Parabolic Equations

(i) Bender - Schmidt Method

The one dimensional heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where} \quad \alpha^2 = \frac{k}{\rho c}$$

The above equation can be written as $u_{xx} = \frac{1}{\alpha^2} u_t$

(or) $u_{xx} = a u_t$ where $a = \frac{1}{\alpha^2}$.

ie, $\boxed{u_{xx} - a u_t = 0} \longrightarrow (1)$

Here $A=1$, $B=0$, $C=0$

$\therefore B^2 - 4AC = 0 - 4 \times 1 \times 0$
$= 0$.

$\therefore$ So, it is parabolic.

Now, we solve the above equation (1) by the method of finite differences

ie, $u_{xx} = a u_t \longrightarrow (1)$

with the boundary conditions

$$\left.\begin{aligned} u(0,t) &= T_0 \\ u(l,t) &= T_l \end{aligned}\right\} \longrightarrow (2)$$

and the initial condition

$$u(x,0) = f(x) \longrightarrow (3)$$

we select a spacing h for the variable x and a spacing k for the time direction.

WKT, $u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$

& $u_t = \frac{u_{i,j+1} - u_{i,j}}{k}$

Using these forward differences in ① we get

$$\boxed{u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i-1,j} \quad \text{where} \quad \lambda = k/ah^2} \longrightarrow ④$$

Eqn. ④ is called explicit formula. It is valid if $0 < \lambda \le 1/2$.

If we take $\lambda = 1/2$ the coefficient of $u_{i,j}$ vanishes.

Hence equation ④ reduces to

$$\boxed{u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right]} \longrightarrow ⑤$$

when $\lambda = \frac{1}{2}$, WKT $\lambda = k/ah^2$

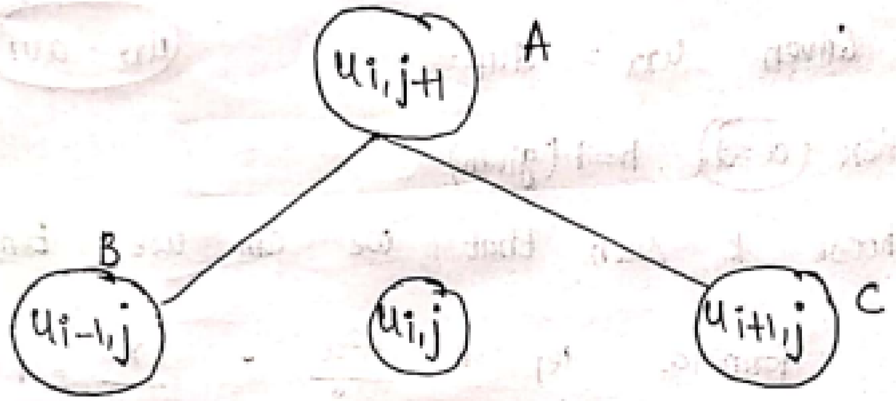
ie, $\frac{1}{2} = \frac{k}{ah^2}$

$\Rightarrow \boxed{k = \frac{ah^2}{2}}$

ie, If for a given h, choose $k = \frac{ah^2}{2}$ such that the explicit formula (4) will reduce to eqn. (5)

Equation (5) is called the Bender - schmidt recurrence equation.

Schematic diagram of eqn (5) is



ie, the value of u at A =

$\frac{1}{2}$ [value of u at B + value of u at C]

Note :-

The boundary condition (2) can be put in difference form as

$$\left. \begin{aligned} u_{0,j} &= T_0 \\ u_{n,j} &= T_l \end{aligned} \right\} \quad j = 1, 2, \ldots$$

where $nh = l$.

The initial condition (3) is

$$u_{i,0} = f(ih), \quad i = 1, 2, \ldots$$

Problems :-

① Find the solution of the parabolic equation $u_{xx} - 2u_t = 0$ when

B.C

$u(0,t)=0$, $u(4,t)=0$,

initial cond.

$u(x,0) = x(4-x)$. Assume $h=1$, find the values upto $t=5$.

Soln :-

Given $u_{xx} = 2u_t$ WKT $u_{xx} = a u_t$

Here $a=2$, $h=1$ (given)

Choose k such that we can use Bender Schimdt formula ie, $k = \frac{h^2 a}{2} = \frac{1 \times 2}{2} = 1$

$\boxed{k=1}$

Using Bender - Schmidt recurrence relation, the values of u_{ij} are tabulated below

x - direction $h=1$ →

j \ i	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	1.5	2	1.5	0
3	0	1	1.5	1	0
4	0	0.75	1	0.75	0
5	0	0.5	0.75	0.5	0

← $u(x,0) = x(4-x)$.

↓ t - direction $k=1$

Explanation of the working.

Range of x is 4

we put $x = ih$, $x_i = ih = i$ ($\because h=1$)

$t = jh$, $t_j = jk = j$ ($\because k=1$).

In the first row, value of

$u(x,0) = x(4-x) = i(4-i)$

For the values of $i = 1,2,3,4$ at $t=0$

the values of $u(x,0)$ are $0, 3, 4, 3, 0$

and these are written in first row.

Since $u(0,t) = 0$ for all values of t, $u(0,j) = 0$

for all values of j. Hence the entries in the first column are zero.

||ly, since $u(4,t)=0$ for all t. ~~[illegible]~~ ie, $u(4,j)=0$

for all values of j. Hence the entries in the last column are zero.

we have Bender - Schmidt relation

$$\boxed{u_{i,j+1} = \frac{1}{2}\left[u_{i+1,j} + u_{i-1,j}\right] \longrightarrow (1)}$$

Put $j=0$ in (1)

$$u_{i,1} = \frac{1}{2}\left[u_{i+1,0} + u_{i-1,0}\right] \rightarrow (2)$$

Put $i=1$ in (2), $u_{1,1} = \frac{1}{2}\left[u_{2,0} + u_{0,0}\right]$

$= \frac{1}{2}[4+0] = 2$

Put $i=2$ in ②, $u_{2,1} = \frac{1}{2}[u_{3,0} + u_{1,0}]$

$= \frac{1}{2}[3+3] = 3$

Put $i=3$ in ②, $u_{3,1} = \frac{1}{2}[u_{4,0} + u_{2,0}]$

$= \frac{1}{2}[0+4] = 2$

Thus the second row is filled.

$|||^{ly}$ put $j = 1,2,3,4$ the other rows are filled.

2) Solve $u_t = u_{xx}$ subject to $u(0,t)=0$, $u(1,t)=0$ and $u(x,0) = \sin \pi x$, $0<x<1$.

Soln: Since h & k are not given we will select them properly & use Bender-schmidt method.

$k = \frac{a}{2}h^2 = \frac{1}{2}h^2$

Given $u_{xx} = u_t$

WKT $\boxed{u_{xx} = a u_t}$

$a=1$

Since range of x is $(0,1)$

take $h = 0.2$

$\therefore k = \frac{(0.2)^2}{2} = 0.02$.

The formula is $\boxed{u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j})}$

$x \rightarrow$ direction $h=0.2$

$\downarrow$ t-direc. $k=0.02$

$j \backslash i$	0	0.2	0.4	0.6	0.8	1
0	0	0.5878	0.9511	0.9511	0.5878	0
0.02	0	0.4756	0.7695	0.7695	0.4756	0
0.04	0	0.3848	0.6225	0.6225	0.3848	0
0.06	0	0.3113	0.5036	0.5036	0.3113	0
0.08	0	0.2518	0.4074	0.4074	0.2518	0
0.1	0	0.2037	0.3296	0.3296	0.2037	0

$u(x,0) = \sin \pi x$

ie, $u(0,0)=0$, $u(0.2,0) = \sin 0.2\pi$,
$u(0.4,0) = \sin 0.4\pi$, $\sin(0.6,0) = \sin 0.6\pi$,
$u(0.8,0) = \sin 0.8\pi$, $\sin(1,0) = \sin\pi = 0$. (*)

Note:- Use radian mode to find all (*)

③ Solve $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ given $\underbrace{u(0,t)=0, \ u(4,t)=0}_{B.C}$,

initial condition

$u(x,0) = x(4-x)$, assuming $h=k=1$.

Find the values of u upto $t=5$.

Soln:- If we want to use Bender - Schmidt formula, we should have $k = \frac{a}{2}h^2$. Given $u_{xx} = u_t$

WKT $u_{xx} = a u_t$ $\quad a=1$

Here, $k=h=1$, $a=1$. These values do not satisfy the condition. Hence, we cannot apply Bender-Schmidt formula.

Hence, we go the explicit formula,

$$u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i-1,j}$$

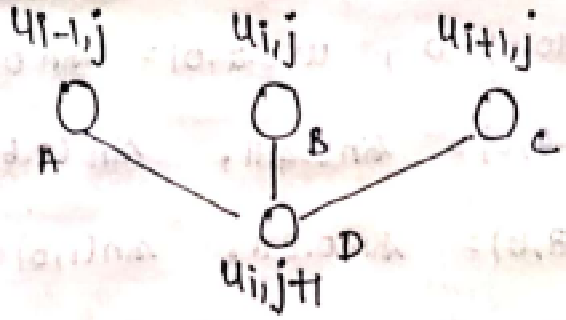
where $\lambda = \dfrac{k}{ah^2}$ ⟶ (1)

Now, $\lambda = \dfrac{k}{ah^2} = \dfrac{1}{1\times 1} = 1$.

Hence, eqn (1) reduces to

$$u_{i,j+1} = u_{i+1,j} - u_{i,j} + u_{i-1,j}$$

ie,



Value of u at D = Value of u at A + Value of u at C − Value of u at B.

x- direction h=1 →

j \ i	0	1	2	3	4
0	0	3	4	3	0
1	0	1	2	1	0
2	0	1	0	1	0
3	0	−1	2	−1	0
4	0	3	−4	3	0
5	0	−7	10	−7	0

← $u(x,0) = x(4-x)$

t dir ↓ k=1 ↓

Note: Since $\lambda=1$ is used in the working, it violates the condition for use of Explicit formula. So the solution is not stable and it is not a practical problem.

(4) (HW) Given $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial f}{\partial t}$, $f(0,t) = f(5,t) = 0$, B.C

I.C $f(x,0) = x^2(25-x^2)$. Find f in the range taking $h=1$ and upto 5 seconds.

Hint: To use Schmidt method, $k = \frac{a}{2}h^2$

here $a=1$, $h=1$ $\therefore$ $k = 1/2$

step-size of time $= 1/2$

step-size of $x = 1$.

(5) Solve $u_{xx} = 32u_t$, taking $h=0.25$, for $t>0$, $0<x<1$ & $u(x,0)=0$, $u(0,t)=0$, $u(1,t)=t$. (B.C)

initial condn.

Soln: The range for x is $(0,1)$, $h=0.25$ (given).

$$k = \frac{a}{2}h^2 = \frac{32}{2}\left(\frac{1}{4}\right)^2 = 1$$

$u_{xx} = 32u_t$

$\boxed{u_{xx} = au_t}$

$a=32$

$k=1$

step-size of time t is 1.

x → direction h=0.25

t direc. k=1

j \ i	0	0.25	0.5	0.75	1
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	0.5	2
3	0	0	0.25	1	3
4	0	0.125	0.5	1.625	4
5	0	0.25	0.875	2.25	5

← u

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