16 Some Examples on Polynomial Reduction

Independent Set $\leq_p$ Vertex Cover. We first give the input reduction, and then show that the two problems are "equivalent" (using the solution for one can solve the other).

Input Reduction. Given an instance of independent set $G = (V, E)$ with $|V| = n$ and parameter k, we create an input graph $G' = G$ for vertex cover, with parameter $k' = n - k$.

Proof of "equivalence".
$\Rightarrow$: Suppose that S is an independent set. Consider an arbitrary edge $e = (u, v)$. Since S is independent, it cannot be the case that both u and v are in S; so one of them must be in $V - S$. It follows that every edge has at least one end in $V - S$, and so $V - S$ is a vertex cover.

$\Leftarrow$: Suppose $V - S$ is a vertex cover. Consider any two nodes u and v in S. If they were joined by edge e, then neither end of e would be in $V - S$, contradicting our assumption that $V - S$ is a vertex cover. It follows that no two nodes in S are joined by an edge, and so S is an independent set.

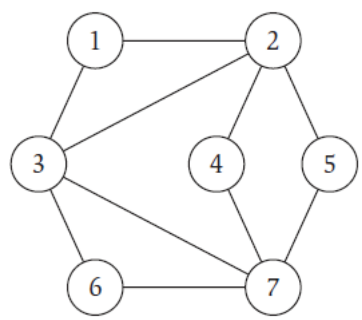

Figure 8.1 A graph whose largest independent set has size 4, and whose smallest vertex cover has size 3.

Vertex Cover $\leq_p$ Set Cover. We start from the input reduction, and then show that the "equivalence".

Input Reduction. Given an instance of Vertex Cover, specified by a graph $G = (V, E)$ and a number k, we formulate an instance of Set Cover in which the ground set U is equal to E. Each time we pick a vertex in the Vertex Cover Problem, we cover all the edges incident to it; thus, for each vertex $i \in V$, we add a set $S_i \subseteq U$ to our Set Cover instance, consisting of all the edges in G incident to i.

Proof of "equivalence".
$\Rightarrow$: If $\{v_{i_1}, \ldots, v_{i_\ell}\}$ is a vertex cover in G of size $\ell \leq k$, then the sets $S_{i_1}, \ldots, S_{i_\ell}$ cover U.

$\Leftarrow$: If $S_{i_1}, \ldots, S_{i_\ell}$ are $\ell \leq k$ sets that cover U, then every edge in G is incident to one of the vertices $v_{i_1}, \ldots, v_{i_\ell}$, and so the set $v_{i_1}, \ldots, v_{i_\ell}$ is a vertex cover in G of size $\ell \leq k$.

3-SAT $\leq_p$ Independent Set We start from the input reduction, and then show that the "equivalence".

Input Reduction. Given an instance of 3-SAT, we construct a graph $G = (V, E)$ consisting of $3k$ nodes grouped into k triangles as shown in Figure 8.3. That is, for $i = 1, 2, \ldots, k$, we construct three vertices v_{i1}, v_{i2}, v_{i3} joined to one another by edges. We give each of these vertices a label; v_{ij} is labeled with the j-th term from the clause C_i of the 3-SAT instance.

Before proceeding, consider what the independent sets of size k look like in this graph: Since two vertices cannot be selected from the same triangle, they consist of all ways of choosing one vertex from each of the triangles. This is implementing our goal of choosing a term in each clause that will evaluate to 1; but we have so far not prevented ourselves from choosing two terms that conflict. We encode conflicts by adding some more edges to the graph: For each pair of vertices whose labels correspond to terms that conflict, we add an edge between them.

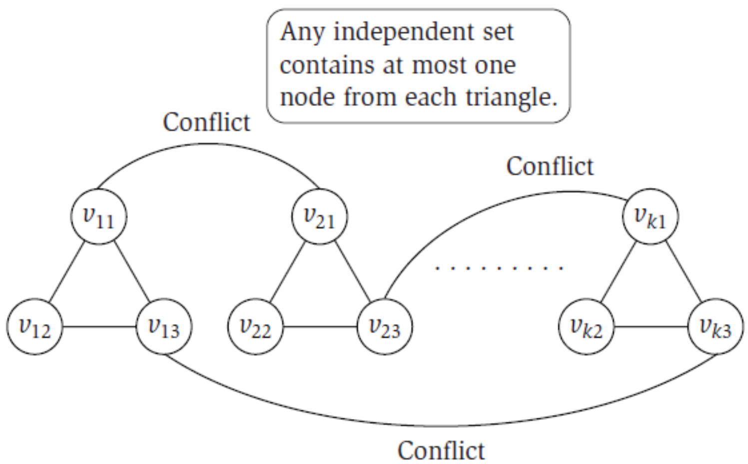

Figure 8.3 The reduction from 3-SAT to Independent Set.

Proof of "equivalence". We now show that the original 3-SAT instance is satisfiable if and only if the graph G we have constructed has an independent set of size (at least) k.

$\Rightarrow$: If the 3-SAT instance is satisfiable, then each triangle in our graph contains at least one node whose label evaluates to 1. Let S be a set consisting of one such node from each triangle. We claim S is independent; for if there were an edge between two nodes $u, v \in S$, then the labels of u and v would have to conflict; but this is not possible, since they both evaluate to 1.

$\Leftarrow$: Suppose our graph G has an independent set S of size at least k. Then, first of all, the size of S is exactly k, and it must consist of one node from each triangle. Now, we claim that there is a truth assignment v for the variables in the 3-SAT instance with the property that the labels of all nodes in S evaluate to 1. Here is how we could construct such an assignment v. For each variable x_i, if neither x_i nor $\bar{x}_i$ appears as a label of a node in S, then we arbitrarily set $v(x_i) = 1$. Otherwise, exactly one of x_i or $\bar{x}_i$ appears as a label of a node in S; for if one node in S were labeled x_i and another were labeled $\bar{x}_i$, then there would be an edge between these two nodes, contradicting our assumption that S is an independent set. Thus, if x_i appears as a label of a node in S, we set $v(x_i) = 1$, and otherwise we set $v(x_i) = 0$. By constructing v in this way, all labels of nodes in S will evaluate to 1.

Transitivity of reductions. If $Z \leq_P Y$, and $Y \leq_P X$, then $Z \leq_P X$.