Unit-5

Numerical Solution of partial differential Equations.

Elliptic Equations

Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(or)

$$\nabla^2 u = 0 \quad \text{(or)}$$

$$u_{xx} + u_{yy} = 0$$

$\downarrow$

Leibmann's iteration process.

Poisson's eqn.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y).$$

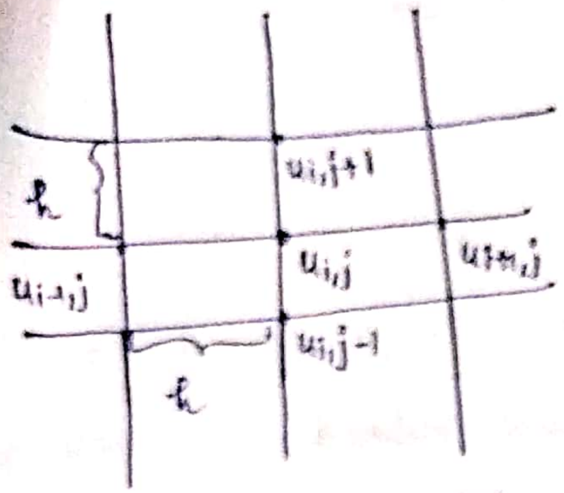
(1) The elliptic equation is ~~Laplace's equ~~ Laplace's equation. ie,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{ie, } \nabla^2 u = 0 \quad \text{(or)} \quad u_{xx} + u_{yy} = 0$$

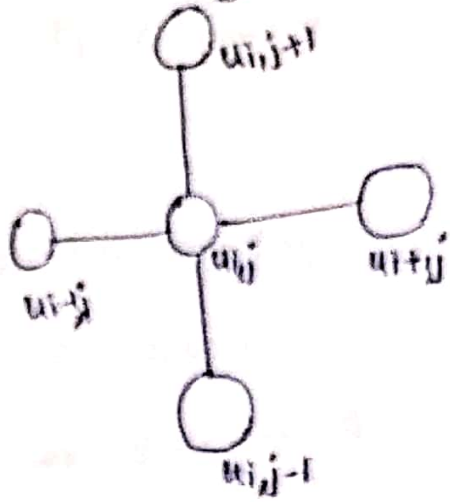
The Standard five point formula

$$\boxed{u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right]}$$

ie, the value of u at any interior point is the arithmetic mean of the values of u at the four lattice points.



(or)

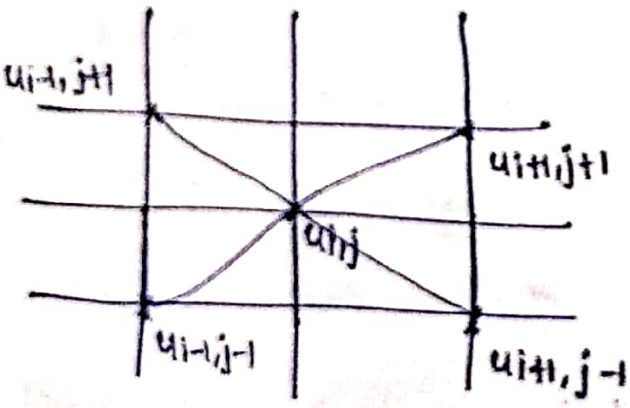


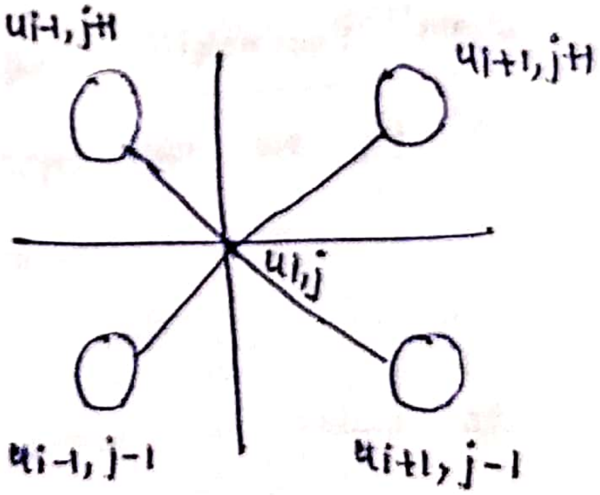
Schematic diagram.

Central value = Average of the other four values.

Diagonal five-point formula:-

$$u_{i,j} = \frac{u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1}}{4}$$





Schematic diagram of diagonal formula.

Note:- The error in the diagonal formula is four times than the error in the standard formula.

Solution of Laplace's Equation (By Liebmann's iteration process)

To solve the Laplace's equation $u_{xx} + u_{yy} = 0$ in bounded square region R with a boundary C when the boundary values of u are given on the boundary.

Note:-

① The iteration formula is

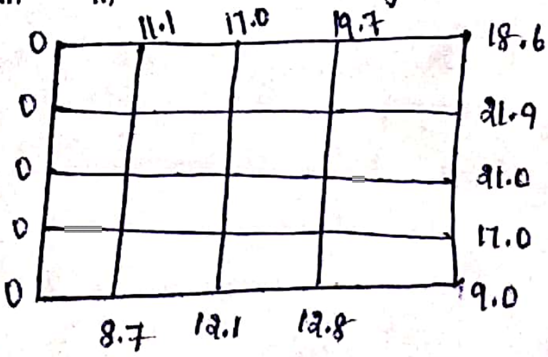
$$u_{i,j}^{(n+1)} = \frac{u_{i+1,j}^{(n)} + u_{i-1,j}^{(n+1)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n+1)}}{4}$$

where the superscript of u denotes the iteration number.

The above eqn. is called LIEBMANN's iteration process. The process is stopped once, we get the values with desired accuracy.

Problems:-

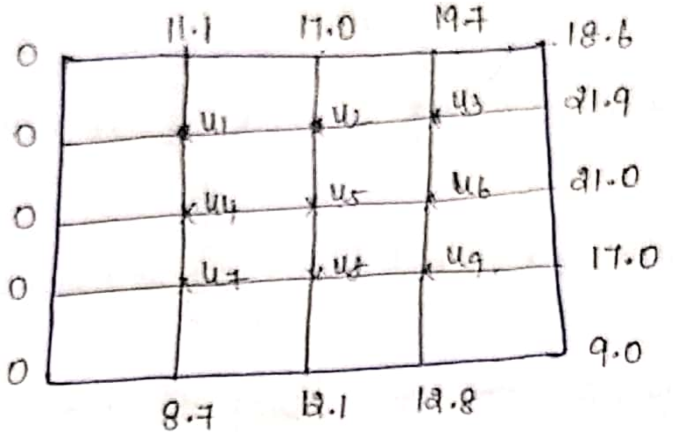
① Find by the Liebmann's method the values at the interior lattice points of a square region of the harmonic function u whose boundary values are as shown in the following figure.



Solution:

Since u is harmonic, it satisfies Laplace equation. $u_{xx}+u_{yy}=0$ in the square → (1)

Let the interior values of u at the 9 grid points be $u_1, u_2, \ldots u_9$. We will find the values of u at the interior mesh points.



Rough values:

Find u_5 first: $u_5 = \frac{1}{4}\left[0+21.0+12.1+17.0\right] = 12.5$ (SFPF)

Knowing u_5, we find u_1, u_3, u_7, u_9 by using diagonal five point formula.

$$u_1 = \frac{0+17.0+0+u_5}{4} = \frac{17.0+12.5}{4} = 7.4 \quad \text{(DFPF)}$$

$$u_3 = \frac{17.0+18.6+u_5+21.0}{4} = \frac{17+18.6+12.5+21}{4} = 17.275 \simeq 17.3 \quad \text{(DFPF)}$$

$$u_7 = \frac{0+u_5+0+12.1}{4} = \frac{12.5+12.1}{4} = 6.2 \quad \text{(DFPF)}$$

$$u_9 = \frac{u_5+21+12.1+9}{4} = \frac{12.5+9+21+12.1}{4} = 13.7 \quad \text{(DFPF)}$$

The remaining 4 values u_2, u_4, u_6, u_8 can be got by using SFPF.

$$\cancel{u_9 = \frac{1}{4}(0+u_1+u_3+u_7) = \frac{1}{4}(7.4+17.3+6.2)}$$

$$u_2 = \frac{1}{4}(17+u_1+u_3+u_5) = \frac{1}{4}(17+7.4+17.3+12.5)$$
$$= 13.6 \text{ (SFPF)}.$$

$$u_4 = \frac{1}{4}(0+u_5+u_1+u_7) = \frac{1}{4}(12.5+7.4+6.2) = 6.5 \text{ (SFPF)}$$

$$u_6 = \frac{1}{4}(u_5+u_3+21+u_9) = \frac{1}{4}(12.5+17.3+21+13.7) = 16.1 \text{ (SFPF)}$$

$$u_8 = \frac{1}{4}(u_5+u_7+u_9+12.1) = \frac{1}{4}(12.5+6.2+13.7+12.1) = 11.1 \text{ (SFPF)}$$

We will now improve the values by using always SFPT.

First iteration:

$$u_1^{(1)} = \frac{1}{4}(0+11.1+u_2+u_4) = \frac{1}{4}(11.1+13.6+6.5) = 7.8$$

$$u_2^{(1)} = \frac{1}{4}(u_1^{(1)}+17+u_3+u_5) = \frac{1}{4}(7.8+17+17.8+12.5)$$
$$= 13.7$$

$$u_3^{(1)} = \frac{1}{4}(u_2^{(1)}+19.7+21.9+u_6) = \frac{1}{4}(13.7+19.7+21.9+16.1) = 17.9$$

$$u_4^{(1)} = \frac{1}{4}(0+u_1^{(1)}+u_5+u_7) = \frac{1}{4}(7.8+12.5+6.2) = 6.6$$

$$u_5^{(1)} = \frac{1}{4}(u_4^{(1)}+u_2^{(1)}+u_6+u_8) = \frac{1}{4}(6.6+13.7+16.1+11.1) = 11.9$$

$$u_6^{(1)} = \frac{1}{4}(u_5^{(1)}+21+u_3^{(1)}+u_9) = \frac{1}{4}(11.9+21+17.9+13.7) = 16.1$$

$$u_7^{(1)} = \frac{1}{4}\left(0 + u_4^{(1)} + u_8 + 8.7\right) = \frac{1}{4}\left(6.6 + 11.1 + 8.7\right) = 6.6$$

$$u_8^{(1)} = \frac{1}{4}\left(u_5^{(1)} + u_7^{(1)} + u_9 + 12.1\right) = \frac{1}{4}\left(11.9 + 6.6 + 13.7 + 12.1\right)$$
$$= 11.1$$

$$u_9^{(1)} = \frac{1}{4}\left(u_8^{(1)} + u_6^{(1)} + 17 + 12.8\right) = \frac{1}{4}\left(11.1 + 16.1 + 17 + 12.8\right)$$
$$= 14.3$$

Second iteration

$u_1^{(2)} = 7.9$

$u_2^{(2)} = 13.7$

$u_3^{(2)} = 17.9$

$u_4^{(2)} = 6.6$

$u_5^{(2)} = 11.9$

$u_6^{(2)} = 16.3$

$u_7^{(2)} = 6.6$

$u_8^{(2)} = 11.2$

$u_9^{(2)} = 14.3$

Third iteration

$u_1^{(3)} = 7.9$

$u_2^{(3)} = 13.7$

$u_3^{(3)} = 17.9$

$u_4^{(3)} = 6.6$

$u_5^{(3)} = 11.9$

$u_6^{(3)} = 16.3$

$u_7^{(3)} = 6.6$

$u_8^{(3)} = 11.2$

$u_9^{(3)} = 14.3$.

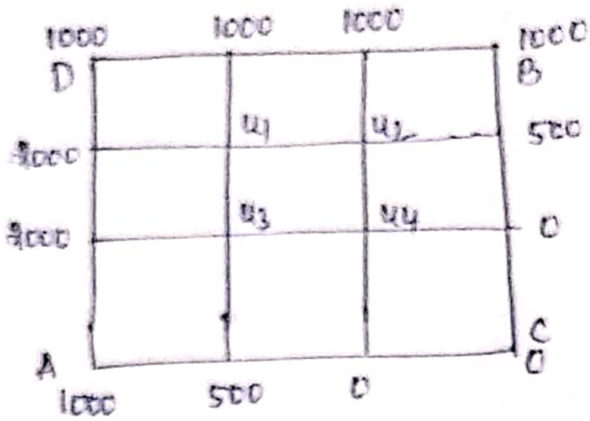
$\therefore$ all the values of u of the third iteration are same as the corresponding values of the second iteration. Hence, we stop the procedure.

$\therefore$ $u_1 = 7.9$ $\quad u_2 = 13.7$ $\quad u_3 = 17.9$

$u_4 = 6.6$ $\quad u_5 = 11.9$ $\quad u_6 = 16.3$

$u_7 = 6.6$ $\quad u_8 = 11.2$ $\quad u_9 = 14.3$.

2) Evaluate the function $u(x,y)$ satisfying $\nabla^2 u = 0$ at the lattice points given the boundary values as follows:



Solution:

$$u_1 = \frac{2000 + 1000 + u_2 + u_3}{4}$$

$$4u_1 = u_2 + u_3 + 3000$$

$$4u_1 - u_2 - u_3 = 3000 \quad \text{①}$$

$$u_2 = \frac{u_1 + 1000 + 500 + u_4}{4}$$

$$4u_2 = u_1 + u_4 + 1500$$

$$4u_2 - u_1 - u_4 = 1500 \quad \text{②}$$

$$u_3 = \frac{2000 + u_1 + u_4 + 500}{4}$$

$$4u_3 = 2500 + u_1 + u_4$$

$$4u_3 - u_1 - u_4 = 2500 \quad \text{③}$$

$$u_4 = \frac{0 + 0 + u_3 + u_2}{4}$$

$$4u_4 = u_3 + u_2$$

$$4u_4 - u_3 - u_2 = 0 \quad \text{④}$$

$$4u_1 - u_2 - u_3 = 3000 \longrightarrow (5)$$

$$u_1 - 4u_2 + u_4 = -1500 \longrightarrow (6)$$

$$u_1 - 4u_3 + u_4 = -2500 \longrightarrow (7)$$

$$u_2 + u_3 - 4u_4 = 0 \longrightarrow 8$$

Solve above eqn by elimination method,

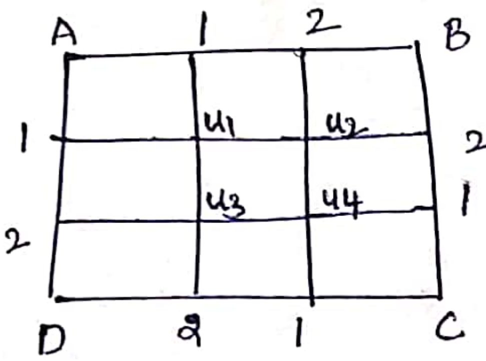
we get $u_1 = 1208.4$

$u_2 = 791.7$

$u_3 = 1041.7$

$u_4 = 458.4.$

(3) Solve $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary conditions as shown below.

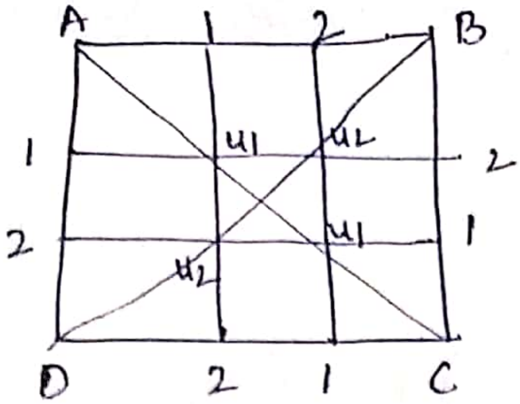


Soln:- we see that it is symmetrical about the diagonals AC & BD.

Let u_1, u_2, u_3, u_4 be the values at the interior grid points.

By symmetry, $u_1 = u_4$ & $u_3 = u_2$.

$\therefore$ we need to find only two values u_1 & u_2.



$$u_1 = \frac{1}{4}(1+1+u_2+u_2)$$

$$= \frac{1}{4}(2+2u_2) = \frac{1}{2}(1+u_2)$$

$$2u_1 - u_2 = 1 \longrightarrow (1).$$

$$u_2 = \frac{1}{4}(u_1+2+2+u_1) = \frac{1}{4}(4+2u_1) = \frac{1}{2}(2+u_1)$$

$$2u_2 - u_1 = 2 \longrightarrow (2)$$

$$(1) \Rightarrow \quad \cancel{2u_1} - u_2 = 1$$

$$(2)\times 2 \Rightarrow \quad 4u_2 - \cancel{2u_1} = 4$$

$$3u_2 = 5$$

$$\boxed{u_2 = 5/3 = 1.6666}$$

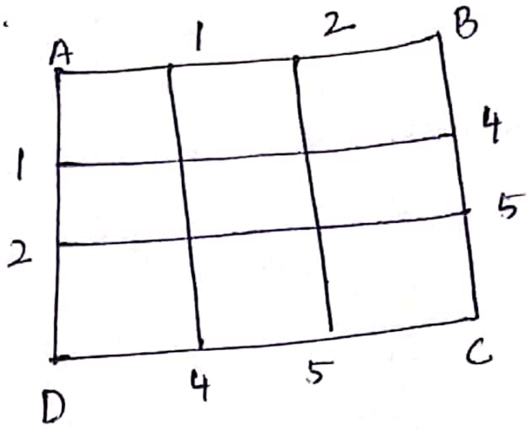
$$(1)\times 2 \Rightarrow \quad 4u_1 - \cancel{2u_2} = 2$$

$$(2) \Rightarrow \quad \cancel{2u_2} - u_1 = 2$$

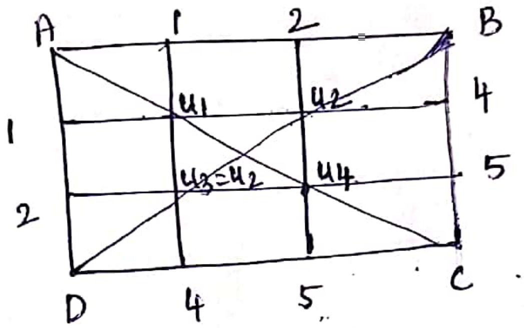
$$3u_1 = 4$$

$$\boxed{u_1 = 4/3 = 1.3333}.$$

(4) Solve $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown in the figure below.



Soln:- The boundary values are symmetrical about the diagonal AC but not about BD.



By symmetry, $\boxed{u_2 = u_3;\ u_1 \neq u_4.}$

we need to find u_1, u_2, u_4 only.

$$u_1 = \frac{1}{4}(1+1+u_2+u_2) = \frac{1}{4}(2+2u_2) = \frac{1}{2}(1+u_2)$$

$$2u_1 - u_2 = 1 \rightarrow (1)$$

$$u_2 = \frac{1}{4}(u_1+2+4+u_4) = \frac{1}{4}(6+u_1+u_4)$$

$$4u_2 = u_1+u_4+6 \Rightarrow 4u_2 - u_1 - u_4 = 6 \rightarrow (2)$$

$$u_4 = \frac{1}{4}(u_2+u_2+5+5) = \frac{1}{4}(2u_2+10) = \frac{1}{2}(u_2+5)$$

$$2u_4 = u_2+5 \Rightarrow 2u_4 - u_2 = 5 \longrightarrow (3)$$

* Solve (1), (2) and (3) we get

$$\boxed{\begin{aligned} u_1 &= 2 \\ u_2 &= 3 \\ u_4 &= 4 \end{aligned}}$$

5) Solve $u_{xx} + u_{yy} = 0$ over the square mesh of sides 4 units satisfying the following boundary conditions:

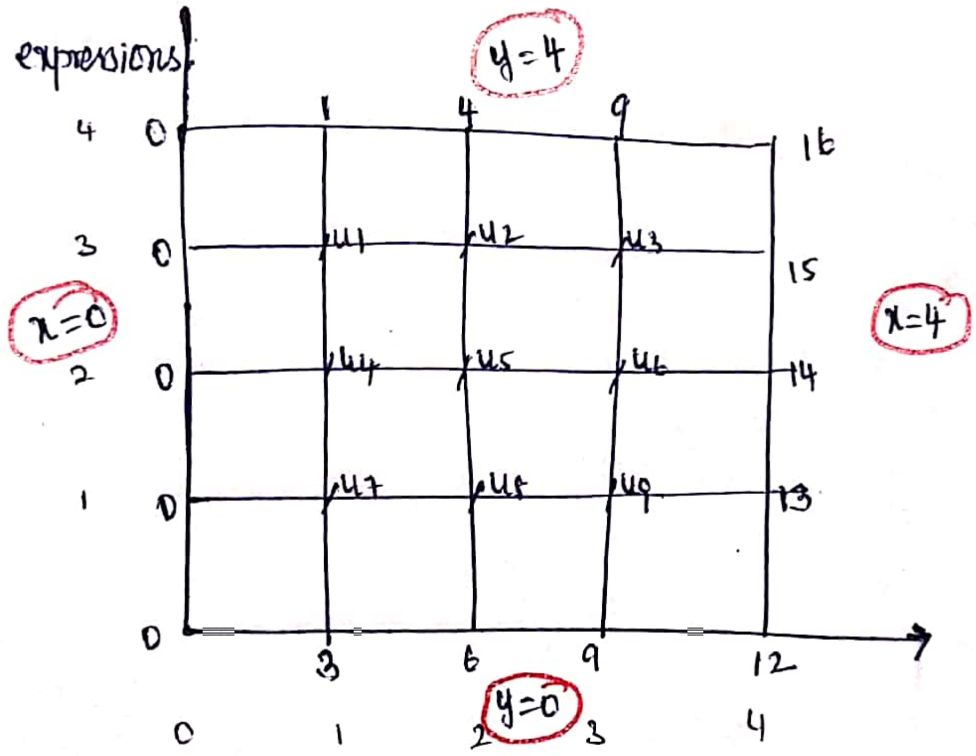
(i) $u(0,y) = 0$ for $0 \le y \le 4$.

(ii) $u(4,y) = 12+y$ for $0 \le y \le 4$

(iii) $u(x,0) = 3x$ for $0 \le x \le 4$

(iv) $u(x,4) = x^2$ for $0 \le x \le 4$.

soln:- we divide the square mesh into 16 sub-squares of side 1 unit and calculate the numerical values of u on the boundary using given analytical expressions.



(i) $u(0,1) = 0$
$u(0,2) = 0$
$u(0,3) = 0$
$u(0,4) = 0$
$u(0,0) = 0$

(ii) $u(4,1) = 13$
$u(4,2) = 14$
$u(4,3) = 15$
$u(4,4) = 16$
$u(4,0) = 12$

(iii) $u(0,0) = 0$
$u(1,0) = 3$
$u(2,0) = 6$
$u(3,0) = 9$
$u(4,0) = 12$

(iv) $u(0,4) = 0$
$u(1,4) = 1$
$u(2,4) = 4$
$u(3,4) = 9$
$u(4,4) = 16$

Rough values

$$u_5 = \frac{1}{4}(0+4+14+6) = 6 \quad (SFPF)$$

$$u_1 = \frac{1}{4}(0+4+0+u_5) = \frac{1}{4}(4+6) = 2.5 \quad (DFPF)$$

$$u_3 = \frac{1}{4}(u_5+14+4+6) = \frac{1}{4}(6+14+4+6) = 10 \quad (DFPF)$$

$$u_7 = \frac{1}{4}(0+6+u_5+0) = \frac{1}{4}(6+6) = 3 \quad (DFPF)$$

$$u_9 = \frac{1}{4}(u_5+14+6+12) = \frac{1}{4}(6+14+6+12) = 9.5 \quad (DFPF).$$

We use SFPF to get the other values of u.

$$u_2 = \frac{1}{4}(u_1+4+u_3+u_5) = \frac{1}{4}(2.5+4+10+6) = 5.625 \quad (SFPF)$$

$$u_4 = \frac{1}{4}(0+u_5+u_1+u_7) = \frac{1}{4}(6+2.5+3) = 3.125 \quad (SFPF)$$

$$u_6 = \frac{1}{4}(u_5+u_3+14+u_9) = \frac{1}{4}(6+10+14+9.5) = 9.875 \quad (SFPF)$$

$$u_8 = \frac{1}{4}(u_7+u_5+u_9+6) = \frac{1}{4}(3+6+9.5+6) = 6.125 \quad (SFPF).$$

Now we proceed for iteration using always SFPF.

First iteration

$$u_1^{(1)} = \frac{1}{4}(0+1+u_2+u_4) = \frac{1}{4}(1+5.625+3.125) = 2.4375$$

$$u_2^{(1)} = \frac{1}{4}(u_1^{(1)}+u_3+4+u_5) =$$

$$\frac{1}{4}(2.4375+10+4+6) = 5.6094$$

$$u_3^{(1)} = \frac{1}{4}\left(u_2^{(1)} + 9 + 15 + u_6\right) = \frac{1}{4}\left(5.6094 + 9 + 15 + 9.875\right)$$
$$= 9.8711.$$

$$u_4^{(1)} = \frac{1}{4}\left(0 + u_1^{(1)} + u_5 + u_7\right) = \frac{1}{4}\left(2.4375 + 6 + 3\right) = 2.8594$$

$u_5^{(1)} = 6.1172$ $\qquad$ $u_8^{(1)} = 6.153$

$u_6^{(1)} = 9.8721$ $\qquad$ $u_9^{(1)} = 9.5063.$

$u_7^{(1)} = 2.9948$

Second iteration :-

$$u_1^{(2)} = \frac{1}{4}\left(0+1 + u_2^{(1)} + u_4^{(1)}\right) = \frac{1}{4}\left(1 + 5.6094 + 2.8594\right) = 2.3672$$

$$u_2^{(2)} = \frac{1}{4}\left(u_1^{(2)} + u_3^{(1)} + 4 + u_5^{(1)}\right) = \frac{1}{4}\left(2.3672 + 9.8711 + 4 + 6.1172\right) = 5.5888$$

$$u_3^{(2)} = \frac{1}{4}\left(u_2^{(2)} + 9 + 15 + u_6^{(1)}\right) = \frac{1}{4}\left(5.5888 + 9 + 15 + 9.8721\right)$$
$$= 9.8652$$

$$u_4^{(2)} = \frac{1}{4}\left(0 + u_1^{(2)} + u_5^{(1)} + u_7^{(1)}\right) = \frac{1}{4}\left(2.3672 + 6.1172 + 2.9948\right) = 2.8698$$

$u_5^{(2)} = 6.1209$ $\qquad$ $u_8^{(2)} = 6.1582$

$u_6^{(2)} = 9.8731$ $\qquad$ $u_9^{(2)} = 9.5078.$

$u_7^{(2)} = 3.0057$

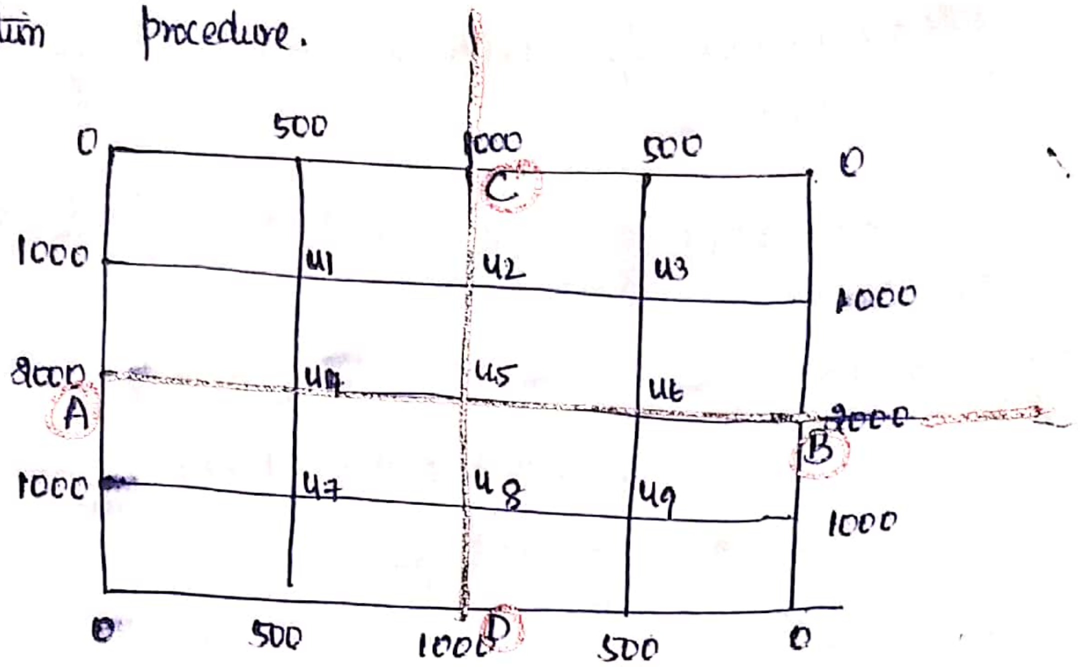
$\therefore$ we conclude correct to one decimal places, we get

$\boxed{u_1 = 2.4}$ $\qquad$ $u_4 = 2.9$ $\qquad$ $u_7 = 3.00$

$u_2 = 5.6$ $\qquad$ $u_5 = 6.1$ $\qquad$ $u_8 = 6.2$

$u_3 = 9.9$ $\qquad$ $u_6 = 9.9$ $\qquad$ $u_9 = 9.5$

4) Solve the equation $\nabla^2 u = 0$ for the following mesh with boundary values as shown, using Leibmann's iteration procedure.



Soln:- Take the central horizontal & vertical lines as AB & CD.

Let $u_1, u_2, \cdots u_9$ be the values of u at the interior grid points of the mesh.

The values of u on the boundary are symmetrical w.r to the lines AB & CD

$\therefore$ Along a line AB, $u_1 = u_7$, $u_2 = u_8$, $u_3 = u_9$

Along line CD, $u_1 = u_3$, $u_4 = u_6$, $u_7 = u_9$

$\therefore$ $u_1 = u_7 = u_9 = u_3$; $u_2 = u_8$; $u_4 = u_6$ &

u_5 is not equal to any value.

$\therefore$ It is enough if we find u_1, u_2, u_4, u_5

Rough values :

$u_5 = \frac{1}{4}(2000 + 2000 + 1000 + 1000) = 1500$ (SFPF)

$u_1 = \frac{1}{4}[0 + 1000 + 2000 + u_5) = \frac{1}{4}(1000 + 2000 + 1500)$
$= 1125$ (DFPF)

$u_2 = \frac{1}{4}(u_1 + 1000 + u_3 + u_5) = \frac{1}{4}(1125 + 1000 + 1125 + 1500) = 1187.5$
(SFPF) $(\because u_1 = u_3)$

$u_4 = \frac{1}{4}(2000 + u_1 + u_5 + u_7) =$
$\frac{1}{4}(2000 + 1500 + 1125 + 1125) =$
1437.5 $(\because u_1 = u_7)$
(SFPF)

Hereafter, we use only SFPF.

Ans:

$\therefore$ $u_1 = 937.6$ $u_2 = 1000.1$ $u_3 = 1250.1$

$u_4 = 1125.1$