2. Show that in a poset category, all arrows are both monic and epic.

objects: elements of P,

unique arrow: $a \to b$ iff $a \le b$

7. A *preorder* is a set P equipped with a binary relation $p \le q$ that is both reflexive and transitive: $a \le a$, and if $a \le b$ and $b \le c$, then $a \le c$. Any preorder P can be regarded as a category by taking the objects to be the elements of P and taking a unique arrow,

$$a \to b \quad \text{if and only if} \quad a \le b. \tag{1.2}$$

The reflexive and transitive conditions on $\le$ ensure that this is indeed a category.

identity

composition

Going in the other direction, any category with at most one arrow between any two objects determines a preorder, simply by defining a binary relation $\le$ on the objects by (1.2).

to prove $f: a \to b$ is monic.

need to prove $\forall\, g, h: c \to a$, $fg = fh \Rightarrow g = h$

Since any arrow $c \to a$ is unique, $g = h$

3. (Inverses are unique.) If an arrow $f : A \to B$ has inverses $g, g' : B \to A$ (i. e., $g \circ f = 1_A$ and $f \circ g = 1_B$, and similarly for g'), then $g = g'$.

if $g \neq g'$,

$\because$ $f \circ g = 1_B = f \circ g'$

$\therefore$ $g' \circ f \circ g = g' \circ f \circ g'$

$(g' \circ f) \circ g = g' \circ (f \circ g')$

$1_A\, g = g' \circ 1_B$

$g = g'$

Single object

morphism for elements

$$\begin{tikzcd} \mathbb{N} \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d] & \mathbb{Z} \times \mathbb{Z} \\ & \mathbb{Z} & \end{tikzcd}$$

inversion

projection of $A \times \text{terminal} \underset{1}{\overset{p}{\cong}} A$

is iso

4. With regard to a commutative triangle,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

in any category $\mathbf{C}$, show

(a) if f and g are isos (resp. monos, resp. epis), so is h;

(b) if h is monic, so is f;

(c) if h is epic, so is g;

(d) (by example) if h is monic, g need not be.

General, h identity, f monic, g epic

B: Greater cardinality.

monic, $A = \mathbb{N} \to \mathbb{Z}$ in monoids

$C =$

h: inclusion

now $h = g \circ f$

(a) If f, g are iso, $\exists\, f^{-1}: B \to A$, $g^{-1}: C \to B$

Construct $f^{-1} g^{-1}$, then $h \circ (\ ^{-1}\ ^{-1}) = h \circ (\ )^{-1}$

2) f, g mono. $\forall\, C \xrightarrow{a,b} A \xrightarrow{f} B$ $\forall a, b$: if $fa = fb \Rightarrow a = b$

$\forall\, D \xrightarrow{i,j} B \xrightarrow{g} C$ $\forall i, j$, if $gi = gj \Rightarrow i = j$

$\forall\, m, n$, if $hm = hn$

$\forall\, m, n : X \to A$ if $hm = hn$

then $(g \circ f)\, m = (g \circ f)\, n$

now $gm, gn : X \to B$

$g(fm) = g(fn)$

$\Rightarrow fm = fn$

$\Rightarrow m = n$

3) $\forall\, i, j : A \to X$. If $ih = jh$

$i(f\boxed{g}) = j(f\boxed{g})$

$(if)\boxed{g} = (jf)\boxed{g}$

$ig = jg$

$i = j$

ii/ h monic, so is f

$h = g \circ f$

$\forall\, m, n : X \to A$ if $fm = fn$

then $gfm = gfn$

$hm = hn$

$m = n$

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\
& C
\end{tikzcd}
$$

B: $\{0,1\}$, A: $\{1\}$, C: $\{1\}$

g don't need

h monic

(d) In **Sets**, put $A = C = \{0\}$, $B = \{0, 1\}$, and all arrows constantly 0. h is monic but g is not.

$\forall\, m, n\ X \to A$ ($\{1\}$) if $hm = hn$,

$\forall\ i,j : X \to \{0,1\}$

$\exists\ i \neq j$ st $g_i = g_j$ ✓

13. In any category with binary products, show directly that

$$A \times (B \times C) \cong (A \times B) \times C.$$

construct an iso. using diagrams

$$\begin{tikzcd}
 & A\times(B\times C) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\
A & & B\times C \arrow[dl, "\pi_1'"'] \arrow[dr, "\pi_2'"] & \\
 & B & & C
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & (A\times B)\times C \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
 & A\times B \arrow[dl, "p_1'"'] \arrow[dr, "p_2'"] & & C \\
A & & B &
\end{tikzcd}$$

compose

$$\begin{tikzcd}
 & & C \\
A\times(B\times C) \arrow[urr, "\pi_2'\circ\pi_2"] \arrow[rr, "f"] \arrow[dr, "\pi_1"'] \arrow[drrr, bend right, "\pi_1'\circ\pi_2"'] & & (A\times B)\times C \arrow[u] \arrow[d, "p_1"] & \\
 & A & A\times B \arrow[l] \arrow[r] & B
\end{tikzcd}$$

similarly.

$g \circ f : A \times (B \times C) \to A \times (B \times C)$

$\overset{?}{=}$ id

$$
\begin{tikzcd}
(A\times B)\times C \arrow[r, "g"] \arrow[urr, bend left, "p_1'\circ p_1"] \arrow[drr, bend left, "p_2'\circ p_1"] \arrow[ddrrr, bend right, "p_2"] & & A\times(B\times C) \arrow[u, "\pi_1"] \arrow[d] \\
& & A \\
& & B\times C \arrow[dl] \arrow[dr] \\
& B & & C
\end{tikzcd}
$$

$$
\begin{tikzcd}
A\times(B\times C) \arrow[r, "g\circ f = id ?"'] & A\times(B\times C) \arrow[r, "\pi_2"] \arrow[dr, "\pi_1"'] & B\times C \arrow[r, "\pi_1'"] \arrow[dr, "\pi_2'"'] & B \\
& & A & C
\end{tikzcd}
$$

$$
\left|\begin{array}{l}
\pi_1 g f = \underline{p_1' \circ p_1} f = \pi_1 = \pi_1 \circ id \\
\pi_1' \pi_2 g f = \pi_1' \pi_2\, id \\
\pi_2' \pi_2 g f = \pi_2' \pi_2\, id
\end{array}\right.
\quad \Rightarrow gf = id
$$

↳ Grp(e). Grp(Top) = "topological group"

Grp(Grp) = "abelian group"

$X \times X \xrightarrow{\mu} X$

terminal $* \xrightarrow{e} X$

inverse $i: X \to X$

terminal group

$G \times G \xrightarrow{\mu} G$

$\boxed{\{e\} \xrightarrow{e} G}$

$G \xrightarrow{\text{inverse}} G$

$= ba$

$\mu((e,b)(a,e)) = \mu(e,b)\mu(e,a)$

? homomorphism

$\mu(a,b) = \mu((a,e)(e,b))$

$\overset{\text{homo}}{=} \mu(a,e)\mu(e,b) = ab$

+ diagram associativity

unit

inverse

$G \cong G \times * \xrightarrow{id \times e} G \times G$

bc of terminal

μ

id

G

has to commute the unit axiom

$\mu(a,e) = a$

没用到 inverse

---

if 有 mathematical structure set with additional data

str $X, Y, Z, \ldots$ morphisms $X \times Y \times Z \to X$

(co)limits (co)limits

categorical version + commutative diagram

general

⊗ no universal property

but can add some structure to make it have the property of product

$\mathcal{C} \to str(\mathcal{C})$

$\mathcal{C}$ = Category, can check str(Cat)

(P projection: more like mapping out of P)

retract of some projection

$$\downarrow \text{lift}$$

P $\xrightarrow{\text{restrict part}}$

$$\begin{tikzcd}
 & & & & Y \arrow[d, "e"] \arrow[d, phantom, shift left=4, "\text{epi}"] \\
A \arrow[r, "r"] \arrow[rrrru, bend left, "l \circ r"] \arrow[rr, bend right, "\text{id}"'] & P \arrow[r, "s"] \arrow[rrru, "\exists\ \text{lift}\ l"] & A \arrow[r, "f"] \arrow[rru, dashed, "\exists\ \text{lift?}"] & & X
\end{tikzcd}$$

$$\boxed{e \circ l \circ r} = f \circ s \circ r = f$$

15. Given a category $\mathbf{C}$ with objects A and B, define the category $\mathbf{C}_{A,B}$ to have objects (X, x_1, x_2), where $x_1 : X \to A$, $x_2 : X \to B$, and with arrows $f : (X, x_1, x_2) \to (Y, y_1, y_2)$ being arrows $f : X \to Y$ with $y_1 \circ f = x_1$ and $y_2 \circ f = x_2$.

Show that $\mathbf{C}_{A,B}$ has a terminal object just in case A and B have a product in $\mathbf{C}$.

$X \times Y$ T x_1, x_2

if $\exists$ $A \times B$.

$$\begin{tikzcd} & X \arrow[dl] \arrow[d, "\exists !"] \arrow[dr] & \\ A & A \times B \arrow[l] \arrow[r] & B \end{tikzcd}$$

if $T = (X, x_1, x_2) \in \mathcal{C}_{A,B}$

$= \left(A \xleftarrow{x_1} X \xrightarrow{x_2} B \right) \in \mathcal{C}_{A,B}$

suppose

$$\begin{tikzcd} & Y \arrow[dl, "y_1"'] \arrow[dr, "y_2"] \arrow[dd, dashed, "\exists !"] & \\ A & & B \\ & X \arrow[ul] \arrow[ur] & \end{tikzcd} \in \mathbb{C}$$

gives an object in $\mathcal{C}_{A,B}$

$$\begin{tikzcd} \left(A \right. & Y \arrow[l, "y_1"'] \arrow[r, "y_2"] \arrow[d, "\exists !"] & \left. B \right) \\ \left(A \right. & X \arrow[l, "x_1"] \arrow[r, "x_2"'] & \left. B \right) = T \end{tikzcd}$$

in $\mathcal{C}_{AB}$

guarantees

objects

$$\begin{tikzcd} & Y \arrow[dl, "y_1"'] \arrow[dr, "y_2"] \arrow[dd, "\exists !"] & \\ A & & B \\ & X \arrow[ul, "x_1"] \arrow[ur, "x_2"'] & \end{tikzcd}$$

commutes

17. In any category **C** with products, define the *graph* of an arrow $f : A \to B$ to be the monomorphism

$$\Gamma(f) = \langle 1_A, f\rangle : A \rightarrowtail A \times B$$

(Why is this monic?). Show that for **C** = **Sets** this determines a functor Γ : **Sets** $\to$ **Rel** to the category **Rel** of relations, as defined in the exercises to Chapter 1. (To get an actual relation $R(f) \subseteq A \times B$, take the image of $\Gamma(f) : A \rightarrowtail A \times B$.)

relation as morphisms

check def or left inverse given by first projection

graph of identity is diagano

composition $(a,c) \in f \circ g$ if $\exists b$ st $(a,b) \in f$, $(b,c) \in$

18. Show that the forgetful functor U : **Mon** $\to$ **Sets** from monoids to sets is representable. Infer that U preserves all (small) products.

$\exists X \in$ Mon st $\mathrm{Hom}_{\mathrm{Mon}}(X,-) \cong U$

$X =$ initial $\mathrm{Hom}_{\mathrm{Mon}}(X,-) \cong 1$

$X =$ terminal $\mathrm{Hom}_{\mathrm{Mon}}(X, M)$

$X = \mathbb{N}$ $\mathrm{Hom}_{\mathrm{Mon}}(\mathbb{N}, M) \cong U(M)$

$f \mapsto f(1)$

b/c $\boxed{\mathbb{N} = \mathrm{Free}(1)}$, so 只要 specify $f(1)$.

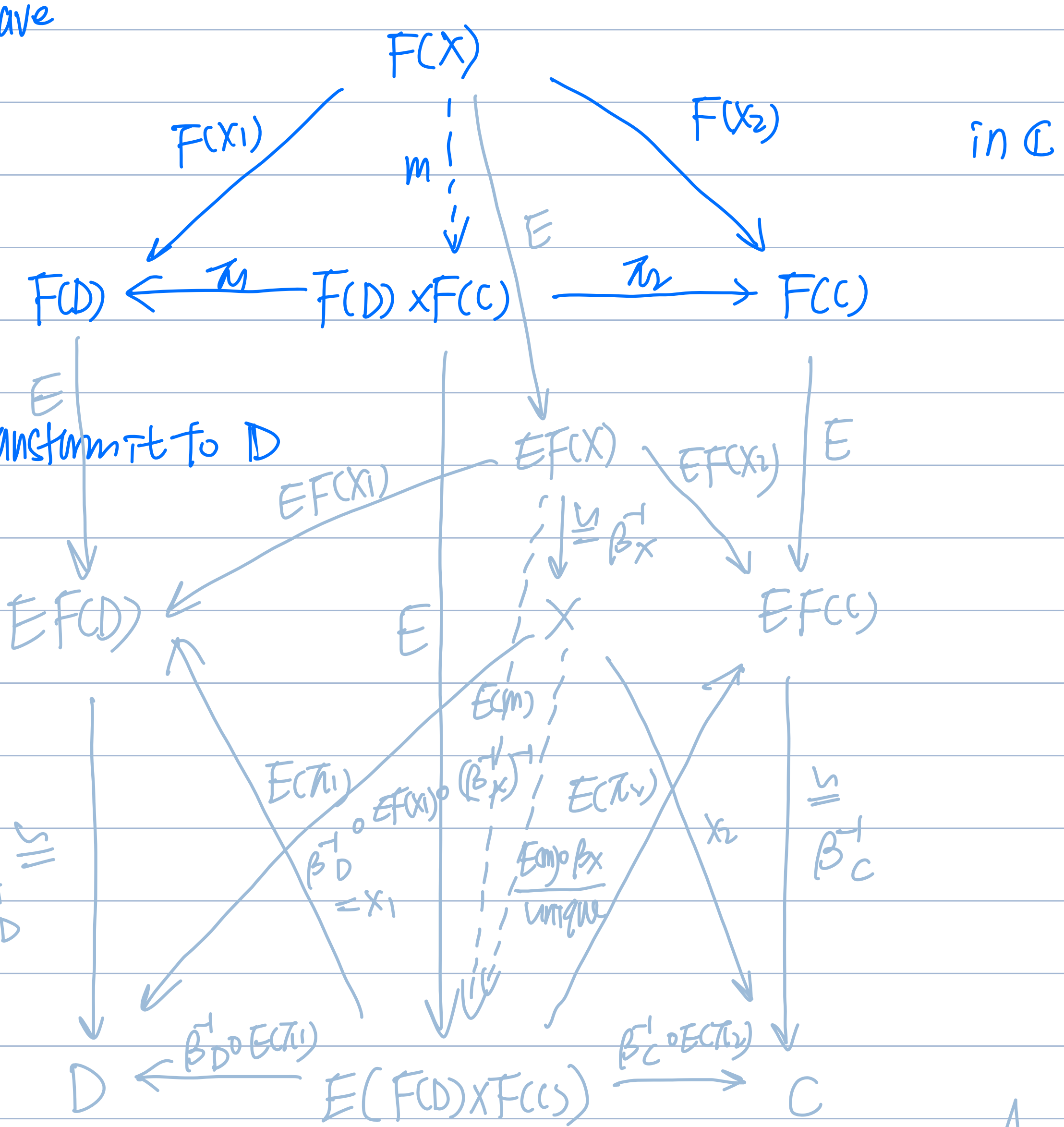
Suppose $\mathbb{C}$ has all binary products.

In $\mathbb{D}$, $\forall D, E \in \mathbb{D}$. want to find the product that satisfy UMP: $\forall X \in \mathbb{D}, x_1 : X \to D, x_2 : X \to E$

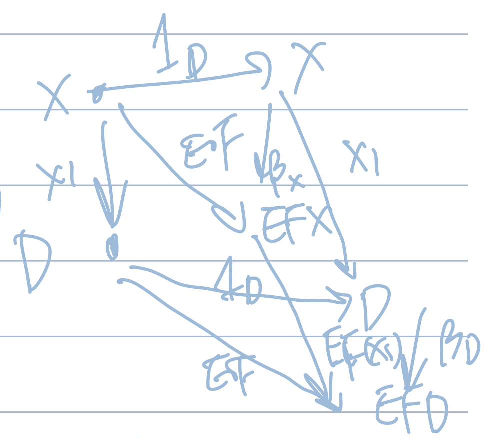
$\because \mathbb{C} \simeq \mathbb{D}$. $\therefore \exists E : \mathbb{C} \to \mathbb{D}$, $F : \mathbb{D} \to \mathbb{C}$ st $\alpha : 1_{\mathbb{C}} \xrightarrow{\sim} F \circ E$, $\beta : 1_{\mathbb{D}} \xrightarrow{\sim} E \circ F$

$\therefore$ Consider $F(D), F(C), F(X), F(x_1), F(x_2) \in \mathbb{C}$. then $\exists F(D) \times F(C) \in \mathbb{C}$

$\therefore$ We have



check: $\beta_D^{-1} \circ E(\pi_1) \circ E(m) \circ \beta_X \overset{?}{=} x_1$



LHS $= \beta_D^{-1} \circ E(\pi_1 \circ m) \circ \beta_X$

$= \beta_D^{-1} \circ E(F(x_1)) \circ \beta_X$

$= \beta_D^{-1} \circ \beta_D \circ x_1$ (by natural transformation)

$= x_1$