Time Series Analysis

Ex2

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(Dept.: Mathematics, Major: statistics)

3.2

Let $\{w_t; t = 0, 1, \cdots\}$ be a white noise process with variance σ_w^2 and let $|\phi| < 1$ be a constant. Consider the process $x_0 = w_0$, and

$$x_t = \phi x_{t-1} + w_t, t = 1, 2, \cdots \tag{1}$$

We might use this method to simulate an AR(1) process from simulated white noise.

(a) Show that $x_t = \sum_{j=0}^{t} \phi^j w_{t-j}$ for any $t = 0, 1, \cdots$

iterating substitute x_{t-1}

$$\begin{aligned} x_t &= \phi x_{t-1} + w_t \\ &= \phi(\phi x_{t-2} + w_{t-1}) + w_t \\ \cdots &= \sum_{j=0}^{t} \phi^j w_{t-j} \end{aligned} \tag{2}$$

for $x_0 = w_0$, we could simplify the process as above.

(b) Find the $\mathbb{E}(x_t)$

take the expectation of (2), where expectation is commute with summation.

$$\mathbb{E}(x_t) = \mathbb{E}(\sum_{j=0}^{t} \phi^j w_{t-j}) = \sum_{j=0}^{t} \mathbb{E}(\phi^j w_{t-j}) = 0 \tag{3}$$

(c) Show that, for $t = 0, 1, \cdots,$

$$var(x_t) = \frac{\sigma_w^2}{1-\phi^2}(1-\phi^{2(t+1)}) \tag{4}$$

since $\mathbb{E}(x_t) = 0$, $var(x_t) = \mathbb{E}(x_t^2)$. And w_t is i.d.d. norm process, which means $w_i \perp w_j$,for $i \neq j$. The var can be shown as

$$var(x_t) = \mathbb{E}(x_t^2) = \mathbb{E}(\sum_{j=0}^{t} \phi^j w_{t-j})^2 = \sigma_w^2 \sum_{j=0}^{t} \phi^{2j} = \frac{\sigma_w^2}{1-\phi^2}(1-\phi^{2(t+1)}) \tag{5}$$

(d) Show that, for $h \geq 0$,

$$cov(x_{t+h}, x_t) = \phi^h var(x_t) \tag{6}$$

in this case, $cov(x_{t+h}, x_t)$ can be shown as,

$$\begin{aligned} cov(x_{t+h}, x_t) &= \mathbb{E}(x_{t+h}x_t) = \mathbb{E}(\sum_{j=0}^{t+h} \phi^j w_{t+h-j} \sum_{k=0}^{t} \phi^k w_{t-k}) \\ &= \mathbb{E}[(w_{t+h} + \phi w_{t+h-1} + \cdots + \phi^h w_t + \cdots + \phi^{t+h} w_0)(w_t + \phi w_{t-1} + \cdots + \phi^t w_0)] \\ &= \sigma_w^2 \sum_{j=0}^{t} \phi^{h+j}\phi^j = \sigma_w^2 \phi^h \sum_{j=0}^{t} \phi^{2j} = \phi^h var(x_t) \end{aligned} \tag{7}$$

(e) Is x_t stationary?

No,

(1) the mean value function, μ_t , defined in (3) is zero and does not depend on time t, and

(2) the autocovariance function, $\gamma(x_{t+h}, x_t)$ can be shown as

$$\gamma(x_{t+h}, x_t) = \frac{cov(x_{t+h}, x_t)}{\sqrt{var(x_{t+h})var(x_t)}} = \frac{\frac{\phi^h \sigma_w^2}{1-\phi^2}(1-\phi^{2(t+1)})}{\sqrt{\frac{\sigma_w^2}{1-\phi^2}(1-\phi^{2(t+1)})\frac{\sigma_w^2}{1-\phi^2}(1-\phi^{2(t+1+h)})}} = \phi^h \sqrt{\frac{1-\phi^{2(t+1)}}{1-\phi^{2(t+1+h)}}} \tag{8}$$

$\gamma(x_{t+h}, x_t)$ defined in (8) depends on t and h, not only just through their difference h.

(f) Argue that, as $t \to \infty$, the process becomes stationary, so in a sense, x_t is 'asymptotically stationary'.

Yes, as $t \to \infty$, $\gamma(x_{t+h}, x_t)$ is asymptotical to ϕ^h, which only depends on difference h.

(g) Comment on how you could use these results to simulate n observations of a stationary Gaussian $AR(1)$ model from simulated iid $N(0, 1)$ values

From (2), suppose $\{w_t\}, t = 1, 2, \cdots, n$ is iid $N(0, 1)$, we could simulate n observations as following,

$$\begin{aligned} &x_0 = w_0 \\ &x_1 = w_1 + \phi w_0 \\ &\cdots \\ &x_t = w_t + \phi w_{t-1} + \cdots + \phi^t w_0 = \sum_{j=0}^{t} \phi^j w_{t-j} \end{aligned} \tag{9}$$

(h) Now suppose $x_0 = \frac{w_0}{1-\phi^2}$. Is this process stationary? Hint: Show $var(x_t)$ is constant.

x_t can be shown as,

$$
\begin{aligned}
x_t &= \phi x_{t-1} + w_t \\
&= \phi(\phi x_{t-2} - w_{t-1}) + w_t \\
&\dots \\
&= \phi^t x_0 + \phi^{t-1} w_1 + \cdots + \phi w_{t-1} + w_t
\end{aligned} \tag{10}
$$

Take var on both side of (10), we can obtain

$$
\begin{aligned}
var(x_t) &= \frac{\phi^{2t}\sigma_w^2}{1-\phi^2} + \phi^{2(t-1)}\sigma_w^2 + \cdots + \sigma_w^2 = \sigma_w^2(1+\phi^2+\cdots+\phi^{2(t-1)}) + \frac{\phi^{2t}\sigma_w^2}{1-\phi^2} \\
&= \sigma_w^2 \frac{1-\phi^{2t}}{1-\phi^2} + \frac{\phi^{2t}\sigma_w^2}{1-\phi^2} = \frac{\sigma_w^2}{1-\phi^2}
\end{aligned} \tag{11}
$$

It can be shown that $var(x_t)$ is constant, so this process is stationary.

3.3

(a) Let $x_t = \phi x_{t-1} + w_t$ where $|\phi| > 1$ and $w_t \sim$iid $N(0, \sigma_w^2)$. Show $\mathbb{E}(x_t) = 0$ and $\gamma_x(h) = \sigma_w^2 \phi^{-2}\phi^{-h}/(1-\phi^{-2})$ for $h \geq 0$

we could modify that argument to obtain a stationary model as follows, $x_{t+1} = \phi x_t + w_{t+1}$,in which case,

$$
\begin{aligned}
x_t &= \phi^{-1}x_{t+1} - \phi^{-1}w_{t+1} = \phi^{-1}(\phi^{-1}x_{t+2} - \phi^{-1}w_{t+2}) - \phi^{-1}w_{t+1} \\
&= \phi^{-k}x_{t+k} - \sum_{j=1}^{k-1}\phi^{-j}w_{t+j}
\end{aligned} \tag{12}
$$

since $|\phi^{-1}| < 1$, we could iterating to the infinity

$$
x_t = -\sum_{j=1}^{\infty}\phi^{-j}w_{t+j} \tag{13}
$$

take the expectation both side, because $\sum_{j=1}^{\infty}\phi^{-j}$ converge

$$
\begin{aligned}
\mathbb{E}(x_t) &= -\mathbb{E}(\sum_{j=1}^{\infty}\phi^{-j}w_{t+j}) \\
&= -\sum_{j=1}^{\infty}\mathbb{E}(\phi^{-j}w_{t+j}) = 0
\end{aligned} \tag{14}
$$

as for $\gamma_x(h)$

$$
\begin{aligned}
\gamma_x(h) &= cov(x_{t+h}, x_t) = cov(-\sum_{j=1}^{\infty}\phi^{-j}w_{t+h+j}, -\sum_{k=1}^{\infty}\phi^{-k}w_{t+k}) \\
&= \mathbb{E}[(\sum_{k=1}^{\infty}\phi^{-k}w_{t+k}))(\sum_{j=1}^{\infty}\phi^{-j}w_{t+h+j})] \\
&= \mathbb{E}[(\phi^{-1}w_{t+1} + \cdots + \phi^{-h-1}w_{t+h+1} + \dots)(\phi^{-1}w_{t+h+1} + \cdots)] \\
&= \sigma_w^2\sum_{j=1}^{\infty}\phi^{-h-j}\phi^{-j} = \sigma_w^2\phi^{-h}\sum_{j=1}^{\infty}\phi^{-2j} \\
&= \sigma_w^2\phi^{-h}\phi^{-2}/(1-\phi^{-2})
\end{aligned} \tag{15}
$$

(b)Let $y_t = \phi^{-1} y_{t-1} + v_t$ where $v_t \sim$ iid $N(0, \sigma_w^2 \phi^{-2})$ and ϕ and σ_w are as in part (a). Argue that y_t is causal with the same mean function and autocovariancefunction as x_t

we can write $y_t = \phi^{-1} y_{t-1} + v_t$ as below,

$$\begin{aligned} y_t &= \phi^{-1} y_{t-1} + v_t = \phi^{-1}(\phi^{-1} y_{t-2} + v_{t-1}) + v_t \\ \cdots &= \sum_{j=0}^{\infty} \phi^{-j} v_{t-j} = \Phi(\mathbf{B}) v_t \end{aligned} \tag{16}$$

since y_t does not depend on the future time, so it is causal. And $|\phi| > 1, |\phi^{-1}| < 1$, y_t is causal with the same mean function.

Then the autocovariance function of y_t, denoted as $\gamma_y(h)$ is followed by

$$\begin{aligned} \gamma_y(h) &= \mathbb{E}(y_{t+h} y_t) = \mathbb{E}[(\sum_{j=0}^{\infty} \phi^{-j} v_{t+h-j})(\sum_{k=0}^{\infty} \phi^{-k} v_{t-k})] \\ &= \mathbb{E}[(v_{t+h} + \phi^{-1} v_{t+h-1} + \cdots + \phi^{-h} v_t + \cdots)(v_t + \phi^{-1} v_{t-1} + \cdots)] \\ &= \sigma_y^2 \sum_{j=0}^{\infty} \phi^{-h-j} \phi^{-j} = \sigma_w^2 \phi^{-2} \phi^{-h} / (1 - \phi^{-2}) = \gamma_x(h) \end{aligned} \tag{17}$$

3.8

Verify the calculations for the autocorrelation function of an ARMA(1,1) process given in Example 3.14. Compare the form with that of the ACF for the ARMA(1,0) and the ARMA(0,1) series. Plot the ACFs of the three series on the same graph for $\phi = 0.6, \theta = 0.9$, and comment on the diagnostic capabilities of the ACF in this case.

ARMA(1,1) process $x_t = \phi x_{t-1} + \theta w_{t-1} + w_t$, where $|\phi| < 1$ and w_t ~iid N(0,σ_w^2)

$$\gamma(h) = cov(x_{t+h}, x_t) = cov(\phi x_{t+h-1} + \theta w_{t+h-1}, x_t) = \phi \gamma(h-1) + \theta cov(w_{t+h-1}, x_t) \tag{18}$$

For a causal ARMA(p,q) model, $\phi(\mathbf{B}) x_t = \theta(\mathbf{B}) w_t$,where $x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j}$

$$(1 - \phi z)(\psi_0 + \psi_1 z + \cdots) = (1 + \theta z) \tag{19}$$

Since $\max(p, q+1) = 2$, the first few values of ARMA(1,1) are

$$\begin{aligned} &\psi_0 = 1 \\ &\psi_1 - \phi \psi_0 = \theta \\ &\psi_j - \phi \psi_{j-1} = 0, j = 2, 3, \cdots \end{aligned} \tag{20}$$

So we obtain $\psi_0 = 1, \psi_1 = \phi + \theta, \psi_j = \phi \psi_{j-1}, j = 2, 3, \cdots$. The general solution is $\psi_j = c\phi^j$. To find the specific solution, use the initial condition $\psi_1 = \phi + \theta$, so $c = \frac{\phi + \theta}{\phi}$. Finally, $\psi_j = (1 + \frac{\theta}{\phi})\phi^j, j \geq 1$ The (18) can be shown as,

$$\begin{aligned} \gamma(h) &= cov(x_{t+h}, x_t) = \sigma_w^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h} = \sigma_w^2 (1 + \frac{\theta}{\phi}) \phi^h + \sigma_w^2 \sum_{j=1}^{\infty} (1 + \frac{\theta}{\phi})^2 \phi^j \phi^{j+h} \\ &= \sigma_w^2 (1 + \frac{\theta}{\phi}) \phi^h + \sigma_w^2 (1 + \frac{\theta}{\phi})^2 \phi^h \sum_{j=1}^{\infty} \phi^{2j} = \sigma_w^2 (1 + \frac{\theta}{\phi}) \phi^h + \sigma_w^2 \phi^h (1 + \frac{\theta}{\phi})^2 \frac{\phi^2}{1 - \phi^2} \\ &= \sigma_w^2 \phi^{h-1} \frac{(\theta + \phi)(1 - \phi^2) + \phi(\theta + \phi)^2}{1 - \phi^2} = \sigma_w^2 \phi^{h-1} \frac{(\theta + \phi)(1 + \phi\theta)}{1 - \phi^2} \end{aligned} \tag{21}$$

From (21), we can write a general homogeneous equation for the ACF

$$
\begin{aligned}
\rho(h) = \frac{\gamma(h)}{\gamma(0)} &= \frac{\gamma(h)}{\sqrt{var(x_t)var(x_{t+h})}} = \frac{\gamma(h)}{\sigma_w^2\sqrt{\sum_{j=0}^{\infty}\phi_j^2\sum_{k=0}^{\infty}\phi_k^2}} \\
&= \frac{\gamma(h)}{\sigma_w^2\sqrt{[1+(1+\frac{\theta}{\phi})^2\phi^2+\cdots][1+(1+\frac{\theta}{\phi})^2\phi^2+\cdots]}} \\
&= \frac{\sigma_w^2\phi^{h-1}\frac{(\theta+\phi)(1+\phi\theta)}{1-\phi^2}}{\sigma_w^2\frac{1+\theta^2+2\theta\phi}{1-\phi^2}} = \frac{\phi^{h-1}(\theta+\phi)(1+\phi\theta)}{1+\theta^2+2\theta\phi}, h \geq 1
\end{aligned} \tag{22}
$$

Find the ARMA(1,0),AR(1),$x_t = \phi x_{t-1} + w_t = \cdots = \sum_{j=0}^{\infty}\phi^j w_{t-j}$ ACF,$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \phi^h, h \geq 0$

Consider the ARMA(0,1),MA(1),model $x_t = w_t + \theta w_{t-1}$ ACF,$\rho(h) = \begin{cases} \frac{\theta}{1+\theta^2}, h = 1 \\ 0, h > 0 \end{cases}$

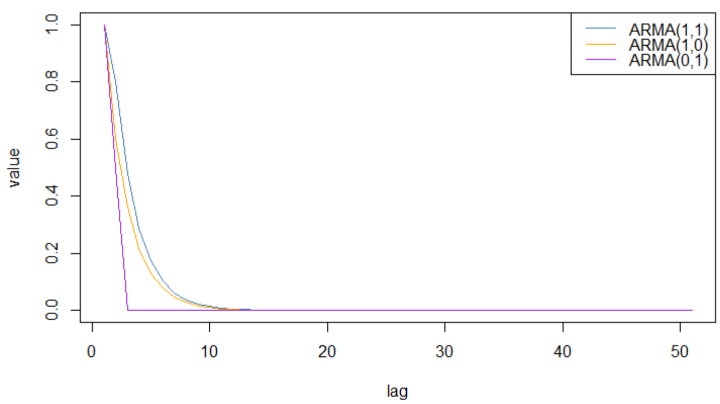

Figure 1: ACFs ARMA(1,1),ARMA(1,0),ARMA(0,1)

Hence, it is unlikely that we will be able to tell the difference betweenan ARMA(1,1)and an ARMA(1,0) based solely on an ACF estimated from a sample.

```
acf_1_1 <- ARMAacf(0.6,0.9,50)
acf_1_0 <- ARMAacf(0.6,0,50)
acf_0_1 <- ARMAacf(0,0.9,50)
plot(acf_1_1,type='l',ylab='value',xlab='lag',col='steelblue')
lines(acf_1_0,type='l',ylab='value',xlab='lag',col='orange')
lines(acf_0_1,type='l',ylab='value',xlab='lag',col='purple')
legend('topright',legend=c('ARMA(1,1)','ARMA(1,0)','ARMA(0,1)'),
col=c('steelblue','orange','purple'),lty=1)
```

3.9

Generate n=100 observations from each of the three models discussed in 3.8. Compute the sample ACF for each model and compare it to the theoretical values in (3)

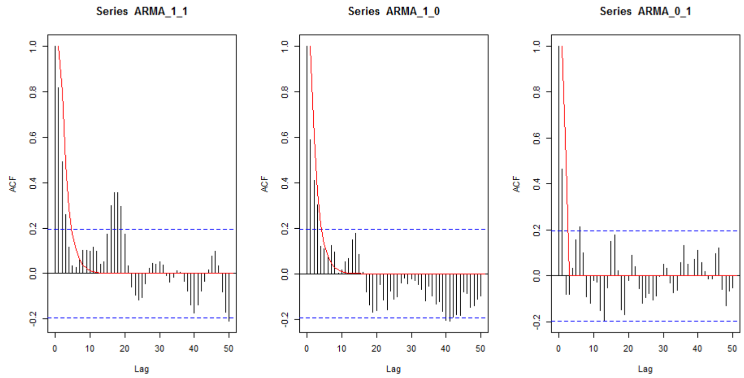

Figure 2: sample ACFs compared theoretical values in ARMA(1,1),ARMA(1,0),ARMA(0,1)(black is sample ACFs, red solid line is theoretical ACFs)

And then compute the sample PACF for each of the generated series and compare the sample ACFs and PACFs

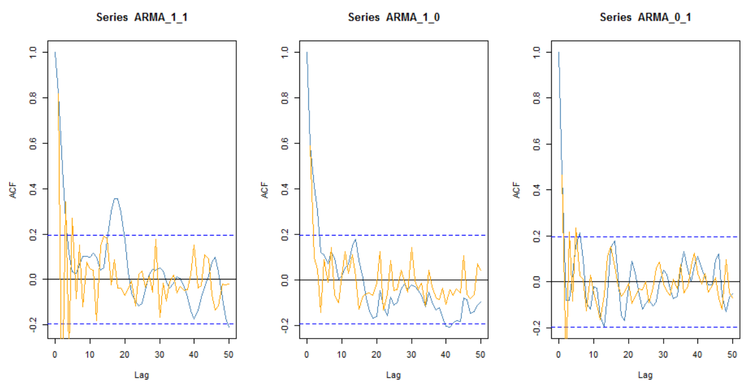

Figure 3: sample PACFs compared ACFs in ARMA(1,1),ARMA(1,0),ARMA(0,1)(blue solid line is sample ACFs, orange solid line is PACFs)

Compared the sample ACFs and PACFs in the Tab (1)

Table 1: ACFs and PACFs

	AR(p)	MA(q)	ARMA(p,q)
ACF	Tails off	Cuts off after lag q	Tails off
PACF	Cuts off after lag p	Tails off	Tails off

```
ARMA_1_1 <- arima.sim(list(order=c(1,0,1),ar=0.6,ma=0.9),n=100)
ARMA_1_0 <- arima.sim(list(order=c(1,0,0),ar=0.6),n=100)
ARMA_0_1 <- arima.sim(list(order=c(0,0,1),ma=0.9),n=100)

par(mfrow=c(1,3))
acf_s1 <- acf(ARMA_1_1,50,type='correlation',plot=F)
pacf_s1 <- pacf(ARMA_1_1,50,type='correlation',plot=F)
```

```
#lines(acf_1_1,col='red')
acf_s2 <- acf(ARMA_1_0,50,type='correlation',plot=F)
pacf_s2 <- pacf(ARMA_1_0,50,type='correlation',plot=F)
#lines(acf_1_0,col='red')
acf_s3 <- acf(ARMA_0_1,50,type='correlation',plot=F)
pacf_s3 <- pacf(ARMA_0_1,50,type='correlation',plot=F)
#lines(acf_0_1,col='red')
par(mfrow=c(1,3))
plot(acf_s1,type='l',col='steelblue')
lines(pacf_s1$acf,type='l',col='orange')
plot(acf_s2,type='l',col='steelblue')
lines(pacf_s2$acf,type='l',col='orange')
plot(acf_s3,type='l',col='steelblue')
lines(pacf_s3$acf,type='l',col='orange'$)
```

3.12

In the context of equation 23,show that,if $\gamma(0) > 0$ and $\gamma(h) \to 0$ as $h \to \infty$, then Γ_n is positive definite

$$\Gamma_n \phi_n = \gamma_n \tag{23}$$

where $\Gamma_n = \{\gamma(k-j)\}_{j,k=1}^{n}$ is an $n \times n$ matrix, $\phi^n = (\phi_{n1}, \cdots, \phi_{nn})'$ is an $n \times 1$ vector, and $\gamma_n = (\gamma(1), \cdots, \gamma(n))'$ is an $n \times 1$ vector. Since the Γ_n is symmetrical matrix, it is nonnegative definite.

$$\Gamma_n = \begin{bmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(n-1) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma(n-1) & \gamma(n-2) & \cdots & \gamma(0) \end{bmatrix}$$

The Γ_n is positive definite matrix means its leading principle submatrices all greater than 0 Since $\gamma_0 > 0$, suppose the *nth* leading principle submatrices determination of Γ_n is L_n,

$$L_1 = \begin{vmatrix} \gamma(0) \end{vmatrix}, L_2 = \begin{vmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{vmatrix}, \cdots, L_n = \begin{vmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(n-1) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma(n-1) & \gamma(n-2) & \cdots & \gamma(0) \end{vmatrix}$$

For ARMA model, the ACFs satisfy general homogeneous equation for the ACF of a causal ARMA process,

$$\gamma(h) - \phi_1\gamma(h-1) - \cdots - \phi_p\gamma(h-p) = 0, h \geq \max(p, q+1) \tag{24}$$

with initial condition, Recall for a causal model, all of the roots of $\phi(\mathbf{B})$ lied out of the unit circle. Let $z_1, z_2, \cdots, z_r$ denote the roots of $\phi(z)$, $|z_j| > 0$,for $i = 1, 2, \cdots r$. So the general solution of (24) is,

$$\gamma(h) = z_1^{-h} P_1(h) + z_2^{-h} P_2(h) + \cdots + z_r^{-h} P_r(h), h \geq p \tag{25}$$

where $P_j(h)$ is a polynomial in h of degree $m_j - 1$. Since $|z_j|^{-1} < 1$ for $i = 1, 2, \cdots r$ in (25), we conclude that $\gamma(h)$ is strictly decreasing,So the leading principle submatrices determination can be shown as,

$$L_1 = \begin{vmatrix}\gamma(0)\end{vmatrix} = \gamma(0) > 0, L_2 = \begin{vmatrix}\gamma(0) & \gamma(1)\\ \gamma(1) & \gamma(0)\end{vmatrix} = \gamma(0)^2 - \gamma(1)^2 > 0$$

$$L_n = \begin{vmatrix}\gamma(0) & \gamma(1) & \cdots & \gamma(n-1)\\ \vdots & \vdots & \ddots & \vdots\\ \gamma(n-1) & \gamma(n-2) & \cdots & \gamma(0)\end{vmatrix}$$

since $h \to \infty$, $\gamma(h) \to 0$. So $\exists M$ s.t. if $h > M$ $\gamma(h) < \varepsilon$,for every $\varepsilon > 0$. Expand L_n as

$$L_n = \begin{vmatrix}\gamma(0) & \gamma(1) & \cdots & \gamma(n-1)\\ \vdots & \vdots & \ddots & \vdots\\ \gamma(n-1) & \gamma(n-2) & \cdots & \gamma(0)\end{vmatrix} = \sum_{1\le i,j\le n} (-1)^{i+j} M_{ij} \overset{n\to\infty}{=} \gamma(0)L_{n-1} + \varepsilon$$

Use mathematical induction, suppose the determination of L_{n-1} is greater than zero, as well as $L_n > 0$. So, all leading principle submatrices determination $L_j > 0, j = 1, 2, \cdots, n$. Γ_n is positive definite matrix.

3.15

For an AR(1) model, determine the general form of the m-step-ahead forecast x_{t+m}^t and show $\mathbb{E}[(x_{t+m} - x_{t+m}^t)^2] = \sigma_w^2 \frac{1-\phi^{2m}}{1-\phi^2}$

AR(1) model process $x_t = \phi x_{t-1} + w_t$, suppose we want the m-step-ahead prediction

$$x_{t+m}^t = \phi_{t1}^{(m)} x_t + \phi_{t2}^{(m)} x_{t-1} + \cdots + \phi_{tt}^{(m)} x_1 \tag{26}$$

where $\{\phi_{n1}^{(m)}, \cdots, \phi_{nn}^{(m)}\}$ satisfy the prediction equations,

$$\sum_{j=1}^{t} \phi_{tj}^{(m)} \gamma(k-j) = \gamma(m+k-1), k = 1, \cdots, t \tag{27}$$

where $\gamma(h) = \frac{\sigma_w^2 \phi^h}{1-\phi^2}, h \ge 0$, the (27) can be shown as,

$$\begin{bmatrix}\gamma(0) & \gamma(1) & \cdots & \gamma(t-1)\\ \gamma(1) & \gamma(0) & \cdots & \gamma(t-2)\\ \vdots & \vdots & \ddots & \vdots\\ \gamma(t-1) & \gamma(t-2) & \cdots & \gamma(0)\end{bmatrix}\begin{bmatrix}\phi_{t1}^{(m)}\\ \phi_{t2}^{(m)}\\ \vdots\\ \phi_{tt}^{(m)}\end{bmatrix} = \begin{bmatrix}\gamma(m)\\ \gamma(m+1)\\ \vdots\\ \gamma(m+t-1)\end{bmatrix}$$

We start our m-step-ahead prediction from one-step-ahead prediction

$$\begin{aligned} &x_{t+1}^t = \phi_{t1} x_t + \phi_{t2} x_{t-1} + \cdots + \phi_{tt} x_1\\ &\mathbb{E}\{[x_t - \phi x_{t-1}]x_{t-1}\} = \mathbb{E}(w_t x_{t-1}) = 0 \end{aligned} \tag{28}$$

by the uniqueness of the coefficients in this case, that $\phi_{t1} = \phi, \phi_{t2} = \phi_{t3} = \cdots = \phi_{tt} = 0$, we could simplify the m-step ahead prediction

$$x_{t+m}^t = \phi x_{t+m-1}^t = \phi^2 x_{t+m-2}^t = \cdots = \phi^m x_t^t = \phi^m x_t \tag{29}$$

So the forecast error can be shown as,

$$\begin{aligned} \mathbb{E}[(x_{t+m} - x_{t+m}^t)^2] &= \mathbb{E}[(x_{t+m} - \phi^m x_t)^2] = \mathbb{E}(x_{t+m}^2 + \phi^{2m} x_t^2 - 2\phi^m x_t x_{t+m})\\ &= \mathbb{E}(x_{t+m}^2) + \phi^{2m}\mathbb{E}(x_t^2) - 2\phi^m \mathbb{E}(x_t x_{t+m}) = \rho(0) + \phi^{2m}\rho(0) - 2\phi^m \rho(m) \end{aligned} \tag{30}$$

Recall, in AR(1) model, $\rho(h) = \frac{\sigma_w^2 \phi^h}{1-\phi^2}$, substitute into (30)

$$\begin{aligned}
\mathbb{E}[(x_{t+m} - x_{t+m}^t)^2] &= \frac{\sigma_w^2}{1-\phi^2} + \phi^{2m}\frac{\sigma_w^2}{1-\phi^2} - 2\phi^m \sigma_w^2 \frac{\phi^m}{1-\phi^2} \\
&= \frac{\sigma_w^2(1-\phi^{2m} - 2\phi^m\phi^m)}{1-\phi} = \frac{\sigma_w^2(1-\phi^{2m})}{1-\phi^2}
\end{aligned} \tag{31}$$