theta t)]_{-1.9}^x = \frac{\exp(\theta x) - \exp(-1.9\theta)}{\exp(2\theta) - \exp(-1.9\theta)}$$

By solving the following equation we get the quantile function

$$u = \frac{\exp(\theta x) - \exp(-1.9\theta)}{\exp(2\theta) - \exp(-1.9\theta)}$$
$$Q_{\theta, 1}(u) = \frac{\log \left( u (\exp(2\theta) - \exp(-1.9\theta)) + \exp(-1.9\theta) \right)}{\theta}$$