Vector Valued Function: $r(t) = f(t)\,\underset{\sim}{i} + g(t)\,\underset{\sim}{j} + h(t)\,\underset{\sim}{k}$

$r'(a) = \lim_{\Delta t \to 0} \dfrac{r(a+\Delta t) - r(a)}{\Delta t}$

$r'(a) = \langle f'(a), g'(a), h'(a) \rangle$

Arc Length: $s = \int_a^b \|r'(t)\|\,dt = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\,dt$

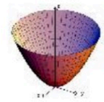
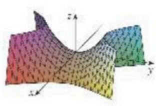
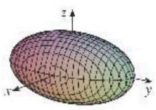
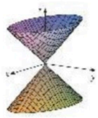
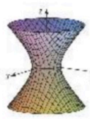
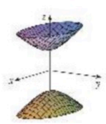
Any Equation in x, y, z with 1 missing variable is a cylinder. eg. $y^2+z^2=1$

Quadratic Surfaces.

Second Degree equation in three variables:

$$\left.\begin{aligned} Ax^2+By^2+Cz^2+J &= 0 \\ Ax^2+By^2+Iz &= 0 \end{aligned}\right\} \text{ std forms.}$$

Quadric surfaces

- **Cylinder** = infinite prism
- **Elliptic paraboloid**: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$ 
- **Hyperbolic paraboloid**: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$ 
- **Ellipsoid**: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ 
- **Elliptic cone**: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ 
- **Hyperboloid of one sheet**: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ 
- **Hyperboloid of two sheets**: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$ 

Limit.

Definition 14 (Limit: A formal definition). Let f be a function of two variables whose domain D contains points arbitrarily close to (a,b). We say that the limit of $f(x,y)$ as (x,y) approaches (a,b) is $L \in \mathbb{R}$, denoted by

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for any number $\epsilon > 0$ there exists a number $\delta > 0$ such that $|f(x,y) - L| < \epsilon$ whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$.

Show Not Exist.

Take limit via different paths that give different limit. eg.

Show Exists.

- using properties of limits/continuity
- Squeeze Theorem.

Theorem 18 (Squeeze). Suppose

- $|f(x,y) - L| \le g(x,y)\ \forall (x,y)$ close to (a,b)
- $\lim_{(x,y)\to(a,b)} g(x,y) = 0$

Then

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

Continuity: f is continuous at (a,b) if

$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Theorem 21 (Continuity and Composition). Suppose $f(x,y)$ is continuous at (a,b) and $g(x)$ is continuous at $f(a,b)$. Then

$$h(x,y) = (g \circ f)(x,y) = g(f(x,y))$$

is continuous at (a,b).

Theorem 3 (Equation of Tangent Plane). Consider the surface S given by $z = f(x,y)$. A normal vector to the tangent plane to S at (a,b) is

$$\langle f_x(a,b), f_y(a,b), -1 \rangle.$$

$\langle 1, 0, f_x(a,b) \rangle \times \langle 0, 1, f_y(a,b) \rangle$

The tangent plane is given by

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Differentiability

Then the increment in z at (a,b) is defined by

$$\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b).$$

Definition 6 (Differentiability - Two Variable). Let $z = f(x,y)$. We say that f is differentiable at (a,b) if we can write

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where ϵ_1 and ϵ_2 are functions of Δx and Δy which vanish (i.e. $\epsilon_1, \epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$).

Linear Approximation.

Theorem 7. Suppose $z = f(x,y)$ is differentiable at (a,b). Let Δx and Δy be small increments in x and y respectively from (a,b). Then

$$\Delta z \approx f_x(a,b)\Delta x + f_y(a,b)\Delta y.$$

f_x exists $\wedge$ f_y exists $\wedge$ continuous $\Longleftrightarrow$ differentiable at (a,b)

at (a,b)

Chain Rule

Theorem 10 (The Chain Rule - General Version). Suppose that u is a differentiable function of n variables $x_1, \dots, x_n$, and each x_j is a differentiable function of m variables $t_1, \dots, t_m$. Then u is a function of $t_1, \dots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, \dots, m$.

Theorem 11 (Implicit Differentiation: Two Independent Variables). Suppose the equation $F(x,y,z) = 0$, where F is differentiable, defines z implicitly as a differentiable function of x and y. Then,

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)}, \quad \frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)}$$

provided $F_z(x,y,z) \neq 0$.

Directional Derivative.

Definition 12 (Directional derivative). The directional derivative of $f(x,y)$ at (x_0,y_0) in the direction of unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$

provided this limit exists.

rate of change of the function at the point (x_0,y_0) in the direction given by $\underset{\sim}{u}$

Theorem 13 (Computing Directional Derivative). If $f(x,y)$ is a differentiable function, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a,b \rangle$ and

$$D_{\mathbf{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b.$$

We can rewrite it in terms of vectors:

$$D_{\mathbf{u}} f(x,y) = \langle f_x, f_y \rangle \cdot \langle a,b \rangle = \langle f_x, f_y \rangle \cdot \hat{\mathbf{u}} = \nabla f(x,y) \cdot \hat{\underset{\sim}{u}}$$

Level Curve vs ∇f

Theorem 1 (Level Curve vs ∇f). Suppose $f(x,y)$ is differentiable function of x and y at (x_0,y_0). Suppose $\nabla f(x_0,y_0) \neq \mathbf{0}$. Then $\nabla f(x_0,y_0)$ is normal to the level curve $f(x,y) = k$ that contains the point (x_0,y_0).

Level Surface vs ∇f

Theorem 3 (Tangent Plane to Level Surface).

$$\nabla F(x_0,y_0,z_0) \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$$

Theorem 4 (Maximizing Rate of Increase/Decrease of f). Suppose f is a differentiable function of two or three variables. Let P denote a given point. Assume $\nabla f(P) \neq \mathbf{0}$.

- $\nabla f(P)$ points in the direction of maximum rate of change of f at P (maximum value of $D_{\mathbf{u}} f(P)$ is $\|\nabla f(P)\|$)
- $-\nabla f(P)$ points in the direction of minimum rate of change of f at P (minimum value of $D_{\mathbf{u}} f(P)$ is $-\|\nabla f(P)\|$)

eg. $f(x,y) = xe^y$, $\nabla f(x,y) = \langle e^y, xe^y \rangle$

At $(2,0)$, f increases fastest in direction $\nabla f(2,0) = \langle 1, 2 \rangle$.

max rate of change is $\|\nabla f(2,0)\| = \sqrt{5}$ //

Local Min/Max

Let (a,b) be a critical point of f, that is

$$f_x(a,b) = 0 \quad \text{and} \quad f_y(a,b) = 0 .$$

Let $D = f_{xx}(a,b)f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2$.

- f has a local maximum at (a,b) if $D > 0$ and $f_{xx}(a,b) < 0$.
- f has a local minimum at (a,b) if $D > 0$ and $f_{xx}(a,b) > 0$.
- f has neither a local maximum nor a local minimum point at (a,b) if $D < 0$.

  (In this case, (a,b) is known as a **saddle point**)

Absolute Max, Min

Step 1. Find the values of f at its critical points in D.
Step 2. Find the extreme values of f on the boundary of D.
Step 3. The largest of the values from **Step 1** and **Step 2** is the absolute maximum value; the smallest of these values is the absolute minimum value.

Legrange Multiplier.

At max/min, the gradient vectors of f and g are parallel.

Theorem 15 (Lagrange Multipliers for Function of Two Variables). Suppose $f(x,y)$ and $g(x,y)$ are differentiable functions such that $\nabla g(x,y) \neq \mathbf{0}$ on the constraint curve $g(x,y) = k$.

Suppose that the minimum/maximum value of $f(x,y)$ subject to the constraint $g(x,y) = k$ occurs at (x_0,y_0).Then

$$\nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0)$$

for some constant λ (called a Lagrange Multiplier).

Extra:

Test Coplaner: $\underset{\sim}{a} \cdot |\underset{\sim}{b} \times \underset{\sim}{c}| = 0 \rightarrow$ coplanar

Volume of parallelepiped $= |\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c})|$

Area of parallelogram $= |\underset{\sim}{a} \times \underset{\sim}{b}|$

Eg. Differentiability.

Example: Using the definition of differentiability, show that $f(x,y) = 2x^2 - xy$ is differentiable at $(1,2)$.

Solution. Let $z = f(x,y)$. Then, the increment of z at $(1,2)$ is

$$\begin{aligned} \Delta z &= f(1+\Delta x, 2+\Delta y) - f(1,2) \\ &= 2\Delta x - \Delta y + 2(\Delta x)^2 - \Delta x \Delta y \end{aligned}$$

$f_x(1,2) = 2$, $f_y(1,2) = -1$

so $\Delta z = f_x(1,2)\Delta x + f_y(1,2)\Delta y + 2(\Delta x)^2 - \Delta x\Delta y$

Then $2(\Delta x)^2 - \Delta x \Delta y = \epsilon_1 \Delta x + \epsilon_2 \Delta y$, we chose $\epsilon_1 = 2\Delta x$, $\epsilon_2 = -\Delta x$.

Hence $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$.

Eg. Squeeze Theorem

Show $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$.

$\left|\frac{3x^2y}{x^2+y^2} - 0\right| = 3\frac{x^2}{x^2+y^2}|y| \le 3|y|$. Since $\lim_{(x,y)\to(0,0)} |y| = 0$, shown

Eg. Find equation of normal line at point $(2,1)$ to the ellipse $\frac{x^2}{4} + y^2 = 2$.

Let $f(x,y) = \frac{x^2}{4} + y^2$, $\nabla f(x,y) = \langle \frac{x}{2}, 2y \rangle$

$\nabla f(2,1) = \langle 1, 2 \rangle$.

Hence the normal is $x = 2+t$, $y = 1+2t$.

Find equation of tangent line.

Since $\nabla f(2,1) = \langle 1, 2 \rangle$, a vector tangential is $\langle 2, -1 \rangle$.

Hence the tangent line is $x = 2+2t$, $y = 1-t$.

Eg. Find tangent plane to the elliptic paraboloid $\boxed{z = 2x^2 + y}$ at pt $(1,1,3)$.

$f_x(1,1) = 4$, $f_y(1,1) = 2$

equation of plane is $z = f(1,1) + 4(x-1) + 2(y-1)$
$= 4x + 2y - 3$.

Eg. Find equation of tangent plane at $(-2,1,3)$ to the ellipsoid $\underbrace{\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3}_{\text{level surface with } k=3 \text{ + } F(x,y,z).}$

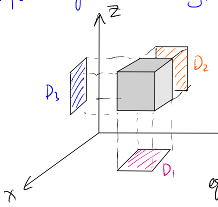
$F(x,y,z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$

get F_x, F_y, F_z

$\downarrow$ use formula involve level surface

$$\frac{d}{dx}(a) = 0$$
$$\frac{d}{dx}(x) = 1$$
$$\frac{d}{dx}(au) = a\frac{du}{dx}$$
$$\frac{d}{dx}(u+v-w) = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}$$
$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{1}{v}\frac{du}{dx} - \frac{u}{v^2}\frac{dv}{dx}$$
$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$
$$\frac{d}{dx}(\sqrt{u}) = \frac{1}{2\sqrt{u}}\frac{du}{dx}$$
$$\frac{d}{dx}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dx}$$
$$\frac{d}{dx}\left(\frac{1}{u^n}\right) = -\frac{n}{u^{n+1}}\frac{du}{dx}$$
$$\frac{d}{dx}[f(u)] = \frac{d}{du}[f(u)]\frac{du}{dx}$$
$$\frac{d}{dx}[\ln u] = \frac{d}{dx}[\log_e u] = \frac{1}{u}\frac{du}{dx}$$
$$\frac{d}{dx}[\log_a u] = \log_a e \frac{1}{u}\frac{du}{dx}$$
$$\frac{d}{dx}e^u = e^u\frac{du}{dx}$$
$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}$$
$$\frac{d}{dx}(u^v) = vu^{v-1}\frac{du}{dx} + \ln u\ u^v\frac{dv}{dx}$$
$$\frac{d}{dx}\sin u = \cos u\frac{du}{dx}$$
$$\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}$$
$$\frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx}$$
$$\frac{d}{dx}\cot u = -\csc^2 u\frac{du}{dx}$$
$$\frac{d}{dx}\sec u = \sec u \tan u\frac{du}{dx}$$
$$\frac{d}{dx}\csc u = -\csc u \cot u\frac{du}{dx}$$

Triple Integral over Type I, II, III regions



Type I: $E=\{(x,y,z):(x,y)\in D, u_1(x,y)\le z\le u_2(x,y)\}$

Type II: $E=\{(x,y,z):(y,z)\in D_2, u_1(y,z)\le x\le u_2(y,z)\}$

Type III: $E=\{(x,y,z):(x,z)\in D_3, u_1(x,z)\le y\le u_2(x,z)\}$

eg. $V=\iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$; D is the projection, u_2 is the top face, u_1 bottom

Cylindrical Coordinate. Use when projection can be described in polar coordinate.

cylindrical → rectangular

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$.

rectangular → cylindrical

$r = x^2+y^2$

$\tan\theta = \frac{y}{x}$ $(x\neq 0)$

$z = z$

Spherical Coordinate

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$\rho^2 = x^2+y^2+z^2$

Convert spherical to rectangular coordinates.

Spherical Coordinates simplify equations for regions involving sphere and cones.

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\, d\rho\, d\theta\, d\phi}$$

Change of Variable

Def: The Jacobian of transformation T given by $x = x(u,v)$ and $y = y(u,v)$ is $\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$\|\vec{r}_u\times\vec{r}_v\| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ ⇒ Hence, $dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

2D: $\boxed{\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}$

3D: $\boxed{\iiint_R f(x,y,z)\,dV = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw}$

3D Jacobian: $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$

Line Integral

Scalar Field. Theorem: Suppose C is given by $r(t) = \langle x(t), y(t)\rangle$, $a\le t\le b$

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = \int_a^b f(x(t), y(t))\,\|r'(t)\|\,dt.$$

In Vector Field: $W = \int_C \vec{F}\cdot\vec{T}\,ds = \boxed{\int_a^b \vec{F}\cdot\vec{r}'(t)\,dt = \int_C \vec{F}\cdot d\vec{r}}$

- Theorem $\int_C \vec{F}\cdot d\vec{r} = -\int_{-C} \vec{F}\cdot d\vec{r}$

Vector Field 3D case:

$$\int_a^b F(x(t), y(t), z(t))\cdot r'(t)\,dt = \int_a^b \langle P, Q, R\rangle\cdot\langle x'(t), y'(t), z'(t)\rangle\,dt$$

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz}$$

Conservative Vector Field. $\vec{F} = \nabla f$, f is the potential function.

Determine Conservative: ∘ Prove $\vec{F} = \nabla f$, or

∘ if $\vec{F}$ is in open and simply connected Region, check:

$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ iff $\vec{F}$ is conservative on D (2D)

$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}$, $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$, $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ (3D)

Fundamental Theorem for Line Integral

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} = f(B) - f(A)}$$

(path Independent in conservative field).

Green's Theorem: $\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$

C is positively oriented, simple closed, piecewise smooth curve. C can be written as ∂D, positively oriented boundary of D.

Area of Plane Region: Area $= \iint_D 1\,dA =$ $A = \int_C x\,dy = -\int_C y\,dx = \frac{1}{2}\left(\int_C x\,dy - y\,dx\right)$

Parametrize: $r(t)$ is for curve, $r(u,v)$ traces out a space.

Surface Integral (scalar Field).

Theorem 7 (Surface Integral Formula).

$$\boxed{\iint_S f(x,y,z)\,dS = \iint_D f(x(u,v), y(u,v), z(u,v))\,\|\mathbf{r}_u\times\mathbf{r}_v\|\,dA.}$$

Surface Integral of Vector Field.

If S is the surface $z = g(x,y)$, $\boxed{\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\left(\sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2+1}\right) dA}$

Surface Area: $A(S) = \iint_S 1\,dS = \iint_D \|r_u\times r_v\|\,dA$

Surface with Orientation:

positive orientation is the one for which the normal vector point outward from E.

negative orientation is Inward pointing

Get unit normal vector $\underset{\sim}{n}$, use parametrization $r(u,v)$ so

$\underset{\sim}{n} = \frac{r_u\times r_v}{\|r_u\times r_v\|}$ and the opposite orientation is $-\underset{\sim}{n}$

Surface Integral of Vector Field.

$\boxed{\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS}$ ⟶ $\iint_S \mathbf{F}\cdot d\mathbf{S} = \boxed{\iint_D \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA}$ { note $dS = \|\vec{r}_u\times\vec{r}_v\|\,dA$. Need to check if $\underset{\sim}{n}$ is indeed given by $\frac{r_u\times r_v}{\|r_u\times r_v\|}$

Special Case: Suppose that S is given by

$$\mathbf{r}(x,y) = \langle x, y, g(x,y)\rangle, \quad (x,y)\in D.$$

- Upward pointing normals: $\frac{\langle -g_x, -g_y, 1\rangle}{\sqrt{g_x^2+g_y^2+1}}$
- Downward pointing normals: $\frac{\langle g_x, g_y, -1\rangle}{\sqrt{g_x^2+g_y^2+1}}$

Divergence and Curl. $\text{div}\,\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla\cdot\vec{F}$

$$\text{curl}\,\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\underset{\sim}{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\underset{\sim}{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\underset{\sim}{k} = \nabla\times\vec{F}$$

Gauss' Theorem:

Theorem 2 (The Divergence Theorem / Gauss' Theorem)**.** Let E be a solid region where the boundary surface S of E is piecewise smooth with positive (outward) orientation.
Let $\mathbf{F}(x,y,z)$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E.
Then,

$$\boxed{\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \text{div}\,\mathbf{F}\,dV.}$$

Stoke's Theorem.

Theorem 4 (Stoke's Theorem)**.** Let C be the boundary curve (simple closed curve) of a surface S with unit normals $\mathbf{n}$. Suppose that C is positively oriented with respect to $\mathbf{n}$.
Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then,

$$\boxed{\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}}$$

Somehow Linking Equations:

$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}\cdot\vec{r}'(t)\,dt = \int_C P\,dx + Q\,dy + R\,dz = \int_C \nabla f\cdot d\vec{r}$ (in conservative)

$= f(b) - f(a)$ (in conservative)

$= \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$ (under Stoke's Theorem Condition)

Pythagorean identities:

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

Reciprocal identities:

$\csc x = \frac{1}{\sin x}$

$\sec x = \frac{1}{\cos x}$

$\cot x = \frac{1}{\tan x}$

Even - odd identities:

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\tan(-x) = -\tan x$

Co - function identities:

$\cos\left(\frac{\pi}{2} - x\right) = \sin x$

$\sin\left(\frac{\pi}{2} - x\right) = \cos x$

$\tan\left(\frac{\pi}{2} - x\right) = \cot x$

$\cot\left(\frac{\pi}{2} - x\right) = \tan x$

$\csc\left(\frac{\pi}{2} - x\right) = \sec x$

$\sec\left(\frac{\pi}{2} - x\right) = \csc x$

Periodicity identities:

$\sin(x\pm 2\pi) = \sin x$

$\cos(x\pm 2\pi) = \cos x$

$\tan(x\pm\pi) = \tan x$

$\cot(x\pm\pi) = \cot x$

$\sec(x\pm 2\pi) = \sec x$

$\csc(x\pm 2\pi) = \csc x$

Sum and difference formulas:

$\sin(x\pm y) = \sin x\cos y \pm \cos x\sin y$

$\cos(x\pm y) = \cos x\cos y \mp \sin x\sin y$

$\tan(x\pm y) = \frac{\tan x \pm \tan y}{1\mp\tan x\tan y}$

Product to sum formulas:

$\sin x\cdot\sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$

$\cos x\cdot\cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$

$\sin x\cdot\cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$

$\cos x\cdot\sin y = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$

Half - angle formulas:

$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{2}}$

$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos x}{2}}$

$\tan\left(\frac{x}{2}\right) = \frac{(1-\cos x)}{\sin x}$

Sum to product:

$\sin x \pm \sin y = 2\sin\left(\frac{x\pm y}{2}\right)\cos\left(\frac{x\mp y}{2}\right)$

$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

Law of sines:

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Law of cosines:

$a^2 = b^2 + c^2 - 2bc\cos A$

$A = \cos^{-1}\left(\frac{b^2+c^2-a^2}{2bc}\right)$

Double - angle formulas:

$\sin 2\theta = 2\cdot\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

Area of triangle:

$\frac{1}{2}ab\sin C$

$\sqrt{s(s-a)(s-b)(s-c)}$,

where $s = \frac{1}{2}(a+b+c)$

	0°	30°	45°	60°	90°
sin(θ)	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos(θ)	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan(θ)	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined