

–Compute  $\vec{U}_m^{*f}$  (add Exchange contribution, set boundary conditions on  $\vec{U}_m^{*f}$ )

–Form RHS  $B = -\beta(P^{\text{iter}} - P^{\text{old}}) - \sum_{m=1}^N \text{Advection}(\theta_m \vec{U}_m^{*f})$

**while** |outer residual| < convergence criteria **do**

–Compute A matrix

$$\begin{aligned} A_r &= \frac{\Delta t^2}{\Delta x^2} \sum_{m=1}^N \frac{\widehat{\theta}_m}{\langle \rho_m^o \rangle} \Big|_r & A_l &= \frac{\Delta t^2}{\Delta x^2} \sum_{m=1}^N \frac{\widehat{\theta}_m}{\langle \rho_m^o \rangle} \Big|_l \\ A_t &= \frac{\Delta t^2}{\Delta y^2} \sum_{m=1}^N \frac{\widehat{\theta}_m}{\langle \rho_m^o \rangle} \Big|_t & A_b &= \frac{\Delta t^2}{\Delta y^2} \sum_{m=1}^N \frac{\widehat{\theta}_m}{\langle \rho_m^o \rangle} \Big|_b \\ A_f &= \frac{\Delta t^2}{\Delta z^2} \sum_{m=1}^N \frac{\widehat{\theta}_m}{\langle \rho_m^o \rangle} \Big|_f & A_{bk} &= \frac{\Delta t^2}{\Delta z^2} \sum_{m=1}^N \frac{\widehat{\theta}_m}{\langle \rho_m^o \rangle} \Big|_{bk} \\ A_p &= \beta + (A_r + A_l + A_t + A_b + A_f + A_{bk}) \end{aligned}$$

–Solve for  $\delta P$

–Update pressure  $P^{\text{iter}} = P^{\text{iter}-1} + \delta P$ , set boundary condition on  $P^{\text{iter}}$

–Compute  $\vec{U}_m^{*f}$  (add Exchange contribution, set boundary conditions on  $\vec{U}_m^{*f}$ , compute  $\langle \rho_m^o \rangle$ )

–Form RHS  $B = -\beta(P^{\text{iter}} - P^{\text{old}}) - \sum_{m=1}^N \text{Advection}(\theta_m \vec{U}_m^{*f})$ ,

$$\text{where outer residual} = \text{Max}(RHS^2), \beta = \sum_{m=1}^N \frac{\theta_m}{\rho_m^o c_m^2}$$

**end while**

–Compute  $\Delta P = P^{\text{iter}} - P^{\text{old}}$

## 2 Derivation of the matrix stencil

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Step 1: Derive the implicit form of the pressure, starting with change equation

$$\frac{dP}{dt} = \frac{\sum_{m=1}^N \frac{\dot{m}}{V\rho_m^o} - \sum_{m=1}^N \nabla \cdot \widehat{\theta}_m \vec{U}_m^{*f}}{\sum_{m=1}^N \frac{\theta_m}{\rho_m^o c_m^2}} \quad (1)$$

and the face-centered velocity

$$\vec{U}_m^{*f} = \left\langle \frac{\rho_m \vec{U}_m}{\rho_m} \right\rangle - \frac{\Delta t}{\langle \rho_m^o \rangle} \nabla^f P^{\text{iter}} + \text{Exchange term} + \Delta t \vec{g}$$

Write eq.1 in terms of  $P^{\text{iter}} = P^{\text{iter}-1} + \delta P$ ,  $\Delta P = P^{\text{iter}} - P^{\text{old}}$  and  $\beta = \sum_{m=1}^N \frac{\theta_m}{\rho_m^o c_m^2}$

$$\beta(P^{\text{iter}} - P^{\text{old}}) = \Delta t \sum_{m=1}^N \frac{\dot{m}}{V\rho_m^o} - \Delta t \nabla \cdot \widehat{\theta}_m \left( \left\langle \frac{\rho_m \vec{U}_m}{\rho_m} \right\rangle - \frac{\Delta t}{\langle \rho_m^o \rangle} \nabla^f P^{\text{iter}} + \text{Exch} + \Delta t \vec{g} \right) \quad (2)$$

Substitute in for  $P^{\text{iter}} = P^{\text{iter}-1} + \delta P$

$$\beta((P^{\text{iter}-1} + \delta P) - P^{\text{old}}) = \Delta t \sum_{m=1}^N \frac{\dot{m}}{V\rho_m^o} - \Delta t \nabla \cdot \widehat{\theta}_m \left( \left\langle \frac{\rho_m \vec{U}_m}{\rho_m} \right\rangle - \frac{\Delta t}{\langle \rho_m^o \rangle} \nabla^f (P^{\text{iter}-1} + \delta P) + \text{Exch} + \Delta t \vec{g} \right)$$

Now factor out  $\delta P$

$$\begin{aligned} \beta \delta P - \Delta t^2 \nabla \cdot \sum_{m=1}^N \frac{\widehat{\theta}_m}{\langle \rho_m^o \rangle} \nabla \delta P &= \beta(P^{\text{iter}} - P^{\text{old}}) + \Delta t \sum_{m=1}^N \frac{\dot{m}}{V\rho_m^o} \\ &\quad - \Delta t \nabla \cdot \sum_{m=1}^N \widehat{\theta}_m \overbrace{\left( \left\langle \frac{\rho_m \vec{U}_m}{\rho_m} \right\rangle - \frac{\Delta t}{\langle \rho_m^o \rangle} \nabla^f P^{\text{iter}} + \text{Exch} + \Delta t \vec{g} \right)}^{\vec{U}_m^{*f}} \end{aligned}$$

The ODE we have to solve is

$$\beta \delta P - \Delta t^2 \nabla \cdot \sum_{m=1}^N \frac{\widehat{\theta}_m}{\langle \rho_m^o \rangle} \nabla \delta P = -\beta(P^{\text{iter}} - P^{\text{old}}) + \Delta t \sum_{m=1}^N \frac{\dot{m}}{V\rho_m^o} - \Delta t \nabla \cdot \left( \sum_{m=1}^N \widehat{\theta}_m \vec{U}_m^{*f} \right)^{\text{iter}} \quad (3)$$

Step 2: Integrate eq. 3 over a cell volume

$$\begin{aligned} \int_V \beta \delta P dV - \Delta t^2 \int_V \nabla \cdot \sum_{m=1}^N \frac{\widehat{\theta}_m}{\langle \rho_m^o \rangle} \nabla \delta P dV &= - \int_V \beta (P^{\text{iter}} - P^{\text{old}}) dV \\ &+ \int_V \Delta t \sum_{m=1}^N \frac{\dot{m}}{V \rho_m^o} \\ &- \Delta t \int_V \nabla \cdot \left( \sum_{m=1}^N \widehat{\theta}_m \vec{U}_m^{*f} \right)^{\text{iter}} dV \end{aligned}$$

Invoke the divergence theorem  $\int_V \nabla \cdot \vec{f} dV = \int_S \vec{f} \cdot d\vec{s}$  to convert the volume to surface integrals.

$$\begin{aligned} V \beta \delta P - \Delta t^2 \int_S \sum_{m=1}^N \frac{\widehat{\theta}_m}{\langle \rho_m^o \rangle} \nabla \delta P \cdot d\vec{s} &= -V \beta (P^{\text{iter}} - P^{\text{old}}) + \Delta t \sum_{m=1}^N \frac{\dot{m}}{\rho_m^o} \\ &- \Delta t \int_S \left( \sum_{m=1}^N \widehat{\theta}_m \vec{U}_m^{*f} \right)^{\text{iter}} \cdot d\vec{s} \end{aligned}$$

Step 3: Write out the contributions for each face of the computational cell

$$\begin{aligned} \Delta t^2 \int_S \sum_{m=1}^N \frac{\widehat{\theta}_m}{\langle \rho_m^o \rangle} \nabla \delta P \cdot d\vec{s} &= \Delta t^2 \left[ \sum_{m=1}^N \frac{\widehat{\theta}_m}{\langle \rho_m^o \rangle} \Big|_R \Delta y \Delta z \frac{(\delta P_R - \delta P)}{\Delta x} - \sum_{m=1}^N \frac{\widehat{\theta}_m}{\langle \rho_m^o \rangle} \Big|_L \Delta y \Delta z \frac{(\delta P - \delta P_L)}{\Delta x} \right. \\ &+ \sum_{m=1}^N \frac{\widehat{\theta}_m}{\langle \rho_m^o \rangle} \Big|_T \Delta x \Delta z \frac{(\delta P_T - \delta P)}{\Delta y} - \sum_{m=1}^N \frac{\widehat{\theta}_m}{\langle \rho_m^o \rangle} \Big|_B \Delta x \Delta z \frac{(\delta P - \delta P_B)}{\Delta y} \\ &\left. + \sum_{m=1}^N \frac{\widehat{\theta}_m}{\langle \rho_m^o \rangle} \Big|_F \Delta x \Delta y \frac{(\delta P_F - \delta P)}{\Delta z} - \sum_{m=1}^N \frac{\widehat{\theta}_m}{\langle \rho_m^o \rangle} \Big|_{BK} \Delta x \Delta y \frac{(\delta P - \delta P_{BK})}{\Delta z} \right] \end{aligned} \quad (4)$$

Form the stencil of Matrix A from eq. 4

$$A_{R/L} = \frac{\Delta y \Delta z}{\Delta x} \sum_{m=1}^N \frac{\widehat{\theta}_m}{\langle \rho_m^o \rangle} \Big|_{R/L} \quad A_{T/B} = \frac{\Delta x \Delta z}{\Delta y} \sum_{m=1}^N \frac{\widehat{\theta}_m}{\langle \rho_m^o \rangle} \Big|_{T/B} \quad A_{F/BK} = \frac{\Delta y \Delta x}{\Delta z} \sum_{m=1}^N \frac{\widehat{\theta}_m}{\langle \rho_m^o \rangle} \Big|_{F/BK}$$

$$\begin{aligned} V \beta \delta P - \Delta t^2 \left[ A_R (\delta P_R - \delta P) - A_L (\delta P - \delta P_L) \right. \\ \left. + A_T (\delta P_T - \delta P) - A_B (\delta P - \delta P_B) \right. \\ \left. + A_F (\delta P_F - \delta P) - A_{BK} (\delta P - \delta P_{BK}) \right] &= -V \beta (P^{\text{iter}} - P^{\text{old}}) + \Delta t \sum_{m=1}^N \frac{\dot{m}}{\rho_m^o} - \Delta t \text{Advection} \left( \sum_{m=1}^N \widehat{\theta}_m \vec{U}_m^{*f} \right)^{\text{iter}} \end{aligned} \quad (5)$$

Factor out  $\delta P$  and divide by  $V$  to nondimensionalize the equation

$$\begin{aligned} \beta - \frac{\Delta t^2}{V} \left[ A_R + A_L + A_T + A_B + A_F + A_{BK} \right] \delta P - \frac{\Delta t^2}{V} \left[ A_R \delta P_R + A_L \delta P_L + A_T \delta P_T + A_B \delta P_B + A_F \delta P_F + A_{BK} \delta P_{BK} \right] &= \\ -\beta (P^{\text{iter}} - P^{\text{old}}) + \frac{\Delta t}{V} \sum_{m=1}^N \frac{\dot{m}}{\rho_m^o} - \frac{\Delta t}{V} \text{Advection} \left( \sum_{m=1}^N \widehat{\theta}_m \vec{U}_m^{*f} \right)^{\text{iter}} \end{aligned}$$

### 3 Nomenclature

<u>Variable</u>	<u>Dimensions</u>	<u>Description</u>
$\Delta t$	$[t]$	Timestep
$\rho$	$[M/L^3]$	Density of material
$\langle \rho_m^o \rangle$	$[M/L^3]$	Microscopic density at the face center
$\vec{U}$	$[L/t]$	Velocity vector
$P$	$[M/L^2]$	Pressure
$\theta_m$		volume fraction of material $m$
$\widehat{\theta}_m$		upwinded volume fraction
<u>Superscript</u>		
$old$		Previous timestep
$n$		Current timestep
$iter$		outer iteration number
<u>Subscript</u>		
$m$		Material index
$o$		Pure material
$l, r, t, b, f, bk$		Left, right, top and bottom, front and back cell faces
<u>Misc</u>		
$\langle \rangle$		value averaged to the face