Algebra review

1. Trigonometry

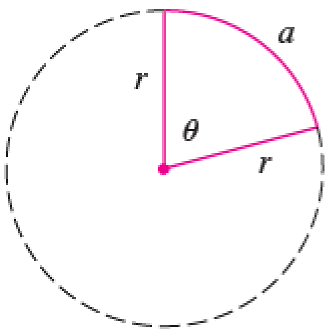


FIGURE 1

1.1 Angles

$$\theta = \frac{a}{r}$$

$$a = \theta r$$

1.2 The Trigonometric functions

$$\sin\theta = \frac{y}{r}, \quad \csc\theta = \frac{r}{y},$$

$$\cos\theta = \frac{x}{r}, \quad \sec\theta = \frac{r}{x},$$

$$\tan\theta = \frac{y}{x}, \quad \cot\theta = \frac{x}{y}$$

1.3 Trigonometric identities

Function	**Identity**
$\csc\theta$	$\dfrac{1}{\sin\theta}$
$\sec\theta$	$\dfrac{1}{\cos\theta}$
$\cot\theta$	$\dfrac{1}{\tan\theta}$
$\tan\theta$	$\dfrac{\sin\theta}{\cos\theta}$
$\cot\theta$	$\dfrac{\cos\theta}{\sin\theta}$

Identity	**Equation**
Pythagorean Identity 1	$\sin^2\theta + \cos^2\theta = 1$
Pythagorean Identity 2	$\tan^2\theta + 1 = \sec^2\theta$
Pythagorean Identity 3	$1 + \cot^2\theta = \csc^2\theta$

- **Sum and Difference Formulas**

$$\begin{aligned}
\sin(x+y) &= \sin x\cos y + \cos x\sin y \\
\sin(x-y) &= \sin x\cos y - \cos x\sin y \\
\cos(x+y) &= \cos x\cos y - \sin x\sin y \\
\cos(x-y) &= \cos x\cos y + \sin x\sin y \\
\tan(x+y) &= \frac{\tan x + \tan y}{1 - \tan x\tan y} \\
\tan(x-y) &= \frac{\tan x - \tan y}{1 + \tan x\tan y}
\end{aligned}$$

- **Double-Angle Identities**

$$\begin{aligned}
\sin(2\theta) &= 2\sin\theta\cos\theta \\
\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\
&= 2\cos^2\theta - 1 \\
&= 1 - 2\sin^2\theta \\
\tan(2\theta) &= \frac{2\tan\theta}{1 - \tan^2\theta}
\end{aligned}$$

- **Half-Angle Identities**

$$
\begin{aligned}
\sin\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1-\cos\theta}{2}} \\
\cos\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1+\cos\theta}{2}} \\
\tan\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \\
&= \frac{\sin\theta}{1+\cos\theta} \\
&= \frac{1-\cos\theta}{\sin\theta}
\end{aligned}
$$

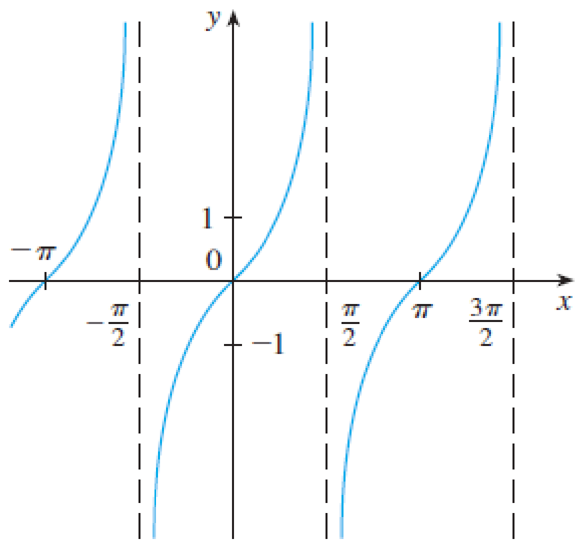
- **product formulas**

$$
\begin{aligned}
\sin x \sin y &= \frac{1}{2}[\cos(x-y) - \cos(x+y)] \\
\cos x \cos y &= \frac{1}{2}[\cos(x-y) + \cos(x+y)] \\
\sin x \cos y &= \frac{1}{2}[\sin(x+y) + \sin(x-y)] \\
\cos x \sin y &= \frac{1}{2}[\sin(x+y) - \sin(x-y)]
\end{aligned}
$$

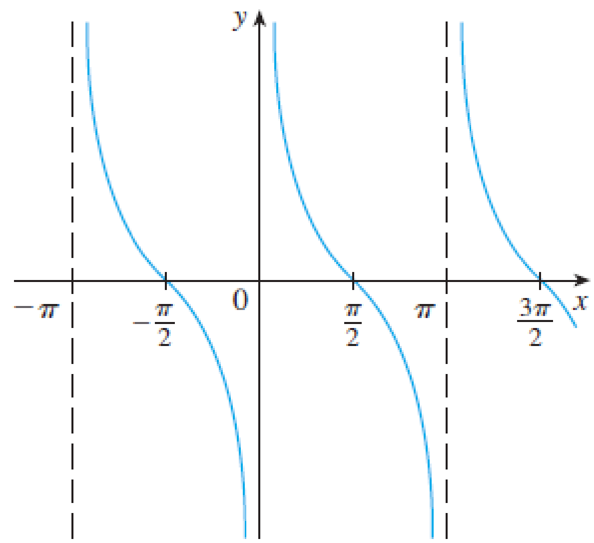
$$
\begin{aligned}
\sin x + \sin y &= 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \\
\sin x - \sin y &= 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) \\
\cos x + \cos y &= 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \\
\cos x - \cos y &= -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)
\end{aligned}
$$

*Hint:

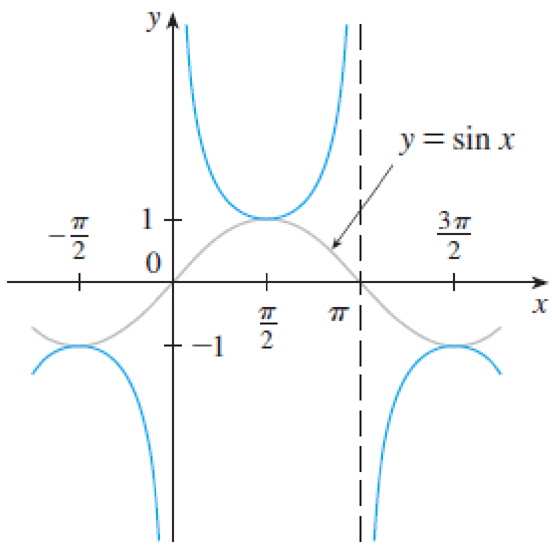
$$
\sin x = \sin\left(\frac{x+y}{2} + \frac{x-y}{2}\right), \quad \sin y = \sin\left(\frac{x+y}{2} - \frac{x-y}{2}\right)
$$



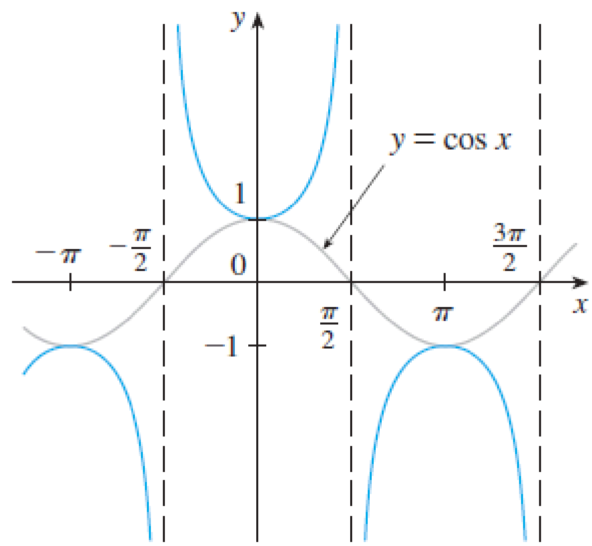
(a) $y = \tan x$



(b) $y = \cot x$



(c) $y = \csc x$



(d) $y = \sec x$

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2. Sigma Notations

Theorem: Properties of Summation

Let c be a constant. Then:

$$\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i$$

$$\sum_{i=m}^{n}(a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$$

$$\sum_{i=m}^{n}(a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$$

Theorem: Common Summation Formulas

Let c be a constant and n a positive integer. Then:

$$1. \sum_{i=1}^{n} 1 = n$$

$$2. \sum_{i=1}^{n} c = nc$$

$$3. \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$4. \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$5. \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

5. Proof by Mathematical Induction

We want to prove: $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

Step 1: Base Case (n = 1)

Left-hand side (LHS): $\sum_{i=1}^{1} i^3 = 1^3 = 1$

Right-hand side (RHS): $\left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = 1$

✅ LHS = RHS, so the base case holds.

Step 2: Inductive Hypothesis

Assume that the formula holds for (n = k), i.e.,

$$\sum_{i=1}^{k} i^3 = \left(\frac{k(k+1)}{2}\right)^2$$

We need to prove that it also holds for (n = k + 1), i.e.,

$$\sum_{i=1}^{k+1} i^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

Step 3: Inductive Step

Start with the left-hand side for $n = k+1$: $\sum_{i=1}^{k+1} i^3 = \left(\sum_{i=1}^{k} i^3\right) + (k+1)^3$

Use the inductive hypothesis:

$$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 = (k+1)^2\left[\left(\frac{k}{2}\right)^2 + (k+1)\right]$$

Simplify inside the brackets:

$$= (k+1)^2\left[\frac{k^2}{4} + (k+1)\right] = (k+1)^2\left[\frac{k^2+4k+4}{4}\right] = (k+1)^2 \cdot \frac{(k+2)^2}{4}$$

So we have:

$$\sum_{i=1}^{k+1} i^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

✅ This matches the desired formula for (n = k + 1).

By the principle of mathematical induction, the identity

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

holds for all positive integers n.

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3. Equations, Inequalities, and Polynomials

3.1 Key Topics

3.1.1 Polynomial Equations

- **Forms**:
  - Linear: $ax + b = 0$
  - Quadratic: $ax^2 + bx + c = 0$
  - Higher-degree: $ax^n + \cdots + k = 0$
- **Solving Methods**:
  - Factoring (common factors, difference of squares, trinomial patterns)
  - Quadratic Formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Completing the Square
- Polynomial Division (Long Division or Synthetic Division)
- Rational Root Theorem: test rational values $\pm \frac{p}{q}$
- Descartes' Rule of Signs: estimate number of positive/negative real roots
- Graphical methods

Rational Root Theorem (Rational Zero Test)

Let the polynomial be:

$$P(x) = a_n x^n + \cdots + a_1 x + a_0$$

Theorem:

If a rational number $\frac{p}{q}$ (in lowest terms) is a root of the polynomial, then:

- p must divide the **constant term** a_0
- q must divide the **leading coefficient** a_n

Steps to Use the Rational Root Theorem:

1. **List all possible rational root candidates**:

$$\text{Candidates} = \pm \frac{\text{factors of } a_0}{\text{factors of } a_n}$$

2. **Test each candidate**

Example:

Let:

$$P(x) = 2x^3 - 3x^2 - 11x + 6$$

- Constant term: $a_0 = 6 \rightarrow$ factors: $\pm 1, \pm 2, \pm 3, \pm 6$
- Leading coefficient: $a_n = 2 \rightarrow$ factors: $\pm 1, \pm 2$
- Possible rational roots:

$$\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$$

Test roots:
Assume we find (x = 2) is a root.
Then use **polynomial division** to factor:

$$P(x) = (x-2)(2x^2 + x - 3)$$

Descartes' Rule of Signs — Explanation

Descartes' Rule of Signs is a method to estimate the number of **positive** and **negative** real roots of a polynomial equation by examining the signs of its coefficients.

Consider a polynomial:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

i. Positive Real Roots

1. Count the number of **sign changes** in the coefficients of $P(x)$
(A sign change occurs when consecutive coefficients have opposite signs, e.g., from + to − or − to +.)
2. The number of **positive real roots** is **equal to the number of sign changes or less than that by an even number** (i.e., number of sign changes − 2k, where $k \geq 0$).

ii. Negative Real Roots

1. Substitute $\boxed{x \to -x}$ to get $\boxed{P(-x)}$
2. Count the number of **sign changes** in the coefficients of $P(-x)$
3. The number of **negative real roots** is equal to the number of sign changes in $P(-x)$ or less by an even number.

Example

Let:

$$P(x) = x^3 - 4x^2 + 5x - 2$$

- Coefficients of $P(x)$ are $[+1, -4, +5, -2]$
  - Number of sign changes: 3
  - Possible positive roots: 3 or 1 (3 minus an even number)

Compute

$$P(-x) = -x^3 - 4x^2 - 5x - 2$$

- Coefficients of $P(-x)$ are $[-1, -4, -5, -2]$
  - Number of sign changes: 0
  - Possible negative roots: 0

Summary

- Sign changes in $P(x)$ give possible positive roots.
- Sign changes in $P(-x)$ give possible negative roots.
- The actual number of roots is the count or less by an even number.

3.1.2 Fundamental Theorem of Algebra

- Every non-zero single-variable polynomial of degree n has exactly n complex roots (counting multiplicity).
- **Corollary**: A real polynomial can be factored into linear and/or irreducible quadratic factors over the reals.

3.1.3 Inequalities

- **Linear/Quadratic Inequalities**:
  - Solve like equations and test intervals (sign charts)
  - Use parabola shape and x-intercepts to determine sign regions
- **Polynomial/Rational Inequalities**:
  - Identify critical points (roots and undefined points)
  - Use sign charts or test points to determine solution intervals
  - Always consider open vs closed intervals based on inequality symbols
  - Write solution sets in interval notation

3.1.4 Special Polynomial Types

- **Even and Odd degree**: affects end behavior
- **Symmetric Polynomials**: e.g., palindromic or reciprocal polynomials
- **Monic Polynomials**: leading coefficient is 1
- **Factored Form**: Useful for analyzing roots and graphing

3.2 Connections to Calculus

- **Graphing**:
  - Roots are x-intercepts
  - Multiplicity affects whether graph crosses or touches x-axis
  - End behavior determined by degree and leading coefficient
- **Limits & Asymptotic Behavior**:
  Let $P(x)$ and $Q(x)$ be polynomials. As $x \to \infty$:
  - If $deg(P) < deg(Q)$: $\lim \frac{P(x)}{Q(x)} = 0$
  - If $deg(P) = deg(Q)$: limit is the ratio of leading coefficients
  - If $deg(P) > deg(Q)$: $\lim \frac{P(x)}{Q(x)} = \infty$ or DNE (slant asymptote may exist)

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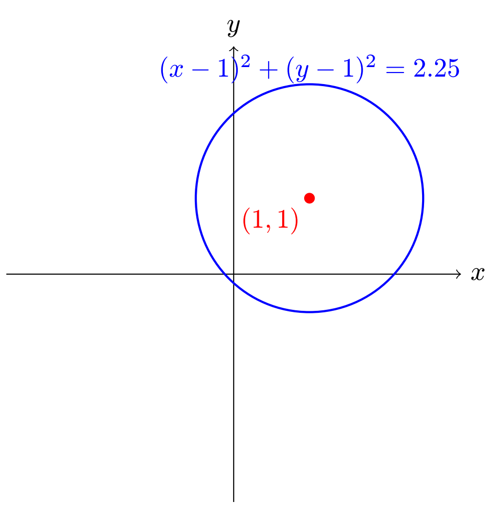
4. Graphs of Second-Degree Equations

4.1 circles

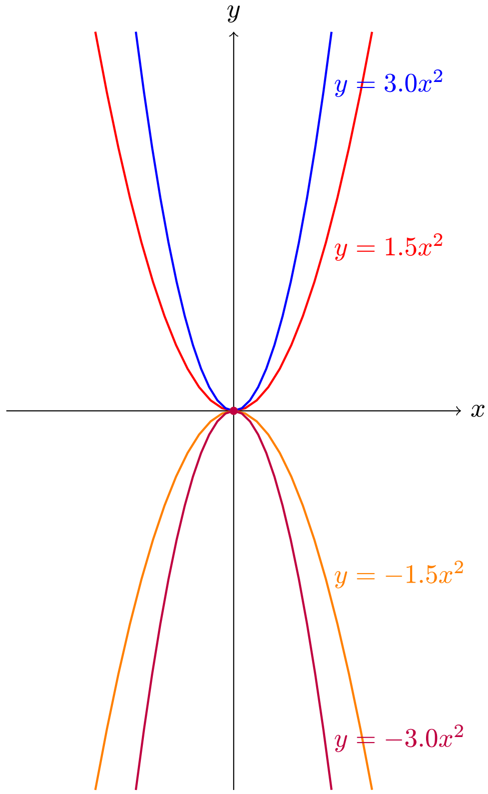
Equation of a Circle :

$$(x-a)^2 + (y-b)^2 = r^2$$

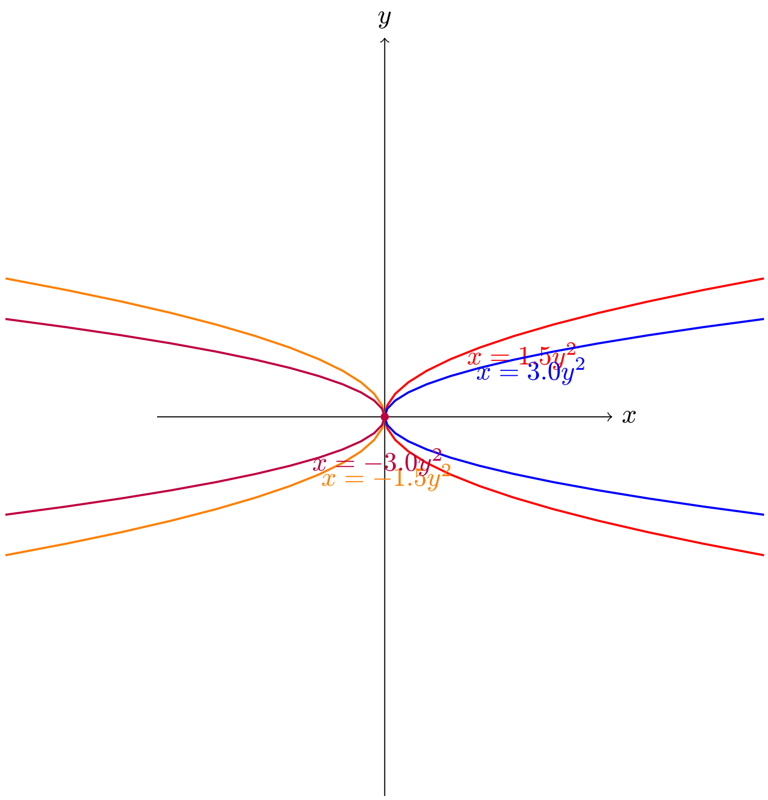


4.2 Parabolas

- $y = ax^2$



- $x = ay^2$

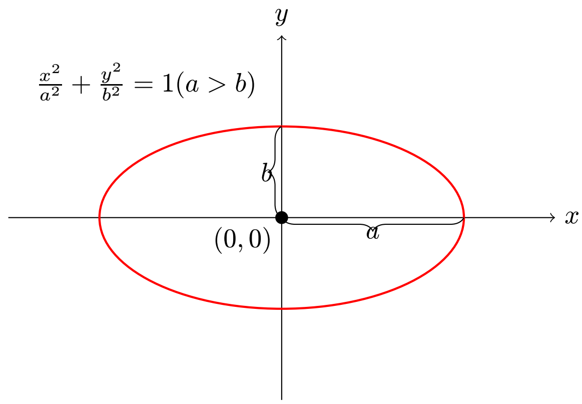


"The larger the value of *a*, the narrower the opening."

4.3 Ellipses

- **Horizontal major axis:**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b)$$



- **Vertical major axis:**

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad (a > b)$$

- **centered at (h, k):** $\boxed{\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1}$

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5. Elementary Functions

5.1 Core Function Types

5.1.1 Exponential Functions

- **Definition**: $f(x) = a^x$ where $a > 0$ $(a \neq 1)$
- **Key Example**: e^x (natural exponential)
- **Properties**:
  - Domain: $\mathbb{R}$
  - Range: $(0, +\infty)$
  - Always increasing if $a > 1$
  - Horizontal asymptote: $y = 0$

Natural Exponential Function

Definition

The natural exponential function is defined as:

$$y = e^x$$

where:

- $e \approx 2.71828$ (Euler's number)
- The function equals its own derivative: $\frac{dy}{dx} = e^x$

Key Properties

1. **Domain**: $(-\infty, \infty)$
2. **Range**: $(0, \infty)$
3. **y-intercept**: (0,1)
4. **Asymptote**: y=0 (horizontal)
5. **Always increasing**
6. **Concave up everywhere**

Special Property at (0,1)

The tangent line at x=0 has:

- Slope = $e^0 = 1$
- Equation: $y = x + 1$

Graph Visualization

5.1.2 Logarithmic Functions

- **Definition**: $f(x) = \log_a x$ (inverse of a^x)
- **Key Example**: $\ln x$ (natural log, base e)
- **Properties**:
  - Domain: $(0, +\infty)$
  - Range: $\mathbb{R}$
  - Vertical asymptote: $x = 0$

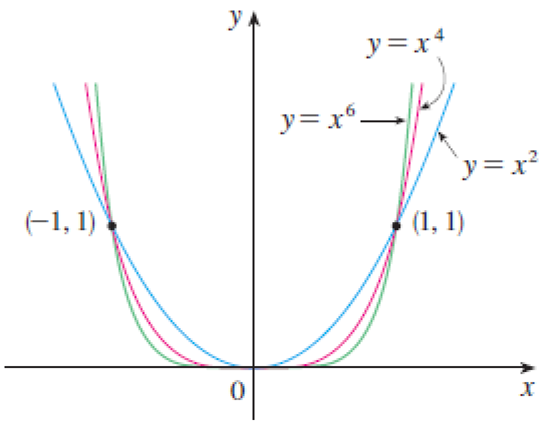
5.1.3 Trigonometric Functions

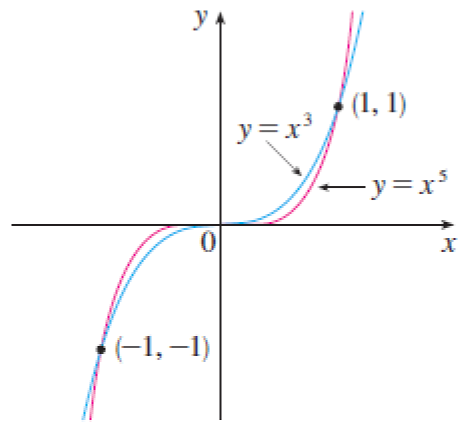
- **Key Examples**:
  - $\sin x$: Periodic with period 2π, range $[-1, 1]$
  - $\cos x$, $\tan x$ (similar analysis)
- **Properties**:
  - Amplitude (height): $|A|$ in $A \sin(Bx)$
  - Period: $\frac{2\pi}{|B|}$

5.1.4 Power Functions

- **Definition**: $f(x) = x^n$ $(n \in \mathbb{R})$
- **Examples**:
  - x^2: Parabola, domain $\mathbb{R}$, range $[0, +\infty)$
  - $\sqrt{x} = x^{1/2}$: Domain $[0, +\infty)$

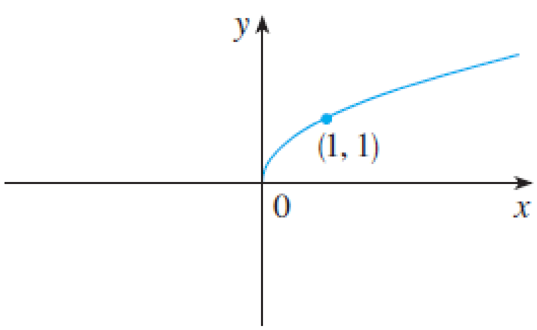
(i) $a = n$, where n is a positive integer:



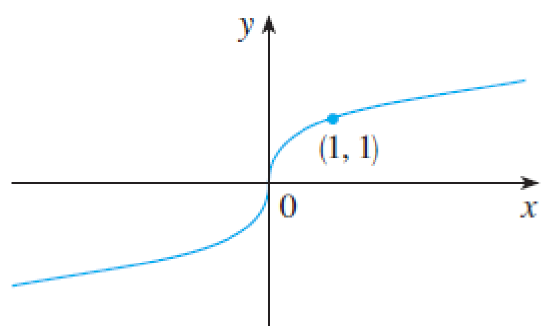


(ii) $a = \frac{1}{n}$, where n is a positive integer:

- The function $f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$ is a root function



(a) $f(x) = \sqrt{x}$



(b) $f(x) = \sqrt[3]{x}$

(iii) $a = -1$:

- ***reciprocal function***: $f(x) = x^{-1} = \frac{1}{x}$

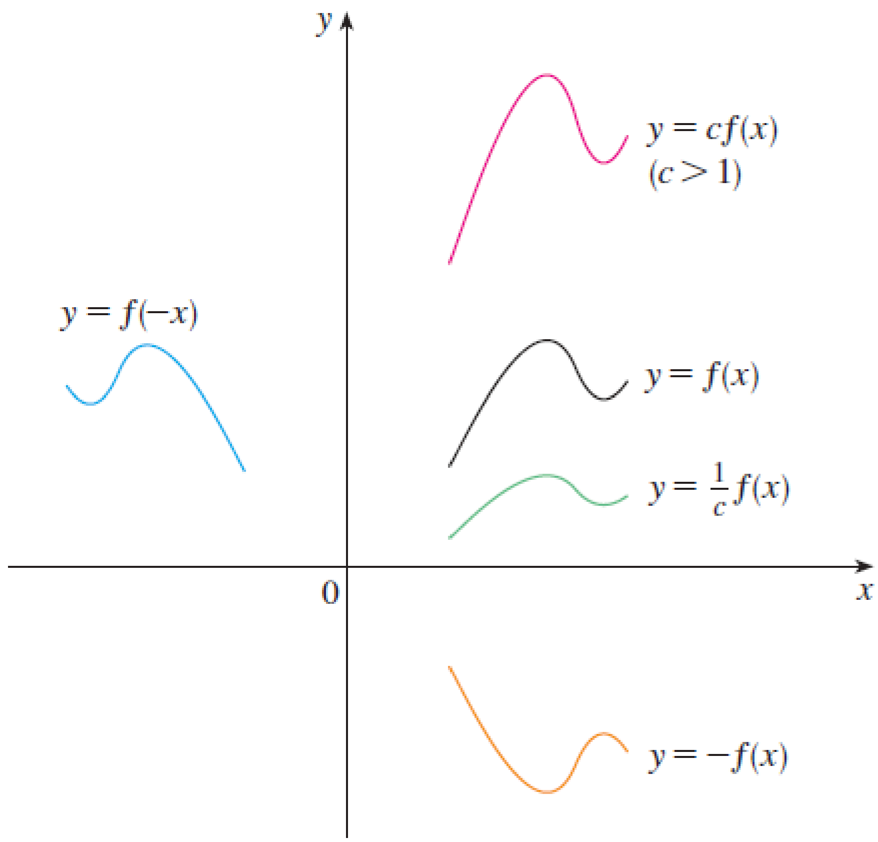
5.2 Function Transformations

For any function $y = f(x)$:

1. **Vertical Shift**: $y = f(x) + k$
   - Up if $k > 0$, down if $k < 0$

1. **Horizontal Shift**: $y = f(x-h)$
    - Right if $h > 0$, left if $h < 0$

1. **Scaling**:
    - Vertical: $y = Af(x)$ (stretch if $|A| > 1$, shrink if $0 < |A| < 1$)
    - Horizontal: $y = f(Bx)$ (compressed if $|B| > 1$, stretched if $0 < |B| < 1$)



2. **Reflection**:
    - $y = -f(x)$ (flip vertically)
    - $y = f(-x)$ (flip horizontally)

5.3 inverse functions and Logarithms

Definition 1

A function f is called a ***one-to-one function*** if it never takes on the same value twice; that is:

$$f(x_1) \neq f(x_2) \quad whenever x_1 \neq x_2$$

Horizontal Line Test A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Definition 2
Let f be a one-to-one function with domain A and range B.
Then its ***inverse function*** f^{-1} has domain B and range A, and is defined by

$$f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y$$

for any $y \in B$.

Definition 3

- $f^{-1}(x) = y \iff f(y) = x$

Definition 4: cancellation equation

- $f^{-1}(f(x)) = x \quad$ for every $x \in A$
- $f(f^{-1}(x)) = x \quad$ for every $x \in B$

How to Find the Inverse Function of a One-to-One Function f

Step 1: Write the equation

$$y = f(x)$$

Step 2: Solve this equation for x in terms of y (if possible): $\boxed{f(y) = x}$

Step 3: Interchange x and y to express f^{-1} as a function of x:

$$y = f^{-1}(x)$$

Example
Find the inverse function of $f(x) = x^3 + 2$.

Solution
According to the inverse function process:

1. Write $y = x^3 + 2$
2. Solve for x in terms of y : $x^3 = y - 2, \quad x = \sqrt[3]{y-2}$
3. Interchange x and y : $y = \sqrt[3]{x-2}$

Therefore, the inverse function is:

$$f^{-1}(x) = \sqrt[3]{x-2}$$

Definition of Logarithm and its Inverse Relationship

- $\log_b x = y \iff b^y = x$
- $\log_b(b^x) = x \quad$ for every $x \in \mathbb{R}$

- $b^{\log_b x} = x$ for every $x > 0$

Laws of Logarithms

If x and y are **positive numbers**, and r is any **real number**, then:

1. $\log_b(xy) = \log_b x + \log_b y$
2. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$
3. $\log_b(x^r) = r\log_b x$

Natural Logarithm and Its Inverse

- $\log_e x = \ln x$
- $\ln x = y \iff e^y = x$
- $\ln(e^x) = x$ for every $x \in \mathbb{R}$
- $e^{\ln x} = x$ for every $x > 0$
- $\ln e = 1$

inverse Trigonometric functions

1. $\arcsin x$

- **Definition**:
  $y = \arcsin x \iff \sin y = x$
- **Domain**: $[-1, 1]$
- **Range**: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

2. $\arccos x$

- **Definition**:
  $y = \arccos x \iff \cos y = x$
- **Domain**: $[-1, 1]$
- **Range**: $[0, \pi]$

3. $\arctan x$

- **Definition**:
  $y = \arctan x \iff \tan y = x$
- **Domain**: $\mathbb{R}$
- **Range**: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

- **The inverse cosecant function:**

$$y = \csc^{-1}(x) \quad (|x| > 1) \quad \Leftrightarrow \quad \csc(y) = x, \quad y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$$

- **The inverse secant function:**

$$y = \sec^{-1}(x) \quad (|x| > 1) \quad \Leftrightarrow \quad \sec(y) = x, \quad y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$$

- **The inverse cotangent function:**

$$y = \cot^{-1}(x) \quad (x \in \mathbb{R}) \quad \Leftrightarrow \quad \cot(y) = x, \quad y \in (0, \pi)$$

Key Connections to Calculus

- **Derivatives**:
  - $\frac{d}{dx} e^x = e^x$
  - $\frac{d}{dx} \ln x = \frac{1}{x}$
  - $\frac{d}{dx} \sin x = \cos x$
- **Limits**:
  - $\lim_{x \to -\infty} e^x = 0$
  - $\lim_{x \to 0^+} \ln x = -\infty$