Poisson's Equation

An equation of the form $\nabla^2 u = f(x,y)$

ie, $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

is called as poisson's equation where $f(x,y)$ is a function of x & y only.

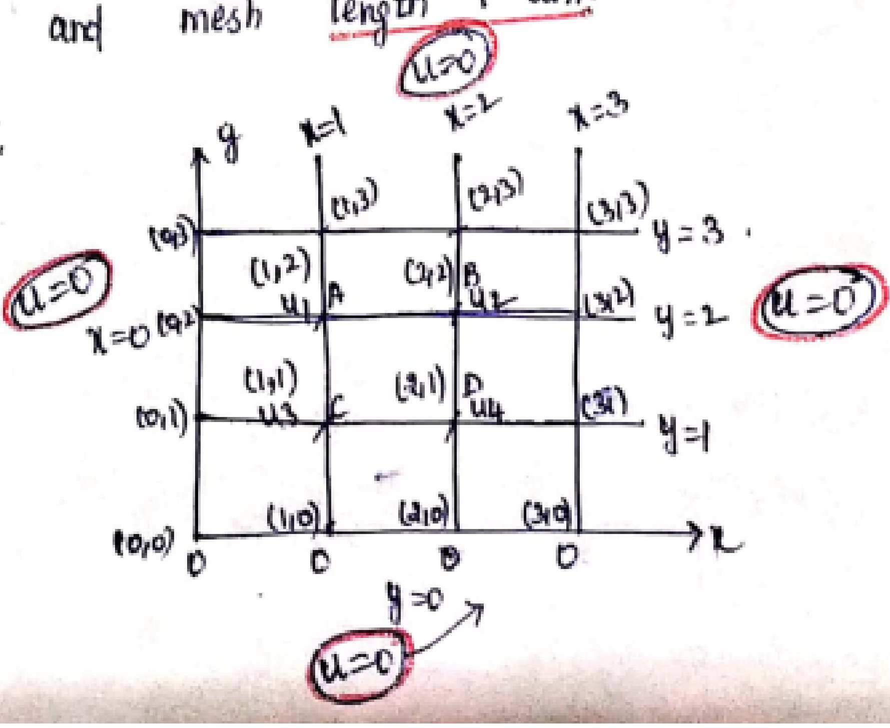
Solve the above eqn,

$$\boxed{u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh)}$$

By applying the above formula at each mesh point, we get a system of linear equation in the pivotal values i,j.

① Solve $\nabla^2 u = -10(x^2+y^2+10)$ over the square mesh with sides $x=0$, $y=0$, $x=3$, $y=3$ with $u=0$ on the boundary and mesh length 1 unit.

Solution :



(2)

The PDE is $\nabla^2 u = -10(x^2+y^2+10)$.

$f(x,y) = -10(x^2+y^2+10)$, $h=1$: (given).

formula:-

$$\boxed{u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh)}$$

$$\therefore u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = (1)^2\left[-10\left((ih)^2+(jh)^2+10\right)\right]$$

$$= -10(i^2h^2 + j^2h^2 + 10)$$

$$= -10(i^2+j^2+10)$$

$\because h=1$

$$\boxed{\therefore u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = -10(i^2+j^2+10)} \longrightarrow (1)$$

Applying the formula (1) at A, $(i=1, j=2)$

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = -10(1+4+10)$$

$$0 + u_2 + u_3 + 0 - 4u_1 = -10(15)$$

$$u_2 + u_3 - 4u_1 = -150 \longrightarrow (1).$$

Applying the formula (1) at B, $(i=2, j=2)$

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = -10(2^2+2^2+10)$$

$$u_1 + 0 + u_4 + 0 - 4u_2 = -10(18)$$

$$u_1 + u_4 - 4u_2 = -180 \longrightarrow (2).$$

Applying the formula (1) at C, $(i=1, j=1)$

$$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = -10(1^2+1^2+10)$$

$$0 + u_4 + 0 + u_1 - 4u_3 = -10(12)$$

$$u_4 + u_1 - 4u_3 = -120 \longrightarrow (3)$$

Applying the formula (1) at D, $i=2, j=1$.

$$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = -10(2^2+1^2+10)$$

$$u_3 + 0 + 0 + u_2 - 4u_4 = -10(15)$$

$$u_3 + u_2 - 4u_4 = -150 \longrightarrow (4).$$

We can solve the eqn (1), (2), (3) & (4) either by direct elimination or by Gauss-seidel method

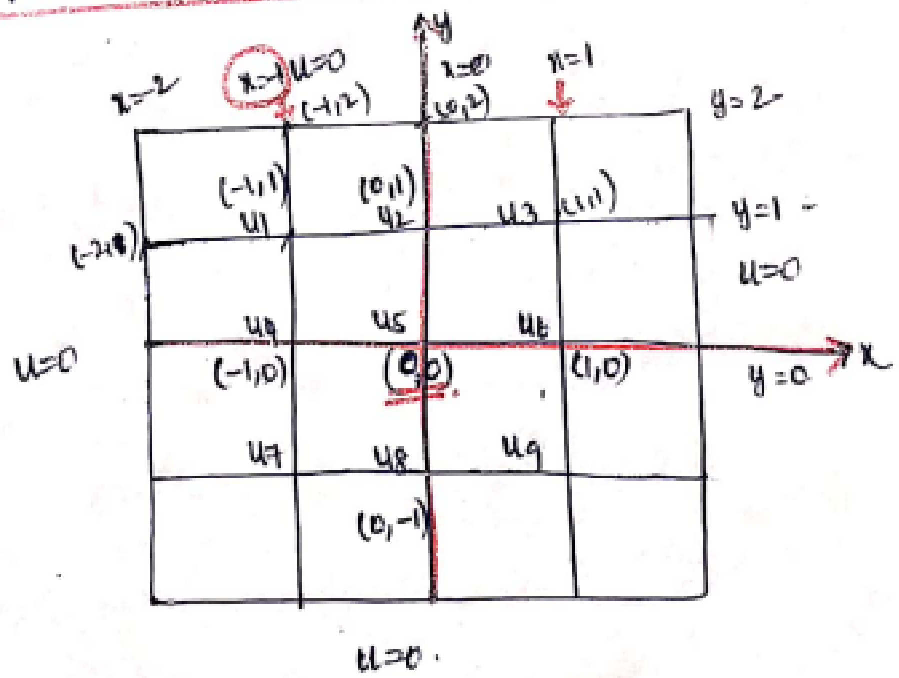
[System of linear equation Gauss-seidel (or) Gauss elimination method]

Ans :-

$$\boxed{\begin{aligned} u_1 &= u_4 = 75 \\ u_2 &= 82.5 \\ u_3 &= 67.5 \end{aligned}}$$

2) Solve $\boxed{\nabla^2 u = 8x^2y^2}$ for square mesh given $u=0$ on the 4 boundaries dividing the square into 16 sub-squares of length 1 unit.

Soln :-



Given the PDE is $\nabla^2 u = 8x^2y^2$

ie, $f(x,y) = 8x^2y^2$

Take the coordinate system with origin at the centre of the square. (4)

Since the PDE and boundary conditions are symmetrical about x, y axes & $y = x$ we have

Along the line x-axis, $u_1 = u_7$, $u_2 = u_8$, $u_3 = u_9$

Along the line y-axis, $u_1 = u_3$, $u_4 = u_6$, $u_7 = u_9$

Along the line $y = x$, $u_4 = u_8$, $u_1 = u_9$, $u_2 = u_6$.

$$\therefore \boxed{u_1 = u_3 = u_7 = u_9}$$

$$\boxed{u_2 = u_4 = u_6 = u_8}$$

$$u_5$$

formula $\therefore$ we need to find only u_1, u_2, u_5, here $h = 1$

$$\boxed{u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh)}$$

$$\therefore u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = 1^2\, 8\left((ih)^2 (jh)^2\right)$$

$$= 8 i^2 h^2 j^2 h^2$$

$$= 8 i^2 j^2$$

$$\boxed{\text{ie, } u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = 8 i^2 j^2} \longrightarrow (1)$$

At $(i = -1, j = 1)$, we have

$$u_{-2,1} + u_{0,1} + u_{-1,0} + u_{-1,2} - 4u_{-1,1} = 8(-1)^2(1)^2$$

$$0 + u_2 + u_4 + 0 - 4u_1 = 8$$

$$u_2 + u_4 - 4u_1 = 8 \quad (\because u_2 = u_4)$$

$$2u_2 - 4u_1 = 8$$

$$\div 2, \quad u_2 - 2u_1 = 4 \longrightarrow (1) \text{ eqn.}$$

At $(i=0, j=1)$.

$$u_{-1,1} + u_{1,1} + u_{0,0} + u_{0,2} - 4u_{0,1} = 8(0)(1)^2 = 0$$

$$u_1 + u_3 + u_5 + 0 - 4u_2 = 0.$$

$$u_1 + u_3 + u_5 - 4u_2 = 0 \quad (\because u_1 = u_3)$$

$$2u_1 + u_5 - 4u_2 = 0. \longrightarrow (2) \text{ eqn.}$$

At $(i=0, j=0)$.

$$u_{-1,0} + u_{1,0} + u_{0,-1} + u_{0,1} - 4u_{0,0} = 8(0)(0) = 0.$$

$$u_4 + u_6 + u_8 + u_2 - 4u_5 = 0.$$

$$4u_2 - 4u_5 = 0 \quad (\because u_2 = u_4 = u_6 = u_8)$$

$$u_2 - u_5 = 0 \longrightarrow (3) \text{ eqn.}$$

From (1), $u_1 = \frac{1}{2}(u_2 - 4)$.

From (3), $u_2 = u_5$

Using in (2), $2\left(\frac{1}{2}(u_2-4)\right) + u_2 - 4u_2 = 0$

$$u_2 - 4 + u_2 - 4u_2 = 0.$$

$$-4 - 2u_2 = 0$$

$$u_2 = -2$$

$\therefore$ $u_5 = -2$

$\therefore$ $u_1 = \frac{1}{2}(-2-4) \Rightarrow u_1 = -3$

$$\therefore \boxed{u_1 = -3, \quad u_2 = -2, \quad u_5 = -2.}$$