7

NATURALITY

We now want to start considering categories and functors more systematically, developing the "category theory" of category theory itself, rather than of other mathematical objects, like groups, or formulas in a logical system. Let me emphasize that, while some of this may look a bit like "abstract nonsense," the idea behind it is that when one has a particular application at hand, the theory can then be specialized to that concrete case. The notion of a functor is a case in point; developing its general theory makes it a clarifying, simplifying, and powerful tool in its many instances.

7.1 Category of categories

We begin by reviewing what we know about the category **Cat** of categories and functors and tying up some loose ends.

We have already seen that **Cat** has finite coproducts $\mathbf{0}$, $\mathbf{C}+\mathbf{D}$; and finite products $\mathbf{1}$, $\mathbf{C}\times\mathbf{D}$. It is very easy to see that there are also all small coproducts and products, constructed analogously. We can therefore show that **Cat** has all limits by constructing equalizers. Thus, let categories $\mathbf{C}$ and $\mathbf{D}$ and parallel functors F and G be given, and define the category $\mathbf{E}$ and functor E,

$$\begin{tikzcd} \mathbf{E} \arrow[r, "E"] & \mathbf{C} \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & \mathbf{D} \end{tikzcd}$$

as follows (recall that for a category $\mathbf{C}$, we write $\mathbf{C}_0$ and $\mathbf{C}_1$ for the collections of objects and arrows, respectively):

$$\mathbf{E}_0 = \{C \in \mathbf{C}_0 \mid F(C) = G(C)\}$$
$$\mathbf{E}_1 = \{f \in \mathbf{C}_1 \mid F(f) = G(f)\}$$

and let $E : \mathbf{E} \to \mathbf{C}$ be the evident inclusion. This is then an equalizer, as the reader can easily check.

The category $\mathbf{E}$ is an example of a *subcategory*, that is, a monomorphism in **Cat** (recall that equalizers are monic). Often, by a subcategory of a category $\mathbf{C}$ one means specifically a collection $\mathbf{U}$ of some of the objects and arrows, $\mathbf{U}_0 \subseteq \mathbf{C}_0$ and $\mathbf{U}_1 \subseteq \mathbf{C}_1$), that is closed under the operations $\mathrm{dom}, \mathrm{cod}, \mathrm{id}$, and $\circ$. There is then an evident inclusion functor

$$i : \mathbf{U} \to \mathbf{C}$$

which is clearly monic.

In general, coequalizers of categories are more complicated to describe—indeed, even for posets, determining the coequalizer of a pair of monotone maps can be quite involved, as the reader should consider.

There are various properties of functors other than being monic and epic that turn out to be quite useful in **Cat**. A few of these are given by the following:

Definition 7.1. A functor $F : \mathbf{C} \to \mathbf{D}$ is said to be

- *injective on objects* if the object part $F_0 : \mathbf{C}_0 \to \mathbf{D}_0$ is injective, it is *surjective on objects* if F_0 is surjective.
- Similarly, F is *injective* (resp. *surjective*) *on arrows* if the arrow part $F_1 : \mathbf{C}_1 \to \mathbf{D}_1$ is injective (resp. surjective).
- F is *faithful* if for all $A, B \in \mathbf{C}_0$, the map

$$F_{A,B} : \mathrm{Hom}_{\mathbf{C}}(A, B) \to \mathrm{Hom}_{\mathbf{D}}(FA, FB)$$

  defined by $f \mapsto F(f)$ is injective.
- Similarly, F is *full* if $F_{A,B}$ is always surjective.

What is the difference between being faithful and being injective on arrows? Consider, for example, the "codiagonal functor" $\nabla : \mathbf{C} + \mathbf{C} \to \mathbf{C}$, as indicated in the following:

$$\begin{tikzcd}
C \arrow[r] \arrow[dr, "1_C"'] & C + C \arrow[d, "\nabla"] & C \arrow[l] \arrow[dl, "1_C"] \\
& C &
\end{tikzcd}$$

∇ is faithful, but not injective on arrows.

A *full subcategory*

$$\mathbf{U} \rightarrowtail \mathbf{C}$$

consists of some objects of $\mathbf{C}$ and *all* of the arrows between them (thus satisfying the closure conditions for a subcategory). For example, the inclusion functor $\mathbf{Sets}_{fin} \rightarrowtail \mathbf{Sets}$ is full and faithful, but the forgetful functor $\mathbf{Groups} \rightarrowtail \mathbf{Sets}$ is faithful but not full.

Example 7.2. There is another "forgetful" functor for groups, namely to the category **Cat** of categories,

$$G : \mathbf{Groups} \to \mathbf{Cat}.$$

Observe that this functor is full and faithful, since a functor between groups $F : G(A) \to G(B)$ is exactly the same thing as a group homomorphism.

And exactly the same situation holds for monoids.

For posets, too, there is a full and faithful, forgetful functor

$$P : \mathbf{Pos} \to \mathbf{Cat}$$

again because a functor between posets $F : P(A) \to P(B)$ is exactly a monotone map. And the same thing holds for the "discrete category" functor $S : \mathbf{Sets} \to \mathbf{Cat}$.

Thus, **Cat** provides a setting for *comparing structures of many different kinds.* For instance, one can have a functor $R : G \to \mathbf{C}$ from a group G to a category $\mathbf{C}$ that is not a group. If $\mathbf{C}$ is a poset, then any such functor must be trivial (why?). But if $\mathbf{C}$ is, say, the category of finite dimensional, real vector spaces and linear maps, then a functor R is exactly a *linear representation* of the group G, representing every element of G as an invertible matrix of real numbers and the group multiplication as matrix multiplication.

What is a functor $g : P \to G$ from a poset to a group? Since G has only one object $*$, it has $g(p) = * = g(q)$ for all $p, q \in P$. For each $p \leq q$, it picks an element $g_{p,q}$ in such a way that

$$g_{p,p} = u \qquad \text{(the unit of } G\text{)}$$

$$g_{q,r} \cdot g_{p,q} = g_{p,r}.$$

For example, take $P = (\mathbb{R}, \leq)$ to be the ordered real numbers and $G = (\mathbb{R}, +)$ the additive group of reals, then *subtraction* is a functor,

$$g : (\mathbb{R}, \leq) \to (\mathbb{R}, +)$$

defined by

$$g_{x,y} = (y - x).$$

Indeed, we have

$$g_{x,x} = (x - x) = 0$$

$$g_{y,z} \cdot g_{x,y} = (z - y) + (y - x) = (z - x) = g_{x,z}.$$

7.2 Representable structure

Let $\mathbf{C}$ be a *locally small category*, so that we have the *representable functors*,

$$\mathrm{Hom}_{\mathbf{C}}(C, -) : \mathbf{C} \to \mathbf{Sets}$$

for all objects $C \in \mathbf{C}$. This functor is evidently faithful if the object C has the property that for any objects X and Y and arrows $f, g : X \rightrightarrows Y$, if $f \neq g$ there is an arrow $x : C \to X$ such that $fx \neq gx$. That is, the arrows in the category are distinguished by their effect on generalized elements based at C. Such an object C is called a *generator* for $\mathbf{C}$.

In the category of sets, for example, the terminal object 1 is a generator. In groups, as we have already discussed, the free group $F(1)$ on one element is a generator. Indeed, the functor represented by $F(1)$ is isomorphic to the forgetful functor $U : \mathbf{Grp} \to \mathbf{Sets}$,

$$\mathrm{Hom}(F(1), G) \cong U(G). \tag{7.1}$$

This isomorphism not only holds for each group G, but also respects group homomorphisms, in the sense that for any such $h : G \to H$, there is a commutative square,

$$\begin{array}{ccc} G & \qquad & \begin{tikzcd} \mathrm{Hom}(F(1), G) \arrow[r, "\cong"] \arrow[d, "h_*"'] & U(G) \arrow[d, "U(h)"] \\ \mathrm{Hom}(F(1), H) \arrow[r, "\cong"'] & U(H) \end{tikzcd} \\ {\scriptstyle h}\downarrow & & \\ H & & \end{array}$$

One says that the isomorphism (7.1) is "natural in G." In a certain sense, this also "explains" why the forgetful functor U preserves all limits, since representable functors necessarily do. The related fact that the forgetful functor is faithful is a precise way to capture the vague idea, which we initially used for motivation, that the category of groups is "concrete."

Recall that there are also *contravariant* representable functors

$$\mathrm{Hom}_{\mathbf{C}}(-, C) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$$

taking $f : A \to B$ to $f^* : \mathrm{Hom}_{\mathbf{C}}(B, C) \to \mathrm{Hom}_{\mathbf{C}}(A, C)$ by $f^*(h) = h \circ f$ for $h : B \to C$.

Example 7.3. Given a group G in a (locally small) category $\mathbf{C}$, the contravariant representable functor $\mathrm{Hom}_{\mathbf{C}}(-, G)$ actually has a group structure, giving a functor

$$\mathrm{Hom}_{\mathbf{C}}(-, G) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Grp}.$$

In **Sets**, for example, for each set X, we can define the operations on the group $\mathrm{Hom}(X, G)$ pointwise,

$$\begin{aligned} u(x) &= u \quad \text{(the unit of } G) \\ (f \cdot g)(x) &= f(x) \cdot g(x) \\ f^{-1}(x) &= f(x)^{-1}. \end{aligned}$$

In this case, we have an isomorphism

$$\mathrm{Hom}(X, G) \cong \Pi_{x \in X} G$$

with the product group. Functoriality in X is given simply by precomposition; thus, for any function $h : Y \to X$, one has

$$\begin{aligned} h^*(f \cdot g)(y) &= (f \cdot g)(h(y)) \\ &= f(h(y)) \cdot g(h(y)) \\ &= h^*(f)(y) \cdot h^*(g)(y) \\ &= (h^*(f) \cdot h^*(g))(y) \end{aligned}$$

and similarly for inverses and the unit. Indeed, it is easy to see that this construction works just as well for any other algebraic structure defined by operations and equations. Nor is there anything special about the category **Sets** here; we can do the same thing in any category with an internal algebraic structure.

For instance, in topological spaces, one has the ring $\mathbb{R}$ of real numbers and, for any space X, the ring

$$\mathcal{C}(X) = \mathrm{Hom}_{\mathrm{Top}}(X, \mathbb{R})$$

of real-valued, continuous functions on X. Just as in the previous case, if

$$h : Y \to X$$

is any continuous function, we then get a ring homomorphism

$$h^* : \mathcal{C}(X) \to \mathcal{C}(Y)$$

by precomposing with h. The recognition of $\mathcal{C}(X)$ as representable ensures that this "ring of real-valued functions" construction is functorial,

$$\mathcal{C} : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Rings}.$$

Note that in passing from $\mathbb{R}$ to $\mathrm{Hom}_{\mathbf{Top}}(X, \mathbb{R})$, all the algebraic structure of $\mathbb{R}$ is retained, but properties determined by conditions that are not strictly equational are not necessarily preserved. For instance, $\mathbb{R}$ is not only a ring, but also a *field*, meaning that every nonzero real number r has a multiplicative inverse r^{-1}; formally,

$$\forall x (x = 0 \vee \exists y.\ y \cdot x = 1).$$

To see that this condition fails in, for example, $\mathcal{C}(\mathbb{R})$, consider the continuous function $f(x) = x^2$. For any argument $y \neq 0$, the multiplicative inverse must be $g(y) = 1/y^2$. But if this function were to be continuous, at 0 it would have to be $\lim_{y \to 0} 1/y^2$ which does not exist in $\mathbb{R}$.

Example 7.4. A very similar situation occurs in the category **BA** of Boolean algebras. Given the Boolean algebra $\mathbf{2}$ with the usual (truth-table) operations $\wedge, \vee, \neg, 0, 1$, for any set X, we make the set

$$\mathrm{Hom}_{\mathbf{Sets}}(X, \mathbf{2})$$

into a Boolean algebra with the pointwise operations:

$$0(x) = 0$$
$$1(x) = 1$$
$$(f \wedge g)(x) = f(x) \wedge g(x)$$
$$\text{etc.}$$

When we define the operations in this way in terms of those on $\mathbf{2}$, we see immediately that $\mathrm{Hom}(X, \mathbf{2})$ is a Boolean algebra too, and that precomposition is a contravariant functor,

$$\mathrm{Hom}(-, \mathbf{2}) : \mathbf{Sets}^{\mathrm{op}} \to \mathbf{BA}$$

into the category $\mathbf{BA}$ of Boolean algebras and their homomorphisms.

Now observe that for any set X, the familiar isomorphism

$$\mathrm{Hom}(X, \mathbf{2}) \cong \mathcal{P}(X)$$

between characteristic functions $\phi : X \to \mathbf{2}$ and subsets $V_\phi = \phi^{-1}(1) \subseteq X$, relates the pointwise Boolean operations in $\mathrm{Hom}(X, \mathbf{2})$ to the subset operations of intersection, union, etc. in $\mathcal{P}(X)$:

$$V_{\phi \wedge \psi} = V_\phi \cap V_\psi$$
$$V_{\phi \vee \psi} = V_\phi \cup V_\psi$$
$$V_{\neg \phi} = X - V_\phi$$
$$V_1 = X$$
$$V_0 = \emptyset$$

In this sense, the set-theoretic Boolean operations on $\mathcal{P}(X)$ are induced by those on $\mathbf{2}$, and the powerset $\mathcal{P}$ is seen to be a contravariant functor to the category of Boolean algebras,

$$\mathcal{P}^{\mathrm{BA}} : \mathbf{Sets}^{\mathrm{op}} \to \mathrm{BA}.$$

As was the case for the covariant representable functor $\mathrm{Hom}_{\mathbf{Grp}}(F(1), -)$ and the forgetful functor U from groups to sets, here the contravariant functors $\mathrm{Hom}_{\mathbf{Sets}}(-, \mathbf{2})$ and $\mathcal{P}^{\mathrm{BA}}$ from sets to Boolean algebras can also be seen to be *naturally* isomorphic, in the sense that for any function $f : Y \to X$, the following square of Boolean algebras and homomorphisms commutes:

$$\begin{tikzcd}
X & \mathrm{Hom}(X, \mathbf{2}) \arrow[r, "\cong"] \arrow[d, "f^*"'] & P(X) \arrow[d, "f^{-1}"] \\
Y \arrow[u, "f"] & \mathrm{Hom}(Y, \mathbf{2}) \arrow[r, "\cong"'] & P(Y)
\end{tikzcd}$$

7.3 Stone duality

Before considering the topic of naturality more systematically, let us take a closer look at the foregoing example of powersets and Boolean algebras.

Recall that an *ultrafilter* in a Boolean algebra B is a proper subset $U \subset B$ such that

- $1 \in U$
- $x, y \in U$ implies $x \wedge y \in U$
- $x \in U$ and $x \leq y$ implies $y \in U$
- if $U \subset U'$ and U' is a filter, then $U' = B$

The maximality condition on U is equivalent to the condition that for every $x \in B$, either $x \in U$ or $\neg x \in U$ but not both (exercise!).

We already know that there is an isomorphism between the set $\mathrm{Ult}(B)$ of ultrafilters on B and the Boolean homomorphisms $B \to \mathbf{2}$,

$$\mathrm{Ult}(B) \cong \mathrm{Hom}_{\mathbf{BA}}(B, \mathbf{2}).$$

This assignment $\mathrm{Ult}(B)$ is functorial and contravariant, and the displayed isomorphism above is natural in B. Indeed, given a Boolean homomorphism $h : B' \to B$, let

$$\mathrm{Ult}(h) = h^{-1} : \mathrm{Ult}(B) \to \mathrm{Ult}(B').$$

Of course, we have to show that the inverse image $h^{-1}(U) \subset B$ of an ultrafilter $U \subset B'$ is an ultrafilter in B. But since we know that $U = \chi_U^{-1}(1)$ for some $\chi_U : B' \to 2$, we have

$$\begin{aligned}\mathrm{Ult}(h)(U) &= h^{-1}(\chi_U^{-1}(1)) \\ &= (\chi_U \circ h)^{-1}(1).\end{aligned}$$

Therefore, $\mathrm{Ult}(h)(U)$ is also an ultrafilter. Thus, we have a contravariant functor of ultrafilters

$$\mathrm{Ult} : \mathbf{BA}^{\mathrm{op}} \to \mathbf{Sets},$$

as well as the contravariant powerset functor coming back

$$\mathcal{P}^{\mathbf{BA}} : \mathbf{Sets}^{\mathrm{op}} \to \mathbf{BA}.$$

The constructions,

$$\begin{tikzcd} \mathbf{BA}^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Ult}"'] & \mathbf{Sets} \arrow[l, shift right, "(\mathcal{P}^{\mathbf{BA}})^{\mathrm{op}}"'] \end{tikzcd}$$

are *not* mutually inverse, however. For in general, $\mathrm{Ult}(\mathcal{P}(X))$ is much larger than X, since there are many ultrafilters in $\mathcal{P}(X)$ that are not "principal," that is, of the form $\{U \subseteq X \mid x \in U\}$ for some $x \in X$. (But what if X is finite?) Instead, there is a more subtle relation between these functors that we consider in more detail later; namely, these are an example of adjoint functors.

For now, consider the following observations. Let

$$\mathcal{U} = \mathrm{Ult} \circ (\mathcal{P}^{\mathbf{BA}})^{\mathrm{op}} : \mathbf{Sets} \to \mathbf{BA}^{\mathrm{op}} \to \mathbf{Sets}$$

so that

$$\mathcal{U}(X) = \{U \subseteq \mathcal{P}(X) \mid U \text{ is an ultrafilter}\}$$

is a *covariant* functor on **Sets**. Now, observe that for any set X, there is a function

$$\eta : X \to \mathcal{U}(X)$$

taking each element $x \in X$ to the principal ultrafilter

$$\eta(x) = \{U \subseteq X \mid x \in U\}.$$

This map is "natural" in X, that is, for any function $f : X \to Y$, the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "f"'] & \mathcal{U}(X) \arrow[d, "\mathcal{U}(f)"] \\
Y \arrow[r, "\eta_Y"'] & \mathcal{U}(Y)
\end{tikzcd}$$

This is so because, for any ultrafilter $\mathcal{V}$ in $\mathcal{P}(X)$,

$$\mathcal{U}(f)(\mathcal{V}) = \{U \subseteq Y \mid f^{-1}(U) \in \mathcal{V}\}.$$

So in the case of the principal ultrafilters $\eta(x)$, we have

$$\begin{aligned}
(\mathcal{U}(f) \circ \eta_X)(x) &= \mathcal{U}(f)(\eta_X(x)) \\
&= \{V \subseteq Y \mid f^{-1}(V) \in \eta_X(x)\} \\
&= \{V \subseteq Y \mid x \in f^{-1}(V)\} \\
&= \{V \subseteq Y \mid fx \in V\} \\
&= \eta_Y(fx) \\
&= (\eta_Y \circ f)(x).
\end{aligned}$$

Finally, observe that there is an analogous natural map at the "other side" of this situation, in the category of Boolean algebras. Specifically, for every Boolean algebra B, there is a homomorphism similar to the function η,

$$\phi_B : B \to \mathcal{P}(\mathrm{Ult}(B))$$

given by

$$\phi_B(b) = \{\mathcal{V} \in \mathrm{Ult}(B) \mid b \in \mathcal{V}\}.$$

It is not hard to see that ϕ_B is always injective. For, given any distinct elements $b, b' \in B$, the Boolean prime ideal theorem implies that there is an ultrafilter $\mathcal{V}$ containing one but not the other. The Boolean algebra $\mathcal{P}(\mathrm{Ult}(B))$, together with the homomorphism ϕ_B, is called the *Stone representation* of B. It presents the arbitrary Boolean algebra B as an algebra of subsets. For the record, we thus have the following step toward a special case of the far-reaching *Stone duality theorem.*

Proposition 7.5. *Every Boolean algebra B is isomorphic to one consisting of subsets of some set X, equipped with the set-theoretical Boolean operations.*

7.4 Naturality

A natural transformation is a morphism of functors. That is right: for fixed categories $\mathbf{C}$ and $\mathbf{D}$, we can regard the functors $\mathbf{C} \to \mathbf{D}$ as the *objects* of a new category, and the arrows between these objects are what we are going to call natural transformations. They are to be thought of as different ways of "relating" functors to each other, in a sense that we now explain.

Let us begin by considering a certain kind of situation that often arises: we have some "construction" on a category $\mathbf{C}$ and some other "construction," and we observe that these two "constructions" are related to each other in a way that is independent of the specific objects and arrows involved. That is, the relation is really between the constructions themselves. To give a simple example, suppose $\mathbf{C}$ has products and consider, for objects $A, B, C \in \mathbf{C}$,

$$(A \times B) \times C \quad \text{and} \quad A \times (B \times C).$$

Regardless of what objects A, B, and C are, we have an isomorphism

$$h : (A \times B) \times C \xrightarrow{\sim} A \times (B \times C).$$

What does it mean that this isomorphism does not really depend on the particular objects A, B, C? One way to explain it is this:

Given any $f : A \to A'$, we get a commutative square

$$\begin{tikzcd}
(A \times B) \times C \arrow[r, "h_A"] \arrow[d] & A \times (B \times C) \arrow[d] \\
(A' \times B) \times C \arrow[r, "h_{A'}"'] & A' \times (B \times C)
\end{tikzcd}$$

So what we really have is an isomorphism between the "constructions"

$$(- \times B) \times C \quad \text{and} \quad - \times (B \times C)$$

without regard to what is in the argument-place of these.

Now, by a "construction," we of course just mean a functor, and by a "relation between constructors" we mean a *morphism of functors* (which is what we are about to define). In the example, it is an isomorphism

$$(- \times B) \times C \;\cong\; - \times (B \times C)$$

of functors $\mathbf{C} \to \mathbf{C}$. In fact, we can of course consider the functors of three arguments:

$$F = (-_1 \times -_2) \times -_3 : \mathbf{C}^3 \to \mathbf{C}$$

and

$$G = -_1 \times (-_2 \times -_3) : \mathbf{C}^3 \to \mathbf{C}$$

and there is an analogous isomorphism

$$F \cong G.$$

But an *isomorphism* is a special morphism, so let us define the general notion first.

Definition 7.6. For categories $\mathbf{C}, \mathbf{D}$ and functors

$$F, G : \mathbf{C} \to \mathbf{D}$$

a *natural transformation* $\vartheta : F \to G$ is a family of arrows in $\mathbf{D}$

$$(\vartheta_C : FC \to GC)_{C \in \mathbf{C}_0}$$

such that, for any $f : C \to C'$ in $\mathbf{C}$, one has $\vartheta_{C'} \circ F(f) = G(f) \circ \vartheta_C$, that is, the following commutes:

$$\begin{tikzcd}
FC \arrow[r, "\vartheta_C"] \arrow[d, "Ff"'] & GC \arrow[d, "Gf"] \\
FC' \arrow[r, "\vartheta_{C'}"'] & GC'
\end{tikzcd}$$

Given such a natural transformation $\vartheta : F \to G$, the $\mathbf{D}$-arrow $\vartheta_C : FC \to GC$ is called the *component of* ϑ at C.

If you think of a functor $F : \mathbf{C} \to \mathbf{D}$ as a "picture" of $\mathbf{C}$ in $\mathbf{D}$, then you can think of a natural transformation $\vartheta_C : FC \to GC$ as a "cylinder" with such a picture at each end.

7.5 Examples of natural transformations

We have already seen several examples of natural transformations in previous sections, namely the isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Grp}}(F(1), G) &\cong U(G) \\ \mathrm{Hom}_{\mathbf{Sets}}(X, \mathbf{2}) &\cong \mathcal{P}(X) \\ \mathrm{Hom}_{\mathbf{BA}}(B, \mathbf{2}) &\cong \mathrm{Ult}(B). \end{aligned}$$

There were also the maps from Stone duality,

$$\eta_X : X \to \mathrm{Ult}(\mathcal{P}(X))$$
$$\phi_B : B \to \mathcal{P}(\mathrm{Ult}(B)).$$

We now consider some further examples.

Example 7.7. Consider the free monoid $M(X)$ on a set X and define a natural transformation $\eta : 1_{\mathrm{Sets}} \to UM$, such that each component $\eta_X : X \to UM(X)$ is given by the "insertion of generators" taking every element x to itself, considered as a word.

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & UM(X) \arrow[d, dotted, "UM(f)"] \\ Y \arrow[r, "\eta_Y"'] & UM(Y) \end{tikzcd}$$

This is natural, because the homomorphism $M(f)$ on the free monoid $M(X)$ is completely determined by what f does to the generators.

Example 7.8. Let $\mathbf{C}$ be a category with products, and $A \in \mathbf{C}$ fixed. A natural transformation from the functor $A \times - : \mathbf{C} \to \mathbf{C}$ to $1_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ is given by taking the component at C to be the second projection

$$\pi_2 : A \times C \to C.$$

From this, together with the pairing operation $\langle -, - \rangle$, one can build up the isomorphism,

$$h : (A \times B) \times C \xrightarrow{\sim} A \times (B \times C).$$

For another such example in more detail, consider the functors

$$\times : \mathbf{C}^2 \to \mathbf{C}$$
$$\bar{\times} : \mathbf{C}^2 \to \mathbf{C}$$

where $\bar{\times}$ is defined on objects by

$$A \mathbin{\bar{\times}} B = B \times A$$

and on arrows by

$$\alpha \mathbin{\bar{\times}} \beta = \beta \times \alpha.$$

Define a "twist" natural transformation $t : \times \to \bar{\times}$ by

$$t_{(A,B)}\langle a, b\rangle = \langle b, a\rangle.$$

To check that the following commutes,

$$\begin{tikzcd}
A \times B \arrow[r, "t_{(A,B)}"] \arrow[d, "\alpha \times \beta"'] & B \times A \arrow[d, "\beta \times \alpha"] \\
A' \times B' \arrow[r, "t_{(A',B')}"'] & B' \times A'
\end{tikzcd}$$

observe that for any generalized elements $a : Z \to A$ and $b : Z \to B$,

$$\begin{aligned}
(\beta \times \alpha)t_{(A,B)}\langle a, b\rangle &= (\beta \times \alpha)\langle b, a\rangle \\
&= \langle \beta b, \alpha a\rangle \\
&= t_{(A',B')}\langle \alpha a, \beta b\rangle \\
&= t_{(A',B')} \circ (\alpha \times \beta)\langle a, b\rangle.
\end{aligned}$$

Thus, $t : \times \to \bar{\times}$ is natural. In fact, each component $t_{(A,B)}$ is an isomorphism with inverse $t_{(B,A)}$. This is a simple case of an isomorphism of functors.

Definition 7.9. The *functor category* $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$ has

Objects: functors $F : \mathbf{C} \to \mathbf{D}$,

Arrows: natural transformations $\vartheta : F \to G$.

For each object F, the natural transformation 1_F has components

$$(1_F)_C = 1_{FC} : FC \to FC$$

and the composite natural transformation of $F \xrightarrow{\vartheta} G \xrightarrow{\phi} H$ has components

$$(\phi \circ \vartheta)_C = \phi_C \circ \vartheta_C.$$

Definition 7.10. A *natural isomorphism* is a natural transformation

$$\vartheta : F \to G$$

which is an isomorphism in the functor category $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$.

Lemma 7.11. *A natural transformation $\vartheta : F \to G$ is a natural isomorphism iff each component $\vartheta_C : FC \to GC$ is an isomorphism.*

Proof. Exercise! □

In our first example, we can therefore say that the isomorphism

$$\vartheta_A : (A \times B) \times C \cong A \times (B \times C)$$

is *natural in* A, meaning that the functors

$$F(A) = (A \times B) \times C$$
$$G(A) = A \times (B \times C)$$

are naturally isomorphic.

Here is a classical example of a natural isomorphism.

Example 7.12. Consider the category

$$\mathrm{Vect}(\mathbb{R})$$

of real vector spaces and linear transformations

$$f : V \to W.$$

Every vector space V has a *dual space*

$$V^* = \mathrm{Vect}(V, \mathbb{R})$$

of linear transformations. And every linear transformation

$$f : V \to W$$

gives rise to a *dual linear transformation*

$$f^* : W^* \to V^*$$

defined by precomposition, $f^*(A) = A \circ f$ for $A : W \to \mathbb{R}$. In brief, $(-)^* = \mathrm{Vect}(-, \mathbb{R}) : \mathbf{Vect}^{\mathrm{op}} \to \mathbf{Vect}$ is the *contravariant representable* functor endowed with vector space structure, just like the examples already considered in Section 7.2.

As in those examples, there is a canonical linear transformation from each vector space to its double dual,

$$\begin{aligned} \eta_V :\ & V \to V^{**} \\ & x \mapsto (\mathrm{ev}_x : V^* \to \mathbb{R}) \end{aligned}$$

where $\mathrm{ev}_x(A) = A(x)$ for every $A : V \to \mathbb{R}$. This map is the component of a natural transformation,

$$\eta : 1_{\mathbf{Vect}} \to {**}$$

since the following always commutes:

$$\begin{tikzcd} V \arrow[r, "\eta_V"] \arrow[d, "f"'] & V^{**} \arrow[d, "f^{**}"] \\ W \arrow[r, "\eta_W"'] & W^{**} \end{tikzcd}$$

in **Vect**. Indeed, given any $v \in V$ and $A : W \to \mathbb{R}$ in W^*, we have

$$\begin{aligned} (f^{**} \circ \eta_V)(v)(A) &= f^{**}(\mathrm{ev}_v)(A) \\ &= \mathrm{ev}_v(f^*(A)) \\ &= \mathrm{ev}_v(A \circ f) \\ &= (A \circ f)(v) \\ &= A(fv) \\ &= \mathrm{ev}_{fv}(A) \\ &= (\eta_W \circ f)(v)(A). \end{aligned}$$

Now, it is a well-known fact in linear algebra that every *finite dimensional* vector space V is isomorphic to its dual space $V \cong V^*$ just for reasons of dimension. However, there is no "natural" way to choose such an isomorphism. On the other hand, the natural transformation,

$$\eta_V : V \to V^{**}$$

is a natural isomorphism when V is finite dimensional.

Thus, the formal notion of naturality captures the informal fact that $V \cong V^{**}$ "naturally," unlike $V \cong V^*$.

A similar situation occurs in **Sets**. Here we take 2 instead of $\mathbb{R}$, and the dual A^* of a set A then becomes

$$A^* = \mathcal{P}(A) \cong \mathbf{Sets}(A, 2)$$

while the dual of a map $f : A \to B$ is the inverse image $f^* : \mathcal{P}(B) \to \mathcal{P}(A)$.

Note that the exponential evaluation corresponds to (the characteristic function of) the membership relation on $A \times \mathcal{P}(A)$.

$$\begin{tikzcd} 2^A \times A \arrow[r, "\epsilon"] \arrow[d, "\cong"'] & 2 \arrow[d, "id"] \\ A \times P(A) \arrow[r, "\tilde{\in}"'] & 2 \end{tikzcd}$$

Transposing again gives a map

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr] & PP(A) = A^{**} \\ & 2^{P(A)} \arrow[u, "\cong"'] \end{tikzcd}$$

which is described by

$$\eta_A(a) = \{U \subseteq A \mid a \in U\}.$$

In **Sets**, one always has A strictly smaller than $\mathcal{P}(A)$, so $\eta_A : A \to A^{**}$ is *never* an isomorphism. Nonetheless, $\eta : 1_{\mathbf{Sets}} \to **$ is a natural transformation, which the reader should prove.

7.6 Exponentials of categories

We now want to show that the category **Cat** of (small) categories and functors is cartesian closed, by showing that any two categories $\mathbf{C}, \mathbf{D}$ have an exponential $\mathbf{D}^{\mathbf{C}}$. Of course, we take $\mathbf{D}^{\mathbf{C}} = \mathrm{Fun}(\mathbf{C}, \mathbf{D})$, the category of functors and natural transformations, for which we need to prove the required universal mapping property (UMP).

Proposition 7.13. **Cat** *is cartesian closed, with the exponentials*

$$\mathbf{D}^{\mathbf{C}} = \mathrm{Fun}(\mathbf{C}, \mathbf{D}).$$

Before giving the proof, let us note the following. Since exponentials are unique up to isomorphism, this gives us a way to verify that we have found the "*right*" definition of a morphism of functors. For the notion of a natural transformation is completely determined by the requirement that it makes the set $\mathrm{Hom}(\mathbf{C}, \mathbf{D})$ into an exponential category. This is an example of how category theory can serve as a conceptual tool for discovering new concepts. Before giving the proof, we need the following.

Lemma 7.14 (bifunctor lemma). *Given categories* $\mathbf{A}, \mathbf{B}$*, and* $\mathbf{C}$*, a map of arrows and objects,*

$$F_0 : \mathbf{A}_0 \times \mathbf{B}_0 \to \mathbf{C}_0$$

$$F_1 : \mathbf{A}_1 \times \mathbf{B}_1 \to \mathbf{C}_1$$

is a functor $F : \mathbf{A} \times \mathbf{B} \to \mathbf{C}$ *iff*

1. *F is functorial in each argument: $F(A,-):\mathbf{B}\to\mathbf{C}$ and $F(-,B):\mathbf{A}\to\mathbf{C}$ are functors for all $A\in\mathbf{A}_0$ and $B\in\mathbf{B}_0$.*
2. *F satisfies the following "interchange law." Given $\alpha : A\to A'\in\mathbf{A}$ and $\beta : B\to B'\in\mathbf{B}$, the following commutes:*

$$\begin{tikzcd}
F(A,B) \arrow[r, "{F(A,\beta)}"] \arrow[d, "{F(\alpha,B)}"'] & F(A,B') \arrow[d, "{F(\alpha,B')}"] \\
F(A',B) \arrow[r, "{F(A',\beta)}"'] & F(A',B')
\end{tikzcd}$$

that is, $F(A',\beta)\circ F(\alpha,B)=F(\alpha,B')\circ F(A,\beta)$ in $\mathbf{C}$.

Proof. (Lemma) In $\mathbf{A}\times\mathbf{B}$, any arrow

$$\langle\alpha,\beta\rangle : \langle A,B\rangle\to\langle A',B'\rangle$$

factors as

$$\begin{tikzcd}
\langle A,B\rangle \arrow[r, "{\langle 1_A,\beta\rangle}"] \arrow[d, "{\langle\alpha,1_B\rangle}"'] & \langle A,B'\rangle \arrow[d, "{\langle\alpha,1_{B'}\rangle}"] \\
\langle A',B\rangle \arrow[r, "{\langle 1_{A'},\beta\rangle}"'] & \langle A',B'\rangle
\end{tikzcd}$$

So (1) and (2) are clearly necessary. To show that they are also sufficient, we can define the (proposed) functor:

$$F(\langle A,B\rangle)=F(A,B)$$
$$F(\langle\alpha,\beta\rangle)=F(A',\beta)\circ F(\alpha,B)$$

The interchange law, together with functoriality in each argument, then ensures that

$$F(\alpha',\beta')\circ F(\alpha,\beta)=F(\langle\alpha',\beta'\rangle\circ\langle\alpha,\beta\rangle)$$

as can be read off from the following diagram:

$$
\begin{tikzcd}
F(A,B) \arrow[d, "F(\alpha,B)"'] \arrow[dr, "F(\alpha,\beta)"] & & \\
F(A',B) \arrow[r, "F(A',\beta)"'] \arrow[d, dotted, "F(\alpha',B)"'] & F(A',B') \arrow[d, "F(\alpha,B')"] \arrow[dr, "F(\alpha',\beta')"] & \\
F(A',B) \arrow[r, dotted, "F(A'',\beta)"'] & F(A'',B') \arrow[r, "F(A'',\beta')"'] & F(A'',B'')
\end{tikzcd}
$$

□

Proof. (Proposition) We need to show:

1. $\epsilon = \mathrm{eval} : \mathrm{Fun}(\mathbf{C},\mathbf{D}) \times \mathbf{C} \to \mathbf{D}$ is functorial.
2. For any category $\mathbf{X}$ and functor

   $$F : \mathbf{X} \times \mathbf{C} \to \mathbf{D}$$

   there is a functor

   $$\tilde{F} : \mathbf{X} \to \mathrm{Fun}(\mathbf{C},\mathbf{D})$$

   such that $\epsilon \circ (\tilde{F} \times 1_{\mathbf{C}}) = F$.
3. Given any functor

   $$G : \mathbf{X} \to \mathrm{Fun}(\mathbf{C},\mathbf{D}),$$

   one has $(\epsilon \circ \widetilde{(G \times 1_{\mathbf{C}})}) = G$.

(1) Using the bifunctor lemma, we show that ϵ is functorial. First, fix $F : \mathbf{C} \to \mathbf{D}$ and consider $\epsilon(F,-) = F : \mathbf{C} \to \mathbf{D}$. This is clearly functorial! Next, fix $C \in \mathbf{C}_0$ and consider $\epsilon(-,C) : \mathrm{Fun}(\mathbf{C},\mathbf{D}) \to \mathbf{D}$ defined by

$$(\vartheta : F \to G) \mapsto (\vartheta_C : FC \to GC).$$

This is also clearly functorial.

For the interchange law, consider any $\vartheta : F \to G \in \mathrm{Fun}(\mathbf{C},\mathbf{D})$ and $(f : C \to C') \in \mathbf{C}$, then we need the following to commute:

$$
\begin{tikzcd}
\epsilon(F,C) \arrow[r, "\vartheta_C"] \arrow[d, "F(f)"'] & \epsilon(G,C) \arrow[d, "G(f)"] \\
\epsilon(F,C') \arrow[r, "\vartheta_{C'}"'] & \epsilon(G,C')
\end{tikzcd}
$$

But this holds because $\epsilon(F, C) = F(C)$ and ϑ is a natural transformation.

The conditions (2) and (3) are now routine. For example, for (2), given

$$F : \mathbf{X} \times \mathbf{C} \to \mathbf{D}$$

let

$$\tilde{F} : \mathbf{X} \to \mathrm{Fun}(\mathbf{C}, \mathbf{D})$$

be defined by

$$\tilde{F}(X)(C) = F(X, C).$$

$\square$

7.7 Functor categories

Let us consider some particular functor categories.

Example 7.15. First, clearly $\mathbf{C}^{\mathbf{1}} = \mathbf{C}$ for the terminal category $\mathbf{1}$. Next, what about $\mathbf{C}^{\mathbf{2}}$, where $\mathbf{2} = \cdot \to \cdot$ is the single arrow category? This is just the arrow category of $\mathbf{C}$ that we already know,

$$\mathbf{C}^{\mathbf{2}} = \mathbf{C}^{\to}.$$

Consider instead the discrete category, $2 = \{0, 1\}$. Then clearly,

$$\mathbf{C}^{2} \cong \mathbf{C} \times \mathbf{C}.$$

Similarly, for any set I (regarded as a discrete category), we have

$$\mathbf{C}^{I} \cong \prod_{i \in I} \mathbf{C}.$$

Example 7.16. "Transcendental deduction of natural transformations"
Given the *possibility* of functor categories $\mathbf{D}^{\mathbf{C}}$, we can determine what the objects and arrows therein *must* be as follows:

Objects: these correspond uniquely to functors of the form

$$\mathbf{1} \to \mathbf{D}^{\mathbf{C}}$$

and hence to functors

$$\mathbf{C} \to \mathbf{D}.$$

Arrows: by the foregoing example, arrows in the functor category correspond uniquely to functors of the form

$$\mathbf{1} \to (\mathbf{D}^{\mathbf{C}})^{\mathbf{2}}$$

thus to functors of the form

$$\mathbf{2} \to \mathbf{D}^{\mathbf{C}}$$

and hence to functors

$$\mathbf{C} \times \mathbf{2} \to \mathbf{D}$$

respectively

$$\mathbf{C} \to \mathbf{D}^{\mathbf{2}}.$$

But a functor from $\mathbf{C}$ into the arrow category $\mathbf{D}^{\mathbf{2}}$ (respectively a functor into $\mathbf{D}$ from the cylinder category $\mathbf{C} \times \mathbf{2}$) is exactly a natural transformation between two functors from $\mathbf{C}$ into $\mathbf{D}$, as the reader can see by drawing a picture of the functor's image in $\mathbf{D}$.

Example 7.17. Recall that a (directed) graph can be regarded as a pair of sets and a pair of functions,

$$\begin{tikzcd} G_1 \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & G_0 \end{tikzcd}$$

where G_1 is the set of edges, G_0 is the set of vertices, and s and t are the source and target operations.

A homomorphism of graphs $h : G \to H$ is a map that preserves sources and targets. In detail, this is a pair of functions $h_1 : G_1 \to H_1$ and $h_0 : G_0 \to H_0$ such that for all edges $e \in G$, we have $sh_1(e) = h_0 s(e)$ and similarly for t as well. But this amounts exactly to saying that the following two diagrams commute:

$$\begin{tikzcd} G_1 \arrow[r, "h_1"] \arrow[d, "s_G"'] & H_1 \arrow[d, "s_H"] \\ G_0 \arrow[r, "h_0"'] & H_0 \end{tikzcd} \qquad \begin{tikzcd} G_1 \arrow[r, "h_1"] \arrow[d, "t_G"'] & H_1 \arrow[d, "t_H"] \\ G_0 \arrow[r, "h_0"'] & H_0 \end{tikzcd}$$

Now consider the category Γ, pictured as follows:

$$\begin{tikzcd} \cdot \arrow[r, shift left] \arrow[r, shift right] & \cdot \end{tikzcd}$$

It has exactly two objects and two distinct, parallel, nonidentity arrows. A graph G is then exactly a functor,

$$G : \Gamma \to \mathbf{Sets}$$

and a homomorphism of graphs $h : G \to H$ is exactly a natural transformation between these functors. Thus, the category of graphs is a functor category,

$$\mathbf{Graphs} = \mathbf{Sets}^{\Gamma}.$$

As we see later, it follows from this fact that **Graphs** is cartesian closed.

Example 7.18. Given a product $\mathbf{C} \times \mathbf{D}$ of categories, take the first product projection

$$\mathbf{C} \times \mathbf{D} \to \mathbf{C}$$

and transpose it to get a functor

$$\Delta : \mathbf{C} \to \mathbf{C}^{\mathbf{D}}.$$

For $C \in \mathbf{C}$, the functor $\Delta(C)$ is the "constant C-valued functor,"

- $\Delta(C)(X) = C$ for all $X \in \mathbf{D}_0$
- $\Delta(x) = 1_C$ for all $x \in \mathbf{D}_1$.

Moreover, $\Delta(f) : \Delta(C) \to \Delta(C')$ is the natural transformation, each component of which is f.

Now suppose we have any functor

$$F : \mathbf{D} \to \mathbf{C}$$

and a natural transformation

$$\vartheta : \Delta(C) \to F.$$

Then, the components of ϑ all look like

$$\vartheta_D : C \to F(D)$$

since $\Delta(C)(D) = C$. Moreover, for any $d : D \to D'$ in $\mathbf{D}$, the usual naturality square becomes a triangle, since $\Delta(C)(d) = 1_C$ for all $d : D \to D'$.

$$\begin{tikzcd}
C \arrow[r, "\vartheta_D"] \arrow[d, "1_C"'] & FD \arrow[d, "Fd"] \\
C \arrow[r, "\vartheta_{D'}"'] & FD'
\end{tikzcd}$$

Thus, such a natural transformation $\vartheta : \Delta(C) \to F$ is exactly a cone to the base F (with vertex C). Similarly, a map of cones $\vartheta \to \varphi$ is a constant natural transformation, that is, one of the form $\Delta(h)$ for some $h : C \to D$, making a commutative triangle

$$\begin{tikzcd}
\Delta(C) \arrow[rr, "\Delta(h)"] \arrow[dr, "\vartheta"'] & & \Delta(D) \arrow[dl, "\varphi"] \\
& F &
\end{tikzcd}$$

Example 7.19. Take posets P, Q and consider the functor category,

$$Q^P.$$

The functors $Q \to P$, as we know, are just monotone maps, but what is a natural transformation?

$$\vartheta : f \to g$$

For each $p \in P$, we must have

$$\vartheta_p : fp \leq gp$$

and if $p \leq q$, then there must be a commutative square involving $fp \leq fq$ and $gp \leq gq$, which, however, is automatic. Thus, the only condition is that $fp \leq gp$ for all p, that is, $f \leq g$ *pointwise*. Since this is just the usual ordering of the poset Q^P, the exponential poset agrees with the functor category. Thus, we have the following.

Proposition 7.20. *The inclusion functor,*

$$\mathbf{Pos} \to \mathbf{Cat}$$

preserves CCC structure.

Example 7.21. What happens if we take the functor *category* of two groups G and H?

$$H^G$$

Do we get an exponential of groups? Let us first ask, what is a natural transformation between two group homomorphisms $f, g : G \to H$? Such a map $\vartheta : f \to g$ would be an element $h \in H$ such that for every $x \in G$, we have

$$g(x) \cdot h = h \cdot f(x)$$

or, equivalently,

$$g(x) = h \cdot f(x) \cdot h^{-1}.$$

Therefore, a natural transformation $\vartheta : f \to g$ is an *inner automorphism* $y \mapsto h \cdot y \cdot h^{-1}$ of H (called *conjugation by* h) that takes f to g. Clearly, every such arrow $\vartheta : f \to g$ has an inverse $\vartheta^{-1} : g \to f$ (conjugation by h^{-1}). But H^G is still not usually a group, simply because there may be many *different* homomorphisms $G \to H$, so the functor category H^G has more than one object.

This suggests enlarging the category of groups to include also categories with more than one object, but still having inverses for all arrows. Such categories are called *groupoids*, and have been studied by topologists (they occur as the collection of paths between different points in a topological space). A groupoid can thus be regarded as a generalized group, in which the domains and codomains of elements x and y must match up, as in any category, for the multiplication $x \cdot y$ to be defined.

It is clear that if G and H are any groupoids, then the functor category H^G is also a groupoid. Thus, we have the following proposition, the detailed proof of which is left as an exception.

Proposition 7.22. *The category* **Grpd** *of groupoids is cartesian closed and the inclusion functor*

$$\mathbf{Grpd} \to \mathbf{Cat}$$

preserves the CCC structure.

7.8 Monoidal categories

As a further application of natural transformations, we can finally give the general notion of a monoidal category, as opposed to the special case of a *strict* one. Recall from Section 4.1 that a strict monoidal category is by definition a monoid in **Cat**, that is, a category **C** equipped with an associative multiplication functor,

$$\otimes : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$$

and a distinguished object I that acts as a unit for $\otimes$. A monoidal category with a discrete category **C** is just a monoid in the usual sense, and every set X gives rise to one of these, with **C** the set of endomorphisms $\mathrm{End}(X)$ under composition. Another example, not discrete, is now had by considering the category $\mathrm{End}(\mathbf{D})$ of endo*functors* of an *arbitrary* category **D**, with their natural transformations as arrows; that is, let,

$$\mathbf{C} = \mathrm{End}(\mathbf{D}), \quad G \otimes F = G \circ F, \quad I = 1_{\mathbf{D}}.$$

This can also be seen to be a strict monoidal category. Indeed, the multiplication is clearly associative and has $1_{\mathbf{D}}$ as unit, so we just need to check that composition is a bifunctor $\mathrm{End}(\mathbf{D}) \times \mathrm{End}(\mathbf{D}) \longrightarrow \mathrm{End}(\mathbf{D})$. Of course, for this we can use the bifunctor lemma. Fixing F and taking any natural transformation $\alpha : G \to G'$, we have, for any object D,

$$\alpha_{FD} : G(FD) \to G'(FD)$$

which is clearly functorial as an operation $\mathrm{End}(\mathbf{D}) \longrightarrow \mathrm{End}(\mathbf{D})$. Fixing G and taking $\beta : F \to F'$ gives

$$G(\beta_D) : G(FD) \to G(F'D)$$

which is also easily seen to be functorial. So it just remains to check the exchange law. This comes down to seeing that the square below commutes, which it plainly does just because α is natural.

$$
\begin{tikzcd}
GFD \arrow[r, "\alpha_{FD}"] \arrow[d, "G\beta_D"'] & G'FD \arrow[d, "G'\beta_D"] \\
GF'D \arrow[r, "\alpha_{F'D}"'] & G'F'D
\end{tikzcd}
$$

Some of the other examples of strict monoidal categories that we have seen involved "product-like" operations such as meets $a \wedge b$ and joins $a \vee b$ in posets. We would like to also capture general products $A \times B$ and coproducts $A + B$ in categories having these; however, these operations are not generally associative on the nose, but only up to isomorphism. Specifically, given any three objects A, B, C in a category with all finite products, we do not have $A \times (B \times C) = (A \times B) \times C$, but instead an isomorphism,

$$A \times (B \times C) \cong (A \times B) \times C.$$

Note, however, that there is exactly one such isomorphism that commutes with all three projections, and it is natural in all three arguments. Similarly, taking a terminal object 1, rather than $1 \times A = A = A \times 1$, we have natural isomorphisms,

$$1 \times A \cong A \cong A \times 1$$

which, again, are uniquely determined by the condition that they commute with the projections. This leads us to the following definition.

Definition 7.23. A *monoidal category* consists of a category $\mathbf{C}$ equipped with a functor

$$\otimes : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$$

and a distinguished object I, together with natural isomorphisms,

$$\alpha_{ABC} : A \otimes (B \otimes C) \xrightarrow{\sim} (A \otimes B) \times C,$$

$$\lambda_A : I \otimes A \xrightarrow{\sim} A, \quad \rho_A : A \otimes I \xrightarrow{\sim} A.$$

Moreover, these are required to always make the following diagrams commute:

$$
\begin{tikzcd}
 & (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha"] & \\
A \otimes (B \otimes (C \otimes D)) \arrow[ur, "\alpha"] \arrow[d, "1 \otimes \alpha"'] & & ((A \otimes B) \otimes C) \otimes D \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "\alpha \otimes 1"']
\end{tikzcd}
$$

("Mac Lane's pentagon")

$$
\begin{tikzcd}
A \otimes (I \otimes A) \arrow[rr, "\alpha"] \arrow[dr, "1 \otimes \lambda_A"'] & & (A \otimes I) \otimes A \arrow[dl, "\rho_A \otimes 1"] \\
 & A \otimes A &
\end{tikzcd}
$$

$$
\begin{tikzcd}
I \otimes I \arrow[rr, equal] \arrow[dr, "\lambda_I"'] & & I \otimes I \arrow[dl, "\rho_I"] \\
 & I &
\end{tikzcd}
$$

In this precise sense, a monoidal category is thus a category that is strict monoidal "up to natural isomorphism"—where the natural isomorphisms are specified and compatible. An example is, of course, a category with all finite products, where the required equations above are ensured by the UMP of products and the selection of the maps α, λ, ρ as the unique ones commuting with projections. We leave the verification as an exercise. The reader familiar with tensor products of vector spaces, modules, rings, etc., will have no trouble verifying that these, too, give examples of monoidal categories.

A further example comes from an unexpected source: linear logic. The logical operations of linear conjunction and disjunction, sometimes written $P \otimes Q$ and $P \oplus Q$, can be modeled in a monoidal category, usually with extra structure $\sigma_{AB} : A \otimes B \xrightarrow{\sim} B \otimes A$ making these operations "symmetric" (up to isomorphism). Here, too, we leave the verification to the reader familiar with this logical system.

The basic theorem regarding monoidal categories is Mac Lane's coherence theorem, which says that "all diagrams commute." Somewhat more precisely, it says that any diagram in a monoidal category constructed, like those above, just from identities, the functor $\otimes$, and the maps α, λ, ρ will necessarily commute. We shall not state the theorem more precisely than this, nor will we give its somewhat technical proof which, surprisingly, uses ideas from proof theory related to Gentzen's cut elimination theorem! The details can be found in Mac Lane's book, *Categories Work*.

7.9 Equivalence of categories

Before examining some particular functor categories in more detail, we consider one very special application of the concept of natural isomorphism. Consider first the following situation.

Example 7.24. Let $\mathbf{Ord}_{\mathrm{fin}}$ be the category of finite ordinal numbers. Thus, the *objects* are the sets $0, 1, 2, \ldots,$ where $0 = \emptyset$ and $n = \{0, \ldots, n-1\}$, while the *arrows* are *all* functions between these sets. Now suppose that for each finite set A we select an ordinal $|A|$ that is its cardinal and an isomorphism,

$$A \cong |A|.$$

Then for each function $f : A \to B$ of finite sets, we have a function $|f|$ by completing the square

$$\begin{tikzcd} A \arrow[r, "\cong"] \arrow[d, "f"'] & {|A|} \arrow[d, dotted, "{|f|}"] \\ B \arrow[r, "\cong"'] & {|B|} \end{tikzcd} \tag{7.2}$$

This clearly gives us a functor

$$|-| : \mathbf{Sets}_{\mathrm{fin}} \to \mathbf{Ord}_{\mathrm{fin}}.$$

Actually, all the maps in the above square are in $\mathbf{Sets}_{\mathrm{fin}}$; so we should also make the inclusion functor

$$i : \mathbf{Ord}_{\mathrm{fin}} \to \mathbf{Sets}_{\mathrm{fin}}$$

explicit. Then we have the selected isos,

$$\vartheta_A : A \xrightarrow{\sim} i|A|$$

and we know by (7.2) that

$$i(|f|) \circ \vartheta_A = \vartheta_B \circ f.$$

This, of course, says that we have a natural isomorphism

$$\vartheta : 1_{\mathbf{Sets}_{\mathrm{fin}}} \to i \circ |-|$$

between two functors of the form

$$\mathbf{Sets}_{\mathrm{fin}} \to \mathbf{Sets}_{\mathrm{fin}}.$$

On the other hand, if we take an ordinal and take its ordinal, we get nothing new,

$$|i(-)| = 1_{\mathbf{Ord}_{\mathrm{fin}}} : \mathbf{Ord}_{\mathrm{fin}} \to \mathbf{Ord}_{\mathrm{fin}}.$$

This is so because, for any finite ordinal n,

$$|i(n)| = n$$

and we can assume that we take $\vartheta_n = 1_n : n \to |i(n)|$, so that also,

$$|i(f)| = f : n \to m.$$

In sum, then, we have a situation where two categories are very similar; but they are *not* the same and they are *not even isomorphic* (why?). This kind of correspondence is what is captured by the notion of equivalence of categories.

Definition 7.25. An *equivalence of categories* consists of a pair of functors

$$E : \mathbf{C} \to \mathbf{D}$$
$$F : \mathbf{D} \to \mathbf{C}$$

and a pair of natural isomorphisms

$$\alpha : 1_{\mathbf{C}} \xrightarrow{\sim} F \circ E \qquad \text{in } \mathbf{C}^{\mathbf{C}}$$
$$\beta : 1_{\mathbf{D}} \xrightarrow{\sim} E \circ F \qquad \text{in } \mathbf{D}^{\mathbf{D}}.$$

In this situation, the functor F is called a *pseudo-inverse* of E. The categories $\mathbf{C}$ and $\mathbf{D}$ are then said to be *equivalent*, written $\mathbf{C} \simeq \mathbf{D}$.

Observe that equivalence of categories is a generalization of isomorphism. Indeed, two categories $\mathbf{C}, \mathbf{D}$ are isomorphic if there are functors.

$$E : \mathbf{C} \to \mathbf{D}$$
$$F : \mathbf{D} \to \mathbf{C}$$

such that

$$1_{\mathbf{C}} = F \circ E$$
$$1_{\mathbf{D}} = E \circ F.$$

In the case of equivalence $\mathbf{C} \simeq \mathbf{D}$, we replace the identity natural transformations by natural isomorphisms. In that sense, equivalence of categories as "isomorphism up to isomorphism."

Experience has shown that the mathematically significant properties of objects are those that are invariant under isomorphisms, and in category theory, identity of objects is a much less important relation than isomorphism. So it is really equivalence of categories that is the more important notion of "similarity" for categories.

In the foregoing example $\mathbf{Sets}_{\text{fin}} \simeq \mathbf{Ord}_{\text{fin}}$, we see that every set is isomorphic to an ordinal, and the maps between ordinals are just the maps between them *as sets*. Thus, we have

1. for every set A, there is some ordinal n with $A \cong i(n)$,
2. for any ordinals n, m, there is an isomorphism,

$$\mathrm{Hom}_{\mathbf{Ord}_{\text{fin}}}(n, m) \cong \mathrm{Hom}_{\mathbf{Sets}_{\text{fin}}}(i(n), i(m))$$

   where $i : \mathbf{Ord}_{\text{fin}} \to \mathbf{Sets}_{\text{fin}}$ is the inclusion functor.

In fact, these conditions are characteristic of equivalences, as the following proposition shows.

Proposition 7.26. *The following conditions on a functor $F : \mathbf{C} \to \mathbf{D}$ are equivalent:*

1. *F is (part of) an equivalence of categories.*
2. *F is full and faithful and "essentially surjective" on objects: for every $D \in \mathbf{D}$ there is some $C \in \mathbf{C}$ such that $FC \cong D$.*

Proof. (1 implies 2) Take $E : \mathbf{D} \to \mathbf{C}$, and

$$\alpha : 1_{\mathbf{C}} \xrightarrow{\sim} EF$$

$$\beta : 1_{\mathbf{D}} \xrightarrow{\sim} FE.$$

In $\mathbf{C}$, for any C, we then have $\alpha_C : C \xrightarrow{\sim} EF(C)$, and

$$\begin{tikzcd}
C \arrow[r, "\alpha_C"] \arrow[d, "f"'] & EF(C) \arrow[d, "EF(f)"] \\
C' \arrow[r, "\alpha_{C'}"'] & EF(C')
\end{tikzcd}$$

commutes for any $f : C \to C'$.

Thus, if $F(f) = F(f')$, then $EF(f) = EF(f')$, so $f = f'$. So F is faithful. Note that by symmetry, E is also faithful.

Now take any arrow

$$h : F(C) \to F(C') \text{ in } \mathbf{D},$$

and consider

$$\begin{tikzcd} C \arrow[r, "\cong"] \arrow[d, "f"'] & EF(C) \arrow[d, "E(h)"] \\ C' \arrow[r, "\cong"'] & EF(C') \end{tikzcd}$$

where $f = (\alpha_{C'})^{-1} \circ E(h) \circ \alpha_C$. Then, we have also $F(f) : F(C) \to F(C')$ and

$$EF(f) = E(h) : EF(C) \to EF(C')$$

by the naturality square

$$\begin{tikzcd} C \arrow[r, "\alpha_C"] \arrow[d, "f"'] & EF(C) \arrow[d, "EF(f)"] \\ C' \arrow[r, "\alpha_{C'}"'] & EF(C') \end{tikzcd}$$

Since E is faithful, $F(f) = h$. So F is also *full*.

Finally, for any object $D \in \mathbf{D}$, we have

$$\beta : 1_{\mathbf{D}} \xrightarrow{\sim} FE$$

so

$$\beta_D : D \cong F(ED), \quad \text{for } ED \in \mathbf{C}_0.$$

(2 implies 1) We need to define $E : \mathbf{D} \to \mathbf{C}$ and natural transformations,

$$\alpha : 1_{\mathbf{C}} \xrightarrow{\sim} EF$$

$$\beta : 1_{\mathbf{D}} \xrightarrow{\sim} FE.$$

Since F is essentially surjective, for each $D \in \mathbf{D}_0$, we can *choose* some $E(D) \in \mathbf{C}_0$ along with some $\beta_D : D \xrightarrow{\sim} FE(D)$. That gives E on objects and the proposed components of $\beta : 1_{\mathbf{D}} \to FE$.

Given $h : D \to D'$ in $\mathbf{D}$, consider

$$\begin{tikzcd} D \arrow[r, "\beta_D"] \arrow[d, "h"'] & FE(D) \arrow[d, dotted, "\beta_{D'} \circ h \circ \beta_D^{-1}"] \\ D' \arrow[r, "\beta_{D'}"'] & FE(D') \end{tikzcd}$$

Since $F : \mathbf{C} \to \mathbf{D}$ is full and faithful, there is a unique arrow

$$E(h) : E(D) \to E(D')$$

with $FE(h) = \beta_{D'} \circ h \circ \beta_D^{-1}$. It is easy to see that then $E : \mathbf{D} \to \mathbf{C}$ is a functor and $\beta : 1_{\mathbf{D}} \xrightarrow{\sim} FE$ is clearly a natural isomorphism.

To find $\alpha : 1_{\mathbf{C}} \to EF$, apply F to any C and consider $\beta_{FC} : F(C) \to FEF(C)$. Since F is full and faithful, the preimage of β_{FC} is an isomorphism,

$$\alpha_C = F^{-1}(\beta_{FC}) : C \xrightarrow{\sim} EF(C)$$

which is easily seen to be natural, since β is. □

7.10 Examples of equivalence

Example 7.27. *Pointed sets and partial maps*

Let **Par** be the category of sets and partial functions. An arrow

$$f : A \rightharpoonup B$$

is a function $|f| : U_f \to B$ for some $U_f \subseteq A$. Identities in **Par** are the same as those in **Sets**, that is, 1_A is the *total* identity function on A. The composite of $f : A \rightharpoonup B$ and $g : B \rightharpoonup C$ is given as follows: Let $U_{(g \circ f)} := f^{-1}(U_g) \subseteq A$, and $|g \circ f| : U_{(g \circ f)} \to C$ is the horizontal composite indicated in the following diagram, in which the square is a pullback:

$$\begin{tikzcd}
{|f|^{-1}(U_g)} \arrow[r] \arrow[d, hook] & U_g \arrow[r, "|g|"'] \arrow[d, hook] & C \\
U_f \arrow[r, "|f|"'] \arrow[d, hook] & B & \\
A & &
\end{tikzcd}$$

It is easy to see that composition is associative and that the identities are units, so we have a category **Par**.

The category of *pointed sets*,

$$\mathbf{Sets}_*$$

has as objects, sets A equipped with a distinguished "point" $a \in A$, that is, pairs,

$$(A, a) \text{ with } a \in A.$$

Arrows are functions that preserve the point, that is, an arrow $f : (A, a) \to (B, b)$ is a function $f : A \to B$ such that $f(a) = b$.

Now we show:

Proposition 7.28. $\mathbf{Par} \simeq \mathbf{Sets}_*$

The functors establishing the equivalence are as follows:

$$F : \mathbf{Par} \to \mathbf{Sets}_*$$

is defined on an object A by $F(A) = (A \cup \{*\}, *)$, where $*$ is a new element that we add to A. We also write $A_* = A \cup \{*\}$. For arrows, given $f : A \rightharpoondown B$, $F(f) : A_* \to B_*$ is defined by

$$f_*(x) = \begin{cases} f(x) & \text{if } x \in U_f \\ * & \text{otherwise.} \end{cases}$$

Then clearly $f_*(*_A) = *_B$, so in fact $f_* : A_* \to B_*$ is "pointed," as required.

Coming back, the functor

$$G : \mathbf{Sets}_* \to \mathbf{Par}$$

is defined on an object (A, a) by $G(A, a) = A - \{a\}$ and for an arrow $f : (A, a) \to (B, b)$

$$G(f) : A - \{a\} \rightharpoondown B - \{b\}$$

is the function with domain

$$U_{G(f)} = A - f^{-1}(b)$$

defined by $G(f)(x) = f(x)$ for every $f(x) \neq b$.

Now $G \circ F$ is the identity on **Par**, because we are just adding a new point and then throwing it away. But $F \circ G$ is only naturally isomorphic to $1_{\mathbf{Sets}_*}$, since we have

$$(A, a) \cong ((A - \{a\}) \cup \{*\}, *).$$

These sets are not equal, since $a \neq *$. It still needs to be checked, of course, that F and G are functorial, and that the comparison $(A, a) \cong ((A - \{a\}) \cup \{*\}, *)$ is natural, but we leave these easy verifications to the reader.

Observe that this equivalence implies that **Par** has all limits, since it is equivalent to a category of "algebras" of a very simple type, namely sets equipped with a single, nullary operation, that is, a "constant." We already know that limits of algebras can always be constructed as limits of the underlying sets, and an easy exercise shows that a category equivalent to one with limits of any type also has such limits.

Example 7.29. *Slice categories and indexed families*

For any set I, the functor category $\mathbf{Sets}^I$ is the category of I-indexed sets. The objects are I-indexed families of sets

$$(A_i)_{i \in I}$$

and the arrows are I-indexed families of functions,

$$(f_i : A_i \to B_i)_{i \in I} : (A_i)_{i \in I} \longrightarrow (B_i)_{i \in I}.$$

This category has an equivalent description that is often quite useful: it is equivalent to the slice category of $\mathbf{Sets}$ over I, consisting of arrows $\alpha : A \to I$ and "commutative triangles" over I (see Section 1.6),

$$\mathbf{Sets}^I \ \simeq \ \mathbf{Sets}/I.$$

Indeed, define functors

$$\Phi : \mathbf{Sets}^I \ \longrightarrow \ \mathbf{Sets}/I$$
$$\Psi : \mathbf{Sets}/I \ \longrightarrow \ \mathbf{Sets}^I$$

on objects as follows:

$$\Phi((A_i)_{i \in I}) = \pi : \coprod_{i \in I} A_i \to I \quad \text{(the indexing projection)},$$

where the coproduct is conveniently taken to be

$$\coprod_{i \in I} A_i = \{(i, a) \mid a \in A_i\}.$$

And coming back, we have

$$\Psi(\alpha : A \to I) = (\alpha^{-1}\{i\})_{i \in I}.$$

The effect on arrows is analogous and easily inferred. We leave it as an exercise to show that these are indeed mutually pseudo-inverse functors. (Why are they *not* inverses?)

The equivalent description of $\mathbf{Sets}^I$ as $\mathbf{Sets}/I$ leads to the idea that, for a general category $\mathcal{E}$, the slice category $\mathcal{E}/X$, for any object X, can also be regarded as the category of "X-indexed objects of $\mathcal{E}$", although the functor category $\mathcal{E}^X$ usually does not make sense. This provides a precise notion of an "X-indexed family of objects E_x of $\mathcal{E}$," namely as a map $E \to X$.

For instance, in topology, there is the notion of a "fiber bundle" as a continuous function $\pi : Y \to X$, thought of as a family of spaces $Y_x = \pi^{-1}(x)$, the "fibers" of π, varying continuously in a parameter $x \in X$. Similarly, in dependent type theory there are "dependent types" $x : X \vdash A(x)$, thought of as families of types indexed over a type. These can be modeled as objects $[\![A]\!] \to [\![X]\!]$ in the slice category $\mathcal{E}/[\![X]\!]$ over the interpretation of the (closed) type X as an object of a category $\mathcal{E}$.

If $\mathcal{E}$ has pullbacks, reindexing of an "indexed family" along an arrow $f : Y \to X$ in $\mathcal{E}$ is represented by the pullback functor $f^* : \mathcal{E}/X \to \mathcal{E}/Y$. This is motivated by the fact that in **Sets** the following diagram commutes (up to natural isomorphism) for any $f : J \to I$:

$$\begin{tikzcd} \mathbf{Sets}^I \arrow[r, "\simeq"] \arrow[d, "\mathbf{Sets}^f"'] & \mathbf{Sets}/I \arrow[d, "f^*"] \\ \mathbf{Sets}^J \arrow[r, "\simeq"'] & \mathbf{Sets}/J \end{tikzcd}$$

where the functor $\mathbf{Sets}^f$ is the reindexing along f:

$$(\mathbf{Sets}^f(A_i))_j = A_{f(j)}.$$

Moreover, there are also functors going in the other direction,

$$\Sigma_f, \Pi_f : \mathbf{Sets}/J \longrightarrow \mathbf{Sets}/I$$

which, in terms of indexed families, are given by taking sums and products of the fibers:

$$(\Sigma_f(A_j))_i = \sum_{f(j)=i} A_j$$

and similarly for Π. These functors can be characterized in terms of the pullback functor f^* (as adjoints, see Section 9.7), and so also make sense in categories more general than **Sets**, where there are no "indexed families" in the usual sense. For instance, in dependent type theory, these operations are formalized by logical rules of inference similar to those for the existential and universal quantifier, and the resulting category of types has such operations of dependent sums and products.

Example 7.30. *Stone duality*
Many examples of equivalences of categories are given by what are called "dualities." Often, classical duality theorems are not of the form $\mathbf{C} \cong \mathbf{D}^{\mathrm{op}}$ (much less $\mathbf{C} = \mathbf{D}^{\mathrm{op}}$), but rather $\mathbf{C} \simeq \mathbf{D}^{\mathrm{op}}$, that is, $\mathbf{C}$ is equivalent to the opposite (or "dual") category of $\mathbf{D}$. This is because the duality is established by a construction that returns the original thing only up to isomorphism, not "on the nose." Here is a simple example, which is a very special case of the far-reaching *Stone duality* theorem.

Proposition 7.31. *The category of finite Boolean algebras is equivalent to the opposite of the category of finite sets,*

$$\mathbf{BA}_{\mathrm{fin}} \simeq \mathbf{Sets}^{\mathrm{op}}_{\mathrm{fin}}.$$

Proof. The functors involved here are the contravariant powerset functor

$$\mathcal{P}^{\mathbf{BA}} : \mathbf{Sets}_{\mathrm{fin}}^{\mathrm{op}} \to \mathbf{BA}_{\mathrm{fin}}$$

on one side (the powerset of a finite set is finite!). Going back, we use the functor,

$$A : \mathbf{BA}_{\mathrm{fin}}^{\mathrm{op}} \to \mathbf{Sets}_{\mathrm{fin}}$$

taking the set of *atoms* of a Boolean algebra,

$$A(\mathcal{B}) = \{a \in \mathcal{B} \mid 0 < a \text{ and } (b < a \Rightarrow b = 0)\}.$$

In the finite case, this is isomorphic to the ultrafilter functor that we have already studied (see Section 7.3).

Lemma 7.32. *For any finite Boolean algebra $\mathcal{B}$, there is an isomorphism between atoms a in $\mathcal{B}$ and ultrafilters $U \subseteq \mathcal{B}$, given by*

$$U \mapsto \bigwedge_{b \in U} b$$

and

$$a \mapsto \uparrow(a).$$

Proof. If a is an atom, then $\uparrow(a)$ is an ultrafilter, since for any b either $a \wedge b = a$ and then $b \in\uparrow(a)$ or $a \wedge b = 0$ and so $\neg b \in\uparrow(a)$.

If $U \subseteq \mathcal{B}$ is an ultrafilter then $0 < \bigwedge_{b \in U} b$, because, since U is finite and closed under intersections, we must have $\bigwedge_{b \in U} b \in U$. If $0 \neq b_0 < \bigwedge_{b \in U} b$ then b_0 is not in U, so $\neg b_0 \in U$. But then $b_0 < \neg b_0$ and so $b_0 = 0$.

Plainly, $U \subseteq\uparrow(\bigwedge_{b \in U} b)$ since $b \in U$ implies $\bigwedge_{b \in U} b \subseteq b$. Now let $\bigwedge_{b \in U} b \leq a$ for some a not in U. Then, $\neg a \in U$ implies that also $\bigwedge_{b \in U} b \leq \neg a$, and so $\bigwedge_{b \in U} b \leq a \wedge \neg a = 0$, which is impossible. □

Since we know that the set of ultrafilters $\mathrm{Ult}(\mathcal{B})$ is contravariantly functorial (it is represented by the Boolean algebra **2**, see Section 7.3), we therefore also have a contravariant functor of atoms $A \cong \mathrm{Ult}$. The explicit description of this functor is this: if $h : \mathcal{B} \to \mathcal{B}'$ and $a' \in A(\mathcal{B}')$, then it follows from the lemma that there is a *unique* atom $a \in \mathcal{B}$ such that $a' \leq h(b)$ iff $a \leq b$ for all $b \in \mathcal{B}$. To find this atom a, take the intersection over the ultrafilter $h^{-1}(\uparrow(a'))$,

$$A(a') = a = \bigwedge_{a' \leq h(b)} b.$$

Thus, we get a function

$$A(h) : A(\mathcal{B}') \to A(\mathcal{B}).$$

Of course, we must still check that this is a pseudo-inverse for $\mathcal{P}^{\mathbf{BA}} : \mathbf{Sets}_{\mathrm{fin}}^{\mathrm{op}} \to \mathbf{BA}_{\mathrm{fin}}$. The required natural isomorphisms,

$$\alpha_X : X \to A(\mathcal{P}(X))$$
$$\beta_{\mathcal{B}} : \mathcal{B} \to \mathcal{P}(A(\mathcal{B}))$$

are explicitly described as follows:

The atoms in a finite powerset $\mathcal{P}(X)$ are just the singletons $\{x\}$ for $x \in X$, thus $\alpha_X(x) = \{x\}$ is clearly an isomorphism.

To define $\beta_{\mathcal{B}}$, let

$$\beta_{\mathcal{B}}(b) = \{a \in A(\mathcal{B}) \mid a \leq b\}.$$

To see that $\beta_{\mathcal{B}}$ is also iso, consider the proposed inverse,

$$(\beta_{\mathcal{B}})^{-1}(B) = \bigvee_{a \in B} a \qquad \text{for } B \subseteq A(\mathcal{B}).$$

The isomorphism then follows from the following lemma, the proof of which is routine.

Lemma 7.33. *For any finite Boolean algebra $\mathcal{B}$,*

1. $b = \bigvee\{a \in A(\mathcal{B}) \mid a \leq b\}$.
2. *If a is an atom and $a \leq b \vee b'$, then $a \leq b$ or $a \leq b'$.*

Of course, one must still check that α and β really are natural transformations. This is left to the reader. □

Finally, we remark that the duality

$$\mathbf{BA}_{\mathrm{fin}} \ \simeq\ \mathbf{Sets}_{\mathrm{fin}}^{\mathrm{op}}$$

extends to one between *all* sets on the one side and the complete, atomic Boolean algebras, on the other,

$$\mathbf{caBA} \ \simeq\ \mathbf{Sets}^{\mathrm{op}},$$

where a Boolean algebra $\mathcal{B}$ is *complete* if every subset $U \subseteq \mathcal{B}$ has a join $\bigvee U \in \mathcal{B}$ and a *complete homomorphism* preserves these joins and $\mathcal{B}$ is *atomic* if every nonzero element $0 \neq b \in \mathcal{B}$ has some $a \leq b$ with a an atom.

Moreover, this is just the discrete case of the full Stone duality theorem, which states an equivalence between the category of *all* Boolean algebras and the opposite of a certain category of topological spaces, called "Stone spaces," and all continuous maps between them. For details, see Johnstone (1982).

7.11 Exercises

1. Consider the (covariant) composite functor,

$$\mathcal{F} = \mathcal{P}^{\mathbf{BA}} \circ \mathrm{Ult}^{\mathrm{op}} : \mathbf{BA} \to \mathbf{Sets}^{\mathrm{op}} \to \mathbf{BA}$$

taking each Boolean algebra B to the powerset algebra of sets of ultrafilters in B. Note that

$$\mathcal{F}(B) \cong \mathrm{Hom}_{\mathbf{Sets}}(\mathrm{Hom}_{\mathbf{BA}}(B, \mathbf{2}), 2)$$

is a sort of "double-dual" Boolean algebra. There is always a homomorphism,

$$\phi_B : B \to \mathcal{F}(B)$$

given by $\phi_B(b) = \{\mathcal{V} \in \mathrm{Ult}(B) \mid b \in \mathcal{V}\}$. Show that for any Boolean homomorphism $h : A \to B$, the following square commutes:

$$\begin{tikzcd} A \arrow[r, "\phi_A"] \arrow[d, "h"'] & \mathcal{F}(A) \arrow[d, "\mathcal{F}(h)"] \\ B \arrow[r, "\phi_B"'] & \mathcal{F}(B) \end{tikzcd}$$

2. Show that the homomorphism $\phi_B : B \to \mathcal{F}(B)$ in the foregoing problem is always injective (use the Boolean prime ideal theorem). This is the classical "Stone representation theorem," stating that every Boolean algebra is isomorphic to a "field of sets," that is, a sub-Boolean algebra of a powerset. Is the functor $\mathcal{F}$ faithful?
3. Prove that for any *finite* Boolean algebra B, the "Stone representation"

$$\phi : B \to \mathcal{P}(\mathrm{Ult}(B))$$

is in fact an isomorphism of Boolean algebras. (Note the similarity to the case of finite dimensional vector spaces.) This concludes the proof that we have an equivalence of categories,

$$\mathbf{BA}_{\mathrm{fin}} \simeq \mathbf{Sets}_{\mathrm{fin}}^{\mathrm{op}}$$

This is the "finite" case of Stone duality.
4. Consider the forgetful functors

$$\mathbf{Groups} \xrightarrow{U} \mathbf{Monoids} \xrightarrow{V} \mathbf{Sets}$$

Say whether each is faithful, full, injective on arrows, surjective on arrows, injective on objects, and surjective on objects.

5. Make every poset $(X, \leq)$ into a topological space by letting $U \subseteq X$ be open just if $x \in U$ and $x \leq y$ implies $y \in U$ (U is "closed upward"). This is called the *Alexandroff topology* on X. Show that it gives a functor

$$A : \mathbf{Pos} \to \mathbf{Top}$$

from posets and monotone maps to spaces and continuous maps by showing that any monotone map of posets $f : P \to Q$ is continuous with respect to this topology on P and Q (the inverse image of an open set must be open).

   Is A faithful? Is it full?

6. Prove that every functor $F : \mathbf{C} \to \mathbf{D}$ can be factored as $D \circ E = F$,

$$\mathbf{C} \xrightarrow{E} \mathbf{E} \xrightarrow{D} \mathbf{D}$$

in the following two ways:

   (a) $E : \mathbf{C} \to \mathbf{E}$ is bijective on objects and full, and $D : \mathbf{E} \to \mathbf{D}$ is faithful;

   (b) $E : \mathbf{C} \to \mathbf{E}$ surjective on objects and $D : \mathbf{E} \to \mathbf{D}$ is injective on objects and full and faithful.

   When do the two factorizations agree?

7. Show that a natural transformation is a natural isomorphism just if each of its components is an isomorphism. Is the same true for monomorphisms?

8. Show that a functor category $\mathbf{D}^{\mathbf{C}}$ has binary products if $\mathbf{D}$ does (construct the product of two functors F and G "objectwise": $(F \times G)(C) = F(C) \times G(C)$).

9. Show that the map of sets

$$\eta_A : A \longrightarrow PP(A)$$
$$a \longmapsto \{U \subseteq A | a \in U\}$$

is the component at A of a natural transformation $\eta : 1_{\mathbf{Sets}} \to PP$, where $P : \mathbf{Sets}^{\mathrm{op}} \to \mathbf{Sets}$ is the (contravariant) powerset functor.

10. Let $\mathbf{C}$ be a locally small category. Show that there is a functor

$$\mathrm{Hom} : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Sets}$$

such that for each object C of $\mathbf{C}$,

$$\mathrm{Hom}(C, -) : \mathbf{C} \to \mathbf{Sets}$$

is the covariant representable functor and

$$\mathrm{Hom}(-, C) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$$

is the contravariant one. (Hint: *use the bifunctor lemma.*)

11. Recall from the text that a *groupoid* is a category in which every arrow is an isomorphism. Prove that the category of groupoids is cartesian closed.
12. Let $\mathbf{C} \cong \mathbf{D}$ be equivalent categories. Show that $\mathbf{C}$ has binary products if and only if $\mathbf{D}$ does.
13. What sorts of properties of categories do *not* respect equivalence? Find one that respects isomorphism, but not equivalence.
14. Complete the proof that $\mathbf{Par} \cong \mathbf{Sets}_*$.
15. Show that equivalence of categories is an equivalence relation.
16. A category is *skeletal* if isomorphic objects are always identical. Show that every category is equivalent to a skeletal subcategory. (Every category has a "skeleton.")
17. Complete the proof that, for any set I, the category of I-indexed families of sets, regarded as the functor category $\mathbf{Sets}^I$, is equivalent to the slice category $\mathbf{Sets}/I$ of sets over I,

$$\mathbf{Sets}^I \simeq \mathbf{Sets}/I.$$

Show that reindexing of families along a function $f : J \to I$, given by precomposition,

$$\mathbf{Sets}^f((A_i)_{i\in I}) = (A_{f(j)})_{j\in J}$$

is represented by pullback, in the sense that the following diagram of categories and functors commutes up to natural isomorphism:

$$\begin{tikzcd}
\mathbf{Sets}^I \arrow[r, "\simeq"] \arrow[d, "\mathbf{Sets}^f"'] & \mathbf{Sets}/I \arrow[d, "f^*"] \\
\mathbf{Sets}^J \arrow[r, "\simeq"'] & \mathbf{Sets}/J
\end{tikzcd}$$

Here $f^* : \mathbf{Sets}/J \to \mathbf{Sets}/I$ is the pullback functor along $f : J \to I$. Finally, infer that $\mathbf{Sets}/2 \simeq \mathbf{Sets} \times \mathbf{Sets}$, and similarly for any n other than 2.

18. Show that a category with finite products is a monoidal category. Infer that the same is true for any category with finite coproducts.