UNIT V (21MAB206T)

ACADEMIC YEAR 2023-2024 (ODD SEMESTER)

NUMERICAL METHODS AND ANALYISIS

TOPICS

- Classification of Second-Order Partial Differential Equations
- Elliptic Equations-Finite Difference Scheme
- Standard Five Point Finite Difference Formula
- Diagonal Five Point Finite Difference Formula
- Liebman’s Iterative Process
- Solution of Laplace Equations by Liebman’s Iterative process
- Solution of Poisson Equation
- One Dimensional Parabolic Equation– Bender-Schmidt Scheme
- Crank-Nicholson scheme

Classification of Second-Order Partial Differential Equations

Classification of Second-Order Partial Differential Equations

- The most general linear partial differential equation of second order can be written as

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

where A, B, C, D, E, F and G are functions of x, y or are real constants.

The partial differential equation is called a

Elliptic equation	if $B^2 - AC < 0$
Parabolic equation	if $B^2 - AC = 0$
Hyperbolic equation	if $B^2 - AC > 0$.

The simplest examples of the above equations are the following:

Parabolic equation:	$u_t = c^2 u_{xx}$,	(One dimensional heat equation).
Hyperbolic equation:	$u_{tt} = c^2 u_{xx}$,	(One dimensional wave equation).
Elliptic equation:	$u_{xx} + u_{yy} = 0$,	(Two dimensional Laplace equation).

- Note: The same differential equation may be elliptic in one region, parabolic in another and hyperbolic in some other region.
- For example:
- $xu_{xx} + u_{yy} = 0$ is elliptic if $x > 0$, hyperbolic if $x < 0$ and parabolic if $x = 0$.

Examples

- Example 1: Classify the following equation
- $x^2 f_{xx} + (1 - y^2) f_{yy} = 0.$
- Solution: Here $A = x^2$, $B = 0$, $C = 1 - y^2$
- $B^2 - 4AC = -4x^2(1 - y^2) = 4x^2(y^2 - 1)$
- x^2 is positive for all x except $x = 0$.
- If $-1 < y < 1,\ y^2 - 1$ is negative.
- $\therefore B^2 - 4AC$ is negative if $-1 < y < 1$ and $x \neq 0$
- $\therefore$ The equation is elliptic in the region $-\infty < x < \infty, -1 < y < 1$.
- $\therefore$ The equation is hyperbolic in the region $-\infty < x < \infty$, $\mathrm{y} < -1 \cup y > 1$.
- $\therefore$ The equation is parabolic for $x = 0$ and for all y or for $x, y = \pm 1$.

- Example 2: Classify the following equation
- $u_{xx} + 4u_{xy} + (x^2 + 4y^2)u_{yy} = \sin(x + y)$
- Solution: Here $A = 1$, $B = 4$, $C = x^2 + 4y^2$
- $B^2 - 4AC = 16 - 4(x^2 + 4y^2) = 4(4 - x^2 - 4y^2)$
- The equation is elliptic if $4 - x^2 - 4y^2 < 0$
- $\Rightarrow x^2 + 4y^2 > 4$
- $\Rightarrow \frac{x^2}{4} + \frac{y^2}{1} > 1$
- $\therefore$ It is elliptic in the region outside the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$
- $\therefore$ It is hyperbolic in the region inside the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$
- $\therefore$ It is parabolic on the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$

Elliptic Equations-Finite Difference Scheme

Elliptic Equations-Finite Difference Scheme

- Some famous elliptic equation are

(a) *Laplace's equation:* $u_{xx} + u_{yy} = \nabla^2 u = 0$, with $u(x, y)$ prescribed on the boundary, that is, $u(x, y) = f(x, y)$ on the boundary.

(b) *Poisson's equation:* $u_{xx} + u_{yy} = \nabla^2 u = G(x, y)$, with $u(x, y)$ prescribed on the boundary, that is, $u(x, y) = g(x, y)$ on the boundary.

Finite difference method We have a two dimensional domain $(x, y) \in R$. We superimpose on this domain R, a rectangular network or mesh of lines with step lengths h and k respectively, parallel to the x- and y-axis. The mesh of lines is called a grid. The points of intersection of the mesh lines are called *nodes* or *grid points* or *mesh points*. The grid points are given by (x_i, y_j), (see Figs. 5.3 a, b), where the mesh lines are defined by

$$x_i = ih,\ i = 0, 1, 2, \ldots;\ y_j = jk,\ j = 0, 1, 2, \ldots$$

If $h = k$, then we have a uniform mesh. Denote the numerical solution at (x_i, y_j) by $u_{i,j}$.

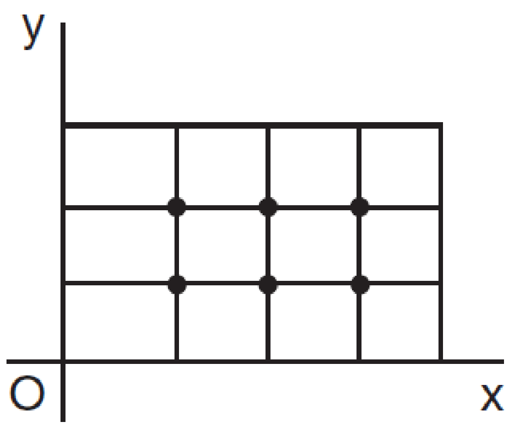


Fig. 5.3a. Nodes in a rectangle.

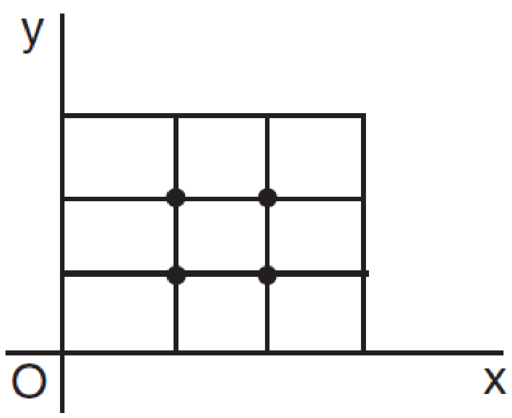


Fig. 5.3b. Nodes in a square.

At the nodes, the partial derivatives in the differential equation are replaced by suitable difference approximations. That is, the partial differential equation is approximated by a difference equation at each nodal point. This procedure is called *discretization* of the partial differential equation. We use the following central difference approximations.

$$(u_x)_{i,j} = \frac{1}{2h}\,(u_{i+1,j} - u_{i-1,j}), \qquad (u_y)_{i,j} = \frac{1}{2k}\,(u_{i,j+1} - u_{i,j-1}),$$

$$(u_{xx})_{i,j} = \frac{1}{h^2}\,(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}), \qquad (u_{yy})_{i,j} = \frac{1}{k^2}\,(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}).$$

Standard five-point formula

Solution of Laplace's equation We apply the Laplace's equation at the nodal point (i, j). Inserting the above approximations in the Laplace's equation, we obtain

$$(u_{xx})_{i,j} + (u_{yy})_{i,j} = \frac{1}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + \frac{1}{k^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) = 0 \tag{5.22}$$

or $$(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + p^2(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) = 0, \text{ where } p = h/k. \tag{5.23}$$

If $h = k$, that is, $p = 1$ (called uniform mesh spacing), we obtain the difference approximation as

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0 \tag{5.24}$$

This approximation is called the *standard five point formula*. We can write this formula as

$$u_{i,j} = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}). \tag{5.25}$$

We observe that $u_{i,j}$ is obtained as the mean of the values at the four neighbouring points in the x and y directions.

Remark 6 The nodes in the mesh are numbered in an orderly way. We number them from left to right and from top to bottom or from bottom to top. A typical numbering is given in Figs.5.5a, 5.5b.

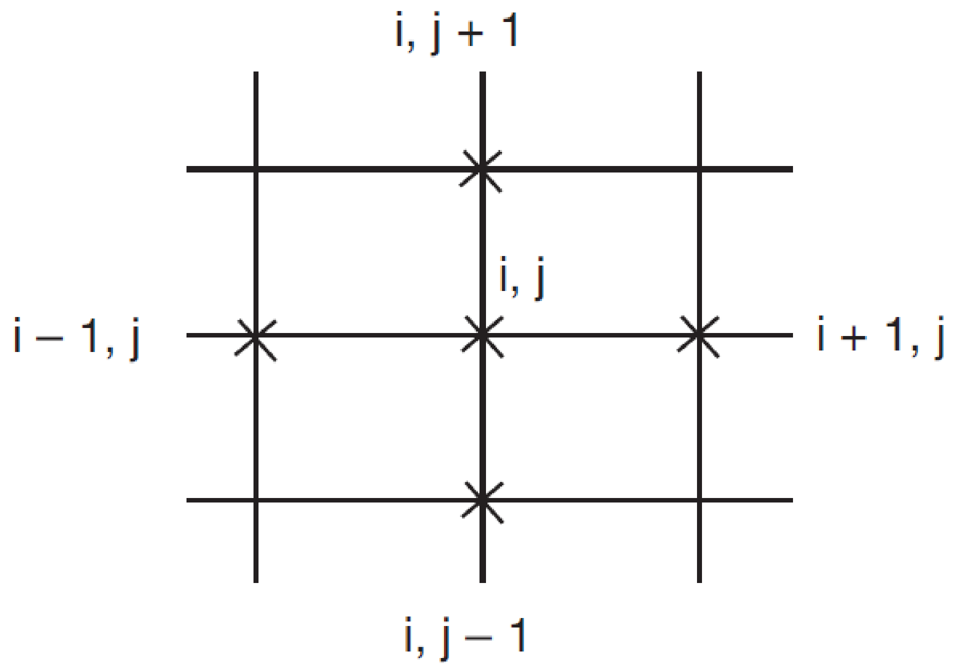


Fig. 5.4. Standard five point formula.

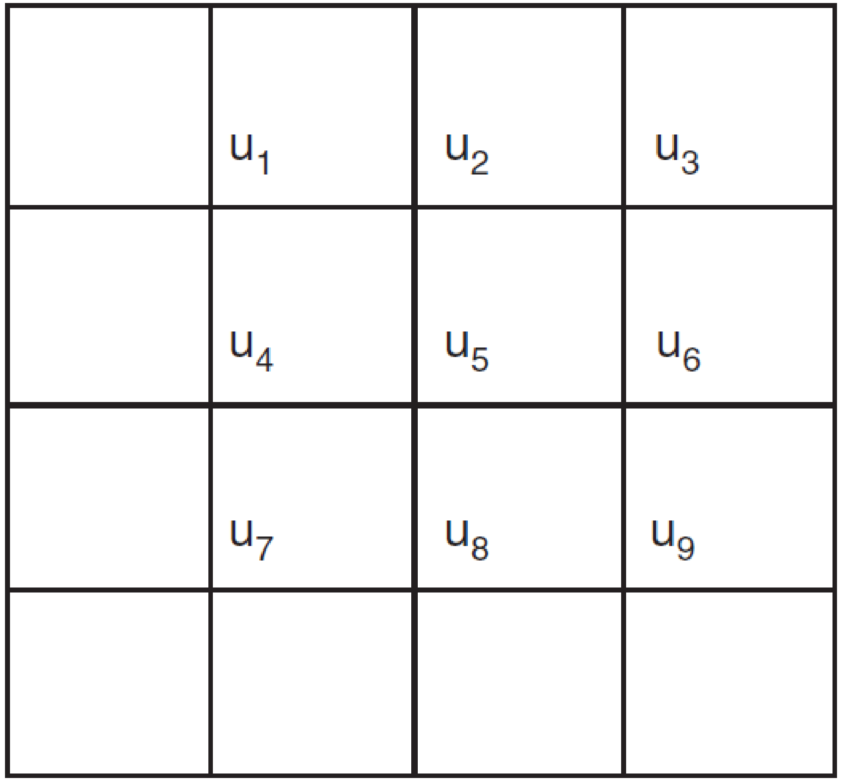


Fig. 5.5a. Numbering of nodes.

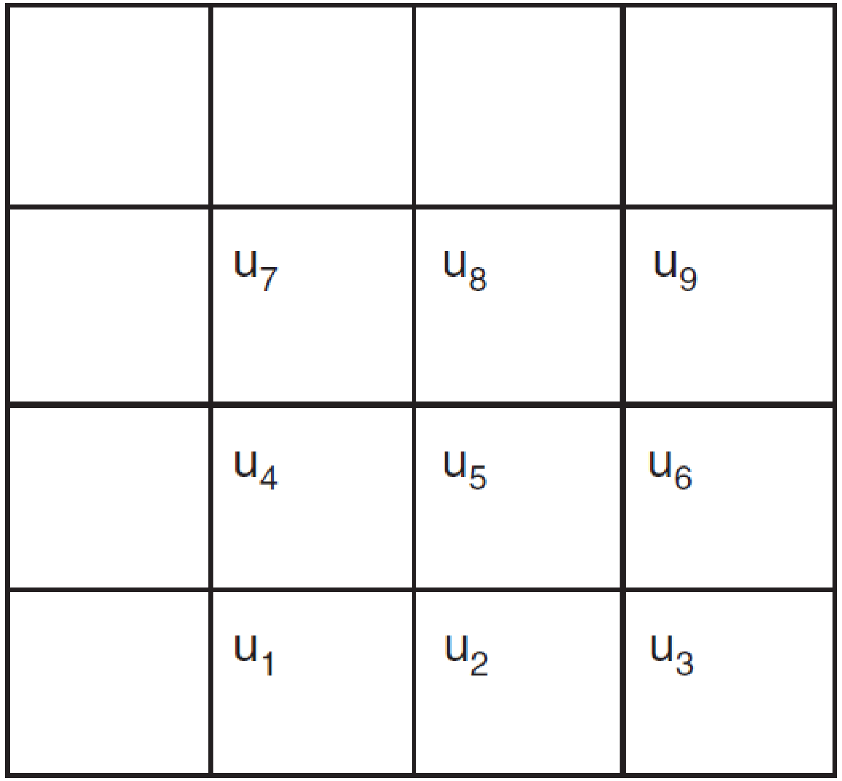


Fig. 5.5b. Numbering of nodes.

Diagonal five-point formula

- Diagonal five-point formula

$$u_{i,j} = \frac{1}{4}\left(u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i-1,j-1}\right).$$

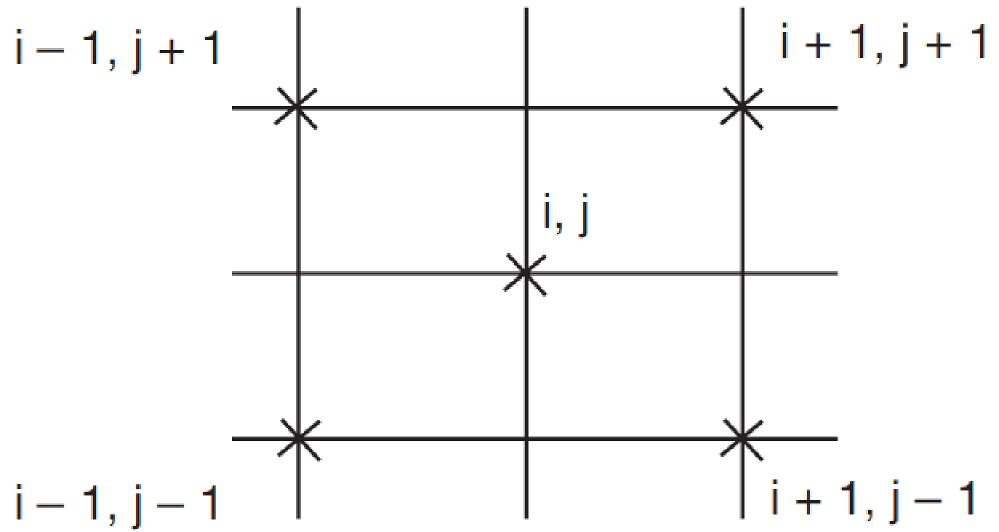
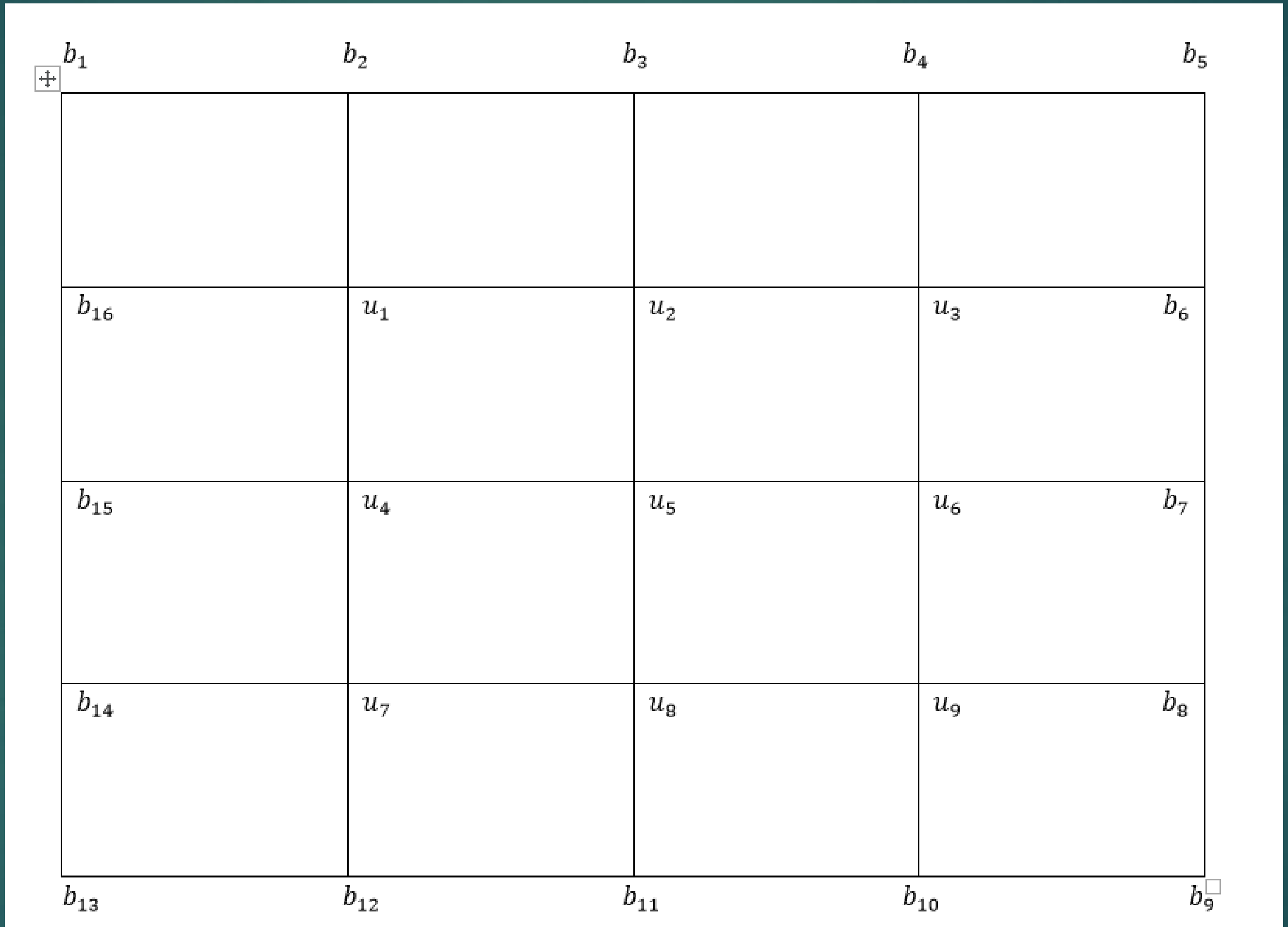


Fig. 5.6. Diagonal five point formula.

Liebman’s Iterative Process

Solution of Laplace's Equation by Liebman's Iterative Process

- To solve the Laplace's equation $u_{xx} + u_{yy} = 0$ in bounded square region R with a boundary C when the boundary values of u are given on the boundary (or at least at the grid points in the boundary).
- First we divide the square region into a network of sub-squares of side h. (See figure in next page.)
- The boundary values of u at the grid points are given and noted by $b_1, b_2, \dots, b_{16}$. The values of u at the interior lattice or grid points are assumed to be $u_1, u_2, \dots, u_9$.
- To start the iteration process, initially we find rough values at interior points and then we improve them by iterative process mostly using standard five point formula.


b_1 b_2 b_3 b_4 b_5
b_{16} u_1 u_2 u_3 b_6
b_{15} u_4 u_5 u_6 b_7
b_{14} u_7 u_8 u_9 b_8
b_{13} b_{12} b_{11} b_{10} b_9

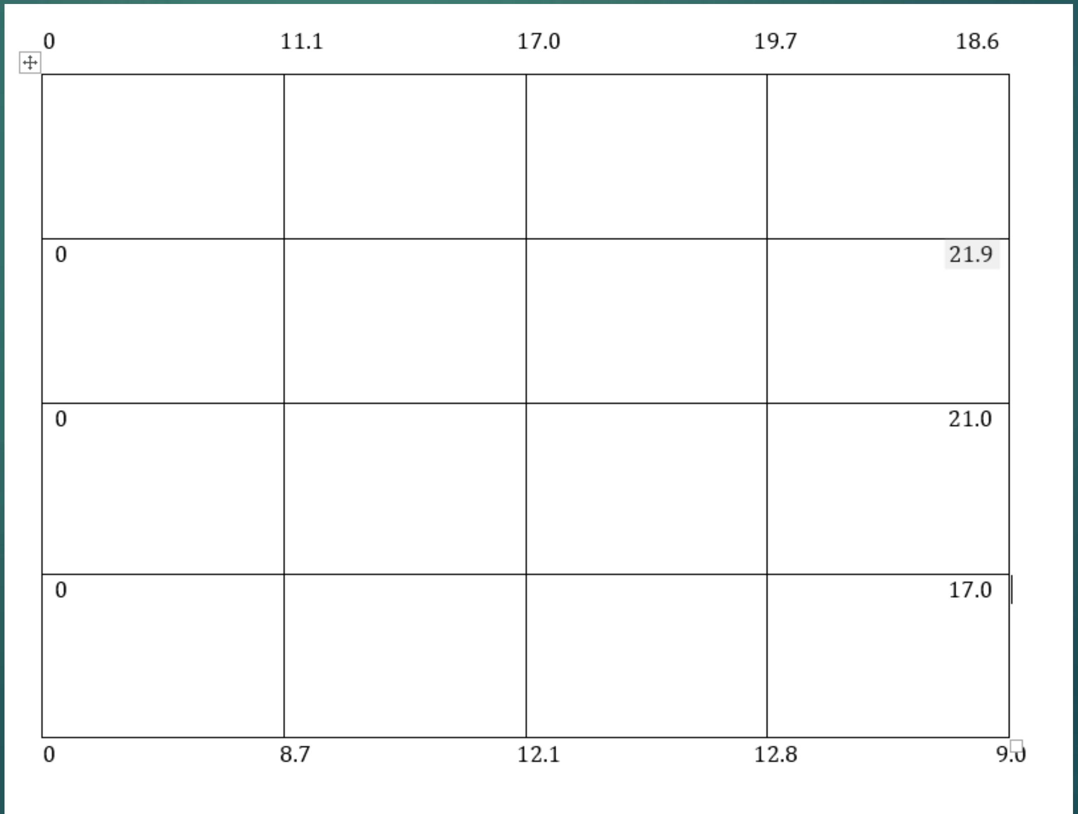
- Find u_5 first using standard five point formula(SFPF)
- $u_5 = \frac{1}{4}(b_3 + b_7 + b_{11} + b_{15})$
- After knowing u_5, we find u_1, u_3, u_7, u_9 by using Diagonal five point formula(DFPF)
- $u_1 = \frac{1}{4}(b_3 + b_1 + b_{15} + u_5)$
- $u_3 = \frac{1}{4}(b_3 + b_7 + b_5 + u_5)$
- $u_7 = \frac{1}{4}(b_{13} + b_{11} + b_{15} + u_5)$
- $u_9 = \frac{1}{4}(b_7 + b_{11} + b_9 + u_5)$

- The remaining 4 values u_2, u_4, u_6, u_8 can be find by using Standard five point formula(SFPF)
- $u_2 = \frac{1}{4}(u_1 + u_3 + u_5 + b_3)$
- $u_4 = \frac{1}{4}(u_1 + u_7 + u_5 + b_{15})$
- $u_6 = \frac{1}{4}(u_9 + u_3 + u_5 + b_7)$
- $u_8 = \frac{1}{4}(u_7 + u_9 + u_5 + b_{11})$

- After knowing all the boundary values of u and rough values of u at every grid point in the interior region R. Now, we iterate the process and improve the values of u with accuracy. Start with u_5 and proceed to get the values of $u_1, u_2, \dots, u_9$ always using SFPF, taking into account the latest available values of u to use in the formula. The iterative formula is
- $u_{i,j}^{(n+1)} = \frac{1}{4}\left(u_{i+1,j}^{(n)} u_{i-1,j}^{(n+1)} + u_{i,j-1}^{(n)} u_{i,j+1}^{(n+1)}\right)$
- Where the superscript of u denotes the iteration number.
- The process is stopped once we get the values with desired accuracy.

Examples

- ***Example1***: Find by the Liebmann's method the values at the interior points of a square region of the harmonic function u whose boundary values are as shown in the following figure.



Solution: Since u is harmonic, it satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in the square.

Let the interior values of u at the 9 grid points be $u_1, u_2, \ldots, u_9$. We will find the rough values of u at the interior mesh points as explained previously. We will then proceed to refine them.

0	11.1	17.0	19.7	18.6
0	u_1	u_2	u_3	21.9
0	u_4	u_5	u_6	21.0
0	u_7	u_8	u_9	17.0
0	8.7	12.1	12.8	9.0

Finding rough values:

$$u_5 = \frac{1}{4}(0 + 17.0 + 21.0 + 12.1) = 12.5 \qquad \text{(SFPF)}$$

$$u_1 = \frac{1}{4}(0 + 12.5 + 17.0 + 0) = 7.4 \qquad \text{(DFPF)}$$

$$u_3 = \frac{1}{4}(18.6 + 12.5 + 17.0 + 1.0) = 17.3 \qquad \text{(DFPF)}$$

$$u_7 = \frac{1}{4}(0 + 12.5 + 12.1 + 0) = 6.2 \qquad \text{(DFPF)}$$

$$u_9 = \frac{1}{4}(9.0 + 12.5 + 12.1 + 21.0) = 13.7 \qquad \text{(DFPF)}$$

$$u_2 = \frac{1}{4}(7.4 + 17.0 + 17.3 + 12.5) = 13.6 \qquad \text{(SFPF)}$$

$$u_4 = \frac{1}{4}(7.4 + 6.2 + 0 + 12.5) = 6.5 \qquad \text{(SFPF)}$$

$$u_6 = \frac{1}{4}(13.7 + 21.0 + 17.3 + 12.5) = 16.1 \qquad \text{(SFPF)}$$

$$u_8 = \frac{1}{4}(13.7 + 12.1 + 6.2 + 12.5) = 11.1 \qquad \text{(SFPF)}$$

Now, we have got the rough values at all interior grid points and already we posses the boundary values at the lattices points. We will now the improve the values by using always SPFP.

First iteration: We obtain all values by SFPF

$$u_1^{(1)} = \frac{1}{4}(0 + 11.1 + u_2 + u_4) = \frac{1}{4}(0 + 11.1 + 13.6 + 6.5) = 7.8$$

$$u_2^{(1)} = \frac{1}{4}(7.8 + 17.0 + 17.3 + 12.5) = 13.7$$

$$u_3^{(1)} = \frac{1}{4}(13.7 + 21.9 + 19.7 + 16.1) = 17.9$$

$$u_4^{(1)} = \frac{1}{4}(0 + 12.5 + 7.8 + 6.2) = 6.6$$

$$u_5^{(1)} = \frac{1}{4}(13.7 + 11.1 + 6.6 + 16.1) = 11.9$$

$$u_6^{(1)} = \frac{1}{4}(17.9 + 13.7 + 11.9 + 21.0) = 16.1$$

$$u_7^{(1)} = \frac{1}{4}(6.6 + 8.7 + 0 + 11.1) = 6.6$$

$$u_8^{(1)} = \frac{1}{4}(11.9 + 12.1 + 6.6 + 13.7) = 11.1$$

$$u_9^{(1)} = \frac{1}{4}(16.1 + 12.8 + 17.0 + 11.1) = 14.3$$

Second iteration:

$$u_1^{(2)} = \frac{1}{4}(0 + 11.1 + 13.7 + 6.6) = 7.9$$

$$u_2^{(2)} = \frac{1}{4}(7.9 + 17.0 + 17.9 + 11.9) = 13.7$$

$$u_3^{(2)} = \frac{1}{4}(13.7 + 21.9 + 19.7 + 16.1) = 17.9$$

$$u_4^{(2)} = \frac{1}{4}(0 + 11.9 + 7.9 + 6.6) = 6.6$$

$$u_5^{(2)} = \frac{1}{4}(13.7 + 11.1 + 6.6 + 16.1) = 11.9$$

$$u_6^{(2)} = \frac{1}{4}(17.9 + 14.3 + 11.9 + 21.0) = 16.3$$

$$u_7^{(2)} = \frac{1}{4}(6.6 + 8.7 + 0 + 11.1) = 6.6$$

$$u_8^{(2)} = \frac{1}{4}(11.9 + 12.1 + 6.6 + 14.3) = 11.2$$

$$u_9^{(2)} = \frac{1}{4}(16.3 + 12.8 + 17.0 + 11.2) = 14.3$$

Third iteration:

$$u_1^{(3)} = \frac{1}{4}(0 + 11.1 + 13.7 + 6.6) = 7.9$$

$$u_2^{(3)} = \frac{1}{4}(7.9 + 17.0 + 17.9 + 11.9) = 13.7$$

$$u_3^{(3)} = \frac{1}{4}(13.7 + 21.9 + 19.7 + 16.3) = 17.9$$

$$u_4^{(3)} = \frac{1}{4}(0 + 11.9 + 7.9 + 6.6) = 6.6$$

$$u_5^{(3)} = \frac{1}{4}(13.7 + 11.2 + 6.6 + 16.3) = 11.9$$

$$u_6^{(3)} = \frac{1}{4}(17.9 + 14.3 + 11.9 + 21.0) = 16.3$$

$$u_7^{(3)} = \frac{1}{4}(6.6 + 8.7 + 0 + 11.2) = 6.6$$

$$u_8^{(3)} = \frac{1}{4}(11.9 + 12.1 + 6.6 + 14.3) = 11.2$$

$$u_9^{(3)} = \frac{1}{4}(16.3 + 12.8 + 17.0 + 11.2) = 14.3$$

Now, all the 9 values of u of the third iteration are same as the corresponding values of the second iteration. Hence we stop the procedure and accept

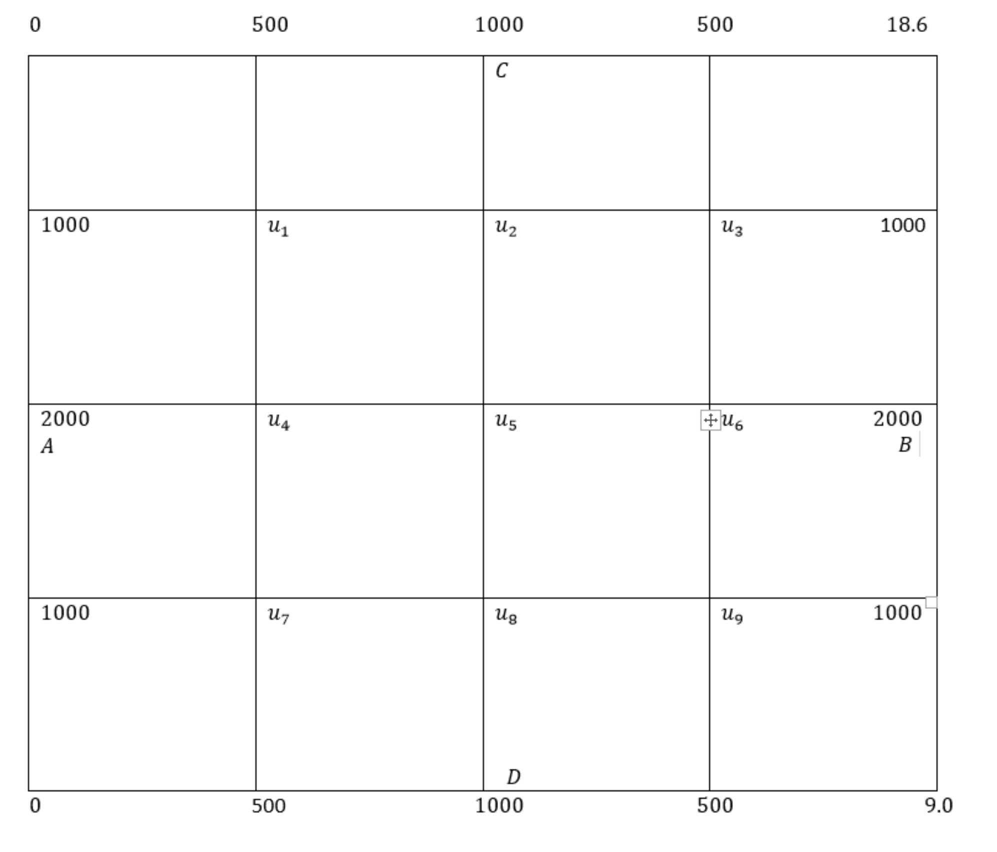
$$u_1 = 7.9, u_2 = 13.7, u_3 = 17.9, u_4 = 6.6,$$
$$u_5 = 11.9, u_6 = 16.3, u_7 = 6.6, u_8 = 11.2, u_9 = 14.3$$

Instead of working out so elaborately, we can write the values of u′s at each grid point and work out the scheme easily. The values of u′s are shown in next slide:

0	11.1	17.0	19.7	18.6
0	u_1 7.4 7.8 7.9 7.9	u_2 13.6 13.7 13.7 13.7	u_3 17.3 17.9 17.9 17.9	21.9
0	u_4 6.5 6.6 6.6 6.6	u_5 12.5 11.9 11.9 11.9	u_6 16.1 16.1 16.3 16.3	21.0
0	u_7 6.2 6.6 6.6 6.6	u_8 11.1 11.1 11.2 11.2	u_9 13.7 14.3 14.3 14.3	17.0
0	8.7	12.1	12.8	9.0

Examples

- ***Example2***: Solve the equation $\nabla^2 u = 0$ for the following mesh, with boundary values as shown, using Leibmann's iteration procedure.



Solution: Take the central horizontal and vertical lines as AB and CD.

Let $u_1, u_2, \dots, u_9$ be the values of u at the interior grid points of the mesh.

The values of u on the boundary are symmetrical w.r.t. the lines AB and CD.

Hence, the values of u inside the mesh will also be symmetrical about AB and CD.

$\therefore u_1 = u_3 = u_7 = u_9; u_2 = u_8; u_4 = u_6$ and u_5 is not equal at any value.

$\therefore$ It is enough to find values at u_1, u_2, u_4, u_5.

It is shown in the following figure.

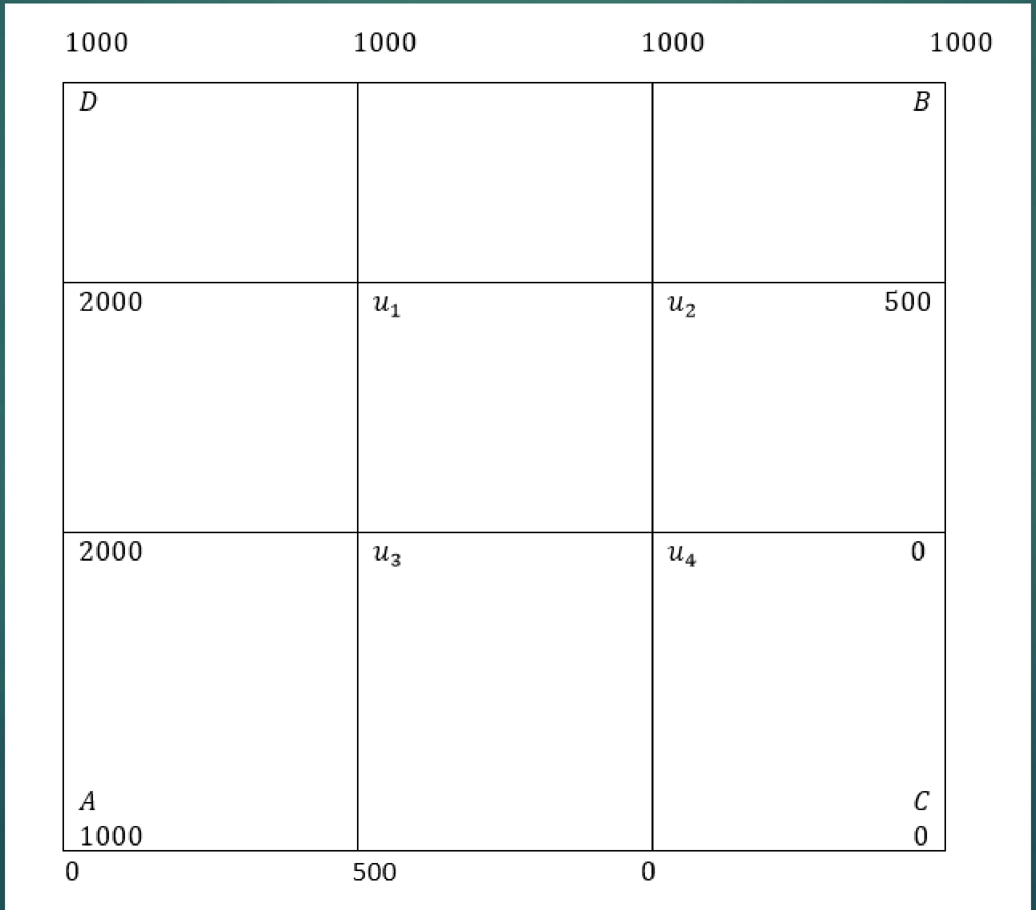
0 500 1000 500 18.6

1000	u_1 1125 940.43 1031.25 939.1 984.38 938.3 960.94 937.7 949.22 937.6 943.36 937.6	u_2 1187.5 1002.93 1093.75 1001.6 1046.88 1000.4 1023.44 1000.2 1011.72 1000.1 1005.86 1000.1	u_3 1000 1125 940.43 1031.25 939.1 984.38 938.3 960.94 937.7 949.22 937.6 943.36 937.6
2000	u_4 1437.5 1252.93 1343.75 1250.8 1296.88 1250.2 1273.44 1250.1 1261.72 1250.1 1255.86 1250.1	u_5 1500 1130.86 1312.5 1127.93 1218.75 1126.6 1171.88 1125.8 1148.44 1125.2 1136.72 1125.1	u_6 2000 1437.5 1252.93 1343.75 1250.8 1296.88 1250.2 1273.44 1250.1 1261.72 1250.1 1255.86 1250.1
1000	u_7 1125 940.43 1031.25 939.1 984.38 938.3 960.94 937.7 949.22 937.6 943.36 937.6	u_8 1187.5 1002.93 1093.75 1001.6 1046.88 1000.4 1023.44 1000.2 1011.72 1000.1 1005.86 1000.1	u_9 1000 1125 940.43 1031.25 939.1 984.38 938.3 960.94 937.7 949.22 937.6 943.36 937.6

0 500 1000 500 9.0

Examples

- ***Example3***: Evaluate the function $u(x, y)$ satisfying $\nabla^2 u = 0$ at the lattice points given the boundary values as follows.



Solution:

Method 1:

We have $4u_1 = 1000 + 2000 + u_3 + u_2 = 3000 + \mathrm{u}_2 + \mathrm{u}_3$ (1)

$4u_2 = 1500 + u_1 + u_4$ (2)

$4u_3 = 2500 + u_4 + u_1$ (3)

$4u_4 = u_2 + u_3$ (4)

i.e., $4u_1 - u_2 - u_3 = 3000$ (5)

$u_1 - 4u_2 + u_4 = -1500$ (6)

$u_1 - 4u_3 + u_4 = -2500$ (7)

$u_2 + u_3 - 4u_4 = 0$ (8)

We eliminate u_1 from (5) and (6) and (6) and (7)

$$15u_2 - u_3 - 4u_4 = 9000$$
$$4u_2 - 4u_3 = -1000$$

Eliminate u_4 from (8) and (9)

$$14u_2 - 2u_3 = 9000$$

From (10) and (11), $u_2 = 791.7$, $u_3 = 1041.7$

From (5), $u_1 = 1208.4$, $u_2 = 791.7$, $u_3 = 1041.7$ $u_4 = 458.4$.

Method 2:

Instead getting 4 equations in u_1, u_2, u_3, u_4 and solving them for $u's$, we can assume some values for u_4 (or any other u) and proceed iterative procedure; we can take $u_4 = 0$ and proceed or we take a value of $u_4 = 400$ (Guess this by seeing the values of u on the vertical line through u_2, u_4).

Rough values:

$$u_1 = \frac{1}{4}(1000 + 2000 + 1000 + 400) = 1100 \qquad \text{(DFPF)}$$

$$u_2 = \frac{1}{4}(u_1 + u_4 + 1500) = 750 \qquad \text{(SFPF)}$$

$$u_3 = \frac{1}{4}(u_1 + u_4 + 2500) = 1000 \qquad \text{(SFPF)}$$

$$u_4 = \frac{1}{4}(u_2 + u_3) = 437.5 \qquad \text{(SFPF)}$$

First iteration

$$u_1^{(1)} = \frac{1}{4}(750 + 1000 + 3000) = 1187.5 \qquad \text{(SFPF)}$$

$$u_2^{(1)} = \frac{1}{4}(1187.5 + 437.5 + 1500) = 781.25 \qquad \text{(SFPF)}$$

$$u_3^{(1)} = \frac{1}{4}(1187.5 + 437.5 + 2500) = 1031.25 \qquad \text{(SFPF)}$$

$$u_4^{(1)} = \frac{1}{4}(781.25 + 1031.25) = 453.125 \qquad \text{(SFPF)}$$

Second iteration

$$u_1^{(2)} = \frac{1}{4}(781.25 + 1031.25 + 3000) = 1203.125 \qquad \text{(SFPF)}$$

$$u_2^{(2)} = \frac{1}{4}(1203.125 + 453.125 + 1500) = 789.1 \qquad \text{(SFPF)}$$

$$u_3^{(2)} = \frac{1}{4}(1203.125 + 453.125 + 2500) = 1039.1 \qquad \text{(SFPF)}$$

$$u_4^{(2)} = \frac{1}{4}(789.1 + 1039.1) = 457.1 \qquad \text{(SFPF)}$$

Third iteration

$$u_1^{(3)} = \frac{1}{4}(789.1 + 1039.1 + 3000) = 1207.1 \qquad \text{(SFPF)}$$

$$u_2^{(3)} = \frac{1}{4}(1207.1 + 457.1 + 1500) = 791.1 \qquad \text{(SFPF)}$$

$$u_3^{(3)} = \frac{1}{4}(1207.1 + 457.1 + 2500) = 1041.1 \qquad \text{(SFPF)}$$

$$u_4^{(3)} = \frac{1}{4}(791.1 + 1041.1) = 458.1 \qquad \text{(SFPF)}$$

Fourth iteration

$$u_1^{(4)} = \frac{1}{4}(791.1 + 1041.1 + 3000) = 1208.1 \qquad \text{(SFPF)}$$

$$u_2^{(4)} = \frac{1}{4}(1208.1 + 458.1 + 1500) = 791.6 \qquad \text{(SFPF)}$$

$$u_3^{(4)} = \frac{1}{4}(1208.1 + 458.1 + 2500) = 1041.6 \qquad \text{(SFPF)}$$

$$u_4^{(4)} = \frac{1}{4}(791.6 + 1041.6) = 458.3 \qquad \text{(SFPF)}$$

Fifth iteration

$$u_1^{(4)} = \frac{1}{4}(791.6 + 1041.6 + 3000) = 1208.3 \qquad \text{(SFPF)}$$

$$u_2^{(4)} = \frac{1}{4}(1208.3 + 458.3 + 1500) = 791.7 \qquad \text{(SFPF)}$$

$$u_3^{(4)} = \frac{1}{4}(1208.3 + 458.3 + 2500) = 1041.7 \qquad \text{(SFPF)}$$

$$u_4^{(4)} = \frac{1}{4}(791.7 + 1041.7) = 458.4 \qquad \text{(SFPF)}$$

In the next iteration, we get the same values upto 3 decimal places. So we stop here.

Hence $u_1 = 1208.3$, $u_2 = 791.7$ $u_3 = 1041.7$ $u_4 = 458.4$

NOTE: If we start with $u_4 = 0$, then we have to do more iteration, so to avoid excess labour, judiciously we assume the value.

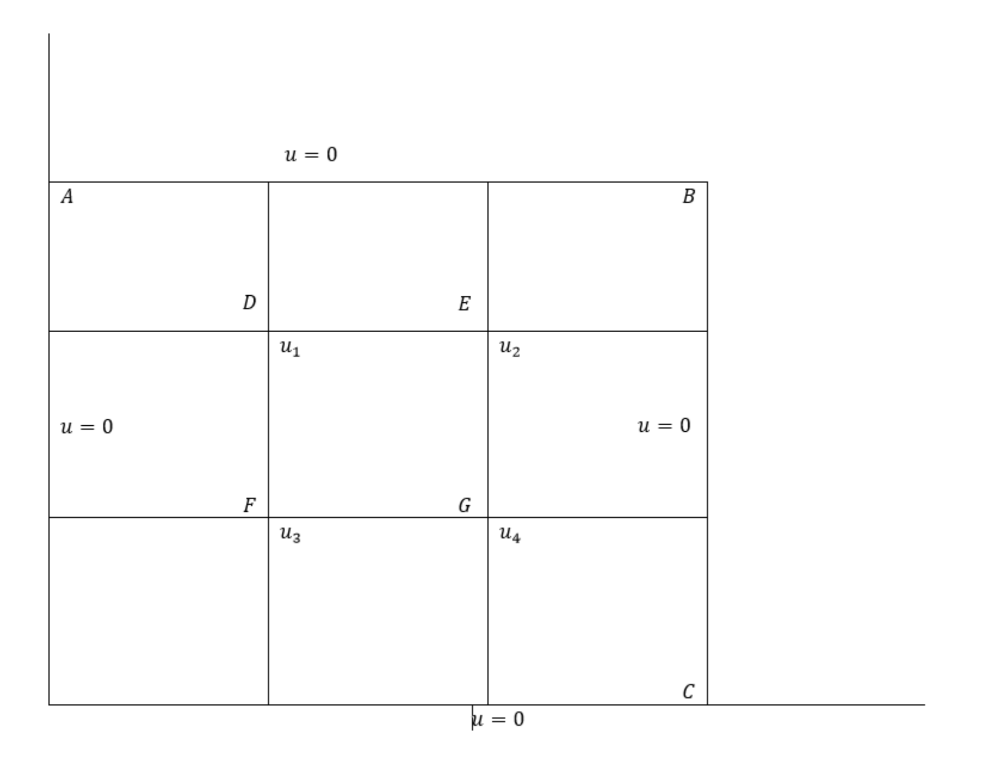
Solution of Poisson Equation

Solution of Poisson Equation

- An equation of the form $\nabla^2 u = f(x, y)$
- i.e., $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ (1)
- Is called as Poisson's equation where $f(x, y)$ is a function of x and y only.
- We will solve the above equation numerically at the points of the square mesh, replacing the derivative by difference quotients. Taking $x = ih$ and $y = jk = jh$ (here) the differential equation reduces to
- $\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2} = f(ih, jh)$
- i.e. $u_{i-1,j} - 2u_{i,j} + u_{i+1,j} + u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = h^2 f(ih, jh)$
- By applying the above formula at each mesh point, we get a system of linear equation in the pivotal values i, j.

Examples

- ***Example1***: Solve $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square mesh with sides $x = 0, y = 0, x = 3, y = 3$ with $u = 0$ on the boundary and mesh length 1 unit.
- ***Solution***:



- The P.D.E is $\nabla^2 u = -10(x^2 + y^2 + 10)$ (1)
- Using the theory, (here $h = 1$)
- $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = -10(i^2 + j^2 + 10)(2)$
- Applying the formula at D $(i = 1, j = 2)$
- $0 + 0 + u_2 + u_3 - 4u_1 = -10(15) = -150$
- $u_2 + u_3 - 4u_1 = -150$ (3)
- Applying at E $(i = 2, j = 2)$
- $u_1 + u_4 - 4u_2 = -180$ (4)
- Applying at F $(i = 1, j = 1)$
- $u_1 + u_4 - 4u_3 = -120$ (5)
- Applying at G $(i = 2, j = 1)$
- $u_2 + u_3 - 4u_4 = -150$ (6)

- We can solve the equation (3), (4), (5), (6) either by direct elimination or by Gauss-Seidal method.
- By direct elimination method, we have
- $(5) - (4)$ gives
- $u_2 - u_3 = 15$
- $(3) + 4(4)$ gives
- $-15u_2 + u_3 + 4u_4 = -870$ (8)
- $(6) + (8)$
- $-7u_2 + u_3 = -510$ (9)
- $(7) + (9)$
- $u_2 = 82.5$

- From (7),
- $u_3 = u_2 - 15 = 67.5$
- From (3),
- $u_1 = 75$
- $u_4 = \frac{150+150}{4} = 75$
- So
- $u_1 = 75 = u_4, u_2 = 82.5, u_3 = 67.5$

One Dimensional Parabolic Equation– Bender-Schmidt Scheme

One Dimensional Parabolic Equation– Bender-Schmidt Scheme

- The one-dimensional heat equation, namely
- $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$
- Setting $\alpha^2 = \frac{1}{a}$, the equation becomes,
- $\frac{\partial^2 u}{\partial x^2} - \frac{1}{a}\frac{\partial u}{\partial t} = 0$
- Here $A = 1, B = 0, C = 0$
- $\therefore B^2 - 4AC = 0.$
- Therefore, it is parabolic at all points.

- Solve $u_{xx} = au_t$
- With boundary conditions
- $u(0, t) = T_0$
- $u(l, t) = T_1$
- And with initial condition $u(x, 0) = f(x), 0 < x < l$
- We select a spacing h for the variable x and a spacing k for the time variable t.
- Then
- $u_{i,j+1} = \lambda u_{i+1,j} + (1 - 2\lambda)u_{i,j} + \lambda u_{i-1,j} \qquad (1)$
- Where $\lambda = \frac{k}{ah^2}$

- Writing the boundary condition as
- $u_{0,j} = T_0$
- $u_{n,j} = T_l$
- Where $nh = l$
- And initial condition as
- $u_{i,0} = f(ih), i = 1,2, \dots,$
- u is known at $t = 0$.
- Equation (1) facilitates to get the values of u at $x = ih$ and time t_{j+k}.
- Equation (1) is called Explicit formula. It is valid if $0 \leq \lambda \leq \frac{1}{2}$.

- If $\lambda = \frac{1}{2}$, the coefficient of $u_{i,j}$ vanishes.
- Hence
- $u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i-1,j}\right]$ (2)
- The value of u at $x = x_i$ at $t = t_{j+1}$ is equal to the average of the values of u the surrounding points x_{i-1} and x_{i+1} at the previous time t_j.
- $\lambda = \frac{1}{2} = \frac{k}{ah^2}$ implies
- $k = \frac{a}{2}h^2$
- Equation (2) is called Bender-Schmidt recurrence equation.

Examples

- ***Example1***: Solve $u_{xx} - 2u_t = 0$ given
- $u(0,t) = 0, u(4,t) = 0, u(x,0) = x(4-x)$. Assume $h = 1$. Find the values of u upto $t = 5$.
- ***Solution***: $u_{xx} = 2u_t$
- $\therefore a = 2$
- To use Bender Schmidt equation
- $k = \frac{a}{2}h^2 = 1$
- Step size in time $= k = 1$. The value of $u_{i,j}$ are tabulated in the next slide.

x −direction →

↓ t −direction

j \ i	0	1	2	3	4	
0	0	3	4	3	0	← $u(x,0) = x(4-x)$
1	0	2	3	2	0	
2	0	1.5	2	1.5	0	
3	0	1	1.5	1	0	
4	0	0.75	1	0.75	0	
5	0	0.5	0.75	0.5	0	

Crank-Nicholson scheme

Crank-Nicholson scheme

- Solve $u_{xx} = au_t$
- With boundary conditions
- $u(0, t) = T_0$
- $u(l, t) = T_1$
- And with initial condition $u(x, 0) = f(x), 0 < x < l$
- Setting $\lambda = \frac{k}{ah^2}$
- We get
- $\lambda\left(u_{i+1,j+1} + u_{i-1,j+1}\right) - 2(\lambda + 1)u_{i,j+1} = 2(\lambda - 1)u_{i,j} - \lambda(u_{i+1,j} + u_{i-1,j})$
- The above equation is called Crank-Nicholson difference scheme.
- For $\lambda = 1$, $k = ah^2$, the Crank-Nicholson formula reduces to
- $u_{i,j+1} = \frac{1}{4}\left[u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j} + u_{i-1,j}\right]$
- We use this simplified formula to solve the problem.

Examples

- ***Example1***: Solve by Crank-Nicholson method the equation $u_{xx} = u_t$ subject to $u(x,0) = 0$, $u(0,t) = 0$ and $u(1,t) = t$ for two time steps.
- ***Solutions:*** x ranges form 0 to 1. Taking $h = \frac{1}{4}$; here $a = 1$
- $\therefore k = ah^2$ to use simple form
- $k = 1\left(\frac{1}{4}\right)^2 = \frac{1}{16}$.
- We use
- $u_{i,j+1} = \frac{1}{4}\left[u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j} + u_{i-1,j}\right] \qquad (1)$

i / j	0	0.25	0.5	0.75	1
0	0	3	4	3	0
1/16	0	u_1	u_2	u_3	1/16
2/16	0	u_4	u_5	u_6	2/16
3/16	0				3/16

↓
t −direction

- Let the unknowns be represented by $u_1, u_2, u_3, \dots,$
- The boundary conditions are marked in the table against $t = 0, x = 0$ and $x = 1$.
- Using the scheme, we get
- $u_1 = \frac{1}{4}(0 + 0 + 0 + u_2)$ i.e. $u_1 = \frac{1}{4}u_2$ (2)
- $u_2 = \frac{1}{4}(0 + 0 + u_1 + u_3)$ i.e. $u_2 = \frac{1}{4}(u_1 + u_3)$ (3)
- $u_3 = \frac{1}{4}\left(0 + 0 + u_2 + \frac{1}{16}\right)$ i.e. $u_3 = \frac{1}{4}\left(u_2 + \frac{1}{16}\right)$ (4)

- Solving the three equations given by (2), (3), (4), we get u_1, u_2, u_3
- Substitute the values of u_3, u_1 in (3)
- $u_2 = \frac{1}{4}\left[\frac{1}{4}u_2 + \frac{1}{4}\left(u_2 + \frac{1}{16}\right)\right]$
- $u_2 = \frac{1}{224} = 0.0045$
- $u_1 = \frac{1}{896} = 0.0011$
- $u_3 = 0.0168$
- Similarly we get u_4, u_5, u_6 by getting 3 equation in 3 unknowns u_4, u_5, u_6.
- We get $u_4 = 0.005899$, $u_5 = 0.01913$, $u_6 = 0.05277$.