# Unit II (21MAB206T)

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21MAB206T- NUMERICAL METHODS AND ANALYSIS

# Topics Discussed

- Finite Differences
  - Different types of Operators and their Relation
  - Difference of polynomial
  - Factorial Polynomials
- Interpolation
  - For equal Intervals:
    - Newton’s Forward Interpolation
    - Newton’s Backward Interpolation
  - For unequal Intervals:
    - Newton’s Divided Difference Interpolation
    - Lagrange’s Interpolation
  - Lagrange Formula for Inverse Interpolation

# Operators

# Introduction

Let $y = f(x)$ be a given function of $x$ and let $y_0, y_1, ....y_n$ be the values of $y$ corresponding to $x_0, x_1, ....x_n$ the values of $x$. Here the independent variable $x$ is called argument and the corresponding dependent value $y$ is called entry. The difference between any two consecutive values of $x$ need not be same.

In general,

| | | | | | |
|---|---|---|---|---|---|
| $x$ | $x_0$ | $x_1$ | $x_2$ | ... | $x_n$ |
| $y$ | $y_0$ | $y_1$ | $y_2$ | ... | $y_n$ |

Here $y_1 - y_0, y_2 - y_1, ..., y_n - y_{n-1}$ is called the first difference of $y$ and it is denoted by $\Delta y$. (i.e). $\Delta y_0 = y_1 - y_0$, $\Delta y_1 = y_2 - y_1$, ..., $\Delta y_{n-1} = y_n - y_{n-1}$, here "$\Delta$" is called forward difference operator.

# Types Of Operators

Suppose the argument $x_0, x_1, \ldots. x_n$ are equally spaced, which means that $x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, \ldots, x_n = x_0 + nh$, here h is called the interval length or length of an interval.

- Forward Difference Operator ($\Delta$):

  $\Delta f(x) = f(x+h) - f(x)$

  (i.e). $\Delta y_0 = \Delta y_1 - \Delta y_0$

  similarly $\Delta^2 f(x) = \Delta(\Delta f(x)) = \Delta f(x+h) - \Delta f(x)$

  $$= f(x+2h) - 2f(x+h) + f(x).$$

- Backward Difference Operator ($\nabla$):

  $\nabla f(x) = f(x) - f(x-h)$

  (i.e). $\nabla y_1 = \nabla y_1 - \nabla y_0$

- Central Difference Operator ($\delta$):

  $\delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$

- Shifting Operator ($E$):

  $Ef(x) = f(x+h), E^2 f(x) = f(x+2h), \ldots, E^n f(x) = f(x+nh)$

- Averaging Operator ($\mu$):

  $\mu f(x) = \frac{1}{2}[f(x+\frac{h}{2}) + f(x-\frac{h}{2})]$

- Difference Operator ($D$):

  $Df(x) = \frac{d}{dx} f(x)$

- Unit Operator 1:

  $1.f(x) = f(x)$

# Properties of Operators

- The operators $\Delta, \nabla, \delta, E, \mu$ *and* $D$ are all linear.

  **Proof:**

  $$\Delta(af(x)+bg(x)) = [af(x+h)+bg(x+h)] - [af(x)+bg(x)]$$
  $$= a[f(x+h)-f(x)] + b[g(x+h)-g(x)]$$
  $$= a\Delta f(x) + b\Delta g(x)$$

- The operator is distributive over addition.

  $$\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x) = \Delta^{n+m} f(x) = \Delta^n \Delta^m f(x)$$

# Relation Between Operators

- Relation between $\Delta$ *and* $E$:

$$\begin{aligned}\Delta f(x) &= f(x+h) - f(x)\\ &= Ef(x) - 1.f(x)\\ &= (E-1)f(x)\\ \Delta &= E - 1.\end{aligned}$$

- Relation between $\nabla$ *and* $E$:

$$\begin{aligned}\nabla f(x) &= f(x) - f(x-h)\\ &= 1.f(x) - E^{-1}f(x)\\ &= (1-E^{-1})f(x)\\ \nabla &= 1 - E^{-1}.\end{aligned}$$

- Relation between $\delta$ *and* $E$:

$$\delta f(x) = f(x+\tfrac{h}{2}) - f(x-\tfrac{h}{2})$$
$$= E^{1/2}f(x) - E^{-1/2}f(x)$$
$$= (E^{1/2} - E^{-1/2})f(x)$$
$$\delta = E^{1/2} - E^{-1/2}.$$

- Relation between $\mu$ *and* $E$:

$$\mu f(x) = f(x+\tfrac{h}{2}) + f(x-\tfrac{h}{2})$$
$$= E^{1/2}f(x) + E^{-1/2}f(x)$$
$$= (E^{1/2} + E^{-1/2})f(x)$$
$$\mu = E^{1/2} + E^{-1/2}.$$

- Relation between $D$ *and* $\Delta$:

$$Df(x) = \frac{d}{dx}f(x)$$
$$D = \frac{1}{h}[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \ldots]$$

# Problems on Operators

- Prove that $E\nabla = \Delta = \nabla E$

**Proof:**

$$(E\nabla)f(x) = E(\nabla f(x)) = E(f(x) - f(x-h))$$
$$= Ef(x) - Ef(x-h)$$
$$= f(x+h) - f(x) = \Delta f(x)$$

$\therefore E\nabla = \Delta$

similarly $(\nabla E)f(x) = \nabla(Ef(x)) = \nabla f(x+h)$

$$= f(x+h) - f(x) = \Delta f(x)$$

$\therefore \nabla E = \Delta$

Hence $E\nabla = \Delta = \nabla E$

- Prove that $\nabla\Delta = \Delta - \nabla = \delta^2$

  **Proof:**

$$\begin{aligned}\nabla\Delta &= (1-E^{-1})(E-1)\\ &= E+E^{-1}-2\\ &= (E^{1/2}-E^{-1/2})^2 = \delta^2\end{aligned}$$

$$\begin{aligned}\Delta-\nabla &= (E-1)-(1-E^{-1})\\ &= E+E^{-1}-2 = \delta^2\end{aligned}$$

- Prove that $(1-\nabla)(1+\Delta) = 1$

  **Proof:**

$$(1-\nabla)(1+\Delta) = E.E^{-1} = 1$$

- Prove that $\mu\delta = \frac{1}{2}(\Delta+\nabla)$

  **Proof:**

$$\frac{1}{2}(\Delta+\nabla) = \frac{1}{2}(E-1+1-E^{-1}) = \frac{1}{2}(E-E^{-1}) = \mu\delta$$

# Difference of a polynomial

*Theorem.* The $n$th differences (forward) of a polynomial of the $n$th degree are constants.

That is, if $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{x-2} + \cdots\cdots + a_n$

Then $\Delta^n f(x) = a_0 n!\, h^n$

where $h$ is the interval of differencing.

$$\Delta f(x) = f(x+h) - f(x)$$
$$= a_0[(x+h)^n - x^n] + a_1[(x+h)^{n-1} - x^{n-1}] + \cdots\cdots + a_n$$
$$= a_0[nhx^{n-1} + \cdots\cdots] + \cdots\cdots$$
$$= a_0 nhx^{n-1} + \text{terms involving powers of } x \text{ less than } (n-1)$$

That is, $\Delta f(x) =$ a polynomial of degree $(n-1)$

$$\Delta^2 f(x) = a_0\, nh\,[(x+h)^{n-1} - x^{n-1}] + \text{terms involving lesser degree}$$
$$= a_0 n(n-1)\, h^2\, x^{n-2} + \text{terms involving degree less than } (n-2)$$

*i.e.*, second difference of a polynomial of degree $n$ is a polynomial of degree $x^{n-2}$.

Proceeding like this

$$\Delta^n f(x) = a_0\, n!\, h^n\, x^0$$
$$= a_0\, n!\, h^n$$

**Note 1.** The converse of the theorem is also true. That is, if the $n$th differences of a tabulated function are constants, then the function is a polynomial of degree $n$.

**2.** The $(n+1)$th and higher differences of a polynomial of degree $n$ are zeros.

- Find the missing term in the following data:

| X: | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| y: | 2 | 4 | 8 | - | 32 | 64 | 128 |

**Solution**:

There are 6 given values. Then we get polynomial of degree five to satisfies the given data.

(i.e). $\triangle^6 y_0 = 0.$

$(E-1)^6 y_0 = 0$

$(E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1)y_0 = 0$

$y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0 = 0$

$128 - 6(64) + 15(32) - 20y_3 + 15(8) - 6(4) + 2 = 0$

$20y_3 = 322$. Hence $y_3 = 16.1$.

Therefore Missing value is 16.1

# Practice Questions

**Prove the following:**

- $1 + \mu^2\delta^2 = (1 + \frac{1}{2}\delta^2)^2$
- $E^{1/2} = \mu + \frac{1}{2}\delta$
- $E^{-1/2} = \mu - \frac{1}{2}\delta$
- $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{\delta^2}{4}}$

- Estimate the production for 1964 and 1966 from the following data:

| X: | 1961 | 1962 | 1963 | 1964 | 1965 | 1966 | 1967 |
|---|---|---|---|---|---|---|---|
| y: | 200 | 220 | 260 | - | 350 | - | 430 |

**Hint**:There are 5 given values. Then we get polynomial of degree four to satisfies the given data.

(i.e). $\Delta^5 y_0 = 0$.

# Factorial Polynomial

A factorial polynomial $x^{(n)}$ is defined as $x^{(n)} = x(x-h)(x-2h)\ldots(x-(n-1)h)$, where $n$ is positive integer.

- Differences of $x^{(n)}$:

$$\triangle x^{(n)} = (x+h)^{(n)} - x^{(n)}$$

$$= (x+h)x(x-h)\ldots(x-(n-2)h) - x(x-h)(x-2h)\ldots(x-(n-1)h)$$

$$= x(x-h)\ldots(x-(n-2)h)[(x+h)-(x-(n-1)h]$$

$$= n.hx^{(n-1)}$$

Similarly $\triangle^2 x^{(n)} = \triangle x^{(n)}[n.hx^{(n-1)}]$

$$= n.(n-1)h^2 x^{(n-2)}$$

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$\triangle^r x^{(n)} = n.(n-1).(n-2)\ldots(n-r+1)h^r.x^{(n-r)}$, where r is positive integer and $r < n$.

- **Problem 1:**

  Express $x^4 + 3x^3 - 5x^2 + 6x - 7$ as a factorial polynomial and get their successive forward differences, taking h=1.

  **Solution**:

$$
\begin{array}{c|ccccc}
0 & 1 & 3 & -5 & 6 & -7 \\
 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 3 & -5 & 6 & -7 \\
 & 0 & 1 & 4 & -1 & \\
\hline
2 & 1 & 4 & -1 & 5 & \\
 & 0 & 2 & 12 & & \\
\hline
3 & 1 & 6 & 11 & & \\
 & 0 & 3 & & & \\
\hline
 & 1 & 9 & & &
\end{array}
$$

Therefore, factorial polynimial is

$$f(x) = 1x^{(4)} + 9x^{(3)} + 11x^{(2)} + 5x^{(1)} - 7$$

$$\Delta f(x) = 4x^{(3)} + 27x^{(2)} + 22x^{(1)} + 5$$

$$\Delta^2 f(x) = 12x^{(2)} + 54x^{(1)} + 22$$

$$\Delta^3 f(x) = 24x^{(1)} + 54$$

$$\Delta^4 f(x) = 24$$

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$$\Delta^r f(x) = 0 \; if \; r > 4.$$

- **Problem 2:**

Express $f(x) = x^3 - 3x^2 + 5x + 7$ as a factorial polynomial and get their successive forward differences, taking h=2.

**Solution**:

$$
\begin{array}{c|cccc}
0 & 1 & -3 & 5 & 7 \\
 & 0 & 0 & 0 & 0 \\
\hline
2 & 1 & -3 & 5 & 7 \\
 & 0 & 2 & -2 & \\
\hline
4 & 1 & -1 & 3 & \\
 & 0 & 4 & & \\
\hline
 & 1 & 3 & &
\end{array}
$$

Therefore, factorial polynimial is

$$f(x) = 1x^{(3)} + 3x^{(2)} + 3x^{(1)} + 7$$

$$\Delta f(x) = 3x^{(2)}.2^1 + 6x^{(1)}.2^1 + 3$$

$$\Delta^2 f(x) = 6x^{(1)}.2^2 + 6.2^2$$

$$\Delta^3 f(x) = 6.2^3 = 48$$

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$$\Delta^r f(x) = 0 \; if \; r > 3.$$

# Interpolation

# Interpolation(for equal Interval)

**Interpolation** means the process of computing intermediate values of a function from a given set of tabular values of the function.

Suppose the following table represents a set of corresponding values of $x$ and $y$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| x | : | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $\cdots$ | $x_n$ |
| y | : | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $\cdots$ | $y_n$ |

We require the value of $y = y_i$ corresponding to a value $x = x_i$, where $x_0 < x_i < x_n$.

**Extrapolation** is used to denote the process of finding the values outside the interval $(x_0, x_n)$.

# Gregory-Newton forward interpolation formula or Newton's forward interpolation formula (for equal intervals)

Let $y = f(x)$ denote a function which takes the values $y_0, y_1, \ldots, y_n$ corresponding to the values $x_0, x_1, \ldots, x_n$ respectively of $x$.

Suppose that the values of $x$ viz. $x_0, x_1, \ldots, x_n$ are equidistant. That is, $x_i - x_{i-1} = h$, for $i = 1, 2, \ldots, n$.

We can prove the formula using symbolic operator methods.

$$\begin{aligned} P_n(x) &= P_n(x_0 + uh) = E^u P_n(x_0) = E^u y_0 \\ &= (1+\Delta)^u y_0 \end{aligned}$$

$$= [1 + {}^uC_1\Delta + {}^uC_2\Delta^2 + {}^uC_3\Delta^3 + \cdots + {}^uC_r\Delta^r + \cdots + {}^uC_n\Delta^n + \cdots]y_0$$

$$= y_0 + \frac{u^{(1)}}{1!}\Delta y_0 + \frac{u^{(2)}}{2!}\Delta^2 y_0 + \frac{u^{(3)}}{3!}\Delta^3 y_0 + \cdots + \frac{u^{(r)}}{r!}\Delta^r y_0 + \cdots + \frac{u^{(n)}}{n!}\Delta^n y_0 + \cdots$$

where $u = \dfrac{x - x_0}{h}$

If $y(x)$ is a polynomial of $n$th degree $\Delta^{n+1} y_0, \ldots$ are zero. Hence

$$\boxed{P_n(x) = P_n(x_0 + uh) = y_0 + \frac{u^{(1)}}{1!}\Delta y_0 + \frac{u^{(2)}}{2!}\Delta^2 y_0 + \frac{u^{(3)}}{3!}\Delta^3 y_0 + \cdots + \frac{u^{(n)}}{n!}\Delta^n y_0}$$

**Note:**

- The first two terms will give the linear interpolation and the first three terms will give a parabolic interpolation and so on.
- Since this formula involves forward differences of $y_0$, we call it Newton's forward interpolation formula. Since this involves the forward differences of $y_0$, this is used to interpolate the values of $y$ nearer to the beginning value of the table.
- This is applicable only if the interval of differencing $h$ is constant.

# Gregory-Newton Backward interpolation formula (for equal intervals)

Newton's forward interpolation formula cannot be used for interpolating a value of $y$ nearer to the end of the table of values. For this purpose, we get another backward interpolation formula.

Suppose $y = f(x)$ takes the values $y_0, y_1, \ldots, y_n$ corresponding to the values $x_0, x_1, \ldots, x_n$ of $x$.

Let $x_i - x_{i-1} = h$ for $i = 1, 2, \ldots, n$.(equal intervals)

$$\therefore \qquad x_i = x_0 + ih, i = 0, 1, \ldots, n.$$

We can prove the formula using symbolic operator methods.

$$\begin{aligned}
P_n(x) &= P_n(x_n + vh) = E^v P_n(x_n) \\
&= (1-\nabla)^{-v} y_n \qquad \textit{since} \qquad E = (1-\nabla)^{-1} \\
&= \left[1 + v\nabla + \frac{v(v+1)}{2!}\nabla^2 + \frac{v(v+1)(v+2)}{3!}\nabla^3 + \cdots\right] y_n
\end{aligned}$$

$$\boxed{P_n(x) = P_n(x_n + vh) = y_n + v\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots}$$

where $v = \dfrac{x - x_n}{h}$

**Note:**

- Since the formula involves the backward difference operator, it is named as backward interpolation formula
- This is used to interpolate the values of y nearer to the end of a set tabular values. This may also be used to extrapolate closure to the right of $y_n$.

# Examples:

**Example:1** Find the values of $y$ at $x = 21$ and $x = 28$ from the following data

| | | | | | |
|---|---|---|---|---|---|
| x | : | 20 | 23 | 26 | 29 |
| y | : | 0.3420 | 0.3907 | 0.4384 | 0.4848 |

**Solution:** Since $x = 21$ is nearer to the beginning of the table, we use Newton's forward formula. Here, $h$ =constant=3.

| $x$ | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ |
|---|---|---|---|---|
| 20 | 0.3420 | | | |
| | | 0.0487 | | |
| 23 | 0.3907 | | −0.0010 | |
| | | 0.0477 | | −0.0003 |
| 26 | 0.4384 | | −0.0013 | |
| | | 0.0464 | | |
| 29 | 0.4848 | | | |

The topmost diagonal gives the forward differences of $y_0$ while the lowermost diagonal gives the backward differences of $y_n$.

There are only 4 data given. Hence the collocation polynomial will be of degree 3.

By Newton's forward interpolation formula,

$$y(x) \approx P_3(x) = y_0 + \frac{u^{(1)}}{1!}\Delta y_0 + \frac{u^{(2)}}{2!}\Delta^2 y_0 + \frac{u^{(3)}}{3!}\Delta^3 y_0 + \cdots \text{ where}$$

$$u = \frac{x - x_0}{h} = \frac{21 - 20}{3} = 0.3333$$

$$y(21) \approx P_3(21) = 0.3420 + (0.3333)(0.0487) + \frac{(0.3333)(-0.6666)}{2}(-0.001) + \frac{(0.3333)(-0.6666)(-1.6666)}{6}(-0.0003)$$

$$\therefore y(21) \approx 0.3583.$$

Since $x = 28$ is nearer to end value, we use Newton's backward interpolation formula.

$$y(x) \approx P_3(x) = P_3(x_n + vh) = y_n + v \nabla y_n + \frac{v(v+1)}{2} \nabla^2 y_n + \frac{v(v+1)(v+2)}{6} \nabla^3 y_n + \cdots$$

$$y(28) \approx P_3(28) = P_3\left[29 + \left(-\frac{1}{3}\right) 3\right] \text{ where } v = \frac{x - x_n}{h} = \frac{28 - 29}{3} = -\frac{1}{3}$$

$$= 0.4848 + \left(-\frac{1}{3}\right)(0.0464) + \frac{\left(-\frac{1}{3}\right)\left(\frac{2}{3}\right)}{2}(-0.0013) + \frac{\left(-\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)}{6}(-0.0003) + \cdots$$

$$=0.4848-0.015465+0.0001444+0.0000185$$

$$y(28) \approx 0.4695.$$

**Example:2** The population of a town is as follows.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Year | x | : | 1941 | 1951 | 1961 | 1971 | 1981 | 1991 |
| Population in lakhs | y | : | 20 | 24 | 29 | 36 | 46 | 51 |

Estimate the population increase during the period 1946 to 1976.

**Solution:** Since six data are given, $P(x)$ is of degree 5. To find $y$ at $x = 1946$ use forward interpolation and to find $y$ at $x = 1976$, use backward interpolation formula. Here, $h$ =constant=10.

| $x$ | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ | $\Delta^5 y$ |
|---|---|---|---|---|---|---|
| 1941 | 20 | | | | | |
| | | 4 | | | | |
| | | | 1 | | | |
| 1951 | 24 | | | 1 | | |
| | | 5 | | | | |
| | | | 2 | | | |
| | | | | | 0 | |
| 1961 | 29 | | | 1 | | |
| | | 7 | | | | |
| | | | 3 | | | −9 |
| | | | | | −9 | |
| 1971 | 36 | | | −8 | | |
| | | 10 | | | | |
| | | | −5 | | | |
| 1981 | 46 | | | | | |
| | | 5 | | | | |
| 1991 | 51 | | | | | |

$$u = \frac{x - x_0}{h} = \frac{1946 - 1941}{10} = \frac{1}{2}$$

$$\begin{aligned}
y(1946) &\approx P_5(1946) = P_5\left[1941 + \frac{1}{2}(10)\right] \\
&= y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{6}\Delta^3 y_0 + \cdots \\
&= 20 + \frac{1}{2}(4) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}(1) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}(1) \\
&+ \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{24}(0) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{120}(-9) \\
&= 20 + 2 - 0.125 + 0.0625 - 0.24609 \\
&= 21.69
\end{aligned}$$

$$
\begin{aligned}
y(1976) &\approx P_5(1976) = P_5\left[1991 - \frac{3}{2}(10)\right] \because v = \frac{x - x_n}{h} = \frac{1976 - 1991}{10} = -\frac{3}{2} \\
&= y_n + v\nabla y_n + \frac{v(v+1)}{2}\nabla^2 y_n + \frac{v(v+1)(v+2)}{6}\nabla^3 y_n + \cdots \\
&= 51 - \frac{3}{2}(5) + \frac{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)}{2}(-5) + \frac{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)}{6}(-8) \\
&+ \frac{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)}{24}(-9) + \frac{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)}{120}(-9) \\
&= 51 - 7.5 - 1.875 - 0.5 - 0.2109375 - 0.10546875 \\
&= 40.8085938
\end{aligned}
$$

Therefore, increase in population during the period

$$= 40.809 - 21.69 = 19.119\text{lakhs}.$$

**Example:3** From the data given below, find the number of students whose weight is between 60 and 70.

| Weight in lbs | : | 0 – 40 | 40 – 60 | 60 – 80 | 80 – 100 | 100 – 120 |
|---|---|---|---|---|---|---|
| No. of students | : | 250 | 120 | 100 | 70 | 50 |

**Solution:**

| $x$ weight | $y$ (No. of students) | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|---|---|---|---|---|---|
| Below 40 | 250 | | | | |
| | | 120 | | | |
| Below 60 | 370 | | −20 | | |
| | | 100 | | −10 | |
| Below 80 | 470 | | −30 | | 20 |
| | | 70 | | 10 | |
| Below 100 | 540 | | −20 | | |
| | | 50 | | | |
| Below 120 | 590 | | | | |

To calculate the number of students whose weight is less than 70.
We will use forward difference formula

$$u = \frac{x - x_0}{h} = \frac{70 - 40}{20} = \frac{3}{2}$$

$$\begin{aligned}
y(70) &= y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{6}\Delta^3 y_0 + \cdots \\
&= 250 + \frac{3}{2}(120) + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{2}(-20) + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{6}(-10) \\
&+ \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{24}(20) \\
&= 250 + 180 - 7.5 + 0.625 + 0.46875 \\
&= 423.59 \\
&\approx 424
\end{aligned}$$

Number of students whose weights is between 60 and 70

$$y(70) - y(60) = 424 - 370 = 54.$$

# Practice Problems:

1. From the following table of half-yearly premium for policies maturing at different ages,estimate the premium for policies maturing at age 46 and 63.

| | | | | | | |
|---|---|---|---|---|---|---|
| Age (x) | : | 45 | 50 | 55 | 60 | 65 |
| Premium (y) | : | 114.84 | 96.16 | 83.32 | 74.48 | 68.48 |

2. The following data are taken from the steam table.

| | | | | | |
|---|---|---|---|---|---|
| Temp. degree/cel.: | 140 | 150 | 160 | 170 | 180 |
| Pressure kgf/$cm^2$: | 3.685 | 4.854 | 6.302 | 8.076 | 10.225 |

Find the pressure at temperature $t = 142^\circ$ and $t = 175^\circ$

3. Find a polynomial of degree two which takes the values

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| x | : | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| y | : | 1 | 2 | 4 | 7 | 11 | 16 | 22 | 29 |

# Interpolation with Unequal Intervals

**Divided differences**

Let the function $y = f(x)$ assume the values $f(x_0), f(x_1), \ldots, f(x_n)$ corresponding to the arguments $x_0, x_1, \ldots, x_n$ respectively where the intervals $x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}$ need not be equal.

**Definitions**

The **first divided differences** of $f(x)$ for the arguments $x_0, x_1$ is defined as $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$. It is denoted by $f(x_0, x_1)$ or $[x_0, x_1]$ or $\underset{x_1}{\Delta} f(x_0)$.

$$f(x_0, x_1) = [x_0, x_1] = \underset{x_1}{\Delta} f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \tag{1}$$

The **second divided differences** of $f(x)$ for the three arguments $x_0, x_1, x_2$ is defined as

$$f(x_0, x_1, x_2) = \underset{x_1, x_2}{\Delta^2} f(x_0) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} \tag{2}$$

The **third divided differences** of $f(x)$ for the four arguments $x_0, x_1, x_2, x_3$ as

$$f(x_0, x_1, x_2, x_3) = \underset{x_1, x_2, x_3}{\Delta^3} f(x_0) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0} \qquad (3)$$

Equations (1), (2), (3) refer to divided differences of order one, two, and three respectively.

**Example:1** Find the divided differences of $f(x) = x^3 + x + 2$ for the arguments 1,3,6,11.

**Solution:** The divided difference table is

| $x$ | $y$ | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ |
|---|---|---|---|---|
| 1 | 4 | | | |
| | | $\frac{32-4}{3-1} = 14$ | | |
| 3 | 32 | | $\frac{64-14}{6-1} = 10$ | |
| | | $\frac{224-32}{6-3} = 64$ | | $\frac{20-10}{11-1} = 1$ |
| 6 | 224 | | $\frac{224-64}{11-3} = 20$ | |
| | | $\frac{1344-224}{11-6} = 224$ | | |
| 11 | 1344 | | | |

**Newton's interpolation formula for unequal intervals (or Newton's divided difference formula)**

$$f(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) + \cdots + (x - x_0)(x - x_1)\cdots(x - x_{n-1})f(x_0, x_1, \ldots, x_n)$$

is called Newton's divided difference interpolation formula for unequal intervals.

**Example 2:** From the following table find $f(x)$ and hence $f(6)$ using Newton's interpolation formula

| | | | | |
|---|---|---|---|---|
| x: | 1 | 2 | 7 | 8 |
| f(x): | 1 | 5 | 5 | 4 |

**Solution:**

Here, intervals are not equal.

| $x$ | $f(x)$ | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ |
|---|---|---|---|---|
| 1 | 1 | | | |
| | | $\frac{5-1}{2-1}=4$ | | |
| | | | $\frac{0-4}{7-1}=-\frac{2}{3}$ | |
| 2 | 5 | | | |
| | | $\frac{5-5}{7-2}=0$ | | $\frac{-\frac{1}{6}+\frac{2}{3}}{8-1}=\frac{1}{14}$ |
| | | | $\frac{-1-0}{8-2}=-\frac{1}{6}$ | |
| 7 | 5 | | | |
| | | $\frac{4-5}{8-7}=-1$ | | |
| 8 | 4 | | | |

By Newton's divided difference formula,

$$
\begin{aligned}
f(x) &= f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) + \cdots \\
&= 1 + (x-1)4 + (x-1)(x-2)\left(-\frac{2}{3}\right) + (x-1)(x-2)(x-7)\left(\frac{1}{14}\right) \\
&= \frac{1}{42}(3x^3 - 58x^2 + 321x - 224) \\
f(6) &= \frac{1}{42}[3 \times 216 - 36 \times 58 + 1926 - 224] \\
&= 6.23809524.
\end{aligned}
$$

# Practice Problems:

1. Form the divided difference table for the following data

| x : | $-2$ | 0 | 3 | 5 | 7 | 8 |
|---|---|---|---|---|---|---|
| y=f(x): | $-792$ | 108 | $-72$ | 48 | $-144$ | $-252$ |

2. Using Newton's divided difference formula, find the values of $f(2)$, $f(8)$ and f(15) given the following table:

| x : | 4 | 5 | 7 | 10 | 11 | 13 |
|---|---|---|---|---|---|---|
| f(x): | 48 | 100 | 294 | 900 | 1210 | 2028 |

3. Find the equation $y = f(x)$ of least degree and passing through the points (-1,-21),(1,15),(2,12),(3,3). Find also $y$ at $x = 0$.

# Lagrange's interpolation formula for unequal intervals

Let $y = f(x)$ be a function such that $f(x)$ takes the values $y_0, y_1, \ldots, y_n$ corresponding to $x_0, x_1, \ldots, x_n$.

$$\begin{aligned} y = f(x) &= \frac{(x - x_1)(x - x_2)\cdots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\cdots(x_0 - x_n)}.y_0 \\ &+ \frac{(x - x_0)(x - x_2)\cdots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\cdots(x_1 - x_n)}.y_1 \\ &+ \cdots\cdots\cdots\cdots \\ &+ \frac{(x - x_0)(x - x_1)\cdots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\cdots(x_n - x_{n-1})}.y_n \end{aligned}$$

This equation is called Lagrange's interpolation formula for unequal intervals.

**NOTE:** Lagrange's interpolation formula can also be used when the values of $x$ are equally spaced.

**Example:1** Using Lagrange's formula of interpolation find $y(9.5)$ given

| | | | | |
|---|---|---|---|---|
| x: | 7 | 8 | 9 | 10 |
| y: | 3 | 1 | 1 | 9 |

**Solution:** By Lagrange's formula,

$$\begin{aligned} y = f(x) &= \frac{(x-8)(x-9)(x-10)}{(7-8)(7-9)(7-10)} \times 3 \\ &+ \frac{(x-7)(x-9)(x-10)}{(8-7)(8-9)(8-10)} \times 1 \\ &+ \frac{(x-7)(x-8)(x-10)}{(9-7)(9-8)(9-10)} \times 1 \\ &+ \frac{(x-7)(x-8)(x-9)}{(10-7)(10-8)(10-9)} \times 9 \end{aligned}$$

$$
\begin{aligned}
f(9.5) &= \frac{(1.5)(0.5)(-0.5)}{(-1)(-2)(-3)} \times 3 \\
&+ \frac{(2.5)(0.5)(-0.5)}{(1)(-1)(-2)} \times 1 \\
&+ \frac{(2.5)(1.5)(-0.5)}{(2)(1)(-1)} \times 1 \\
&+ \frac{(2.5)(1.5)(0.5)}{(3)(2)(1)} \times 9 \\
&= 0.1875 - 0.3125 + 0.9375 + 2.8125 \\
&= 3.625
\end{aligned}
$$

**Example:2** Find the parabola of the form $y = ax^2 + bx + c$ passing through the points $(0,0), (1,1)$, and $(2,20)$.

**Solution:**

By Lagrange's interpolation formula,

$$\begin{aligned}
y = f(x) &= \frac{(x-1)(x-2)}{(0-1)(0-2)} \times 0 \\
&+ \frac{(x-0)(x-2)}{(1-0)(1-2)} \times 1 \\
&+ \frac{(x-0)(x-1)}{(2-0)(2-1)} \times 20 \\
&= 0 - x(x-2) + 10x(x-1) \\
y &= 9x^2 - 8x.
\end{aligned}$$

# Inverse Interpolation

For a given set of table values, we interpolate the value of $x$ for a given value of $y$. In such case, we will take $y$ as independent variable and $x$ as dependent variable. Replacing $'y'$ by $'x'$ and $'x'$ by $'y'$ in Lagrange's interpolation formula, we get Inverse Lagrange's interpolation formula.

$$\begin{aligned} x &= \frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)}.x_0 \\ &+ \frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)}.x_1 \\ &+ \cdots \\ &+ \frac{(y-y_0)(y-y_1)\cdots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\cdots(y_n-y_{n-1})}.x_n \end{aligned}$$

This formula is called formula of inverse interpolation.

**Example:1** Find the age corresponding to the annuity value 13.6 given the table

| Age (x) | : | 30 | 35 | 40 | 45 | 50 |
|---|---|---|---|---|---|---|
| Annuity value (y): | | 15.9 | 14.9 | 14.1 | 13.3 | 12.5 |

**Solution:** By using inverse interpolation formula,

$$\begin{aligned} x &= \frac{(13.6-14.9)(13.6-14.1)(13.6-13.3)(13.6-12.5)}{(15.9-14.9)(15.9-14.1)(15.9-13.3)(15.9-12.5)}\times 30 \\ &+ \frac{(13.6-15.9)(13.6-14.1)(13.6-13.3)(13.6-12.5)}{(14.9-15.9)(14.9-14.1)(14.9-13.3)(14.9-12.5)}\times 35 \\ &+ \frac{(13.6-15.9)(13.6-14.9)(13.6-13.3)(13.6-12.5)}{(14.1-15.9)(14.1-14.9)(14.1-13.3)(14.1-12.5)}\times 40 \\ &+ \frac{(13.6-15.9)(13.6-14.9)(13.6-14.1)(13.6-12.5)}{(13.3-15.9)(13.3-14.9)(13.3-14.1)(13.3-12.5)}\times 45 \\ &+ \frac{(13.6-15.9)(13.6-14.9)(13.6-14.1)(13.6-13.3)}{(12.5-15.9)(12.5-14.9)(12.5-14.1)(12.5-13.3)}\times 50 \\ \therefore x(y=13.6) &= 43. \end{aligned}$$

# Practice Problems:

1. Use Lagrange's formula to fit a polynomial to the data

| | | | | | |
|---|---|---|---|---|---|
| x | : | −1 | 0 | 2 | 3 |
| y | : | −8 | 3 | 1 | 12 |

   and hence find $y(x = 1)$.

2. From the data given below, find the value of $x$ when $y = 13.5$

| | | | | | | |
|---|---|---|---|---|---|---|
| x | : | 93.0 | 96.2 | 100.0 | 104.2 | 108.7 |
| f(x) | : | 11.38 | 12.80 | 14.70 | 17.07 | 19.91 |