Importance sampling

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1 Importance sampling

Let $X_1, X_2, \dots$ be i.i.d. uniform variables on $(-1.9, 2)$ and define

$$S_n = 30 + \sum_{k=1}^{n} X_k$$

Consider the ruin probability

$$p(n) = P(\exists k \leq n : S_k \leq 0)$$

We use monte carlo integration to estimate $p(100)$. The key insight is that

$$p(n) = \mathbb{E}\left[I_{(\exists k \leq n : S_k \leq 0)}\right]$$

As a result we have by the strong law of large numbers

$$p(100) \approx \frac{1}{N} \sum_{i=1}^{N} I_{((\exists k \leq 100 : S_k \leq 0), i)}$$

Where N is the number of realizations of the experiment.
Define:

$$\varphi(\theta) = \int_{-1.9}^{2} \exp\left(\theta z dz\right) = \frac{\exp\left(2\theta\right) - \exp\left(-1.9\theta\right)}{\theta}$$

Now let

$$g_{\theta,n}(x) = \frac{1}{\varphi(\theta)^n} \exp\left(\theta \sum_{k=1}^{n} x_k\right)$$

for $x \in (-1.9, 2)^n$. Now we easily see that

$$g_{\theta,n} = \prod_{i=1}^{n} \left(g_{\theta,1}\right)_i$$

With $i = 1, \dots n$ different realizations of $g_{\theta,1}$. So we can manage be sampling from $g_{\theta,1}$. This we do be the quantile method. Firstly, the distribution function is

$$G_{\theta,1}(x) = \frac{1}{\varphi(\theta)} \int_{0}^{x} \exp\left(\theta t\right) dt = \frac{1}{\varphi(\theta)\theta} \left[\exp(\theta t)\right]_0^x = \frac{\exp(\theta x) - 1}{\exp(2\theta) - \exp(-1.9\theta)}$$

By solving the following equation we get the quantile function

$$u = \frac{\exp\left(\theta x\right) - 1}{\exp(2\theta) - \exp(-1.9\theta)}$$

$$Q_{\theta,1}(u) = \frac{\log\Big(u\left(\exp(2\theta) - \exp(-1.9\theta)\right) + 1\Big)}{\theta}$$