Time Series Analysis

Homework3

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4.6

A first-order autoregressive model is generated from the white noise series w_t using the generating equations

$$x_t = \phi x_{t-1} + w_t$$

where ϕ, for $|\phi| < 1$, is a parameter and the w_t are independent random variables with mean zero and variance σ_w^2

(a)

Show that the power spectrum of x t is given by

$$f_x(w) = \frac{\sigma_w^2}{1+\phi^2 - 2\phi cos(2\pi w)}$$

As we know, the inverse transform of the spectral density if x_t is ARMA(p,q),$\phi(B)x_t = \theta(B)w_t$, as in our model $x_t - \phi x_{t-1} = w_t$, its spectral density is given by

$$\begin{aligned} f_x(w) &= \sigma_w^2 \frac{|\theta(e^{-2\pi i w})|^2}{|\phi(e^{-2\pi i w})|^2} = \sigma_w^2 \frac{1}{|1-\phi e^{-2\pi i w}|^2} = \sigma_w^2 \frac{1}{|1-\phi e^{-2\pi i w}||1-\phi e^{2\pi i w}|} \\ &= \sigma_w^2 \frac{1}{1+\phi^2 - \phi(e^{2\pi i w} + e^{-2\pi i w})} = \sigma_w^2 \frac{1}{1+\phi^2 - 2\phi \cos(2\pi w)} \end{aligned} \quad (1)$$

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(b)

Verify the autocovariance function of this process is

$$\gamma_x(h) = \frac{\sigma_w^2 \phi^{|h|}}{1-\phi^2}$$

$h = 0, \pm 1, \pm 2, \cdots$ by showing that the inverse transform of $\gamma_x(h)$ is the spectrum derived in part (a).

First, since $|\phi| < 1$, $\{x_t\}$ is stationary

$$\begin{aligned} x_t &= \phi x_{t-1} + w_t \\ &= \phi(\phi x_{t-2} + w_{t-1}) + w_t \\ &= \cdots \\ &= \sum_{j=0}^{\infty} \phi^j w_{t-j} \end{aligned} \tag{2}$$

and we can show the autocovariance $\gamma(h)$ from (2)

$$\begin{aligned} \gamma(h) = cov(x_{t+h}, x_t) &= cov(\sum_{j=0}^{\infty} \phi^j w_{t+h-j}, \sum_{j=0}^{\infty} \phi^j w_{t-j}) \\ &= \sum_{k=0}^{\infty} \phi^{h+2k} \sigma_w^2 = \frac{\phi^h \sigma_w^2}{1-\phi^2} \end{aligned} \tag{3}$$

Then the inverse transform of the spectral density

$$\begin{aligned} f_x(w) &= \sum_{h=-\infty}^{\infty} \gamma_x(h) e^{-2\pi iwh} = \sum_{h=-\infty}^{\infty} \frac{\phi^{|h|}\sigma_w^2}{1-\phi^2} e^{-2\pi iwh} \\ &= \lim_{h\to\infty} (1 + \underbrace{\phi e^{-2\pi iw} + \cdots + \phi^h e^{-2\pi iwh}}_{h>0} + \underbrace{\phi e^{2\pi iw} + \cdots + \phi^h e^{2\pi iwh}}_{h<0}) \frac{\sigma_w^2}{1-\phi^2} \\ &= \lim_{h\to\infty} \left[\frac{1-\phi^h e^{-2\pi iwh}}{1-\phi e^{-2\pi iw}} + \frac{\phi e^{2\pi iw}(1-\phi^h e^{2\pi iwh})}{1-\phi e^{2\pi iw}} \right] \frac{\sigma_w^2}{1-\phi^2} \\ &= \frac{1-\phi e^{2\pi iw} + \phi e^{2\pi iw} - \phi^2}{(1-\phi e^{-2\pi iw})(1-\phi e^{2\pi iw})} \frac{\sigma_w^2}{1-\phi^2} = \sigma_w^2 \frac{1}{1+\phi^2 - 2\phi\cos(2\pi w)} = f_x(w) \end{aligned} \tag{4}$$

4.8

Suppose x_t and y_t are stationary zero-meantime series with x_t independent of y_s for all s and t. Consider the product series

$$z_t = x_t y_y$$

Prove the spectral density for z_t can be written as

$$f_z(w) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f_x(w-v) f_y(v) dv$$

The autocovariance function of z_t is

$$\gamma_z(h) = cov(z_{t+h}, z_t) = cov(x_{t+h}y_{t+h}, x_t y_t) = \mathbb{E}(x_{t+h}y_{t+h}y_t x_t) - \mathbb{E}(x_{t+h}y_{t+h})\mathbb{E}(x_t y_t) \tag{5}$$

Since x_t and y_t are stationary zero-mean and independent time series, we could simplify the (5) as

$$\begin{aligned}\gamma_z(h) &= \mathbb{E}(x_{t+h}y_{t+h}y_t x_t) - \Big[[\mathbb{E}(x_{t+h})\mathbb{E}(y_{t+h}) + cov(x_{t+h}, y_{t+h})][\mathbb{E}(x_t)\mathbb{E}(y_t) + cov(x_t, y_t)]\Big] \\ &= \mathbb{E}(x_{t+h}y_{t+h}y_t x_t) = \mathbb{E}(x_{t+h}x_t y_{t+h}y_t) = \gamma_x(h)\gamma_y(h)\end{aligned} \tag{6}$$

As a result, we can have the spectral density for z_t

$$\begin{aligned}\gamma_x(h)\gamma_y(h) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} f_x(z)e^{2\pi izh}dz \cdot \int_{-\frac{1}{2}}^{\frac{1}{2}} f_y(v)e^{2\pi ivh}dv \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} f_x(z)f_y(v)e^{2\pi i(z+v)h}dvdz \\ &\overset{z+v=w}{=} \int_{-\frac{1}{2}}^{\frac{1}{2}} \underbrace{\int_{-\frac{1}{2}}^{\frac{1}{2}} f_x(w-v)f_y(v)dv}_{f_z(w)} e^{2\pi iwh}dw\end{aligned} \tag{7}$$

4.11

Let the observed series x t be composed of a periodic signal and noise so it can be written as

$$x_t = \beta_1 \cos(2\pi w_k t) + \beta_2 \sin(2\pi w_k t) + w_t$$

where w_t is a white noise process with variance σ_w^2. The frequency ω_k is assumed to be known and of the form k/n in this problem. Suppose we consider estimating β_1, β_2 and σ_w^2 by least squares, or equivalently, by maximum likelihood if the w t are assumed to be Gaussian.

(a)

Prove, for a fixed ω_k , the minimum squared error is attained by

$$\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = 2n^{-1/2} \begin{pmatrix} d_c(\omega_k) \\ d_s(\omega_k) \end{pmatrix}$$

where the cosine and sine transforms (4.31) and (4.32) appear on the right-hand side.

For a known ω_k, the squared error is attained by

$$Q(\beta_1,\beta_2) = \sum_{t=1}^{n} \hat{\omega}_t^2 = \sum_{t=1}^{n} (x_t - \hat{x}_t)^2 = \sum_{t=1}^{n} \left[x_t - \hat{\beta}_1 \cos(2\pi\omega_k t) - \hat{\beta}_2 \sin(2\pi\omega_k t) \right]^2 \quad (8)$$

Take partial derivative in (8) and set it to zero, we can obtain that

$$\begin{aligned} \frac{\partial Q}{\partial \beta_1} &= 2\sum_{t=1}^{n} \left[x_t - \hat{\beta}_1 \cos(2\pi\omega_k t) - \hat{\beta}_2 \sin(2\pi\omega_k t) \right] (-\cos(2\pi\omega_k t) = 0 \\ \frac{\partial Q}{\partial \beta_2} &= 2\sum_{t=1}^{n} \left[x_t - \hat{\beta}_1 \cos(2\pi\omega_k t) - \hat{\beta}_2 \sin(2\pi\omega_k t) \right] (-\sin(2\pi\omega_k t) = 0 \end{aligned} \quad (9)$$

Then set $\omega_k = k/n$ and we could prove that

$$\begin{aligned} &\sum_{t=1}^{n} \cos^2(2\pi\omega_k t) = \frac{n}{2} \\ &\sum_{t=1}^{n} \sin^2(2\pi\omega_k t) = \frac{n}{2} \\ &\sum_{t=1}^{n} \cos(2\pi\omega_k t)\sin(2\pi\omega_k t) = 0 \end{aligned} \quad (10)$$

We here only show the computing of $\sum_{t=1}^{n} \cos^2(2\pi\omega_k t)$, others are as the same procedure

$$\begin{aligned} \sum_{t=1}^{n} \cos^2(2\pi\omega_k t) &= \sum_{t=1}^{n} \left[\frac{\cos(4\pi\omega_k t)}{2} + \frac{1}{2} \right] \\ &= \frac{n}{2} + \frac{1}{4} \sum_{t=1}^{n} \left[e^{4\pi\omega_k it} + e^{-4\pi\omega_k it} \right] \\ &= \frac{n}{2} + \frac{1}{4} \left[\frac{e^{4\pi\omega_k i}(1 - e^{4\pi\omega_k in})}{1 - e^{4\pi\omega_k i}} + \frac{e^{-4\pi\omega_k i}(1 - e^{-4\pi\omega_k in})}{1 - e^{-4\pi\omega_k i}} \right] \\ &= \frac{n}{2} + \frac{1}{4} \left[\frac{(e^{4\pi\omega_k i} + e^{-4\pi\omega_k i}) + (e^{4\pi\omega_k in} + e^{-4\pi\omega_k in}) - (e^{4\pi\omega_k i(n+1)} + e^{-4\pi\omega_k i(n+1)}) - 2}{(1 - e^{4\pi\omega_k i})(1 - e^{-4\pi\omega_k i})} \right] \\ &= \frac{n}{2} + \frac{1}{4} \left[\frac{2\cos 4\pi\omega_k + 2\cos 4\pi\omega_k n - 2\cos 4\pi\omega_k (n+1) - 2}{(1 - e^{4\pi\omega_k i})(1 - e^{-4\pi\omega_k i})} \right] \\ &\overset{\omega_k = k/n}{=} \frac{n}{2} + \frac{1}{4} \frac{2\cos\frac{4\pi k}{n} + 2\cos 4\pi k - 2\cos\frac{4\pi k(n+1)}{n} - 2}{(1 - e^{4\pi\omega_k i})(1 - e^{-4\pi\omega_k i})} = \frac{n}{2} \end{aligned}$$

(11)

Substitute the (10) into (9), we obtain that

$$\sum_{t=1}^{n}\left[x_t\cos(2\pi\omega_k t)-\beta_1\cos^2(2\pi\omega_k t)-\beta_2\sin(2\pi\omega_k t)\cos(2\pi\omega_k t)\right]=\sum_{t=1}^{n}x_t\cos(2\pi\omega_k t)-\frac{n}{2}\beta_1=0$$
$$\sum_{t=1}^{n}\left[x_t\sin(2\pi\omega_k t)-\beta_1\cos(2\pi\omega_k t)\sin(2\pi\omega_k t)-\beta_2\sin^2(2\pi\omega_k t)\right]=\sum_{t=1}^{n}x_t\sin(2\pi\omega_k t)-\frac{n}{2}\beta_2=0 \tag{12}$$

we obtain

$$\hat{\beta}_1=\frac{2}{n}\sum_{t=1}^{n}x_t\cos(2\pi\omega_k t)=2n^{-1/2}d_c(\omega_k)$$
$$\hat{\beta}_2=\frac{2}{n}\sum_{t=1}^{n}x_t\sin(2\pi\omega_k t)=2n^{-1/2}d_s(\omega_k) \tag{13}$$

(b)

Prove that the error sum of squares can be written as

$$SSE=\sum_{t=1}^{n}x_t^2-2I_x(\omega_k)$$

so that the value of ω_k that minimizes squared error is the same as the value that maximizes the periodogram $I_x(\omega_k)$ estimator (4.28).

$$
\begin{aligned}
SSE &= \sum_{t=1}^{n}(x_t-\hat{x}_t)^2 = \sum_{t=1}^{n}\left[x_t-\hat{\beta}_1\cos(2\pi\omega_k t)-\hat{\beta}_2\sin(2\pi\omega_k t)\right]^2 \\
&= \sum_{t=1}^{n}\Big[x_t^2+\hat{\beta}_1^2\cos^2(2\pi\omega_k t)+\hat{\beta}_2^2\sin^2(2\pi\omega_k t) \\
&-2\hat{\beta}_1 x_t\cos(2\pi\omega_w t)-2\hat{\beta}_2 x_t\sin(2\pi\omega_w t)+\hat{\beta}_1\hat{\beta}_2\sin(2\pi\omega_k t)\cos(2\pi\omega_k t)\Big] \\
&= \sum_{t=1}^{n}x_t^2+\sum_{t=1}^{n}\Big[\frac{4}{n}d_c^2(\omega_k)\cos^2(2\pi\omega_k t)+\frac{4}{n}d_s^2(\omega_k)\sin^2(2\pi\omega_k t) \\
&-\frac{4}{\sqrt{n}}d_c(\omega_k)x_t\cos(2\pi\omega_k t)-\frac{4}{\sqrt{n}}d_s(\omega_k)x_t\sin(2\pi\omega_k t) \\
&+\frac{4}{n}d_c(\omega_k)d_s(\omega_k)\underbrace{\sin(2\pi\omega_k t)\cos(2\pi\omega_k t)}_{\Sigma=0}\Big] \\
&= \sum_{t=1}^{n}x_t^2+\frac{4}{n}d_c^2(\omega_k)\underbrace{\sum_{t=1}^{n}\cos^2(2\pi\omega_k t)}_{n/2}+\frac{4}{n}d_s^2(\omega_k)\underbrace{\sum_{t=1}^{n}\sin^2(2\pi\omega_k t)}_{n/2} \\
&-\frac{4}{\sqrt{n}}d_c(\omega_k)\underbrace{\sum_{t=1}^{n}x_t\cos(2\pi\omega_k t)}_{n^{1/2}d_c(w_k)}-\frac{4}{\sqrt{n}}d_s(\omega_k)\underbrace{\sum_{t=1}^{n}x_t\sin(2\pi\omega_k t)}_{n^{1/2}d_s(\omega_k)} \\
&= \sum_{t=1}^{n}x_t^2-2[d_c^2(\omega_k)+d_s^2(\omega_k)] = \sum_{t=1}^{n}x_t^2-2I_x(\omega_k)
\end{aligned}
\tag{14}
$$

So that the value of ω_k that minimizes squared error is the same as the value that maximizes the periodogram $I_x(\omega_k)$ estimator.

(c)

Under the Gaussian assumption and fixed ω_k, show that the F-test of no regression leads to an F-statistic that is a monotone function of $I_x(\omega_k)$.

Under the Gaussian assumption and fixed ω_k,show that the F-test of no regression leads to an F-statistic that is a monotone function of $I_x(\omega_k)$.

First, we compute the variance of $\hat{\beta}_1$ and $\hat{\beta}_2$

$$
\begin{aligned}
var(\hat{\beta}_1) = var(\frac{2}{n}\sum_{t=1}^{n}x_t\cos(2\pi\omega_k t)) = \frac{4\sigma_w^2}{n^2}\sum_{t=1}^{n}\cos^2(2\pi\omega_k t) = \frac{2\hat{\sigma}_w^2}{n} \\
var(\hat{\beta}_2) = var(\frac{2}{n}\sum_{t=1}^{n}x_t\sin(2\pi\omega_k t)) = \frac{4\sigma_w^2}{n^2}\sum_{t=1}^{n}\sin^2(2\pi\omega_k t) = \frac{2\hat{\sigma}_w^2}{n}
\end{aligned}
\tag{15}
$$

We could write the *square sum of estimation* of x_t

$$
\begin{aligned}
SSR = \sum_{t=1}^{n} \hat{x}_t^2 &= \sum_{t=1}^{n} \left[\cos^2(2\pi\omega_k t)\hat{\beta}_1^2 + \sin^2(2\pi\omega_k t)\hat{\beta}_2^2 + 2\cos(2\pi\omega_k t)\sin(2\pi\omega_k t)\hat{\beta}_1\hat{\beta}_2 \right] \\
&= \frac{n}{2} 4n^{-1} d_c^2(\omega_k) + \frac{n}{2} 4n^{-1} d_s^2(\omega_k) = 2I_x(\omega_k)
\end{aligned} \tag{16}
$$

The test of no regression leads to an F-statistic

$$
\begin{aligned}
F_{2,n-3} &= \frac{SSR/2}{SSE/(n-3)} = \frac{n-3}{2} \frac{2I_x(\omega_k)}{\sum_{t=1}^{n} x_t^2 - 2I_x(\omega_k)} \\
&= \frac{n-3}{2} \frac{2I_x(\omega_k)}{n\hat{\rho}(0) - 2I_x(\omega_k)} \overset{IDFT}{=} \frac{n-3}{2} \frac{2I_x(\omega_k)}{n\sum_j d(\omega_j) e^{2\pi i w_j * 0} - 2I_x(\omega_k)} \\
&\overset{fixed\,\omega_k}{=} \frac{n-3}{2} \frac{2I_x(\omega_k)}{nd(\omega_k) - 2I_x(\omega_k)} = \frac{n-3}{2} \frac{2I_x(\omega_k)}{n\sqrt{I_x(\omega_k)} - 2I_x(\omega_k)} \\
&= \frac{n-3}{\frac{n}{\sqrt{I_x(\omega_k)}} - 2}
\end{aligned} \tag{17}
$$

So $F_{2,n-3}$ is monotone function of $I_x(\omega_k)$.

4.17

Suppose x t is a mean-zero,stationary process with spectral density $f_x(\omega)$. If we replace the original series by the tapered series

$$
y_t = h_t x_t \tag{18}
$$

for $t = 1, 2, \cdots, n$, use the modified DFT

$$
d_y(w_j) = n^{-1/2} \sum_{t=1}^{n} h_t x_t e^{-2\pi i \omega_j t} \tag{19}
$$

and let $I_y(\omega_j) = |d_y(\omega_j)|^2$, we obtain

$$
\begin{aligned}
\mathbb{E}[I_y(\omega_j)] &= \mathbb{E}|d_y(\omega_j)|^2 \\
&= n^{-1}\sum_{s=1}^{n} h_s\mathbb{E}(x_s)e^{-2\pi i w_j s}\sum_{t=1}^{n} h_t\mathbb{E}(x_t)e^{2\pi i\omega_j t} \\
&= n^{-1}\sum_{s=1}^{n}\sum_{t=1}^{n} h_s h_t\mathbb{E}(x_s x_t)e^{-2\pi i\omega_j(s-t)} \\
&= n^{-1}\sum_{s=1}^{n}\sum_{t=1}^{n} h_s h_t\gamma(s-t)e^{-2\pi i\omega_j(s-t)} \\
&= n^{-1}\sum_{s=1}^{n}\sum_{t=1}^{n} h_s h_t\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i\omega(s-t)}f_x(\omega)d\omega h_s h_t e^{-2\pi i\omega_j(s-t)} \\
&\overset{Fubini}{=} \int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{f_x(\omega)}{n}\sum_{s=1}^{n}\sum_{t=1}^{n} h_s h_t e^{2\pi i\omega(s-t)}e^{-2\pi i\omega_j(s-t)}d\omega \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{f_x(\omega)}{n}\sum_{s=1}^{n}\sum_{t=1}^{n} h_s h_t e^{2\pi i(s-t)(\omega-\omega_j)}d\omega \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} f_x(\omega)\left[n^{-1/2}\sum_{s=1}^{n} h_s e^{2\pi i s(\omega-\omega_j)}\right]\left[n^{-1/2}\sum_{t=1}^{n} h_t e^{-2\pi i s(\omega-\omega_j)}\right]d\omega \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} f_x(\omega)W_n(\omega_j-\omega)d\omega
\end{aligned}
\tag{20}
$$

In the case that $h_t = 1$ for all t, $I_y(\omega_j) = I_x(\omega_j)$

$$
\begin{aligned}
W_n(\omega) = |H_n(\omega)|^2 &= n^{-1}\sum_{s=1}^{n} h_s e^{-2\pi i\omega s}\sum_{t=1}^{n} h_t e^{2\pi i\omega t} \\
&= \left|n^{-1}\left[\frac{e^{-2\pi i\omega}(1-e^{-2\pi i\omega n})}{1-e^{-2\pi i\omega}}\right]\left[\frac{e^{2\pi i\omega}(1-e^{2\pi i\omega n})}{1-e^{2\pi i\omega}}\right]\right| \\
&= \left|\frac{[(e^{2\pi i\omega}+e^{-2\pi i\omega})+(e^{2\pi i\omega n}+e^{-2\pi i\omega n})-(e^{2\pi i\omega(n+1)}+e^{-2\pi i\omega(n+1)})-2]}{n[2-(e^{2\pi i\omega}+e^{-2\pi i\omega})]}\right| \\
&= \left|\frac{\cos(2\pi\omega)+\cos(2\pi\omega n)-\cos(2\pi\omega(n+1))-1}{n(1-1\cos(2\pi\omega))}\right| = \frac{\sin^2(\pi\omega n)}{n\sin^2(\pi\omega)}
\end{aligned}
\tag{21}
$$

If we consider the averaged periodogram in (20), namely, for frequencies of the form $\omega^* = \omega_j + k/n$, let

$$
\mathbb{B} = \{\omega^* : \omega_j - \frac{m}{n} \le \omega^* \le \omega_j + \frac{m}{n}\}
\tag{22}
$$

where $L = 2m+1$

$$
\bar{f}_x(\omega) = \frac{1}{L}\sum_{k=-m}^{m} I_x(\omega_j + k/n)
\tag{23}
$$

$W_n(\omega)$, in (20) will take the form

$$
W_n(\omega) = \frac{1}{nL}\sum_{k=-m}^{m}\frac{\sin^2[n\pi(\omega+k/n)]}{\sin^2[\pi(\omega+k/n)]}
\tag{24}
$$

4.23

Suppose we wish to test the noise alone hypothesis $H_0 : x_t = n_t$ against the signal-plus-noise hypothesis $H_1 : x_t = s_t + n_t$, where s_t and n_t are uncorrelated zero-mean stationary processes with spectra $f_s(\omega)$ and $f_n(\omega)$. Suppose that we want the test over a band of $L = 2m+1$ frequencies of the form $\omega_{j:n} + k/n$, for k = $0,\pm 1,\pm 2,\cdots,\pm m$ near some fixed frequency ω. Assume that both the signal and noise spectra are approximately constant over the interval.

(a)

Prove the approximate likelihood-based test statistic for testing H_0 against H_1 is proportional to

$$T = \sum_k |d_x(\omega_{j:n} + k/n)|^2 \left(\frac{1}{f_n(\omega)} - \frac{1}{f_s(\omega) + f_n(\omega)} \right)$$

We first show the spectral density of x_t under H_0 and H_1

$$\begin{aligned} f_{x:H_0}(\omega) &= \sum_{h=-\infty}^{\infty} \gamma_{x:H_0}(h) e^{-2\pi i\omega h} = f_n(\omega) \\ f_{x:H_1}(\omega) &= \sum_{h=-\infty}^{\infty} \gamma_{x:H_1}(h) e^{-2\pi i\omega h} = \sum_{h=-\infty}^{\infty} (\gamma_n(h) + \gamma_s(h)) e^{-2\pi i\omega h} = f_s(\omega) + f_n(\omega) \end{aligned} \tag{25}$$

Recall

$$d_x(w_{j:n}) \sim AN(0, \sigma_x^2/2) \tag{26}$$

Besides, $d_x(w_{j:n})$ and $d_x(w_{k:n})$ are asymptotically independent. So we further write the likelihood function of x_t under H_0 in the bandwidth of L/n

$$\begin{aligned} L_0 &= \prod_{k=-m}^{m} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{|d_x(\omega_{j:n}+k/n)|^2}{2\sigma^2}} \\ &= \left[\frac{1}{\sqrt{2\pi\sigma^2}} \right]^{2m+1} \exp\left(-\frac{\sum_{k=-m}^{m} |d_x(\omega_{j:n} + k/n)|^2}{2\sigma^2}\right) \\ &\overset{\sigma^2 = f_n(\omega)}{=} \left[\frac{1}{\sqrt{2\pi f_n(\omega)}} \right]^{2m+1} \exp\left(-\frac{\sum_{k=-m}^{m} |d_x(\omega_{j:n} + k/n)|^2}{2 f_n(\omega)}\right) \end{aligned} \tag{27}$$

The same way, we could write the likelihood function under H_1 assuming the independency of spectral density in one bandwidth.

$$L_1 = \left[\frac{1}{\sqrt{2\pi(f_n(\omega) + f_s(\omega))}} \right]^{2m+1} \exp\left(-\frac{\sum_{k=-m}^{m} |d_x(\omega_{j:n} + k/n)|^2}{2(f_n(\omega) + f_s(\omega))}\right) \tag{28}$$

We use the LR test for testing H_0 against H_1

$$
\begin{aligned}
\chi^2 &= -2\ln L_0 + 2\ln L_1 \\
&= -2\Bigg[-\frac{2m+1}{2}\ln(2\pi f_n(\omega)) - \frac{\sum_{k=-m}^{m}|d_x(\omega_{j:n}+k/n)|^2}{2f_n(\omega)}\Bigg] \\
&+2\Bigg[-\frac{2m+1}{2}\ln[2\pi(f_n(\omega)+f_s(\omega))] - \frac{\sum_{k=-m}^{m}|d_x(\omega_{j:n}+k/n)|^2}{2[f_n(\omega)+f_s(\omega)]}\Bigg] \\
&\propto \frac{\sum_{k=-m}^{m}|d_x(\omega_{j:n}+k/n)|^2}{f_n(\omega)} - \frac{\sum_{k=-m}^{m}|d_x(\omega_{j:n}+k/n)|^2}{f_n(\omega)+f_s(\omega)} \\
&= \sum_{k=-m}^{m}|d_x(\omega_{j:n}+k/n)|^2\Bigg[\frac{1}{f_n(\omega)} - \frac{1}{f_n(\omega)+f_s(\omega)}\Bigg]
\end{aligned}
\tag{29}
$$

(b)

Find the approximate distributions of T under H_0 and H_1.

Under $H_0 : x_t = n_t$, we could show that

$$
\begin{aligned}
T_{x_t:n_t} &= \sum_k |d_{x_t:n_t}(\omega_{j:n}+k/n)|^2\left(\frac{1}{f_n(\omega)} - \frac{1}{f_n(\omega)+f_s(\omega)}\right) \\
&= (2m+1)\bar{f}_{x_t:n_t}(\omega)\left(\frac{1}{f_n(\omega)} - \frac{1}{f_n(\omega)+f_s(\omega)}\right) \dot{\sim} \frac{2f_s(\omega)\chi^2_{2L}}{f_n(\omega)+f_s(\omega)}
\end{aligned}
\tag{30}
$$

Under $H_1 : x_t = n_t + s_t$

$$
\begin{aligned}
T_{x_t:n_t+s_t} &= \sum_k |d_{x_t:n_t+s_t}(\omega_{j:n}+k/n)|^2\left(\frac{1}{f_n(\omega)} - \frac{1}{f_n(\omega)+f_s(\omega)}\right) \\
&= (2m+1)\bar{f}_{x_t:n_t+s_t}(\omega)\left(\frac{1}{f_n(\omega)} - \frac{1}{f_n(\omega)+f_s(\omega)}\right) \dot{\sim} \frac{2f_s(\omega)\chi^2_{2L}}{f_n(\omega)}
\end{aligned}
\tag{31}
$$

(c)

Define the false alarm and signal detection probabilities as $P_F = P\{T > K|H_0\}$ and $P_d = P\{T > k|H_1\}$, respectively. Express these probabilities in terms of the signal-to-noise ratio $f_s(\omega)/f_n(\omega)$ and appropriate chi-squared integrals.

The false alarm probabilities

$$
\begin{aligned}
P_F = P\{T > K|H_0\} &\overset{\eta_1 \sim \chi^2_{2L}}{=} P\{\frac{2f_s(\omega)\eta_1}{f_n(\omega)+f_s(\omega)} > K|H_0\} \\
&= P\{\eta_1 > \frac{(f_n(\omega)+f_s(\omega))K}{2f_s(\omega)}|H_0\} \\
&= \int_{\frac{(f_n(\omega)+f_s(\omega))K}{2f_s(\omega)}}^{\infty} \frac{e^{\frac{-x}{2}}x^{L-1}}{2^L\Gamma(L)}dx
\end{aligned}
\tag{32}
$$

And the signal detection probabilities

$$
\begin{aligned}
P_d = P\{T > k|H_1\} &\overset{\eta_2 \sim \chi^2_{2L}}{=} P\{\frac{2f_s(\omega)\eta_2}{f_n(\omega)}k|H_1\} \\
&= P\{\eta_2 > \frac{(f_n(\omega))k}{2f_s(\omega)}|H_1\} \\
&= \int_{\frac{f_n(\omega)k}{2f_s(\omega)}}^{\infty} \frac{e^{\frac{-x}{2}}x^{L-1}}{2^L\Gamma(L)}dx
\end{aligned}
\tag{33}
$$

4.27

Consider the bivariate time series records containing monthly U.S. production (*prod*) as measured by the Federal Reserve Board Production Index and the monthly unemployment series (*unemp*).

(a)

Compute the spectrum and the log spectrum for each series, and identify statistically significant peaks. Explain what might be generating the peaks. Compute the coherence, and explain what is meant when a high coherence is observed at a particular frequency.

To compute and graph the periodogram, we use FFT for our calculation for original and logged series in Figure 1 and Figure 2

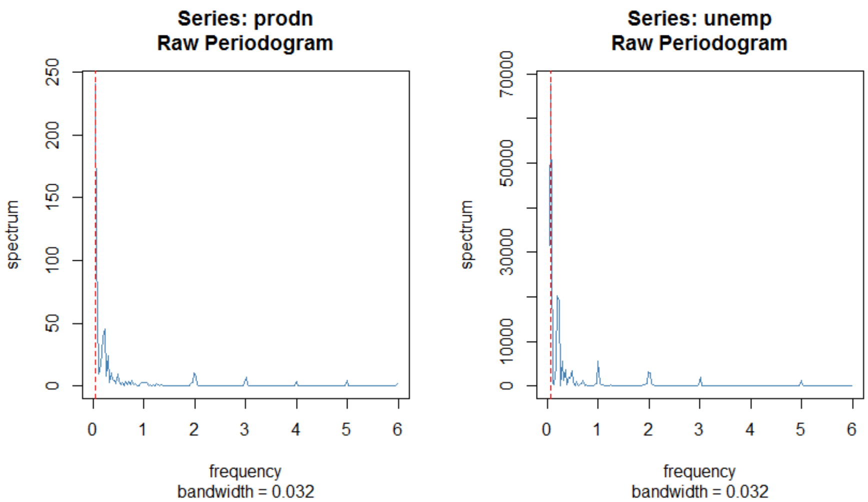

Figure 1. Periodogram of the original U.S. production and the unemployment series, n=372, where the frequency axis is labeled in multiples of $\Delta = 1/12$. We present the highest spectrum density with red vertical dash line

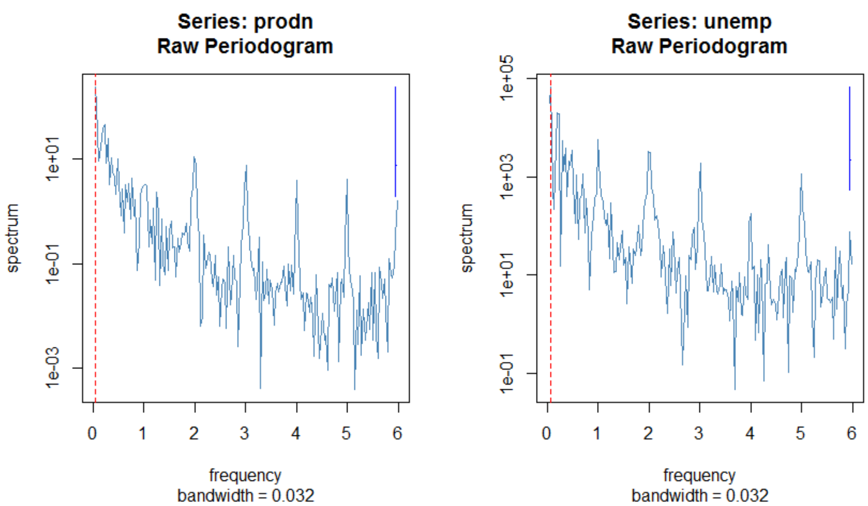

Figure 2. Periodogram of the logged U.S. production and the unemployment series, n=372, where the frequency axis is labeled in multiples of $\Delta = 1/12$. We present the highest spectrum density with red vertical dash line

We note that the value of Δ is the reciprocal of the value of frequency for the data of a time series object. Further, we obtain the most significant is 0.032 (31.25 year) and 0.064 (15.625 year) respectively.

Then we compute the coherence of the U.S. production and the unemployment where the cross-spectrum is defined as

$$\begin{aligned} f_{xy}(\omega) &= \sum_{h=-\infty}^{\infty} \gamma_{xy}(h) e^{-2\pi i \omega h} \\ &= \sum_{h=-\infty}^{\infty} \mathbb{E}[(x_{t+h}-\mu_x)(y_t-\mu_y)] e^{-2\pi i \omega h} \end{aligned} \tag{34}$$

Figure 3 show the squared coherence between *prodn* and *unemp*.

Figure 3. Squared coherency between U.S. production and unemployment series;L=11,n=372, and $\alpha = 0.01$, the red dash horizontal line is $C_{0.001}$

In Figure 3, we line the approximate value that must be exceeded for the original squared to be able to reject $\rho^2_{y,x}(\omega) = 0$ at an priori specified frequency. we may reject the hypothesis of no coherence for values of $\rho^2_{y,x}(\omega) = 0$ that exceed $C_{0.001} = 0.45755$.

In this case, the two series are obviously strongly coherent at $\omega = 1$ and fair coherent at its harmonics $\omega = k\Delta$ for $k = 2, 3, \cdots$ which display the behavior of cycle dependency. Finally, we conclude that the coherence is persistent at the year harmonic frequencies.

(b)

What would be the effect of applying the filter $u_t = x_t - x_{t-1}$ followed by $v_t = u_t - u_{t-12}$ to the series given above? Plot the predicted frequency responses of the simple difference filter and of the seasonal difference of the first difference.

We present the frequency responses of the simple difference filter as

$$|A_{ux}(\omega)|^2 = (1-e^{-2\pi i \omega})(1-e^{2\pi i \omega}) = 2[1-\cos(2\pi\omega)] \tag{35}$$

$$|A_{vu}(\omega)|^2 = (1-e^{-24\pi i\omega})(1-e^{24\pi i\omega}) = 2[1-\cos(24\pi\omega)] \tag{36}$$

We could show the frequency response in (37)

$$v_t = x_t - x_{t-1} - x_{t-12} + x_{t-13} \tag{37}$$

which implies that

$$A_{vx}(\omega) = 1 - e^{-2\pi i\omega} - e^{-24\pi i\omega} + -e^{-26\pi i\omega} \tag{38}$$

and the squared frequency response becomes

$$\begin{aligned}|A_{vx}(\omega)|^2 &= (1-e^{-2\pi i\omega}-e^{-24\pi i\omega}+-e^{-26\pi i\omega})(1-e^{2\pi i\omega}-e^{24\pi i\omega}+-e^{26\pi i\omega}) \\ &= 4-2(e^{2\pi i\omega}+e^{-2\pi i\omega})+(e^{22\pi i\omega}+e^{-22\pi i\omega})-2(e^{24\pi i\omega}+e^{-24\pi i\omega})+(e^{26\pi i\omega}+e^{-26\pi i\omega} \\ &= 4-4\cos(2\pi i\omega)+2\cos(22\pi i\omega)-4\cos(24\pi i\omega)+2\cos(26\pi i\omega)\end{aligned} \tag{39}$$

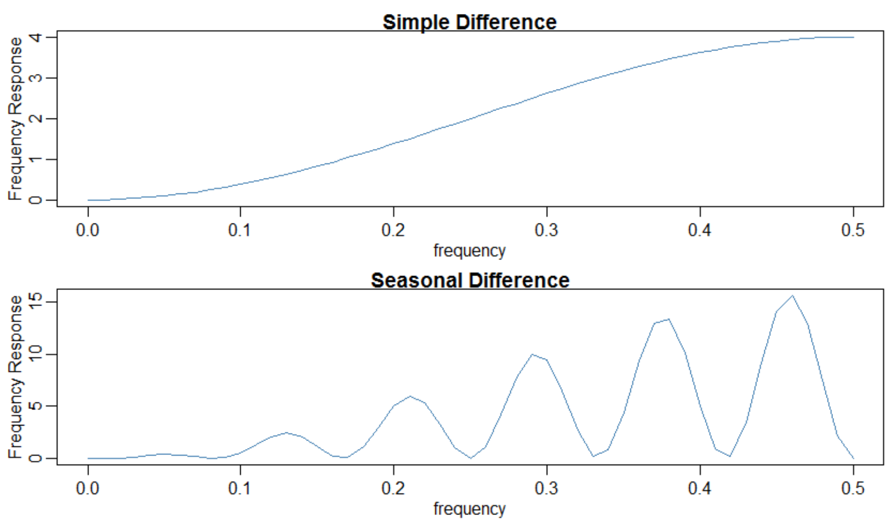

Figure 4. Squared frequency response functions of the first difference (top) and twelve-month difference (bottom) filters

The top panel of Figure 4 shows that the first difference filter will attenuate the lower frequencies and enhance the higher frequencies because the multiplier of the spectrum, $|A_{ux}(\omega)|^2$, is large for the higher frequencies and small for the lower frequencies. In the bottom panel, we show the frequency response of seasonal filter, $|A_{ux}(\omega)|^2$ which also attenuate the lower frequency. However, it enhance the high frequency with cycle with oscillatory behavior.

(c)

Apply the filters successively to one of the two series and plot the output.Examine the output after taking a first difference and comment on whether stationarity is a reasonable assumption.Why or whynot? Plot after taking the seasonal difference of the first difference.What can be noticed about the output that is consistent with what you have predicted from the frequency response? Verify by computing the spectrum of the output after filtering.

We apply the simple and seasonal filter on the U.S. production and the employment and plot the output in Figure 5 and Figure 6

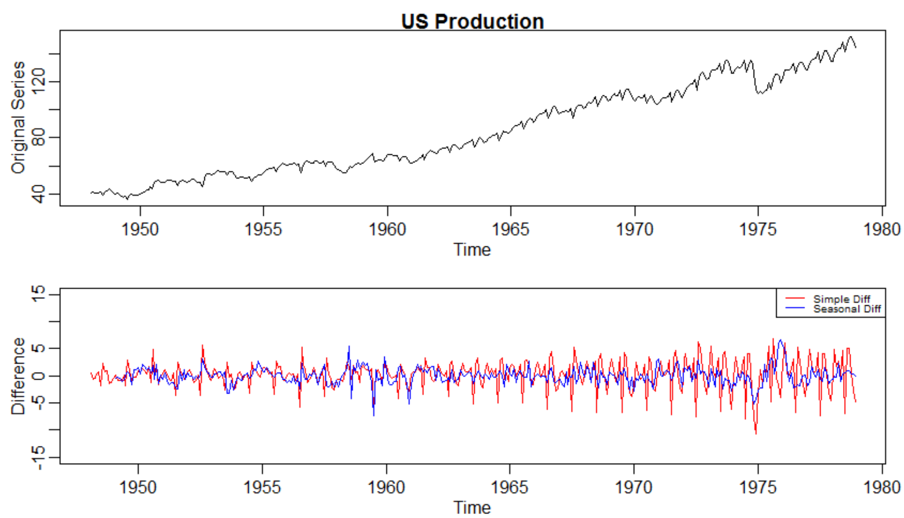

Figure 5. U.S. Production(top) compared with the differenced U.S. Production(bottom),red solid line represent the first difference and blue solid line represent the subsequent seasonal difference

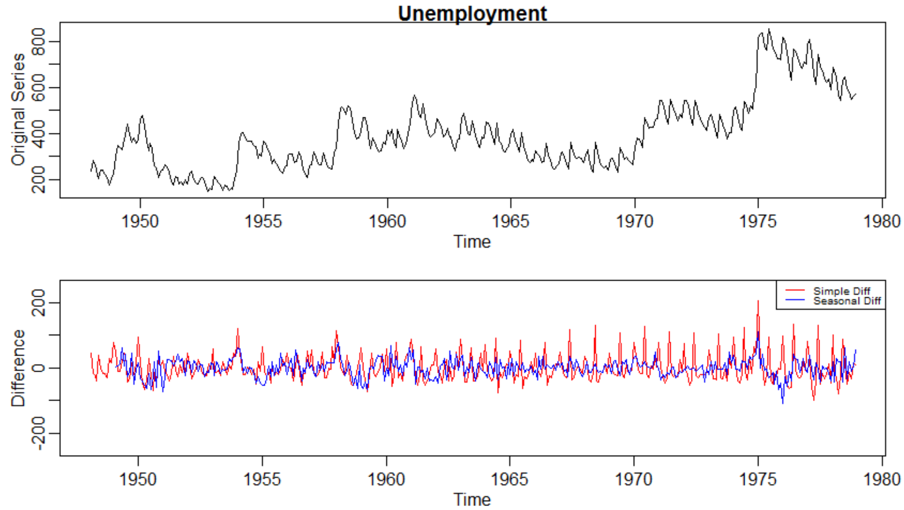

Figure 6. Unemployment(top) compared with the differenced Unemployment(bottom),red solid line represent the first difference and blue solid line represent the subsequent seasonal difference

For example, we choose U.S. Production series for manipulation to reveal more details. First, we comment on the first difference result to see if it is stationary by plot ACF and PACF in **??**

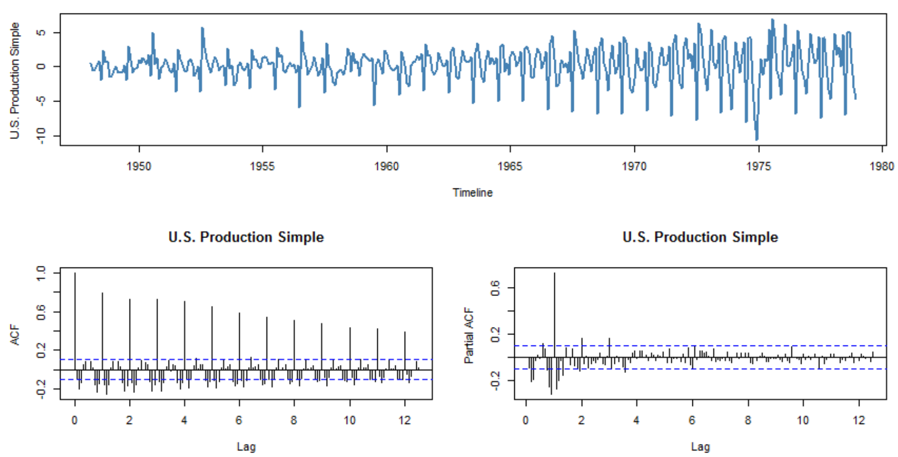

Figure 7. The first differenced U.S. Production(top) with ACF and PACF in the bottom panel

Further, we plot the output after taking the seasonal difference of the first difference and show its ACF and PACF, repectively.

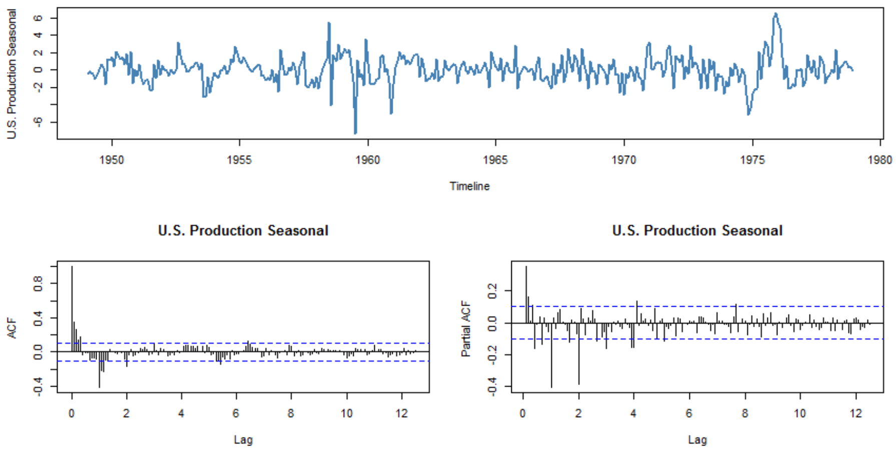

Figure 8. The seasonal differenced U.S. Production(top) with ACF and PACF in the bottom panel

From Figure 7 and Figure 8, it is fair obvious that the first differenced series is not stationary since its ACF decay slow and display several peaks at higher lag. However, in the plot of ACF and PACF of the seasonal differenced series, we notice the ACF and PACF both decay quite quick and show less significant density at the higher lag. So, we conclude that the first differenced is non-stationary but the seasonal difference after first differenced is stationary.

Finally, we verify the theoretical frequency response by computing the periodogram of the output after filtering compared with the theoretical values. In Figure 9, we show the frequency response of our filters, simple filter and mixed filer (seasonal and first differenced mixed)

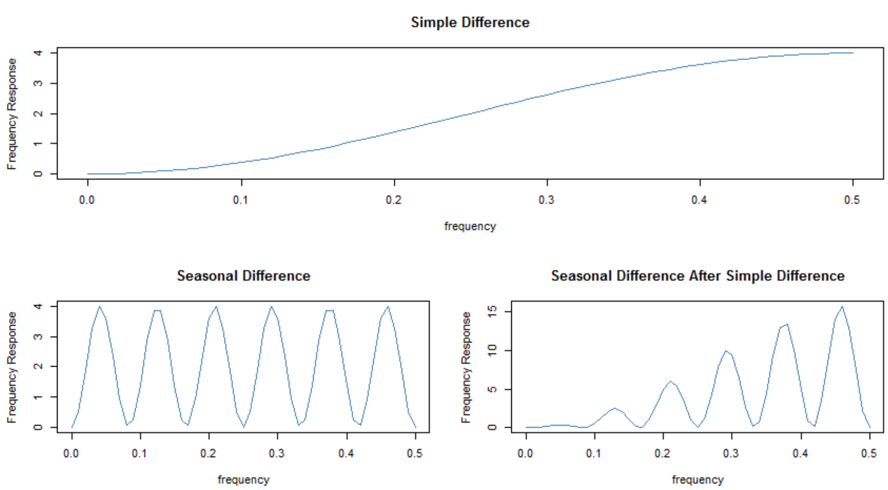

Figure 9. The first differenced frequency response(top) and the seasonal differenced with the seasonal differenced after the simple first difference

We apply the filters on the U.S. Production and show the theoretical spectrum density with its approximation, periodogram values respectively in Figure 10 and Figure 11.

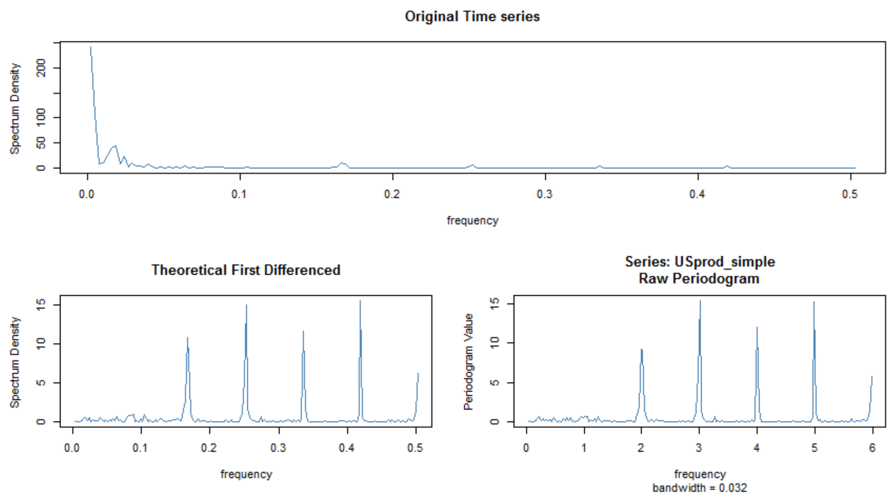

Figure 10. The original spectrum density of time series (top) and the theoretical spectrum density values of simple filter and the periodogram values after simple filter (bottom)

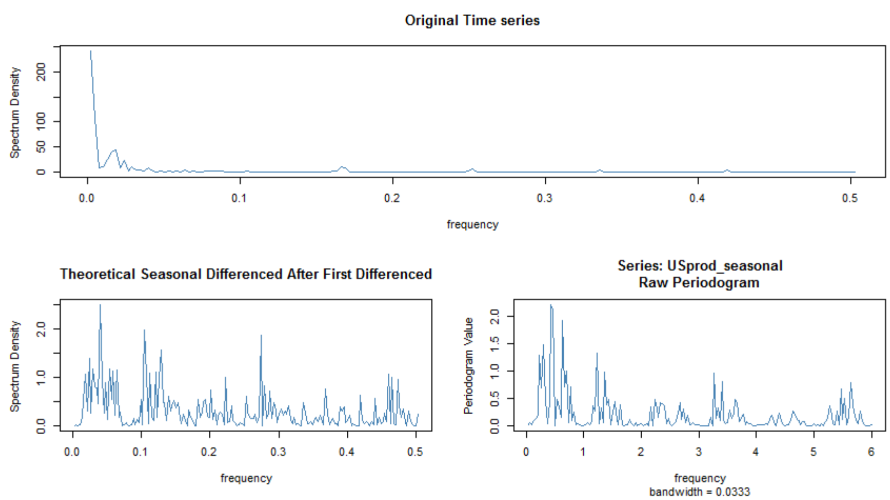

Figure 11. The original spectrum density of time series (top) and the theoretical spectrum density values of mixed filter and the periodogram values after mixed filter (bottom)

From Figure 10 and Figure 11, we complete our verification by comparison the theoretical values with sample values. Except for little discrepancy caused by different bandwidth, we conclude these two results are identical and thereby verify our previous computation.

4.30

Suppose x_t is a stationary series, and we apply two filtering operations in succession, say,

$$y_t = \sum_r a_r x_{t-r} \qquad z_t = \sum_s b_s y_{t-s} \tag{40}$$

(a)

Show the spectrum of the output is

$$f_z(\omega) = |A(\omega)|^2 |B(\omega)|^2 f_x(\omega)$$

where $A(\omega)$ and $B(\omega)$ are the Fourier transforms of the filter sequences a_t and b_t, respectively.

For the process in (40), $y_t = \sum_r a_r y_{t-r}$ has the spectrum $f_x(\omega)$ related to the spectrum of the input x_t by $f_y(\omega) = |A(\omega)|^2 f_x(\omega)$.

The autocovariance function of the filtered output z_t in (40) is

$$\begin{aligned}
\gamma_z(h) &= cov(z_{t+h}, z_t) \\
&= cov(\sum_{s_1} b_{s_1} y_{t+h-s_1}, \sum_{s_2} b_{s_2} y_{t-s_2}) \\
&= \sum_{s_1}\sum_{s_2} b_{s_1} b_{s_2} \gamma_y(h - s_1 + s_2) \\
&= \sum_{s_1}\sum_{s_2} b_{s_1} \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i\omega(h-s_1+s_2) f_y(\omega)} \right| b_{s_2} d\omega \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{s_1} b_{s_1} e^{-2\pi i\omega s_1}\right)\left(\sum_{s_2} b_{s_2} e^{-2\pi i\omega s_2}\right) e^{2\pi i\omega h} f_y(\omega) d\omega \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{s_1} b_{s_1} e^{-2\pi i\omega s_1}\right)\left(\sum_{s_2} b_{s_2} e^{-2\pi i\omega s_2}\right) e^{2\pi i\omega h} |A(\omega)|^2 f_x(\omega) d\omega \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \underbrace{|B(\omega)|^2 |A(\omega)|^2 f_x(\omega)}_{f_z(\omega)} e^{2\pi i\omega h} d\omega
\end{aligned} \tag{41}$$

(b)

What would be the effect of applying the filter

$$u_t = x_t - x_{t-1} \qquad v_t = u_t - u_{t-12}$$

to a time series?

Consider the Fourier transformation of the coefficient a, b

$$|A_{ux}(\omega)|^2 = (1 - e^{-2\pi i\omega})(1 - e^{2\pi i\omega}) = 2[1 - \cos(2\pi\omega)] \tag{42}$$

$$|A_{vu}(\omega)|^2 = (1 - e^{-24\pi i\omega})(1 - e^{24\pi i\omega}) = 2[1 - \cos(24\pi\omega)] \tag{43}$$

We could show the frequency response in (37)

$$v_t = x_t - x_{t-1} - x_{t-12} + x_{t-13} \tag{44}$$

which implies that

$$A_{vx}(\omega) = 1 - e^{-2\pi i\omega} - e^{-24\pi i\omega} + -e^{-26\pi i\omega} \tag{45}$$

and the squared frequency response becomes

$$\begin{aligned}|A_{vx}(\omega)|^2 &= (1 - e^{-2\pi i\omega} - e^{-24\pi i\omega} + -e^{-26\pi i\omega})(1 - e^{2\pi i\omega} - e^{24\pi i\omega} + -e^{26\pi i\omega}) \\ &= 4 - 2(e^{2\pi i\omega} + e^{-2\pi i\omega}) + (e^{22\pi i\omega} + e^{-22\pi i\omega}) - 2(e^{24\pi i\omega} + e^{-24\pi i\omega}) + (e^{26\pi i\omega} + e^{-26\pi i\omega} \\ &= 4 - 4\cos(2\pi i\omega) + 2\cos(22\pi i\omega) - 4\cos(24\pi i\omega) + 2\cos(26\pi i\omega)\end{aligned} \tag{46}$$

(c)

Plot the predicted frequency responses of the simple difference filter and of the seasonal difference of the first difference. Filters like these are called seasonal adjustment filters in economics because they tend to attenuate frequencies at multiples of the monthly periods. The difference filter tends to attenuate low-frequency trends.

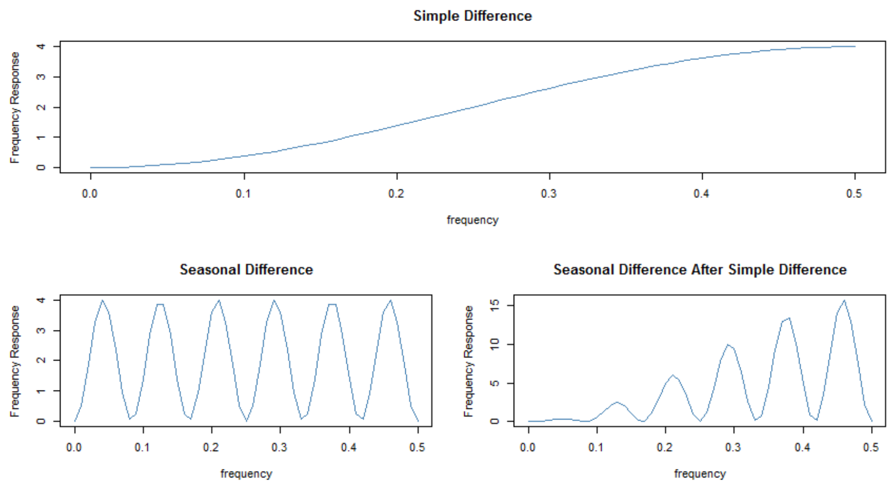

Figure 12. The first differenced frequency response(top) and the seasonal differenced with the seasonal differenced after the simple first difference

Rcode

4.27

```
# loading required packages
library(astsa)
length(prodn)
# customization layout
par(mfrow=c(1,2))
# spectrum analysis of U.S. production
prod.pre <- mvspec(prodn,col='steelblue',log='no')
abline(v=prod.pre$freq[which.max(prod.pre$spec)],
       col='red',
       lty=2)
# spectrum analysis of unemployment
unemp.pre <- mvspec(unemp,col='steelblue',log='no')
abline(v=unemp.pre$freq[which.max(unemp.pre$spec)],
       col='red',
       lty=2)
soi.pre$spec[40]
soi.pre <- mvspec(soi,log='no')
length(soi.pre$sp)
```

```
# cross spectrum analysis
sr <- mvspec(cbind(prodn,unemp),
             kernel('daniell',5),
             plot=F)

f <- qf(0.999,2,sr$df-2)
C <- f/(10+f)
plot(sr,
     plot.type = "coh",
     ci.lty = 2,
     main='U.S. production&unemployment rate')
abline(h = C,
       col='red',
       lty=2)
# create simple and seasonal filters
Plot_simple_seasonal <- function(tseries,title,yrange)
{
  par(mfrow=c(2,1), mar=c(3,3,1,1), mgp=c(1.6,.6,0))
  simple_series <- diff(tseries)
  seasonal_series <- diff(simple_series,12)
  plot(tseries,
       main=title,
       ylab='Original Series')
  plot(simple_series,
        col='red',
       ylim = yrange,ylab='Difference',
       lwd=1)
  lines(seasonal_series,
        col='blue',
        lwd=1)
  legend('topright',
         legend = c('Simple Diff',
                    'Seasonal Diff'),
         col=c('red','blue'),cex=0.6,lty=1)
  return(list(s1=simple_series,s2=seasonal_series))
}
```

```r
# apply mixed filter on U.S. production and unemployment
prodn_diff <- Plot_simple_seasonal(prodn,'US Production',c
   (-15,15))
unemp_diff <- Plot_simple_seasonal(unemp,'Unemployment',c
   (-250,250))
w <- seq(0, 0.5, by=0.1)
# compute frequency response
FRdiff_ux <- abs(1-exp(2i*pi*w))^2
FRdiff_vu <- abs(1-exp(24i*pi*w))^2
FRdiff_vx <- abs(1-exp(2i*pi*w)-exp(24i*pi*w)+exp(26i*pi*w))^2
# plot the u2x,v2u frequency response
plot(w, FRdiff_ux,
     type='l',
     xlab='frequency',
     col='steelblue',
     main='Simple Difference',
     ylab='Frequency Response')
plot(w, FRdiff_vu,
     type='l',
     xlab='frequency',
     col='steelblue',
     main='Seasonal Difference',
     ylab='Frequency Response')

# the filter output
USprod_simple <- prodn_diff$s1
USprod_seasonal <- prodn_diff$s2
par(mfrow=c(1,2))
mvspec(USprod_simple,log='no',col='steelblue')
mvspec(USprod_seasonal,log='no',col='steelblue')
plot_diff <- function(fit,ylab)
{
  layout(matrix(c(1,1,1,1,1,1,1,1,2,2,3,3,2,2,3,3),ncol=4,byrow=
   TRUE))
  plot.ts(fit,
          col='steelblue',
```

```
          ylab=ylab,xlab='Timeline',
          lwd=2)
  acf(fit,150,main=ylab)
  pacf(fit,150,main=ylab)

}
plot_diff(USprod_simple,'U.S. Production Simple')
plot_diff(USprod_seasonal,'U.S. Production Seasonal')

# plot the output of mixed filter
plot(w, FRdiff_ux,
     type='l',
     xlab='frequency',
     col='steelblue',
     main='Simple Difference',
     ylab='Frequency Response')
plot(w, FRdiff_vu,
     type='l',
     xlab='frequency',
     col='steelblue',
     main='Seasonal Difference',
     ylab='Frequency Response')
plot(w, FRdiff_vx,
     type='l',
     xlab='frequency',
     col='steelblue',
     main='Seasonal Difference After Simple Difference',
     ylab='Frequency Response')

M <- length(prodn)
x_spec <- mvspec(prodn,log='no',plot=F)
M_spec <- length(x_spec$spec)
layout(matrix(c(1,1,1,1,1,1,1,1,2,2,3,3,2,2,3,3),ncol=4,byrow=
   TRUE))
# original
plot(1:M_spec/M,x_spec$spec,type='l',
```

```
      main='Original Time series',
      ylab='Spectrum Density',
      col='steelblue',
      xlab='frequency')
w <- 1:M_spec/M
FRdiff_ux <- abs(1-exp(2i*pi*w))^2
FRdiff_vu <- abs(1-exp(24i*pi*w))^2
FRdiff_vx <- abs(1-exp(2i*pi*w)-exp(24i*pi*w)+exp(26i*pi*w))^2
# theoretical simple
plot(1:M_spec/M,
      FRdiff_ux*x_spec$spec,type='l',
      main='Theoretical First Differenced',
      ylab='Spectrum Density',
      col='steelblue',
      xlab='frequency')
# spectrum density of differenced series simple
mvspec(USprod_simple,log='no',
        ylab='Periodogram Value',
        col='steelblue')



layout(matrix(c(1,1,1,1,1,1,1,1,2,2,3,3,2,2,3,3),ncol=4,byrow=
   TRUE))
# original
plot(1:M_spec/M,x_spec$spec,type='l',
      main='Original Time series',
      ylab='Spectrum Density',
      col='steelblue',
      xlab='frequency')

# theoretical seasonal
plot(1:M_spec/M,
      FRdiff_vx*x_spec$spec,type='l',
      main='Theoretical Seasonal Differenced After First
   Differenced',
```

```
     ylab='Spectrum Density',
     col='steelblue',
     xlab='frequency')
# spectrum density of Differenced series
mvspec(USprod_seasonal,log='no',
       ylab='Periodogram Value',
       col='steelblue')
```