Small strain elastic-plastic model

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Let $\boldsymbol{F}$ be the deformation gradient, $\boldsymbol{\sigma}$ be the Cauchy stress, and $\boldsymbol{d}$ be the rate of deformation tensor. We first decompose the deformation gradient into a stretch and a rotations using $\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U}$. The rotation $\boldsymbol{R}$ is then used to rotate the stress and the rate of deformation into the material configuration to give us

$$\widehat{\boldsymbol{\sigma}} = \boldsymbol{R}^T \cdot \boldsymbol{\sigma} \cdot \boldsymbol{R} \; ; \;\; \widehat{\boldsymbol{d}} = \boldsymbol{R}^T \cdot \boldsymbol{d} \cdot \boldsymbol{R} \tag{1}$$

This is equivalent to using a Green-Naghdi objective stress rate. In the following all equations are with respect to the hatted quantities and we drop the hats for convenience.

1 Elastic relation

Let us split the Cauchy stress into a volumetric and a deviatoric part

$$\boldsymbol{\sigma} = p \; \boldsymbol{1} + \boldsymbol{s} \; ; \;\; p = \frac{1}{3} \, \text{tr}(\boldsymbol{\sigma}) \; . \tag{2}$$

Taking the time derivative gives us

$$\dot{\boldsymbol{\sigma}} = \dot{p} \; \boldsymbol{1} + \dot{\boldsymbol{s}} \; . \tag{3}$$

We assume that the elastic response of the material is isotropic. The constitutive relation for a hypoelastic material of grade 0 can be expressed as

$$\dot{\boldsymbol{\sigma}} = \left[\lambda \, \text{tr}(\boldsymbol{d}^e) - 3 \, \kappa \, \alpha \, \frac{d}{dt}(T - T_0)\right] \; \boldsymbol{1} + 2 \, \mu \, \boldsymbol{d}^e \; ; \;\; \boldsymbol{d} = \boldsymbol{d}^e + \boldsymbol{d}^p \tag{4}$$

where $\boldsymbol{d}^e, \boldsymbol{d}^p$ are the elastic and plastic parts of the rate of deformation tensor, λ, μ are the Lame constants, κ is the bulk modulus, α is the coefficient of thermal expansion, T_0 is the reference temperature, and T is the current temperature. If we split $\boldsymbol{d}^e$ into volumetric and deviatoric parts as

$$\boldsymbol{d}^e = \frac{1}{3} \, \text{tr}(\boldsymbol{d}^e) \; \boldsymbol{1} + \boldsymbol{\eta}^e \tag{5}$$

we can write

$$\dot{\boldsymbol{\sigma}} = \left[\left(\lambda + \frac{2}{3}\,\mu\right) \, \text{tr}(\boldsymbol{d}^e) - 3 \, \kappa \, \alpha \, \frac{d}{dt}(T - T_0)\right] \; \boldsymbol{1} + 2 \, \mu \, \boldsymbol{\eta}^e = \kappa \, \left[\text{tr}(\boldsymbol{d}^e) - 3 \, \alpha \, \frac{d}{dt}(T - T_0)\right] \; \boldsymbol{1} + 2 \, \mu \, \boldsymbol{\eta}^e \tag{6}$$

Therefore, we have

$$\dot{\boldsymbol{s}} = 2\,\mu\,\boldsymbol{\eta}^e\,. \tag{7}$$

and

$$\dot{p} = \kappa\,\left[\text{tr}(\boldsymbol{d}^e) - 3\,\alpha\,\frac{d}{dt}(T - T_0)\right]\,. \tag{8}$$

We will use a standard elastic-plastic stress update algorithm to integrate the rate equation for the deviatoric stress. However, we will assume that the volumetric part of the Cauchy stress can be computed using an equation of state. Then the final Cauchy stress will be given by

$$\boldsymbol{\sigma} = \left[p(J) - J\,\frac{dp(J)}{dJ}\,\alpha\,(T - T_0)\right]\,\boldsymbol{1} + \boldsymbol{s}\,;\quad J = \det(\boldsymbol{F})\,. \tag{9}$$

(Note that we assume that the plastic part of the deformation is volume preserving. This is not true for Gurson type models and will lead to a small error in the computed value of $\boldsymbol{\sigma}$.)

2 Flow rule

We assume that the flow rule is given by

$$\boldsymbol{d}^p = \dot{\gamma}\,\boldsymbol{r} \tag{10}$$

We can split $\boldsymbol{d}^p$ into a trace part and a trace free part, i.e.,

$$\boldsymbol{d}^p = \frac{1}{3}\,\text{tr}(\boldsymbol{d}^p)\,\boldsymbol{1} + \boldsymbol{\eta}^p \tag{11}$$

Then, using the flow rule, we have

$$\boldsymbol{d}^p = \frac{1}{3}\,\dot{\gamma}\,\text{tr}(\boldsymbol{r})\,\boldsymbol{1} + \boldsymbol{\eta}^p\,. \tag{12}$$

Therefore we can write the flow rule as

$$\boldsymbol{\eta}^p = \dot{\gamma}\,\left(-\frac{1}{3}\,\text{tr}(\boldsymbol{r})\,\boldsymbol{1} + \boldsymbol{r}\right)\,. \tag{13}$$

Note that

$$\boldsymbol{d} = \boldsymbol{d}^e + \boldsymbol{d}^p \quad \Longrightarrow \quad \text{tr}(\boldsymbol{d}) = \text{tr}(\boldsymbol{d}^e) + \text{tr}(\boldsymbol{d}^p)\,. \tag{14}$$

Also,

$$\boldsymbol{d} = \boldsymbol{d}^e + \boldsymbol{d}^p \quad \Longrightarrow \quad \frac{1}{3}\,\text{tr}(\boldsymbol{d})\,\boldsymbol{1} + \boldsymbol{\eta} = \frac{1}{3}\,\text{tr}(\boldsymbol{d}^e)\,\boldsymbol{1} + \boldsymbol{\eta}^e + \frac{1}{3}\,\text{tr}(\boldsymbol{d}^p)\,\boldsymbol{1} + \boldsymbol{\eta}^p\,. \tag{15}$$

Therefore,

$$\boldsymbol{\eta} = \boldsymbol{\eta}^e + \boldsymbol{\eta}^p\,. \tag{16}$$

3 Isotropic and Kinematic hardening and porosity evolution rules

We assume that the strain rate, temperature, and porosity can be fixed at the beginning of a timestep and consider only the evolution of plastic strain and the back stress while calculating the current stress.

We assume that the plastic strain evolves according to the relation

$$\dot{\varepsilon^p} = \dot{\gamma}\, h^\alpha \tag{17}$$

We also assume that the back stress evolves according to the relation

$$\dot{\widehat{\boldsymbol{\beta}}} = \dot{\gamma}\, \boldsymbol{h}^\beta \tag{18}$$

where $\hat{\boldsymbol{\beta}}$ is the back stress. If $\boldsymbol{\beta}$ is the deviatoric part of $\widehat{\boldsymbol{\beta}}$, then we can write

$$\dot{\boldsymbol{\beta}} = \dot{\gamma}\, \text{dev}(\boldsymbol{h}^\beta)\ . \tag{19}$$

The porosity ϕ is assumed to evolve according to the relation

$$\dot{\phi} = \dot{\gamma}\, h^\phi\ . \tag{20}$$

4 Yield condition

The yield condition is assumed to be of the form

$$f(\boldsymbol{s}, \boldsymbol{\beta}, \varepsilon^p, \phi, \dot{\varepsilon}, T, \dots) = f(\boldsymbol{\xi}, \varepsilon^p, \phi, \dot{\varepsilon}, T, \dots) = 0 \tag{21}$$

where $\boldsymbol{\xi} = \boldsymbol{s} - \boldsymbol{\beta}$ and $\boldsymbol{\beta}$ is the deviatoric part of $\widehat{\boldsymbol{\beta}}$. The Kuhn-Tucker loading-unloading conditions are

$$\dot{\gamma} \geq 0\ ;\quad f \leq 0\ ;\quad \dot{\gamma}\, f = 0 \tag{22}$$

and the consistency condition is $\dot{f} = 0$.

5 Temperature increase due to plastic dissipation

The temperature increase due to plastic dissipation is assume to be given by the rate equation

$$\dot{T} = \frac{\chi}{\rho\, C_p}\, \sigma_y\, \dot{\varepsilon^p}\ . \tag{23}$$

The temperature is updated using

$$T_{n+1} = T_n + \frac{\chi_{n+1}\, \Delta t}{\rho_{n+1}\, C_p}\, \sigma_y^{n+1}\, \dot{\varepsilon^p}_{n+1}\ . \tag{24}$$

6 Continuum elastic-plastic tangent modulus

To determine whether the material has undergone a loss of stability we need to compute the acosutic tensor which needs the computation of the continuum elastic-plastic tangent modulus.

To do that recall that

$$\boldsymbol{\sigma} = p\,\boldsymbol{1} + \boldsymbol{s} \quad \implies \quad \dot{\boldsymbol{\sigma}} = \dot{p}\,\boldsymbol{1}\;. \tag{25}$$

We assume that

$$\dot{p} = J\,\frac{\partial p}{\partial J}\,\text{tr}(\boldsymbol{d}) \qquad \text{and} \quad \dot{\boldsymbol{s}} = 2\,\mu\,\boldsymbol{\eta}^e\;. \tag{26}$$

Now, the consistency condition requires that

$$\dot{f}(\boldsymbol{s},\boldsymbol{\beta},\varepsilon^p,\phi,\dot{\varepsilon},T,\dots) = 0\;. \tag{27}$$

Keeping $\dot{\varepsilon}$ and T fixed over the time interval, we can use the chain rule to get

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{s}} : \dot{\boldsymbol{s}} + \frac{\partial f}{\partial \boldsymbol{\beta}} : \dot{\boldsymbol{\beta}} + \frac{\partial f}{\partial \varepsilon^p}\,\dot{\varepsilon^p} + \frac{\partial f}{\partial \phi}\,\dot{\phi} = 0\;. \tag{28}$$

The needed rate equations are

$$\begin{aligned}
\dot{\boldsymbol{s}} &= 2\,\mu\,\boldsymbol{\eta}^e = 2\,\mu\,(\boldsymbol{\eta} - \boldsymbol{\eta}^p) = 2\,\mu\,[\boldsymbol{\eta} - \dot{\gamma}\,\text{dev}(\boldsymbol{r})] \\
\dot{\boldsymbol{\beta}} &= \dot{\gamma}\,\text{dev}(\boldsymbol{h}^\beta) \\
\dot{\varepsilon^p} &= \dot{\gamma}\,h^\alpha \\
\dot{\phi} &= \dot{\gamma}\,h^\phi
\end{aligned} \tag{29}$$

Plugging these into the expression for $\dot{f}$ gives

$$2\,\mu\,\frac{\partial f}{\partial \boldsymbol{s}} : [\boldsymbol{\eta} - \dot{\gamma}\,\text{dev}(\boldsymbol{r})] + \dot{\gamma}\,\frac{\partial f}{\partial \boldsymbol{\beta}} : \text{dev}(\boldsymbol{h}^\beta) + \dot{\gamma}\,\frac{\partial f}{\partial \varepsilon^p}\,h^\alpha + \dot{\gamma}\,\frac{\partial f}{\partial \phi}\,h^\phi = 0 \tag{30}$$

or,

$$\dot{\gamma} = \frac{2\,\mu\,\frac{\partial f}{\partial \boldsymbol{s}} : \boldsymbol{\eta}}{2\,\mu\,\frac{\partial f}{\partial \boldsymbol{s}} : \text{dev}(\boldsymbol{r}) - \frac{\partial f}{\partial \boldsymbol{\beta}} : \text{dev}(\boldsymbol{h}^\beta) - \frac{\partial f}{\partial \varepsilon^p}\,h^\alpha - \frac{\partial f}{\partial \phi}\,h^\phi}\;. \tag{31}$$

Plugging this expression for $\dot{\gamma}$ into the equation for $\dot{\boldsymbol{s}}$, we get

$$\dot{\boldsymbol{s}} = 2\,\mu\,\left[\boldsymbol{\eta} - \left(\frac{2\,\mu\,\frac{\partial f}{\partial \boldsymbol{s}} : \boldsymbol{\eta}}{2\,\mu\,\frac{\partial f}{\partial \boldsymbol{s}} : \text{dev}(\boldsymbol{r}) - \frac{\partial f}{\partial \boldsymbol{\beta}} : \text{dev}(\boldsymbol{h}^\beta) - \frac{\partial f}{\partial \varepsilon^p}\,h^\alpha - \frac{\partial f}{\partial \phi}\,h^\phi}\right)\,\text{dev}(\boldsymbol{r})\right]\;. \tag{32}$$

At this stage, note that a symmetric $\boldsymbol{\sigma}$ implies a symmetric $\boldsymbol{s}$ and hence a symmetric $\boldsymbol{\eta}$. Also we assume that $\boldsymbol{r}$ is symmetric (and hence $\text{dev}(\boldsymbol{r})$), which is true if the flow rule is associated. Then we can write

$$\boldsymbol{\eta} = \mathsf{I}^{4s} : \boldsymbol{\eta} \qquad \text{and} \qquad \text{dev}(\boldsymbol{r}) = \mathsf{I}^{4s} : \text{dev}(\boldsymbol{r}) \tag{33}$$

where I^{4s} is the fourth-order symmetric identity tensor. Also note that if $\boldsymbol{A}$, $\boldsymbol{C}$, $\boldsymbol{D}$ are second order tensors and B is a fourth order tensor, then

$$(\boldsymbol{A} : \mathsf{B} : \boldsymbol{C})\,(\mathsf{B} : \boldsymbol{D}) \equiv A_{ij}\,B_{ijkl}\,C_{kl}\,B_{mnpq}\,D_{pq} = (B_{mnpq}\,D_{pq})\,(A_{ij}\,B_{ijkl})\,C_{kl} \equiv [(\mathsf{B} : \boldsymbol{D}) \otimes (\boldsymbol{A} : \mathsf{B})] : \boldsymbol{C}\;. \tag{34}$$

Therefore we have

$$\dot{\boldsymbol{s}} = 2\,\mu\,\left[\boldsymbol{\mathsf{I}}^{4s} : \boldsymbol{\eta} - \left(\cfrac{2\,\mu\,[\boldsymbol{\mathsf{I}}^{4s} : \text{dev}(\boldsymbol{r})] \otimes [\frac{\partial f}{\partial \boldsymbol{s}} : \boldsymbol{\mathsf{I}}^{4s}]}{2\,\mu\,\frac{\partial f}{\partial \boldsymbol{s}} : \text{dev}(\boldsymbol{r}) - \frac{\partial f}{\partial \boldsymbol{\beta}} : \text{dev}(\boldsymbol{h}^{\beta}) - \frac{\partial f}{\partial \varepsilon^p}\,h^{\alpha} - \frac{\partial f}{\partial \phi}\,h^{\phi}}\right) : \boldsymbol{\eta}\right]\,. \tag{35}$$

Also,

$$\boldsymbol{\mathsf{I}}^{4s} : \text{dev}(\boldsymbol{r}) = \text{dev}(\boldsymbol{r}) \qquad \text{and} \qquad \frac{\partial f}{\partial \boldsymbol{s}} : \boldsymbol{\mathsf{I}}^{4s} = \frac{\partial f}{\partial \boldsymbol{s}}\,. \tag{36}$$

Hence we can write

$$\dot{\boldsymbol{s}} = 2\,\mu\,\left[\boldsymbol{\mathsf{I}}^{4s} - \left(\cfrac{2\,\mu\,\text{dev}(\boldsymbol{r}) \otimes \frac{\partial f}{\partial \boldsymbol{s}}}{2\,\mu\,\frac{\partial f}{\partial \boldsymbol{s}} : \text{dev}(\boldsymbol{r}) - \frac{\partial f}{\partial \boldsymbol{\beta}} : \text{dev}(\boldsymbol{h}^{\beta}) - \frac{\partial f}{\partial \varepsilon^p}\,h^{\alpha} - \frac{\partial f}{\partial \phi}\,h^{\phi}}\right)\right] : \boldsymbol{\eta} \tag{37}$$

or,

$$\dot{\boldsymbol{s}} = \boldsymbol{\mathsf{B}}^{ep} : \boldsymbol{\eta} = \boldsymbol{\mathsf{B}}^{ep} : \left[\boldsymbol{d} - \frac{1}{3}\,\text{tr}(\boldsymbol{d})\,\boldsymbol{1}\right] \tag{38}$$

where

$$\boldsymbol{\mathsf{B}}^{ep} := 2\,\mu\,\left[\boldsymbol{\mathsf{I}}^{4s} - \left(\cfrac{2\,\mu\,\text{dev}(\boldsymbol{r}) \otimes \frac{\partial f}{\partial \boldsymbol{s}}}{2\,\mu\,\frac{\partial f}{\partial \boldsymbol{s}} : \text{dev}(\boldsymbol{r}) - \frac{\partial f}{\partial \boldsymbol{\beta}} : \text{dev}(\boldsymbol{h}^{\beta}) - \frac{\partial f}{\partial \varepsilon^p}\,h^{\alpha} - \frac{\partial f}{\partial \phi}\,h^{\phi}}\right)\right]\,. \tag{39}$$

Adding in the volumetric component gives

$$\begin{aligned}
\dot{\boldsymbol{\sigma}} &= \dot{p}\,\boldsymbol{1} + \dot{\boldsymbol{s}} \\
&= J\,\frac{\partial p}{\partial J}\,\text{tr}(\boldsymbol{d})\,\boldsymbol{1} + \boldsymbol{\mathsf{B}}^{ep} : \left[\boldsymbol{d} - \frac{1}{3}\,\text{tr}(\boldsymbol{d})\,\boldsymbol{1}\right] \\
&= \left[3\,J\,\frac{\partial p}{\partial J}\,\boldsymbol{1} - \boldsymbol{\mathsf{B}}^{ep} : \boldsymbol{1}\right]\,\frac{\boldsymbol{d} : \boldsymbol{1}}{3} + \boldsymbol{\mathsf{B}}^{ep} : \boldsymbol{d} \\
&= J\,\frac{\partial p}{\partial J}\,(\boldsymbol{1} \otimes \boldsymbol{1}) : \boldsymbol{d} - \frac{1}{3}\,[\boldsymbol{\mathsf{B}}^{ep} : (\boldsymbol{1} \otimes \boldsymbol{1})] : \boldsymbol{d} + \boldsymbol{\mathsf{B}}^{ep} : \boldsymbol{d}\,.
\end{aligned} \tag{40}$$

Therefore,

$$\dot{\boldsymbol{\sigma}} = \left[J\,\frac{\partial p}{\partial J}\,(\boldsymbol{1} \otimes \boldsymbol{1}) - \frac{1}{3}\,[\boldsymbol{\mathsf{B}}^{ep} : (\boldsymbol{1} \otimes \boldsymbol{1})] + \boldsymbol{\mathsf{B}}^{ep}\right] : \boldsymbol{d} = \boldsymbol{\mathsf{C}}^{ep} : \boldsymbol{d}\,. \tag{41}$$

The quantity $\boldsymbol{\mathsf{C}}^{ep}$ is the continuum elastic-plastic tangent modulus. We also use the continuum elastic-plastic tangent modulus in the implicit version of the code. However, for improved accuracy and faster convergence, an algorithmically consistent tangent modulus should be used instead. That tangent modulus can be calculated in the usual manner and is left for development and implementation as an additional feature in the future.

7 Stress update

A standard return algorithm is used to compute the updated Cauchy stress. Recall that the rate equation for the deviatoric stress is given by

$$\dot{\boldsymbol{s}} = 2\,\mu\,\boldsymbol{\eta}^e\,. \tag{42}$$

Integrating the rate equation using a Backward Euler scheme gives

$$\boldsymbol{s}_{n+1} - \boldsymbol{s}_n = 2\,\mu\,\Delta\,t\,\boldsymbol{\eta}^e_{n+1} = 2\,\mu\,\Delta\,t\,(\boldsymbol{\eta}_{n+1} - \boldsymbol{\eta}^p_{n+1}) \tag{43}$$

Now, from the flow rule, we have

$$\boldsymbol{\eta}^p = \dot{\gamma}\,\left(\boldsymbol{r} - \frac{1}{3}\,\mathrm{tr}(\boldsymbol{r})\,\boldsymbol{1}\right)\,. \tag{44}$$

Define the deviatoric part of $\boldsymbol{r}$ as

$$\mathrm{dev}(\boldsymbol{r}) := \boldsymbol{r} - \frac{1}{3}\,\mathrm{tr}(\boldsymbol{r})\,\boldsymbol{1}\,. \tag{45}$$

Therefore,

$$\boldsymbol{s}_{n+1} - \boldsymbol{s}_n = 2\,\mu\,\Delta\,t\,\boldsymbol{\eta}_{n+1} - 2\,\mu\,\Delta\gamma_{n+1}\,\mathrm{dev}(\boldsymbol{r}_{n+1})\,. \tag{46}$$

where $\Delta\gamma := \dot{\gamma}\,\Delta t$. Define the trial stress

$$\boldsymbol{s}^{\mathrm{trial}} := \boldsymbol{s}_n + 2\,\mu\,\Delta\,t\,\boldsymbol{\eta}_{n+1}\,. \tag{47}$$

Then

$$\boldsymbol{s}_{n+1} = \boldsymbol{s}^{\mathrm{trial}} - 2\,\mu\,\Delta\gamma_{n+1}\,\mathrm{dev}(\boldsymbol{r}_{n+1})\,. \tag{48}$$

Also recall that the back stress is given by

$$\dot{\boldsymbol{\beta}} = \dot{\gamma}\,\mathrm{dev}\boldsymbol{h}^{\beta} \tag{49}$$

The evolution equation for the back stress can be integrated to get

$$\boldsymbol{\beta}_{n+1} - \boldsymbol{\beta}_n = \Delta\gamma_{n+1}\,\mathrm{dev}(\boldsymbol{h})^{\beta}_{n+1}\,. \tag{50}$$

Now,

$$\boldsymbol{\xi}_{n+1} = \boldsymbol{s}_{n+1} - \boldsymbol{\beta}_{n+1}\,. \tag{51}$$

Plugging in the expressions for $\boldsymbol{s}_{n+1}$ and $\boldsymbol{\beta}_{n+1}$, we get

$$\boldsymbol{\xi}_{n+1} = \boldsymbol{s}^{\mathrm{trial}} - 2\,\mu\,\Delta\gamma_{n+1}\,\mathrm{dev}(\boldsymbol{r}_{n+1}) - \boldsymbol{\beta}_n - \Delta\gamma_{n+1}\,\mathrm{dev}(\boldsymbol{h})^{\beta}_{n+1}\,. \tag{52}$$

Define

$$\boldsymbol{\xi}^{\mathrm{trial}} := \boldsymbol{s}^{\mathrm{trial}} - \boldsymbol{\beta}_n\,. \tag{53}$$

Then

$$\boldsymbol{\xi}_{n+1} = \boldsymbol{\xi}^{\mathrm{trial}} - \Delta\gamma_{n+1}(2\,\mu\,\mathrm{dev}(\boldsymbol{r}_{n+1}) + \mathrm{dev}(\boldsymbol{h})^{\beta}_{n+1})\,. \tag{54}$$

Similarly, the evolution of the plastic strain is given by

$$\varepsilon^p_{n+1} = \varepsilon^p_n + \Delta\gamma_{n+1}\,h^{\alpha}_{n+1} \tag{55}$$

and the porosity evolves as

$$\phi_{n+1} = \phi_n + \Delta\gamma_{n+1}\,h^{\phi}_{n+1}\,. \tag{56}$$

The yield condition is discretized as

$$f(\boldsymbol{s}_{n+1}, \boldsymbol{\beta}_{n+1}, \varepsilon^p_{n+1}, \phi_{n+1}, \dot{\varepsilon}_{n+1}, T_{n+1}, \dots) = f(\boldsymbol{\xi}_{n+1}, \varepsilon^p_{n+1}, \phi_{n+1}, \dot{\varepsilon}_{n+1}, T_{n+1}, \dots) = 0\,. \tag{57}$$

Important: We assume that the derivatives with respect to $\dot{\varepsilon}$ and T are small enough to be neglected.

7.1 Newton iterations

We now have the following equations that have to be solved for $\Delta\gamma_{n+1}$:

$$
\begin{aligned}
\boldsymbol{\xi}_{n+1} &= \boldsymbol{\xi}^{\text{trial}} - \Delta\gamma_{n+1}(2\,\mu\,\text{dev}(\boldsymbol{r}_{n+1}) + \text{dev}(\boldsymbol{h})^{\beta}_{n+1}) \\
\varepsilon^p_{n+1} &= \varepsilon^p_n + \Delta\gamma_{n+1}\; h^{\alpha}_{n+1} \\
\phi_{n+1} &= \phi_n + \Delta\gamma_{n+1}\; h^{\phi}_{n+1} \\
f(\boldsymbol{\xi}_{n+1}, \varepsilon^p_{n+1}, \phi_{n+1}, \dot{\varepsilon}_{n+1}, T_{n+1}, \dots) &= 0\,.
\end{aligned}
\tag{58}
$$

Recall that if $g(\Delta\gamma) = 0$ is a nonlinear equation that we have to solve for $\Delta\gamma$, an iterative Newton method can be expressed as

$$
\Delta\gamma^{(k+1)} = \Delta\gamma^{(k)} - \left[\frac{dg}{d\Delta\gamma}\right]^{-1}_{(k)} g^{(k)}\,.
\tag{59}
$$

Define

$$
\delta\gamma := \Delta\gamma^{(k+1)} - \Delta\gamma^{(k)}\,.
\tag{60}
$$

Then, the iterative scheme can be written as

$$
g^{(k)} + \left[\frac{dg}{d\Delta\gamma}\right]^{(k)} \delta\gamma = 0\,.
\tag{61}
$$

In our case we have

$$
\begin{aligned}
\mathbf{a}(\Delta\gamma) &= 0 = -\boldsymbol{\xi} + \boldsymbol{\xi}^{\text{trial}} - \Delta\gamma(2\,\mu\,\text{dev}(\boldsymbol{r}) + \text{dev}(\boldsymbol{h})^{\beta}) \\
b(\Delta\gamma) &= 0 = -\varepsilon^p + \varepsilon^p_n + \Delta\gamma\; h^{\alpha} \\
c(\Delta\gamma) &= 0 = -\phi + \phi_n + \Delta\gamma\; h^{\phi} \\
f(\Delta\gamma) &= 0 = f(\boldsymbol{\xi}, \varepsilon^p, \phi, \dot{\varepsilon}, T, \dots)
\end{aligned}
\tag{62}
$$

Therefore,

$$
\begin{aligned}
\frac{d\mathbf{a}}{d\Delta\gamma} &= -\frac{\partial\boldsymbol{\xi}}{\partial\Delta\gamma} - (2\,\mu\,\text{dev}(\boldsymbol{r}) + \text{dev}(\boldsymbol{h})^{\beta}) - \Delta\gamma\,\left(2\,\mu\,\frac{\partial\text{dev}(\boldsymbol{r})}{\partial\Delta\gamma} + \frac{\partial\text{dev}(\boldsymbol{h})^{\beta}}{\partial\Delta\gamma}\right) \\
&= -\frac{\partial\boldsymbol{\xi}}{\partial\Delta\gamma} - (2\,\mu\,\text{dev}(\boldsymbol{r}) + \text{dev}(\boldsymbol{h})^{\beta}) - \Delta\gamma\,\left(2\,\mu\,\frac{\partial\text{dev}(\boldsymbol{r})}{\partial\boldsymbol{\xi}} : \frac{\partial\boldsymbol{\xi}}{\partial\Delta\gamma} + 2\,\mu\,\frac{\partial\text{dev}(\boldsymbol{r})}{\partial\varepsilon^p}\,\frac{\partial\varepsilon^p}{\partial\Delta\gamma} + 2\,\mu\,\frac{\partial\text{dev}(\boldsymbol{r})}{\partial\phi}\,\frac{\partial\phi}{\partial\Delta\gamma} + \right. \\
&\qquad \left. \frac{\partial\text{dev}(\boldsymbol{h})^{\beta}}{\partial\boldsymbol{\xi}} : \frac{\partial\boldsymbol{\xi}}{\partial\Delta\gamma} + \frac{\partial\text{dev}(\boldsymbol{h})^{\beta}}{\partial\varepsilon^p}\,\frac{\partial\varepsilon^p}{\partial\Delta\gamma} + \frac{\partial\text{dev}(\boldsymbol{h})^{\beta}}{\partial\phi}\,\frac{\partial\phi}{\partial\Delta\gamma}\right) \\
\frac{db}{d\Delta\gamma} &= -\frac{\partial\varepsilon^p}{\partial\Delta\gamma} + h^{\alpha} + \Delta\gamma\,\left(\frac{\partial h^{\alpha}}{\partial\boldsymbol{\xi}} : \frac{\partial\boldsymbol{\xi}}{\partial\Delta\gamma} + \frac{\partial h^{\alpha}}{\partial\varepsilon^p}\,\frac{\partial\varepsilon^p}{\partial\Delta\gamma} + \frac{\partial h^{\alpha}}{\partial\phi}\,\frac{\partial\phi}{\partial\Delta\gamma}\right) \\
\frac{dc}{d\Delta\gamma} &= -\frac{\partial\phi}{\partial\Delta\gamma} + h^{\phi} + \Delta\gamma\,\left(\frac{\partial h^{\phi}}{\partial\boldsymbol{\xi}} : \frac{\partial\boldsymbol{\xi}}{\partial\Delta\gamma} + \frac{\partial h^{\phi}}{\partial\varepsilon^p}\,\frac{\partial\varepsilon^p}{\partial\Delta\gamma} + \frac{\partial h^{\phi}}{\partial\phi}\,\frac{\partial\phi}{\partial\Delta\gamma}\right) \\
\frac{df}{d\Delta\gamma} &= \frac{\partial f}{\partial\boldsymbol{\xi}} : \frac{\partial\boldsymbol{\xi}}{\partial\Delta\gamma} + \frac{\partial f}{\partial\varepsilon^p}\,\frac{\partial\varepsilon^p}{\partial\Delta\gamma} + \frac{\partial f}{\partial\phi}\,\frac{\partial\phi}{\partial\Delta\gamma}\,.
\end{aligned}
\tag{63}
$$

Now, define

$$
\Delta\boldsymbol{\xi} := \frac{\partial\boldsymbol{\xi}}{\partial\Delta\gamma}\,\delta\gamma\,;\quad \Delta\varepsilon^p := \frac{\partial\varepsilon^p}{\partial\Delta\gamma}\,\delta\gamma\,;\quad \Delta\phi := \frac{\partial\phi}{\partial\Delta\gamma}\,\delta\gamma\,.
\tag{64}
$$

Then

$$
\begin{aligned}
&\mathbf{a}^{(k)} - \Delta\boldsymbol{\xi} - [2\,\mu\,\text{dev}(\boldsymbol{r}^{(k)}) + \text{dev}(\boldsymbol{h})^{\beta(k)}]\,\delta\gamma \\
&\qquad\qquad - 2\,\mu\,\Delta\gamma\,\left(\frac{\partial\text{dev}(\boldsymbol{r}^{(k)})}{\partial\boldsymbol{\xi}} : \Delta\boldsymbol{\xi} + \frac{\partial\text{dev}(\boldsymbol{r}^{(k)})}{\partial\varepsilon^p}\,\Delta\varepsilon^p + \frac{\partial\text{dev}(\boldsymbol{r}^{(k)})}{\partial\phi}\,\Delta\phi\right) \\
&\qquad\qquad - \Delta\gamma\,\left(\frac{\partial\text{dev}(\boldsymbol{h})^{\beta(k)}}{\partial\boldsymbol{\xi}} : \Delta\boldsymbol{\xi} + \frac{\partial\text{dev}(\boldsymbol{h})^{\beta(k)}}{\partial\varepsilon^p}\,\Delta\varepsilon^p + \frac{\partial\text{dev}(\boldsymbol{h})^{\beta(k)}}{\partial\phi}\,\Delta\phi\right) = 0 \\
&b^{(k)} - \Delta\varepsilon^p + h^\alpha\,\delta\gamma + \Delta\gamma\,\left(\frac{\partial h^{\alpha(k)}}{\partial\boldsymbol{\xi}} : \Delta\boldsymbol{\xi} + \frac{\partial h^{\alpha(k)}}{\partial\varepsilon^p}\,\Delta\varepsilon^p + \frac{\partial h^{\alpha(k)}}{\partial\phi}\,\Delta\phi\right) = 0 \\
&c^{(k)} - \Delta\phi + h^\phi\,\delta\gamma + \Delta\gamma\,\left(\frac{\partial h^{\phi(k)}}{\partial\boldsymbol{\xi}} : \Delta\boldsymbol{\xi} + \frac{\partial h^{\phi(k)}}{\partial\varepsilon^p}\,\Delta\varepsilon^p + \frac{\partial h^{\phi(k)}}{\partial\phi}\,\Delta\phi\right) = 0 \\
&f^{(k)} + \frac{\partial f^{(k)}}{\partial\boldsymbol{\xi}} : \Delta\boldsymbol{\xi} + \frac{\partial f^{(k)}}{\partial\varepsilon^p}\,\Delta\varepsilon^p + \frac{\partial f^{(k)}}{\partial\phi}\,\Delta\phi = 0
\end{aligned}
\tag{65}
$$

Because the derivatives of $\boldsymbol{r}^{(k)}, \boldsymbol{h}^{\alpha(k)}, \boldsymbol{h}^{\beta(k)}, \boldsymbol{h}^{\phi(k)}$ with respect to $\boldsymbol{\xi}, \varepsilon^p, \phi$ may be difficult to calculate, we instead use a semi-implicit scheme in our implementation where the quantities $\boldsymbol{r}$, h^α, $\boldsymbol{h}^\beta$, and h^ϕ are evaluated at t_n. Then the problematic derivatives disappear and we are left with

$$
\begin{aligned}
\mathbf{a}^{(k)} - \Delta\boldsymbol{\xi} - [2\,\mu\,\text{dev}(\boldsymbol{r}_n) + \text{dev}(\boldsymbol{h})^\beta_n]\,\delta\gamma &= 0 \\
b^{(k)} - \Delta\varepsilon^p + h^\alpha_n\,\delta\gamma &= 0 \\
c^{(k)} - \Delta\phi + h^\phi_n\,\delta\gamma &= 0 \\
f^{(k)} + \frac{\partial f^{(k)}}{\partial\boldsymbol{\xi}} : \Delta\boldsymbol{\xi} + \frac{\partial f^{(k)}}{\partial\varepsilon^p}\,\Delta\varepsilon^p + \frac{\partial f^{(k)}}{\partial\phi}\,\Delta\phi &= 0
\end{aligned}
\tag{66}
$$

We now force $\mathbf{a}^{(k)}$, $b^{(k)}$, and $c^{(k)}$ to be zero at all times, leading to the expressions

$$
\begin{aligned}
\Delta\boldsymbol{\xi} &= -[2\,\mu\,\text{dev}(\boldsymbol{r}_n) + \text{dev}(\boldsymbol{h})^\beta_n]\,\delta\gamma \\
\Delta\varepsilon^p &= h^\alpha_n\,\delta\gamma \\
\Delta\phi &= h^\phi_n\,\delta\gamma \\
f^{(k)} + \frac{\partial f^{(k)}}{\partial\boldsymbol{\xi}} : \Delta\boldsymbol{\xi} + \frac{\partial f^{(k)}}{\partial\varepsilon^p}\,\Delta\varepsilon^p + \frac{\partial f^{(k)}}{\partial\phi}\,\Delta\phi &= 0
\end{aligned}
\tag{67}
$$

Plugging the expressions for $\Delta\boldsymbol{\xi}$, $\Delta\varepsilon^p$, $\Delta\phi$ from the first three equations into the fourth gives us

$$
f^{(k)} - \frac{\partial f^{(k)}}{\partial\boldsymbol{\xi}} : [2\,\mu\,\text{dev}(\boldsymbol{r}_n) + \text{dev}(\boldsymbol{h})^\beta_n]\,\delta\gamma + h^\alpha_n\,\frac{\partial f^{(k)}}{\partial\varepsilon^p}\,\delta\gamma + h^\phi_n\,\frac{\partial f^{(k)}}{\partial\phi}\,\delta\gamma = 0 \tag{68}
$$

or

$$
\Delta\gamma^{(k+1)} - \Delta\gamma^{(k)} = \delta\gamma = \frac{f^{(k)}}{\dfrac{\partial f^{(k)}}{\partial\boldsymbol{\xi}} : [2\,\mu\,\text{dev}(\boldsymbol{r}_n) + \text{dev}(\boldsymbol{h}^\beta_n)] - h^\alpha_n\,\dfrac{\partial f^{(k)}}{\partial\varepsilon^p} - h^\phi_n\,\dfrac{\partial f^{(k)}}{\partial\phi}}\,. \tag{69}
$$

7.2 Algorithm

The following stress update algorithm is used for each (plastic) time step:

1. Initialize:

$$k = 0\ ;\ \ (\varepsilon^p)^{(k)} = \varepsilon^p_n\ ;\ \ \phi^{(k)} = \phi_n\ ;\ \ \boldsymbol{\beta}^{(k)} = \boldsymbol{\beta}_n\ ;\ \ \Delta\gamma^{(k)} = 0\ ;\ \ \boldsymbol{\xi}^{(k)} = \boldsymbol{\xi}^{\text{trial}}\ . \tag{70}$$

2. Check yield condition:

$$f^{(k)} := f(\boldsymbol{\xi}^{(k)}, (\varepsilon^p)^{(k)}, \phi^{(k)}, \dot{\varepsilon}_n, T_n, \dots) \tag{71}$$

   If $f^{(k)} <$ tolerance then go to step 5 else go to step 3.

3. Compute updated $\delta\gamma^{(k)}$ using

$$\delta\gamma^{(k)} = \frac{f^{(k)}}{\dfrac{\partial f^{(k)}}{\partial\boldsymbol{\xi}} : [2\,\mu\,\text{dev}(\boldsymbol{r}_n) + \text{dev}(\boldsymbol{h}^\beta_n)] - h^\alpha_n\,\dfrac{\partial f^{(k)}}{\partial\varepsilon^p} - h^\phi_n\,\dfrac{\partial f^{(k)}}{\partial\phi}}\ . \tag{72}$$

   Compute

$$\begin{aligned}
\Delta\boldsymbol{\xi}^{(k)} &= -[2\,\mu\,\text{dev}(\boldsymbol{r}_n) + \text{dev}(\boldsymbol{h}^\beta_n)]\,\delta\gamma^{(k)} \\
(\Delta\varepsilon^p)^{(k)} &= h^\alpha_n\,\delta\gamma^{(k)} \\
\Delta\phi^{(k)} &= h^\phi_n\,\delta\gamma^{(k)}
\end{aligned} \tag{73}$$

4. Update variables:

$$\begin{aligned}
(\varepsilon^p)^{(k+1)} &= (\varepsilon^p)^{(k)} + (\Delta\varepsilon^p)^{(k)} \\
\phi^{(k+1)} &= \phi^{(k)} + \Delta\phi^{(k)} \\
\boldsymbol{\xi}^{(k+1)} &= \boldsymbol{\xi}^{(k)} + \Delta\boldsymbol{\xi}^{(k)} \\
\Delta\gamma^{(k+1)} &= \Delta\gamma^{(k)} + \delta\gamma^{(k)}
\end{aligned} \tag{74}$$

   Set $k \leftarrow k + 1$ and go to step 2.

5. Update and calculate back stress and the deviatoric part of Cauchy stress:

$$\varepsilon^p_{n+1} = (\varepsilon^p)^{(k)}\ ;\ \ \phi_{n+1} = \phi^{(k)}\ ;\ \ \boldsymbol{\xi}_{n+1} = \boldsymbol{\xi}^{(k)}\ ;\ \ \Delta\gamma_{n+1} = \Delta\gamma^{(k)} \tag{75}$$

   and

$$\begin{aligned}
\widehat{\boldsymbol{\beta}}_{n+1} &= \widehat{\boldsymbol{\beta}}_n + \Delta\gamma_{n+1}\,\boldsymbol{h}^\beta(\boldsymbol{\xi}_{n+1}, \varepsilon^p_{n+1}, \phi_{n+1}) \\
\boldsymbol{\beta}_{n+1} &= \widehat{\boldsymbol{\beta}}_{n+1} - \frac{1}{3}\,\text{tr}(\widehat{\boldsymbol{\beta}}_{n+1})\ \boldsymbol{1} \\
\boldsymbol{s}_{n+1} &= \boldsymbol{\xi}_{n+1} + \boldsymbol{\beta}_{n+1}
\end{aligned} \tag{76}$$

6. Update the temperature and the Cauchy stress

$$\begin{aligned}
T_{n+1} &= T_n + \frac{\chi_{n+1}\,\Delta t}{\rho_{n+1}\,C_p}\,\sigma_y^{n+1}\,\dot{\varepsilon}^p{}_{n+1} = T_n + \frac{\chi_{n+1}\,\Delta\gamma_{n+1}}{\rho_{n+1}\,C_p}\,\sigma_y^{n+1}\,h^\alpha_{n+1} \\
p_{n+1} &= p(J_{n+1}) \\
\kappa_{n+1} &= \left[\frac{dp(J)}{dJ}\right]_{n+1} \\
\boldsymbol{\sigma}_{n+1} &= [p_{n+1} - J\,\kappa_{n+1}\,\alpha\,(T_{n+1} - T_0)]\ \boldsymbol{1} + \boldsymbol{s}_{n+1}
\end{aligned} \tag{77}$$

8 Examples

Let us now look at a few examples.

8.1 Example 1

Consider the case of J_2 plasticity with the yield condition

$$f := \sqrt{\frac{3}{2}}\,\|\boldsymbol{s}-\boldsymbol{\beta}\| - \sigma_y(\varepsilon^p,\dot{\varepsilon},T,\dots) = \sqrt{\frac{3}{2}}\,\|\boldsymbol{\xi}\| - \sigma_y(\varepsilon^p,\dot{\varepsilon},T,\dots) \le 0 \tag{78}$$

where $\|\boldsymbol{\xi}\|_= \sqrt{\boldsymbol{\xi}:\boldsymbol{\xi}}$. Assume the associated flow rule

$$\boldsymbol{d}^p = \dot{\gamma}\,\boldsymbol{r} = \dot{\gamma}\,\frac{\partial f}{\partial \boldsymbol{\sigma}} = \dot{\gamma}\,\frac{\partial f}{\partial \boldsymbol{\xi}}\,. \tag{79}$$

Then

$$\boldsymbol{r} = \frac{\partial f}{\partial \boldsymbol{\xi}} = \sqrt{\frac{3}{2}}\,\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|} \tag{80}$$

and

$$\boldsymbol{d}^p = \sqrt{\frac{3}{2}}\,\dot{\gamma}\,\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\,;\quad \|\boldsymbol{d}^p\| = \sqrt{\frac{3}{2}}\,\dot{\gamma}\,. \tag{81}$$

The evolution of the equivalent plastic strain is given by

$$\dot{\varepsilon^p} = \dot{\gamma}\,h^\alpha = \sqrt{\frac{2}{3}}\,\|\boldsymbol{d}^p\| = \dot{\gamma}\,. \tag{82}$$

This definition is consistent with the definition of equivalent plastic strain

$$\varepsilon^p = \int_0^t \dot{\varepsilon}^p\,d\tau = \int_0^t \sqrt{\frac{2}{3}}\,\|\boldsymbol{d}^p\|\;d\tau\,. \tag{83}$$

The evolution of porosity is given by (there is no evolution of porosity)

$$\dot{\phi} = \dot{\gamma}\,h^\phi = 0 \tag{84}$$

The evolution of the back stress is given by the Prager kinematic hardening rule

$$\dot{\widehat{\boldsymbol{\beta}}} = \dot{\gamma}\,\boldsymbol{h}^\beta = \frac{2}{3}\,H'\,\boldsymbol{d}^p \tag{85}$$

where $\widehat{\boldsymbol{\beta}}$ is the back stress and H' is a constant hardening modulus. Also, the trace of $\boldsymbol{d}^p$ is

$$\text{tr}(\boldsymbol{d}^p) = \sqrt{\frac{3}{2}}\,\dot{\gamma}\,\frac{\text{tr}(\boldsymbol{\xi})}{\|\boldsymbol{\xi}\|}\,. \tag{86}$$

Since $\boldsymbol{\xi}$ is deviatoric, $\text{tr}(\boldsymbol{\xi}) = 0$ and hence $\boldsymbol{d}^p = \boldsymbol{\eta}^p$. Hence, $\widehat{\boldsymbol{\beta}} = \boldsymbol{\beta}$ (where $\boldsymbol{\beta}$ is the deviatoric part of $\widehat{\boldsymbol{\beta}}$), and

$$\dot{\boldsymbol{\beta}} = \sqrt{\frac{2}{3}}\,H'\,\dot{\gamma}\,\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\,. \tag{87}$$

These relation imply that

$$\boxed{\begin{aligned} \boldsymbol{r} &= \sqrt{\frac{3}{2}}\ \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|} \\ h^\alpha &= 1 \\ h^\phi &= 0 \\ \boldsymbol{h}^\beta &= \sqrt{\frac{2}{3}}\ H'\ \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\ . \end{aligned}} \tag{88}$$

We also need some derivatives of the yield function. These are

$$\begin{aligned} \frac{\partial f}{\partial \boldsymbol{\xi}} &= \boldsymbol{r} \\ \frac{\partial f}{\partial \varepsilon^p} &= -\frac{\partial \sigma_y}{\partial \varepsilon^p} \\ \frac{\partial f}{\partial \phi} &= 0\ . \end{aligned} \tag{89}$$

Let us change the kinematic hardening model and use the Armstrong-Frederick model instead, i.e.,

$$\dot{\boldsymbol{\beta}} = \dot{\gamma}\ \boldsymbol{h}^\beta = \frac{2}{3}\ H_1\ \boldsymbol{d}^p - H_2\ \boldsymbol{\beta}\ \|\boldsymbol{d}^p\|\ \ . \tag{90}$$

Since

$$\boldsymbol{d}^p = \sqrt{\frac{3}{2}}\ \dot{\gamma}\ \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|} \tag{91}$$

we have

$$\|\boldsymbol{d}^p\| = \sqrt{\frac{3}{2}}\ \dot{\gamma}\ \frac{\|\boldsymbol{\xi}\|}{\|\boldsymbol{\xi}\|} = \sqrt{\frac{3}{2}}\ \dot{\gamma}\ . \tag{92}$$

Therefore,

$$\dot{\boldsymbol{\beta}} = \sqrt{\frac{2}{3}}\ H_1\ \dot{\gamma}\ \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|} - \sqrt{\frac{3}{2}}\ H_2\ \dot{\gamma}\ \boldsymbol{\beta}\ . \tag{93}$$

Hence we have

$$\boxed{\boldsymbol{h}^\beta = \sqrt{\frac{2}{3}}\ H_1\ \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|} - \sqrt{\frac{3}{2}}\ H_2\ \boldsymbol{\beta}\ .} \tag{94}$$

8.2 Example 2

Let us now consider a Gurson type yield condition with kinematic hardening. In this case the yield condition can be written as

$$f := \frac{3\ \boldsymbol{\xi} : \boldsymbol{\xi}}{2\ \sigma_y^2} + 2\ q_1\ \phi^*\ \cosh\left(\frac{q_2\ \mathrm{tr}(\boldsymbol{\sigma})}{2\ \sigma_y}\right) - [1 + q_3\ (\phi^*)^2] \tag{95}$$

where ϕ is the porosity and

$$\phi^* = \begin{cases} \phi & \text{for } \phi \le \phi_c \\ \phi_c - \dfrac{\phi_u^* - \phi_c}{\phi_f - \phi_c}\,(\phi - \phi_c) & \text{for } \phi > \phi_c \end{cases} \tag{96}$$

Final fracture occurs for $\phi = \phi_f$ or when $\phi^*_u = 1/q_1$.

Let us use an associated flow rule

$$\boldsymbol{d}^p = \dot{\gamma}\, \boldsymbol{r} = \dot{\gamma}\, \frac{\partial f}{\partial \boldsymbol{\sigma}}\ . \tag{97}$$

Then

$$\boldsymbol{r} = \frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{3\, \boldsymbol{\xi}}{\sigma_y^2} + \frac{q_1\, q_2\, \phi^*}{\sigma_y}\, \sinh\left(\frac{q_2\, \text{tr}(\boldsymbol{\sigma})}{2\, \sigma_y}\right)\, \boldsymbol{1}\ . \tag{98}$$

In this case

$$\text{tr}(\boldsymbol{r}) = \frac{3\, q_1\, q_2\, \phi^*}{\sigma_y}\, \sinh\left(\frac{q_2\, \text{tr}(\boldsymbol{\sigma})}{2\, \sigma_y}\right) \neq 0 \tag{99}$$

Therefore,

$$\boldsymbol{d}^p \neq \boldsymbol{\eta}^p\ . \tag{100}$$

For the evolution equation for the plastic strain we use

$$(\boldsymbol{\sigma} - \widehat{\boldsymbol{\beta}}) : \boldsymbol{d}^p = (1-\phi)\, \sigma_y\, \dot{\varepsilon}^p \tag{101}$$

where $\dot{\varepsilon}^p$ is the effective plastic strain rate in the matrix material. Hence,

$$\dot{\varepsilon}^p = \dot{\gamma}\, h^\alpha = \dot{\gamma}\, \frac{(\boldsymbol{\sigma} - \widehat{\boldsymbol{\beta}}) : \boldsymbol{r}}{(1-\phi)\, \sigma_y}\,. \tag{102}$$

The evolution equation for the porosity is given by

$$\dot{\phi} = (1-\phi)\, \text{tr}(\boldsymbol{d}^p) + A\, \dot{\varepsilon^p} \tag{103}$$

where

$$A = \frac{f_n}{s_n\sqrt{2\pi}} \exp[-1/2(\varepsilon^p - \varepsilon_n)^2/s_n^2] \tag{104}$$

and f_n is the volume fraction of void nucleating particles, ε_n is the mean of the normal distribution of nucleation strains, and s_n is the standard deviation of the distribution.

Therefore,

$$\dot{\phi} = \dot{\gamma}\, h^\phi = \dot{\gamma}\, \left[(1-\phi)\, \text{tr}(\boldsymbol{r}) + A\, \frac{(\boldsymbol{\sigma} - \widehat{\boldsymbol{\beta}}) : \boldsymbol{r}}{(1-\phi)\, \sigma_y}\right]\ . \tag{105}$$

If the evolution of the back stress is given by the Prager kinematic hardening rule

$$\dot{\widehat{\boldsymbol{\beta}}} = \dot{\gamma}\, \boldsymbol{h}^\beta = \frac{2}{3}\, H'\, \boldsymbol{d}^p \tag{106}$$

where $\widehat{\boldsymbol{\beta}}$ is the back stress, then

$$\dot{\widehat{\boldsymbol{\beta}}} = \frac{2}{3}\, H'\, \dot{\gamma}\, \boldsymbol{r}\ . \tag{107}$$

Alternatively, if we use the Armstrong-Frederick model, then

$$\dot{\widehat{\boldsymbol{\beta}}} = \dot{\gamma}\, \boldsymbol{h}^\beta = \frac{2}{3}\, H_1\, \boldsymbol{d}^p - H_2\, \widehat{\boldsymbol{\beta}}\, \|\boldsymbol{d}^p\|\ . \tag{108}$$

Plugging in the expression for $\boldsymbol{d}^p$, we have

$$\dot{\widehat{\boldsymbol{\beta}}} = \dot{\gamma}\ \left[\frac{2}{3}\ H_1\ \boldsymbol{r} - H_2\ \widehat{\boldsymbol{\beta}}\ \|\boldsymbol{r}\|\right]\ . \tag{109}$$

Therefore, for this model,

$$\boxed{\begin{aligned} \boldsymbol{r} &= \frac{3\ \boldsymbol{\xi}}{\sigma_y^2} + \frac{q_1\ q_2\ \phi^*}{\sigma_y}\ \sinh\left(\frac{q_2\ \text{tr}(\boldsymbol{\sigma})}{2\ \sigma_y}\right)\ \boldsymbol{1} \\ h^\alpha &= \frac{(\boldsymbol{\sigma} - \boldsymbol{\beta}) : \boldsymbol{r}}{(1-\phi)\ \sigma_y} \\ h^\phi &= (1-\phi)\ \text{tr}(\boldsymbol{r}) + A\ \frac{(\boldsymbol{\sigma} - \widehat{\boldsymbol{\beta}}) : \boldsymbol{r}}{(1-\phi)\ \sigma_y} \\ \boldsymbol{h}^\beta &= \frac{2}{3}\ H_1\ \boldsymbol{r} - H_2\ \widehat{\boldsymbol{\beta}}\ \|\boldsymbol{r}\| \end{aligned}} \tag{110}$$

The other derivatives of the yield function that we need are

$$\begin{aligned} \frac{\partial f}{\partial \boldsymbol{\xi}} &= \frac{3\ \boldsymbol{\xi}}{\sigma_y^2} \\ \frac{\partial f}{\partial \varepsilon^p} &= \frac{\partial f}{\partial \sigma_y}\ \frac{\partial \sigma_y}{\partial \varepsilon^p} = -\left[\frac{3\ \boldsymbol{\xi} : \boldsymbol{\xi}}{\sigma_y^3} + \frac{q_1\ q_2\ \phi^*\ \text{tr}(\boldsymbol{\sigma})}{\sigma_y^2}\ \sinh\left(\frac{q_2\ \text{tr}(\boldsymbol{\sigma})}{2\ \sigma_y}\right)\right]\ \frac{\partial \sigma_y}{\partial \varepsilon^p} \\ \frac{\partial f}{\partial \phi} &= 2\ q_1\ \frac{d\phi^*}{d\phi}\ \cosh\left(\frac{q_2\ \text{tr}(\sigma)}{2\ \sigma_y}\right) - 2\ q_3\ \phi^*\ \frac{d\phi^*}{d\phi}\ . \end{aligned} \tag{111}$$