Parabolic equations

(ii) Crank - Nicholson Difference Method.

Consider the one dimensional heat equation

$$u_{xx} = a u_t \longrightarrow (1)$$

To solve (1) we replace the partial derivatives by difference quotient.

At $u_{i,j}$, we have the following finite difference approximation for u_{xx} on the j^{th} row,

$$u_{xx} \simeq \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$|||^{ly}$ at $u_{i,j+1}$, we have

$$u_{xx} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}$$

Averaging these two approximation we get

$$u_{xx} \simeq \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{2h^2} \longrightarrow (2)$$

For u_t, we use the forward difference approximation

$$u_t \simeq \frac{u_{i,j+1} - u_{i,j}}{k} \longrightarrow (3)$$

Using (2) & (3) in (1), we get

$$\boxed{\begin{aligned}\frac{1}{2}\lambda u_{i+1,j+1} + \frac{1}{2}\lambda u_{i-1,j+1} - (\lambda+1)u_{i,j+1} &= \\ -\frac{1}{2}\lambda u_{i+1,j} - \frac{1}{2}\lambda u_{i-1,j} + (\lambda-1)u_{i,j} \\ \text{where } \lambda = \frac{k}{ah^2}.\end{aligned}} \quad (*)$$

This eqn can be rewritten as

$$\lambda(u_{i+1,j+1} + u_{i-1,j+1}) - 2(\lambda+1)u_{i,j+1} = 2(\lambda-1)u_{i,j} - \lambda(u_{i+1,j} + u_{i-1,j})$$

The equation (*) is called Crank-Nicholson differ… Scheme or method.

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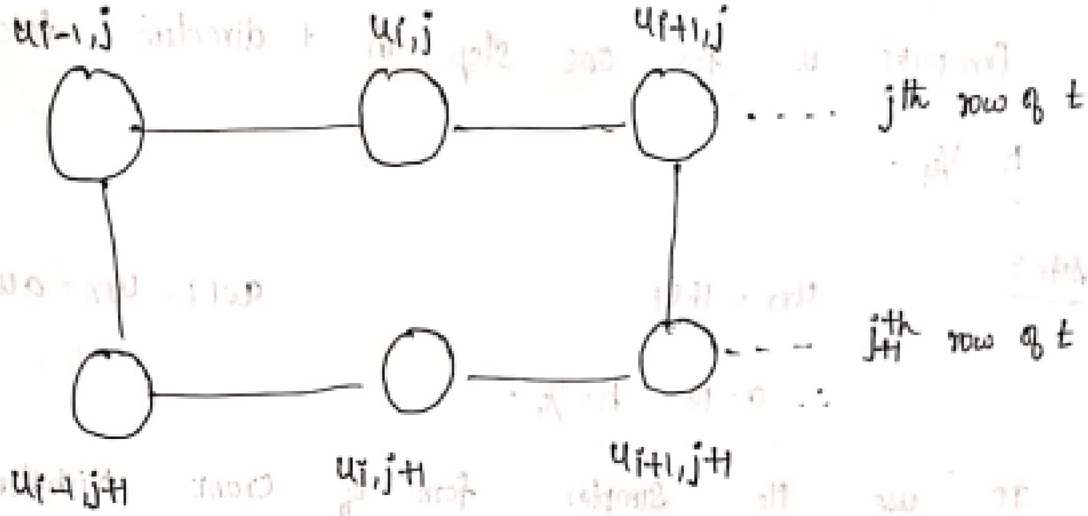
Note :

1. The above formula is shown digramatically as follows.

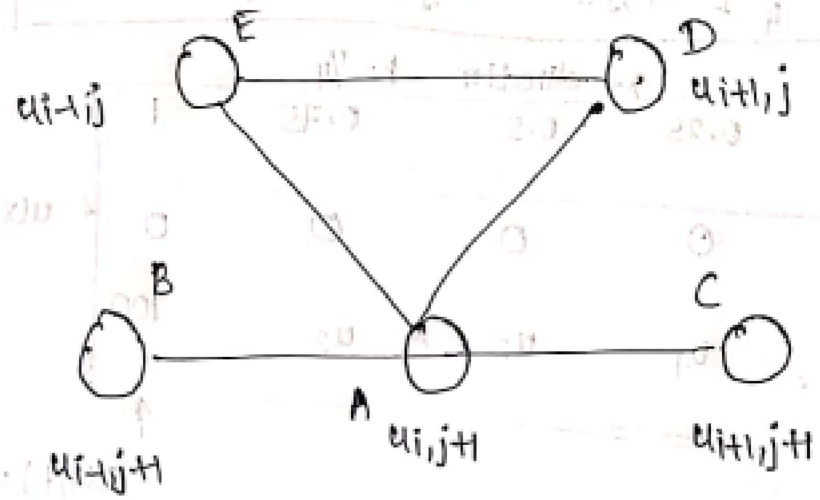


2. Setting $\lambda = 1$, ie $\boxed{k = ah^2}$

The crank - Nicholsom formula reduces to

$$\boxed{u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}\right]}$$

It is represented diagrametically as follows.



The value of u at A = average of the values at B, C, D, E.

1. Using Crank-Nicholson's scheme, solve

$u_{xx} = 16u_t$, $0<x<1$, $t>0$ given

Initial Cond. B.C

$u(x,0)=0$, $u(0,t)=0$, $u(1,t)=100t$.

Compute u for one step in t direction taking

$h = 1/4$.

Soln: $u_{xx} = 16u_t$ w.k.t. $u_{xx} = au_t$

$\therefore a=16$, $h=1/4$.

To use the simpler form of Crank-Nicholson scheme, we choose k in such way that $k=ah^2$.

i.e, $k = 16(1/4)^2 = 1$

$\therefore k=1$

The Crank Nicholson scheme is

$$\boxed{u_{i,j+1} = \frac{1}{4}\left[u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j} + u_{i-1,j}\right]} \longrightarrow (1)$$

x- direction $h=1/4$

j \ i	0	0.25	0.5	0.75	1
0	0	0	0	0	0
1	0	u_1	u_2	u_3	100

← $u(x,0)=0$

t direc. k=1

$u(1,t)=100t$

$u(0,t)=0$.

To find u_1, u_2, u_3 we use (1) at these points.

$u_1 = \frac{1}{4}(0+0+0+u_2)$ $\therefore$ $u_1 = \frac{1}{4}u_2 \longrightarrow (2)$

$u_2 = \frac{1}{4}(0+0+u_1+u_3)$ $\therefore$ $u_2 = \frac{1}{4}(u_1+u_3) \longrightarrow (3)$

$u_3 = \frac{1}{4}(0+0+u_2+100)$ $\therefore$ $u_3 = \frac{1}{4}(u_2+100) \longrightarrow (4)$

Sub. u_1, u_3 values in (3).

$$u_2 = \frac{1}{4}\left[\frac{1}{4}(u_2) + \frac{1}{4}(u_2+100)\right]$$

$$= \frac{1}{16}u_2 + \frac{1}{16}u_2 + \frac{25}{4}$$

$$u_2 = \frac{1}{8}u_2 + \frac{25}{4} \Rightarrow u_2 = \frac{50}{7} = 7.1429.$$

$\therefore$ $u_1 = 1.7857$ and $u_3 = 26.7857$.

$\therefore$ The values are $\boxed{u_1 = 1.7857}$ $\boxed{u_2 = 7.1429}$ $\boxed{u_3 = 26.7857}$.

2) ~~To~~ Solve by Crank-Nicholsom method the equation $u_{xx} = u_t$ subject to $u(x,0)=0$, $u(0,t)=0$ & $u(1,t)=t$, for two time steps.

Soln:- $u_{xx} = u_t$ (given)

WKT $u_{xx} = a u_t$ $\therefore$ $a=1$

x ranges from 0 to 1. Take $h = \frac{1}{4}$.

~~To~~ To use the simpler form of Crank-Nicholsom scheme, we choose k such that $\boxed{k = ah^2}$.

$\therefore$ $k = (1)(\frac{1}{4})^2 = \frac{1}{16}$ $\therefore$ $k = \frac{1}{16}$.

⑥

The Crank-Nicholson scheme

$$u_{i,j+1} = \frac{1}{4}\left[u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j} + u_{i+1,j} \right] \longrightarrow (1)$$

x – direction $h = 1/4$

t direc. $k = \frac{1}{16}$

$j \backslash i$	0	0.25	0.5	0.75	1
0	0	0	0	0	0
$\frac{1}{16}$	0	u_1	u_2	u_3	$\frac{1}{16}$
$\frac{2}{16}$	0	u_4	u_5	u_6.	$\frac{2}{16}$

$\leftarrow u(x,0) = 0$

$u(0,t) = 0$ $u(1,t) = t$.

To find u_1, u_2, u_3 we use (1) at these points.

$$u_1 = \frac{1}{4}(0+0+0+u_2) \quad \therefore u_1 = \tfrac{1}{4}u_2 \quad (2)$$

$$u_2 = \frac{1}{4}(0+0+u_1+u_3) \quad \therefore u_2 = \tfrac{1}{4}(u_1+u_3) \quad (3)$$

$$u_3 = \frac{1}{4}(0+0+u_2+1/16) \quad \therefore u_3 = \tfrac{1}{4}(u_2+1/16) \quad (4)$$

Sub u_1 & u_3 in eqn (3) ∴

$$\therefore u_2 = \frac{1}{4}\left[\frac{1}{4}(u_2) + \frac{1}{4}(u_2 + 1/16) \right]$$

$$= \frac{1}{4}\left[\frac{u_2}{4} + \frac{u_2}{4} + \frac{1}{64} \right]$$

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$$= \frac{1}{4}\left[\frac{2u_2}{4} + \frac{1}{64}\right] = \frac{u_2}{8} + \frac{1}{256}.$$

$$u_2 - \frac{u_2}{8} = \frac{1}{256}.$$

$$\frac{7u_2}{8} = \frac{1}{256} \Rightarrow u_2 = \frac{1}{224} = 0.0045$$

$$\therefore u_1 = \frac{1}{896} = 0.0011, \quad u_3 = 0.0168.$$

Similarly, u_4, u_5, u_6 can be got again getting 3 equations in 3 unknowns u_4, u_5, u_6.

$$\therefore u_4 = 0.005899$$

$$u_5 = 0.01913$$

$$u_6 = 0.05277$$

(5) Using Crank-Nicholson method, solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

subject to $u(x,0) = 0$, $u(0,t) = 0$ & $u(1,t) = t$.

taking $h = 1/4$ & $k = \frac{1}{8}$.

Soln:- Given $u_t = u_{xx}$.

WKT $u_{xx} = a u_t$ $\quad \boxed{\therefore a = 1}$

If $h = 1/4$, $k = \frac{1}{8}$, $\lambda = \frac{k}{ah^2} = \frac{1/8}{(1)(1/4)^2} = 2$

Using in general Crank - Nicholson formula, (8)

$$\boxed{\lambda(u_{i+1,j+1} + u_{i-1,j+1}) - 2(\lambda+1)u_{i,j+1} = 2(\lambda-1)u_{i,j} - \lambda(u_{i+1,j} + u_{i-1,j})}$$

Sub $\lambda = 2$, we get

$$\boxed{u_{i+1,j+1} + u_{i-1,j+1} - 3u_{i,j+1} = u_{i,j} - u_{i+1,j} - u_{i-1,j}} \longrightarrow (*)$$

x — direction

	j \ i	0	0.25	0.5	0.75	1	
t	0	0	0	0	0	0	← $u(x,0)=0$
direc. $t = 1/8$	$\frac{1}{8}$	0	u_1	u_2	u_3	$\frac{1}{8}$	
		↑ $u(0,t)=0$				↑ $u(1,t)=t$	

To find u_1, u_2, u_3 we use (*) at these points

$\therefore\ 0 + u_2 - 3u_1 = 0 \qquad \therefore u_2 = 3u_1 \quad (1)$

$u_1 + u_3 - 3u_2 = 0 \qquad \therefore 3u_2 = u_1 + u_3 \quad (2)$

$u_2 + \frac{1}{8} - 3u_3 = 0 \qquad \therefore 3u_3 = u_2 + \frac{1}{8} \quad (3)$

Adding (1) & (3), $3(u_1 + u_3) = 2u_2 + \frac{1}{8} \longrightarrow (4)$

From (2) & (4), $3(3u_2) = 2u_2 + \frac{1}{8}$

$9u_2 = 2u_2 + 1/8 \Rightarrow 7u_2 = \frac{1}{8} \Rightarrow u_2 = 1/56$

$\therefore$ $u_2 = \frac{1}{56} = 0.01786$

Sub u_2 in eqn (1), $u_1 = \frac{1}{168} = 0.00595$

Sub u_2 in eqn (3) $u_3 = 8/168 = 0.04762$.