Importance sampling

Anders G. Kristensen

09-11-2022

1 Monte Carlo integration

1.1 Direct implementation

Let $X_1, X_2, \dots$ be i.i.d. uniform variables on $(-1.9, 2)$ and define

$$S_n = 30 + \sum_{k=1}^{n} X_k$$

Consider the ruin probability

$$p(n) = P(\exists k \leq n : S_k \leq 0)$$

We use monte carlo integration to estimate $p(100)$. The key insight is that

$$p(n) = \mathbb{E}\left[I_{(\exists k \leq n : S_k \leq 0)}\right]$$

As a result we have by the strong law of large numbers

$$p(100) \approx \frac{1}{N} \sum_{i=1}^{N} I_{\left((\exists k \leq 100 : S_k \leq 0), i\right)}$$

Where N is the number of realizations of the experiment.

1.2 Importance sampling

Define:

$$\varphi(\theta) = \int_{-1.9}^{2} \exp\left(\theta z dz\right) = \frac{\exp\left(2\theta\right) - \exp\left(-1.9\theta\right)}{\theta}$$

Now let

$$g_{\theta,n}(x) = \frac{1}{\varphi(\theta)^n} \exp\left(\theta \sum_{k=1}^{n} x_k\right)$$

for $x \in (-1.9, 2)^n$. Now we easily see that

$$g_{\theta,n} = \prod_{k=1}^{n} \left(g_{\theta,1}\right)_k$$

With $k = 1, \dots n$ different realizations of $g_{\theta,1}$. So we can manage be sampling from $g_{\theta,1}$. This we do be the quantile method. Firstly, the distribution function is

$$G_{\theta,1}(x) = \frac{1}{\varphi(\theta)} \int_{-1.9}^{x} \exp\left(\theta t\right) dt = \frac{1}{\varphi(\theta)\theta} \left[\exp(\theta t)\right]_{-1.9}^{x} = \frac{\exp(\theta x) - \exp(-1.9\theta)}{\exp(2\theta) - \exp(-1.9\theta)}$$

By solving the following equation we get the quantile function

$$u = \frac{\exp(\theta x) - \exp(-1.9\theta)}{\exp(2\theta) - \exp(-1.9\theta)}$$

$$Q_{\theta,1}(u) = \frac{\log\Big(u\left(\exp(2\theta) - \exp(-1.9\theta)\right) + \exp(-1.9\theta)\Big)}{\theta}$$

To simulate from $g_{\theta,1}$ we simulate from the standard uniform distribution and then transform the result with $Q_{\theta,1}(u)$. Analogously to earlier we check whether the event we are taking indicator occurs with our modified S_n: $S_{n,IS}$ Now define

$$w^*(X_i) = q^*(X_i)/g^*_{\theta,1}(X_i)$$

Where $q^*, g^*_{\theta,1}$ are the unnormalized versions of the uniform density and the distribution from $g_{\theta,1}$ respectively. Thus our weights are

$$w^*(X_i) = \exp(-\theta X_i)$$

Let

$$w(X_i) = \frac{w^*(X_i)}{\sum_{i=1}^n w^*(X_i)}$$

From the lecture notes we now have

$$p(100)_{IS} = \sum_{i=1}^{N} w(X_i) \cdot I_{\left(\exists k \leq n : S_{k,IS} \leq 0\right)}$$

1.3 Asymptotic results

1.3.1 Direct implementation

By the central limit theorem we know that it for the original estimator holds that

$$\hat{p}(100) \overset{\text{approx.}}{\sim} \mathcal{N}\left(p(100), \frac{\sigma^2_{MC}}{N}\right)$$

We use $\hat{\sigma}^2_{MC}$ the empirical standard deviation. Thus a 95% confidence-band for $\hat{p}(100)$ is:

$$\hat{p}(100) \pm \frac{\hat{\sigma}_{MC}}{\sqrt{n}}$$

Bearing in mind that this is an asymptotic result.

1.3.2 Importance sampling

From the notes we know that by the Δ-method the asymptotic distribution of $p(100)_{IS}$ is

$$\hat{p}(100)_{IS} \overset{\text{approx.}}{\sim} \mathcal{N}\left(p(100), c^{-2}\frac{\sigma^2_{IS} + p(100)^2\sigma^2_w - 2p(100)\gamma}{N}\right)$$

Where c is the normalization constant for our densities, we can estimate this as

$$\hat{c} = \frac{1}{N}\sum_{i=1}^{N} w^*(X_i)$$

Likewise, we have empirical estimates

$$\begin{aligned}\hat{\sigma}^2_{IS} &= V(I_{\left(\exists k \leq n : S_{k,IS} \leq 0\right)} w^*(X_i)) \\ \hat{\gamma} &= cov(I_{\left(\exists k \leq n : S_{k,IS} \leq 0\right)} w^*(X_i), w^*(X_i)) \\ \hat{\sigma}^2_w &= V(w^*(X_i))\end{aligned}$$