Cam-Clay model based on Borja et al. 1997

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1 Introduction

Introduce the equations and how they differ from Fossum-Brannon.

2 Quantities that are needed in a Uintah implementation

2.1 Elasticity

The elastic strain energy density in Borja's model has the form

$$W(\varepsilon_v^e, \varepsilon_s^e) = W_{\text{vol}}(\varepsilon_v^e) + W_{\text{dev}}(\varepsilon_v^e, V e_s^e)$$

where

$$\begin{aligned} W_{\text{vol}}(\varepsilon_v^e) &= -p_0 \widetilde{\kappa} \exp\left(-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right) \\ W_{\text{dev}}(\varepsilon_v^e, \varepsilon_s^e) &= \tfrac{3}{2}\,\mu\,(\varepsilon_s^e)^2 \end{aligned}$$

where ε_{v0}^e is the volumetric strain corresponding to a mean normal compressive stress p_0 (tension positive), $\widetilde{\kappa}$ is the elastic compressibility index, and the shear modulus is given by

$$\mu = \mu_0 + \frac{\alpha}{\widetilde{\kappa}} W_{\text{vol}}(\varepsilon_v^e) = \mu_0 - \alpha p_0 \exp\left(-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right).$$

The parameter α determines the extent of coupling between the volumetric and deviatoric responses. For consistency with isotropic elasticity, Rebecca Brannon suggests that $\alpha = 0$ (citation?).

The stress invariants p and q are defined as

$$\begin{aligned} p &= \frac{\partial W}{\partial \varepsilon_v^e} = p_0\left[1 + \tfrac{3}{2}\,\frac{\alpha}{\widetilde{\kappa}}\,(\varepsilon_s^e)^2\right]\exp\left(-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right) = p_0\,\beta\,\exp\left(-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right) \\ q &= \frac{\partial W}{\partial \varepsilon_s^e} = 3\left[\mu_0 - \alpha p_0 \exp\left(-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right)\right]\varepsilon_s^e = 3\mu\,\varepsilon_s^e\,. \end{aligned}$$

The derivatives of the stress invariants are

$$\begin{aligned} \frac{\partial p}{\partial \varepsilon_v^e} &= -\frac{p_0}{\widetilde{\kappa}}\left[1 + \tfrac{3}{2}\,\frac{\alpha}{\widetilde{\kappa}}\,(\varepsilon_s^e)^2\right]\exp\left(-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right) = -\frac{p}{\widetilde{\kappa}} \\ \frac{\partial p}{\partial \varepsilon_s^e} &= \frac{\partial q}{\partial \varepsilon_v^e} = \frac{3\alpha p_0 \varepsilon_s^e}{\widetilde{\kappa}}\exp\left(-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right) = \frac{3\alpha p}{\beta\widetilde{\kappa}}\,\varepsilon_s^e \\ \frac{\partial q}{\partial \varepsilon_s^e} &= 3\left[\mu_0 - \alpha p_0 \exp\left(-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right)\right] = 3\mu\,. \end{aligned}$$

2.2 Plasticity

For plasticity we use a Cam-Clay yield function of the form

$$f = \left(\frac{q}{M}\right)^2 + p(p - p_c)$$

where M is the slope of the critical state line and the consolidation pressure p_c is an internal variable that evolves according to

$$\frac{1}{p_c}\,\frac{dp_c}{dt} = \frac{1}{\widetilde{\lambda} - \widetilde{\kappa}}\,\frac{d\varepsilon_v^p}{dt}\,.$$

The derivatives of f that are of interest are

$$\begin{aligned} \frac{\partial f}{\partial p} &= 2p - p_c \\ \frac{\partial f}{\partial q} &= \frac{2q}{M^2}\,. \end{aligned}$$

If we integrate the equation for p_c from t_n to t_{n+1}, we can show that

$$(p_c)_{n+1} = (p_c)_n \exp\left[\frac{(\varepsilon_v^e)_{\text{trial}} - (\varepsilon_v^e)_{n+1}}{\widetilde{\lambda} - \widetilde{\kappa}}\right].$$

The derivative of p_c that is of interest is

$$\frac{\partial p_c}{\partial(\varepsilon_v^e)_{n+1}} = -\frac{(p_c)_n}{\widetilde{\lambda} - \widetilde{\kappa}} \exp\left[\frac{(\varepsilon_v^e)_{\text{trial}} - (\varepsilon_v^e)_{n+1}}{\widetilde{\lambda} - \widetilde{\kappa}}\right].$$

3 Why these quantities are needed: stress update based Rich Reguiero's notes

The volumetric and deviatoric components of the elastic strain $\boldsymbol{\epsilon}^e$ are defined as follows:

$$\boldsymbol{e}^e = \boldsymbol{\epsilon}^e - \tfrac{1}{3}\varepsilon_v^e \boldsymbol{1} = \boldsymbol{\epsilon}^e - \tfrac{1}{3}\text{tr}(\boldsymbol{\epsilon}^e)\,\boldsymbol{1} \quad \text{and} \quad \varepsilon_s^e = \sqrt{\tfrac{2}{3}}\,\|\boldsymbol{e}^e\| = \sqrt{\tfrac{2}{3}}\,\sqrt{\boldsymbol{e}^e : \boldsymbol{e}^e}\,.$$

The stress tensor is decomposed into a volumetric and a deviatoric component

$$\boldsymbol{\sigma} = p\,\boldsymbol{1} + \sqrt{\tfrac{2}{3}}\,q\,\boldsymbol{n} \quad \text{with} \quad \boldsymbol{n} = \frac{\boldsymbol{e}^e}{\|\boldsymbol{e}^e\|} = \sqrt{\tfrac{2}{3}}\,\frac{\boldsymbol{e}^e}{\varepsilon_s^e}.$$

The models used to determine p and q are

$$\begin{aligned} p &= p_0\beta \exp\left[-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right] \quad \text{with} \quad \beta = 1 + \tfrac{3}{2}\,\frac{\alpha}{\widetilde{\kappa}}\,(\varepsilon_s^e)^2 \\ q &= 3\mu\varepsilon_s^e\,. \end{aligned}$$

The strains are updated using

$$\boldsymbol{\epsilon}^e = \boldsymbol{\epsilon}^e_{\text{trial}} - \Delta\gamma\,\frac{\partial f}{\partial \boldsymbol{\sigma}} \quad \text{where} \quad \boldsymbol{\epsilon}^e_{\text{trial}} = \boldsymbol{\epsilon}^e_n + \Delta\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^e_n + (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_n)\,.$$

Remark 1: The interface with MPMICE, among other things in Uintah, requires the computation of the quantity dp/dJ. Since J does not appear in the above equation we proceed as explained below.

$$\begin{aligned} J &= \det(\boldsymbol{F}) = \det(\boldsymbol{1} + \nabla_0\,\mathbf{u}) = \det(\boldsymbol{1} + \boldsymbol{\epsilon}) \\ &= 1 + \text{tr}\boldsymbol{\epsilon} + \tfrac{1}{2}\left[(\text{tr}\boldsymbol{\epsilon})^2 - \text{tr}(\boldsymbol{\epsilon}^2)\right] + \det(\boldsymbol{\epsilon})\,. \quad = 1 + \varepsilon_v + \tfrac{1}{2}\left[\varepsilon_v^2 - \text{tr}(\boldsymbol{\epsilon}^2)\right] + \det(\boldsymbol{\epsilon})\,. \end{aligned}$$

Also,

$$J = \frac{\rho_0}{\rho} = \frac{V}{V_0} \quad \text{and} \quad \varepsilon_v = \frac{V - V_0}{V_0} = \frac{V}{V_0} - 1 = J - 1\,.$$

We use the relation $J = 1 + \varepsilon_v$ while keeping in mind that this is *true only for infinitesimal strains and plastic incompressibility* for which ε_v^2, $\text{tr}(\boldsymbol{\epsilon}^2)$, and $\det(\boldsymbol{\epsilon})$ are zero. Under these conditions

$$\frac{\partial p}{\partial J} = \frac{\partial p}{\partial \varepsilon_v}\,\frac{\partial \varepsilon_v}{\partial J} = \frac{\partial p}{\partial \varepsilon_v} \quad \text{and} \quad \frac{\partial p}{\partial \rho} = \frac{\partial p}{\partial \varepsilon_v}\,\frac{\partial \varepsilon_v}{\partial J}\,\frac{\partial J}{\partial \rho} = -\frac{J}{\rho}\,\frac{\partial p}{\partial \varepsilon_v}\,.$$

Remark 2: MPMICE also needs the density at a given pressure. For the Borja model, with $\varepsilon_v = J - 1 = \rho_0/\rho - 1$, we have

$$\rho = \rho_0\left[1 + \varepsilon_{v0} + \widetilde{\kappa}\ln\left(\frac{p}{p_0\beta}\right)\right]^{-1}\,.$$

Remark 3: The quantity q is related to the deviatoric part of the Cauchy stress, $\boldsymbol{s}$ as follows:

$$q = \sqrt{3J_2} \quad \text{where} \quad J_2 = \tfrac{1}{2}\,\boldsymbol{s} : \boldsymbol{s}\,.$$

The shear modulus relates the deviatoric stress $\boldsymbol{s}$ to the deviatoric strain $\boldsymbol{e}^e$. We assume a relation of the form

$$\boldsymbol{s} = 2\mu\boldsymbol{e}^e\,.$$

Note that the above relation assumes a linear elastic type behavior. Then we get the Borja shear model:

$$q = \sqrt{\tfrac{3}{2}\,\boldsymbol{s} : \boldsymbol{s}} = \sqrt{\tfrac{3}{2}}\,(2\mu)\,\sqrt{\boldsymbol{e}^e : \boldsymbol{e}^e} = \sqrt{\tfrac{3}{2}}\,(2\mu)\,\sqrt{\tfrac{3}{2}}\,\varepsilon_s^e = 3\mu\varepsilon_s^e\,.$$

3.1 Elastic-plastic stress update

For elasto-plasticity we start with a yield function of the form

$$f = \left(\frac{q}{M}\right)^2 + p(p - p_c) \le 0 \quad \text{where} \quad \frac{1}{p_c}\,\frac{dp_c}{dt} = \frac{1}{\widetilde{\lambda} - \widetilde{\kappa}}\,\frac{d\varepsilon_v^p}{dt}\,.$$

Integrating the ODE for p_c with the initial condition $p_c(t_n) = (p_c)_n$, at $t = t_{n+1}$,

$$(p_c)_{n+1} = (p_c)_n \exp\left[\frac{(\varepsilon_v^p)_{n+1} - (\varepsilon_v^p)_n}{\widetilde{\lambda} - \widetilde{\kappa}}\right].$$

From the additive decomposition of the strain into elastic and plastic parts, and if the elastic trial strain is defined as

$$(\varepsilon_v^e)_{\text{trial}} := (\varepsilon_v^e)_n + \Delta\varepsilon_v$$

we have

$$\varepsilon_v^p = \varepsilon_v - \varepsilon_v^e \quad \Longrightarrow \quad (\varepsilon_v^p)_{n+1} - (\varepsilon_v^p)_n = (\varepsilon_v)_{n+1} - (\varepsilon_v^e)_{n+1} - (\varepsilon_v)_n + (\varepsilon_v^e)_n = \Delta\varepsilon_v + (\varepsilon_v^e)_n - (\varepsilon_v^e)_{n+1} = (\varepsilon_v^e)_{\text{trial}} - (\varepsilon_v^e)_{n+1}\,.$$

Therefore we can write

$$(p_c)_{n+1} = (p_c)_n \exp\left[\frac{(\varepsilon_v^e)_{\text{trial}} - (\varepsilon_v^e)_{n+1}}{\widetilde{\lambda} - \widetilde{\kappa}}\right].$$

The flow rule is assumed to be given by

$$\frac{\partial \boldsymbol{\epsilon}^p}{\partial t} = \gamma\,\frac{\partial f}{\partial \boldsymbol{\sigma}}\,.$$

Integration of the PDE with backward Euler gives

$$\boldsymbol{\epsilon}_{n+1}^p = \boldsymbol{\epsilon}_n^p + \Delta t\,\gamma_{n+1}\left[\frac{\partial f}{\partial \boldsymbol{\sigma}}\right]_{n+1} = \boldsymbol{\epsilon}_n^p + \Delta\gamma\left[\frac{\partial f}{\partial \boldsymbol{\sigma}}\right]_{n+1}\,.$$

This equation can be expressed in terms of the trial elastic strain as follows.

$$\boldsymbol{\epsilon}_{n+1} - \boldsymbol{\epsilon}_{n+1}^e = \boldsymbol{\epsilon}_n - \boldsymbol{\epsilon}_n^e + \Delta\gamma\left[\frac{\partial f}{\partial \boldsymbol{\sigma}}\right]_{n+1}$$

or

$$\boldsymbol{\epsilon}_{n+1}^e = \Delta\boldsymbol{\epsilon} + \boldsymbol{\epsilon}_n^e - \Delta\gamma\left[\frac{\partial f}{\partial \boldsymbol{\sigma}}\right]_{n+1} = \boldsymbol{\epsilon}_{\text{trial}}^e - \Delta\gamma\left[\frac{\partial f}{\partial \boldsymbol{\sigma}}\right]_{n+1}\,.$$

In terms of the volumetric and deviatoric components

$$(\varepsilon_v^e)_{n+1} = \text{tr}(\boldsymbol{\epsilon}_{n+1}^e) = \text{tr}(\boldsymbol{\epsilon}_{\text{trial}}^e) - \Delta\gamma\,\text{tr}\left[\frac{\partial f}{\partial \boldsymbol{\sigma}}\right]_{n+1} = (\varepsilon_v^e)_{\text{trial}} - \Delta\gamma\,\text{tr}\left[\frac{\partial f}{\partial \boldsymbol{\sigma}}\right]_{n+1}$$

and

$$\boldsymbol{e}_{n+1}^e = \boldsymbol{e}_{\text{trial}}^e - \Delta\gamma\left[\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)_{n+1} - \tfrac{1}{3}\,\text{tr}\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)_{n+1}\boldsymbol{1}\right].$$

With $\boldsymbol{s} = \boldsymbol{\sigma} - p\boldsymbol{1}$, we have

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{\partial f}{\partial \boldsymbol{s}} : \frac{\partial \boldsymbol{s}}{\partial \boldsymbol{\sigma}} + \frac{\partial f}{\partial p}\,\frac{\partial p}{\partial \boldsymbol{\sigma}} = \frac{\partial f}{\partial \boldsymbol{s}} : [\mathbf{I}^{(s)} - \tfrac{1}{3}\,\boldsymbol{1} \otimes \boldsymbol{1}] + \frac{\partial f}{\partial p}\,\boldsymbol{1} = \frac{\partial f}{\partial \boldsymbol{s}} - \tfrac{1}{3}\,\text{tr}\left[\frac{\partial f}{\partial \boldsymbol{s}}\right]\boldsymbol{1} + \frac{\partial f}{\partial p}\,\boldsymbol{1}$$

and

$$\tfrac{1}{3}\,\text{tr}\left[\frac{\partial f}{\partial \boldsymbol{\sigma}}\right]\boldsymbol{1} = \tfrac{1}{3}\left(\text{tr}\left[\frac{\partial f}{\partial \boldsymbol{s}}\right] - \text{tr}\left[\frac{\partial f}{\partial \boldsymbol{s}}\right] + 3\frac{\partial f}{\partial p}\right)\boldsymbol{1} = \frac{\partial f}{\partial p}\,\boldsymbol{1}\,.$$

Remark 4: Note that, because $\sigma = \sigma(p, q, p_c)$ the chain rule should contain a contribution from p_c:

$$\frac{\partial f}{\partial \sigma} = \frac{\partial f}{\partial q}\,\frac{\partial q}{\partial \sigma} + \frac{\partial f}{\partial p}\,\frac{\partial p}{\partial \sigma} + \frac{\partial f}{\partial p_c}\,\frac{\partial p_c}{\partial \sigma}\,.$$

However, the Borja implementation does not consider that extra term. Also note that for the present model

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(p(\varepsilon_v^e, \varepsilon_s^e, \varepsilon_v^p, \varepsilon_s^p), \boldsymbol{s}(\varepsilon_v^e, \varepsilon_s^e, \varepsilon_v^p, \varepsilon_s^p), p_c(\varepsilon_v^p))$$

Therefore, for situations where $\text{tr}(\partial f/\partial \boldsymbol{s}) = \boldsymbol{0}$, we have

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} - \tfrac{1}{3}\,\text{tr}\left[\frac{\partial f}{\partial \boldsymbol{\sigma}}\right]\boldsymbol{1} = \frac{\partial f}{\partial \boldsymbol{s}} - \tfrac{1}{3}\,\text{tr}\left[\frac{\partial f}{\partial \boldsymbol{s}}\right]\boldsymbol{1} = \frac{\partial f}{\partial \boldsymbol{s}}\,.$$

The deviatoric strain update can be written as

$$\boldsymbol{e}^e_{n+1} = \boldsymbol{e}^e_{\text{trial}} - \Delta\gamma\left(\frac{\partial f}{\partial \boldsymbol{s}}\right)_{n+1}$$

and the shear invariant update is

$$(\varepsilon_s^e)_{n+1} = \sqrt{\tfrac{2}{3}}\,\sqrt{\boldsymbol{e}^e_{n+1} : \boldsymbol{e}^e_{n+1}} = \sqrt{\tfrac{2}{3}}\,\sqrt{\boldsymbol{e}^e_{\text{trial}} : \boldsymbol{e}^e_{\text{trial}} - 2\Delta\gamma\left[\frac{\partial f}{\partial \boldsymbol{s}}\right]_{n+1} : \boldsymbol{e}^e_{\text{trial}} + (\Delta\gamma)^2\left[\frac{\partial f}{\partial \boldsymbol{s}}\right]_{n+1} : \left[\frac{\partial f}{\partial \boldsymbol{s}}\right]_{n+1}}$$

The derivative of f can be found using the chain rule (for smooth f):

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{\partial f}{\partial p}\frac{\partial p}{\partial \boldsymbol{\sigma}} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial \boldsymbol{\sigma}} = (2p - p_c)\,\frac{\partial p}{\partial \boldsymbol{\sigma}} + \frac{2q}{M^2}\,\frac{\partial q}{\partial \boldsymbol{\sigma}}\,.$$

Now, with $p = 1/3\,\text{tr}(\boldsymbol{\sigma})$ and $q = \sqrt{3/2\,\mathbf{s} : \mathbf{s}}$, we have

$$\begin{aligned}
\frac{\partial p}{\partial \boldsymbol{\sigma}} &= \frac{\partial}{\partial \boldsymbol{\sigma}}\left[\tfrac{1}{3}\,\text{tr}(\boldsymbol{\sigma})\right] = \tfrac{1}{3}\,\boldsymbol{1} \\
\frac{\partial q}{\partial \boldsymbol{\sigma}} &= \frac{\partial}{\partial \boldsymbol{\sigma}}\left[\sqrt{\tfrac{3}{2}\,\boldsymbol{s} : \boldsymbol{s}}\right] = \sqrt{\tfrac{3}{2}}\,\frac{1}{\sqrt{\boldsymbol{s} : \boldsymbol{s}}}\,\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{\sigma}} : \boldsymbol{s} = \sqrt{\tfrac{3}{2}}\,\frac{1}{\|\boldsymbol{s}\|}\,\left[\mathsf{I}^{(s)} - \tfrac{1}{3}\boldsymbol{1}\otimes\boldsymbol{1}\right] : \boldsymbol{s} = \sqrt{\tfrac{3}{2}}\,\frac{\boldsymbol{s}}{\|\boldsymbol{s}\|}\,.
\end{aligned}$$

Therefore,

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{2p - p_c}{3}\,\boldsymbol{1} + \sqrt{\tfrac{3}{2}}\,\frac{2q}{M^2}\,\frac{\boldsymbol{s}}{\|\boldsymbol{s}\|}\,.$$

Recall that

$$\boldsymbol{\sigma} = p\,\boldsymbol{1} + \sqrt{\tfrac{2}{3}}\,q\,\boldsymbol{n} = p\,\boldsymbol{1} + \boldsymbol{s}\,.$$

Therefore,

$$\boldsymbol{s} = \sqrt{\tfrac{2}{3}}\,q\,\boldsymbol{n} \quad \text{and} \quad \|\boldsymbol{s}\| = \sqrt{\boldsymbol{s} : \boldsymbol{s}} = \sqrt{\tfrac{2}{3}\,q^2\,\boldsymbol{n} : \boldsymbol{n}} = \sqrt{\tfrac{2}{3}\,q^2\,\frac{\boldsymbol{e}^e : \boldsymbol{e}^e}{\|\boldsymbol{e}^e\|^2}} = \sqrt{\tfrac{2}{3}\,q^2} = \sqrt{\tfrac{2}{3}}\,q\,.$$

So we can write

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{2p - p_c}{3}\,\boldsymbol{1} + \sqrt{\tfrac{3}{2}}\,\frac{2q}{M^2}\,\boldsymbol{n}\,. \tag{1}$$

Using the above relation we have

$$\frac{\partial f}{\partial p} = \tfrac{1}{3}\,\text{tr}\left[\frac{\partial f}{\partial \boldsymbol{\sigma}}\right] = 2p - p_c \quad \text{and} \quad \frac{\partial f}{\partial \boldsymbol{s}} = \frac{\partial f}{\partial \boldsymbol{\sigma}} - \frac{\partial f}{\partial p}\boldsymbol{1} = \sqrt{\tfrac{3}{2}}\,\frac{2q}{M^2}\,\boldsymbol{n}\,.$$

The strain updates can now be written as

$$\begin{aligned}
(\varepsilon_v^e)_{n+1} &= (\varepsilon_v^e)_{\text{trial}} - \Delta\gamma\,[2p_{n+1} - (p_c)_{n+1}] \\
\boldsymbol{e}^e_{n+1} &= \boldsymbol{e}^e_{\text{trial}} - \sqrt{\tfrac{3}{2}}\,\Delta\gamma\left(\frac{2q_{n+1}}{M^2_{n+1}}\right)\boldsymbol{n}_{n+1} \\
(\varepsilon_s^e)_{n+1} &= \sqrt{\tfrac{2}{3}}\,\sqrt{\boldsymbol{e}^e_{\text{trial}} : \boldsymbol{e}^e_{\text{trial}} - \sqrt{6}\,(\Delta\gamma)^2\left(\frac{2q_{n+1}}{M^2_{n+1}}\right)\boldsymbol{n}_{n+1} : \boldsymbol{e}^e_{\text{trial}} + \tfrac{3}{2}\,(\Delta\gamma)^4\left(\frac{2q_{n+1}}{M^2_{n+1}}\right)^2}\,.
\end{aligned}$$

From the second equation above,

$$\boldsymbol{n}_{n+1} : \boldsymbol{e}^e_{\text{trial}} = \boldsymbol{n}_{n+1} : \boldsymbol{e}^e_{n+1} + \sqrt{\tfrac{3}{2}}\,\Delta\gamma \left(\frac{2q_{n+1}}{M^2_{n+1}}\right)\boldsymbol{n}_{n+1} : \boldsymbol{n}_{n+1} = \frac{\boldsymbol{e}^e_{n+1} : \boldsymbol{e}^e_{n+1}}{\left\lVert \boldsymbol{e}^e_{n+1}\right\rVert} + \sqrt{\tfrac{3}{2}}\,\Delta\gamma \left(\frac{2q_{n+1}}{M^2_{n+1}}\right) = \left\lVert \boldsymbol{e}^e_{n+1}\right\rVert + \sqrt{\tfrac{3}{2}}\,\Delta\gamma \left(\frac{2q_{n+1}}{M^2_{n+1}}\right).$$

Also notice that

$$\boldsymbol{e}^e_{\text{trial}} : \boldsymbol{e}^e_{\text{trial}} = \boldsymbol{e}^e_{n+1} : \boldsymbol{e}^e_{n+1} + 2\,\sqrt{\tfrac{3}{2}}\,\Delta\gamma \left(\frac{2q_{n+1}}{M^2_{n+1}}\right)\boldsymbol{e}^e_{n+1} : \boldsymbol{n}_{n+1} + \left[\sqrt{\tfrac{3}{2}}\,\Delta\gamma \left(\frac{2q_{n+1}}{M^2_{n+1}}\right)\right]^2$$

or,

$$\left\lVert \boldsymbol{e}^e_{\text{trial}}\right\rVert^2 = \left[\left\lVert \boldsymbol{e}^e_{n+1}\right\rVert + \sqrt{\tfrac{3}{2}}\,\Delta\gamma \left(\frac{2q_{n+1}}{M^2_{n+1}}\right)\right]^2 .$$

Therefore,

$$\boldsymbol{n}_{n+1} : \boldsymbol{e}^e_{\text{trial}} = \left\lVert \boldsymbol{e}^e_{\text{trial}}\right\rVert$$

and we have

$$(\varepsilon^e_s)_{n+1} = \sqrt{\tfrac{2}{3}}\,\sqrt{\left\lVert \boldsymbol{e}^e_{\text{trial}}\right\rVert^2 - \sqrt{6}\,(\Delta\gamma)^2 \left(\frac{2q_{n+1}}{M^2_{n+1}}\right)\left\lVert \boldsymbol{e}^e_{\text{trial}}\right\rVert + \tfrac{3}{2}\,(\Delta\gamma)^4 \left(\frac{2q_{n+1}}{M^2_{n+1}}\right)^2} = \sqrt{\tfrac{2}{3}}\,\left\lVert \boldsymbol{e}^e_{\text{trial}}\right\rVert - \Delta\gamma \left(\frac{2q_{n+1}}{M^2_{n+1}}\right).$$

The elastic strain can therefore be updated using

$$\begin{aligned}
(\varepsilon^e_v)_{n+1} &= (\varepsilon^e_v)_{\text{trial}} - \Delta\gamma\,[2p_{n+1} - (p_c)_{n+1}]\\
(\varepsilon^e_s)_{n+1} &= (\varepsilon^e_s)_{\text{trial}} - \Delta\gamma \left(\frac{2q_{n+1}}{M^2_{n+1}}\right).
\end{aligned}$$

The consistency condition is needed to close the above equations

$$f = \left(\frac{q_{n+1}}{M}\right)^2 + p_{n+1}[p_{n+1} - (p_c)_{n+1}] = 0\,.$$

The unknowns are $(\varepsilon^e_v)_{n+1}$, $(\varepsilon^e_s)_{n+1}$ and $\Delta\gamma$. Note that we can express the three equations as

$$\begin{aligned}
(\varepsilon^e_v)_{n+1} &= (\varepsilon^e_v)_{\text{trial}} - \Delta\gamma \left[\frac{\partial f}{\partial p}\right]_{n+1}\\
(\varepsilon^e_s)_{n+1} &= (\varepsilon^e_s)_{\text{trial}} - \Delta\gamma \left[\frac{\partial f}{\partial q}\right]_{n+1}\\
f_{n+1} &= 0\,.
\end{aligned} \tag{2}$$

3.2 Newton iterations

The three nonlinear equations in the three unknowns can be solved using Newton iterations for smooth yield functions. Let us define the residual as

$$\underline{\mathrm{r}}(\underline{\mathrm{x}}) = \begin{bmatrix} (\varepsilon^e_v)_{n+1} - (\varepsilon^e_v)_{\text{trial}} + \Delta\gamma \left[\frac{\partial f}{\partial p}\right]_{n+1} \\ (\varepsilon^e_s)_{n+1} - (\varepsilon^e_s)_{\text{trial}} + \Delta\gamma \left[\frac{\partial f}{\partial q}\right]_{n+1} \\ f_{n+1} \end{bmatrix} =: \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \quad \text{where} \quad \underline{\mathrm{x}} = \begin{bmatrix} (\varepsilon^e_v)_{n+1} \\ (\varepsilon^e_s)_{n+1} \\ f_{n+1} \end{bmatrix} =: \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The Newton root finding algorithm is :

To code the algorithm we have to find the derivatives of the residual with respect to the primary variables. Let's do the terms one by one. For the first row,

$$\begin{aligned}
\frac{\partial r_1}{\partial x_1} &= \frac{\partial}{\partial \varepsilon^e_v}\left[\varepsilon^e_v - (\varepsilon^e_v)_{\text{trial}} + \Delta\gamma\,(2p - p_c)\right] = 1 + \Delta\gamma\left(2\frac{\partial p}{\partial \varepsilon^e_v} - \frac{\partial p_c}{\partial \varepsilon^e_v}\right)\\
\frac{\partial r_1}{\partial x_2} &= \frac{\partial}{\partial \varepsilon^e_s}\left[\varepsilon^e_v - (\varepsilon^e_v)_{\text{trial}} + \Delta\gamma\,(2p - p_c)\right] = 2\Delta\gamma\,\frac{\partial p}{\partial \varepsilon^e_s}\\
\frac{\partial r_1}{\partial x_3} &= \frac{\partial}{\partial \Delta\gamma}\left[\varepsilon^e_v - (\varepsilon^e_v)_{\text{trial}} + \Delta\gamma\,(2p - p_c)\right] = 2p - p_c = \frac{\partial f}{\partial p}
\end{aligned}$$

Require: $\underline{\underline{x}}^0$

  $k \leftarrow 0$

  while $\underline{r}(\underline{\underline{x}}^k) \neq 0$ **do**

$$\underline{\underline{x}}^{k+1} \Leftarrow \underline{\underline{x}}^k - \left[\left(\frac{\partial \underline{r}}{\partial \underline{\underline{x}}}\right)^{-1}\right]_{\underline{\underline{x}}^k} \cdot \underline{r}(\underline{\underline{x}}^k)$$

    $k \leftarrow k+1$

  end while

where

$$\frac{\partial p}{\partial \varepsilon_v^e} = -\frac{p_0\,\beta}{\widetilde{\kappa}}\,\exp\left[-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right] = \frac{p}{\widetilde{\kappa}} \quad , \quad \frac{\partial p_c}{\partial \varepsilon_v^e} = \frac{(p_c)_n}{\widetilde{\kappa} - \widetilde{\lambda}}\,\exp\left[\frac{\varepsilon_v^e - (\varepsilon_v^e)_{\text{trial}}}{\widetilde{\kappa} - \widetilde{\lambda}}\right] \quad \text{and}$$

$$\frac{\partial p}{\partial \varepsilon_s^e} = \frac{3\,p_0\,\alpha\,\varepsilon_s^e}{\widetilde{\kappa}}\,\exp\left[-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right].$$

For the second row,

$$\begin{aligned}
\frac{\partial r_2}{\partial x_1} &= \frac{\partial}{\partial \varepsilon_v^e}\left[\varepsilon_s^e - (\varepsilon_s^e)_{\text{trial}} + \Delta\gamma\,\frac{2q}{M^2}\right] = \frac{2\Delta\gamma}{M^2}\,\frac{\partial q}{\partial \varepsilon_v^e} \\
\frac{\partial r_2}{\partial x_2} &= \frac{\partial}{\partial \varepsilon_s^e}\left[\varepsilon_s^e - (\varepsilon_s^e)_{\text{trial}} + \Delta\gamma\,\frac{2q}{M^2}\right] = 1 + \frac{2\Delta\gamma}{M^2}\,\frac{\partial q}{\partial \varepsilon_s^e} \\
\frac{\partial r_2}{\partial x_3} &= \frac{\partial}{\partial \Delta\gamma}\left[\varepsilon_s^e - (\varepsilon_s^e)_{\text{trial}} + \Delta\gamma\,\frac{2q}{M^2}\right] = \frac{2q}{M^2} = \frac{\partial f}{\partial q}
\end{aligned}$$

where

$$\frac{\partial q}{\partial \varepsilon_v^e} = -\frac{3p_0\,\alpha\,\varepsilon_s^e}{\widetilde{\kappa}}\,\exp\left[-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right] = \frac{\partial p}{\partial \varepsilon_s^e} \quad \text{and} \quad \frac{\partial q}{\partial \varepsilon_s^e} = 3\mu_0 + 3p_0\,\alpha\,\exp\left[-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right] = 3\mu\,.$$

For the third row,

$$\begin{aligned}
\frac{\partial r_3}{\partial x_1} &= \frac{\partial}{\partial \varepsilon_v^e}\left[\frac{q^2}{M^2} + p\,(p - p_c)\right] = \frac{2q}{M^2}\,\frac{\partial q}{\partial \varepsilon_v^e} + (2p - p_c)\,\frac{\partial p}{\partial \varepsilon_v^e} - p\,\frac{\partial p_c}{\partial \varepsilon_v^e} = \frac{\partial f}{\partial q}\,\frac{\partial q}{\partial \varepsilon_v^e} + \frac{\partial f}{\partial p}\,\frac{\partial p}{\partial \varepsilon_v^e} - p\,\frac{\partial p_c}{\partial \varepsilon_v^e} \\
\frac{\partial r_3}{\partial x_2} &= \frac{\partial}{\partial \varepsilon_s^e}\left[\frac{q^2}{M^2} + p\,(p - p_c)\right] = \frac{2q}{M^2}\,\frac{\partial q}{\partial \varepsilon_s^e} + (2p - p_c)\,\frac{\partial p}{\partial \varepsilon_s^e} = \frac{\partial f}{\partial q}\,\frac{\partial q}{\partial \varepsilon_s^e} + \frac{\partial f}{\partial p}\,\frac{\partial p}{\partial \varepsilon_s^e} \\
\frac{\partial r_3}{\partial x_3} &= \frac{\partial}{\partial \Delta\gamma}\left[\frac{q^2}{M^2} + p\,(p - p_c)\right] = 0\,.
\end{aligned}$$

We have to invert a matrix in the Newton iteration process. Let us see whether we can make this quicker to do. The Jacobian matrix has the form

$$\frac{\partial \underline{r}}{\partial \underline{\underline{x}}} = \begin{bmatrix} \frac{\partial r_1}{\partial x_1} & \frac{\partial r_1}{\partial x_2} & \frac{\partial r_1}{\partial x_3} \\ \frac{\partial r_2}{\partial x_1} & \frac{\partial r_2}{\partial x_2} & \frac{\partial r_2}{\partial x_3} \\ \frac{\partial r_3}{\partial x_1} & \frac{\partial r_3}{\partial x_2} & \frac{\partial r_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \underline{\underline{A}} & \underline{\underline{B}} \\ \underline{\underline{C}} & 0 \end{bmatrix}$$

where

$$\underline{\underline{A}} = \begin{bmatrix} \frac{\partial r_1}{\partial x_1} & \frac{\partial r_1}{\partial x_2} \\ \frac{\partial r_2}{\partial x_1} & \frac{\partial r_2}{\partial x_2} \end{bmatrix}, \quad \underline{\underline{B}} = \begin{bmatrix} \frac{\partial r_1}{\partial x_3} \\ \frac{\partial r_2}{\partial x_3} \end{bmatrix}, \quad \text{and} \quad \underline{\underline{C}} = \begin{bmatrix} \frac{\partial r_3}{\partial x_1} & \frac{\partial r_3}{\partial x_2} \end{bmatrix}.$$

We can also break up the $\underline{\underline{x}}$ and $\underline{r}$ matrices:

$$\Delta\underline{\underline{x}} = \underline{\underline{x}}^{k+1} - \underline{\underline{x}}^k = \begin{bmatrix} \Delta\underline{\underline{x}}^{vs} \\ \Delta x_3 \end{bmatrix}, \quad \underline{r} = \begin{bmatrix} \underline{r}^{vs} \\ r_3 \end{bmatrix} \quad \text{where} \quad \underline{r}^{vs} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \quad \text{and} \quad \Delta\underline{\underline{x}}^{vs} = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}.$$

Then

$$\begin{bmatrix} \Delta\underline{\underline{x}}^{vs} \\ \Delta x_3 \end{bmatrix} = -\begin{bmatrix} \underline{\underline{A}} & \underline{\underline{B}} \\ \underline{\underline{C}} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \underline{r}^{vs} \\ r_3 \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} \underline{\underline{A}} & \underline{\underline{B}} \\ \underline{\underline{C}} & 0 \end{bmatrix} \begin{bmatrix} \Delta\underline{\underline{x}}^{vs} \\ \Delta x_3 \end{bmatrix} = -\begin{bmatrix} \underline{r}^{vs} \\ r_3 \end{bmatrix}$$

or

$$\underline{\underline{\mathsf{A}}}\,\Delta\underline{\mathsf{x}}^{vs} + \underline{\underline{\mathsf{B}}}\,\Delta x_3 = -\underline{\mathsf{r}}^{vs} \quad \text{and} \quad \underline{\underline{\mathsf{C}}}\,\Delta\underline{\mathsf{x}}^{vs} = -r_3\,.$$

From the first equation above,

$$\Delta\underline{\mathsf{x}}^{vs} = -\underline{\underline{\mathsf{A}}}^{-1}\,\underline{\mathsf{r}}^{vs} - \underline{\underline{\mathsf{A}}}^{-1}\,\underline{\underline{\mathsf{B}}}\,\Delta x_3\,.$$

Plugging in the second equation gives

$$r_3 = \underline{\underline{\mathsf{C}}}\,\underline{\underline{\mathsf{A}}}^{-1}\,\underline{\mathsf{r}}^{vs} + \underline{\underline{\mathsf{C}}}\,\underline{\underline{\mathsf{A}}}^{-1}\,\underline{\underline{\mathsf{B}}}\,\Delta x_3\,.$$

Rearranging,

$$\Delta x_3 = x_3^{k+1} - x_3^{k} = \frac{-\underline{\underline{\mathsf{C}}}\,\underline{\underline{\mathsf{A}}}^{-1}\,\underline{\mathsf{r}}^{vs} + r_3}{\underline{\underline{\mathsf{C}}}\,\underline{\underline{\mathsf{A}}}^{-1}\,\underline{\underline{\mathsf{B}}}}\,.$$

Using the above result,

$$\Delta\underline{\mathsf{x}}^{vs} = -\underline{\underline{\mathsf{A}}}^{-1}\,\underline{\mathsf{r}}^{vs} - \underline{\underline{\mathsf{A}}}^{-1}\,\underline{\underline{\mathsf{B}}}\left(\frac{-\underline{\underline{\mathsf{C}}}\,\underline{\underline{\mathsf{A}}}^{-1}\,\underline{\mathsf{r}}^{vs} + r_3}{\underline{\underline{\mathsf{C}}}\,\underline{\underline{\mathsf{A}}}^{-1}\,\underline{\underline{\mathsf{B}}}}\right).$$

We therefore have to invert only a 2×2 matrix.

3.3 Tangent calculation: elastic

We want to find the derivative of the stress with respect to the strain:

$$\frac{\partial\boldsymbol{\sigma}}{\partial\boldsymbol{\epsilon}} = \boldsymbol{1}\otimes\frac{\partial p}{\partial\boldsymbol{\epsilon}} + \sqrt{\tfrac{2}{3}}\,\boldsymbol{n}\otimes\frac{\partial q}{\partial\boldsymbol{\epsilon}} + \sqrt{\tfrac{2}{3}}\,q\,\frac{\partial\boldsymbol{n}}{\partial\boldsymbol{\epsilon}}\,. \tag{3}$$

For the first term above,

$$\frac{\partial p}{\partial\boldsymbol{\epsilon}} = p_0\,\exp\left[-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right]\frac{\partial\beta}{\partial\boldsymbol{\epsilon}} - p_0\,\frac{\beta}{\widetilde{\kappa}}\,\exp\left[-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right]\frac{\partial\varepsilon_v^e}{\partial\boldsymbol{\epsilon}} = p_0\,\exp\left[-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right]\left(\frac{\partial\beta}{\partial\boldsymbol{\epsilon}} - \frac{\beta}{\widetilde{\kappa}}\,\frac{\partial\varepsilon_v^e}{\partial\boldsymbol{\epsilon}}\right).$$

Now,

$$\frac{\partial\beta}{\partial\boldsymbol{\epsilon}} = \frac{3\alpha}{\widetilde{\kappa}}\,\varepsilon_s^e\,\frac{\partial\varepsilon_s^e}{\partial\boldsymbol{\epsilon}}\,.$$

Therefore,

$$\frac{\partial p}{\partial\boldsymbol{\epsilon}} = \frac{p_0}{\widetilde{\kappa}}\,\exp\left[-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right]\left(3\alpha\,\varepsilon_s^e\,\frac{\partial\varepsilon_s^e}{\partial\boldsymbol{\epsilon}} - \beta\,\frac{\partial\varepsilon_v^e}{\partial\boldsymbol{\epsilon}}\right).$$

We now have to figure out the other derivatives in the above expression. First,

$$\frac{\partial\varepsilon_s^e}{\partial\boldsymbol{\epsilon}} = \sqrt{\tfrac{2}{3}}\,\frac{1}{\sqrt{\boldsymbol{e}^e : \boldsymbol{e}^e}}\,\frac{\partial\boldsymbol{e}^e}{\partial\boldsymbol{\epsilon}} : \boldsymbol{e}^e = \sqrt{\tfrac{2}{3}}\,\frac{1}{\|\boldsymbol{e}^e\|}\left(\frac{\partial\boldsymbol{\epsilon}^e}{\partial\boldsymbol{\epsilon}} - \tfrac{1}{3}\boldsymbol{1}\otimes\frac{\partial\varepsilon_v^e}{\partial\boldsymbol{\epsilon}}\right) : \boldsymbol{e}^e\,.$$

For the special situation where all the strain is elastic, $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^e$, and (see Wikipedia article on tensor derivatives)

$$\frac{\partial\boldsymbol{\epsilon}^e}{\partial\boldsymbol{\epsilon}} = \frac{\partial\boldsymbol{\epsilon}}{\partial\boldsymbol{\epsilon}} = \mathsf{I}^{(s)} \quad \text{and} \quad \frac{\partial\varepsilon_v^e}{\partial\boldsymbol{\epsilon}} = \frac{\partial\varepsilon_v}{\partial\boldsymbol{\epsilon}} = \boldsymbol{1}\,.$$

That gives us

$$\frac{\partial\varepsilon_s^e}{\partial\boldsymbol{\epsilon}} = \sqrt{\tfrac{2}{3}}\,\frac{1}{\|\boldsymbol{e}^e\|}\left(\mathsf{I}^{(s)} - \tfrac{1}{3}\boldsymbol{1}\otimes\boldsymbol{1}\right) : \boldsymbol{e}^e = \sqrt{\tfrac{2}{3}}\,\frac{1}{\|\boldsymbol{e}^e\|}\left[\boldsymbol{e}^e - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{e}^e)\boldsymbol{1}\right].$$

But $\mathrm{tr}(\boldsymbol{e}^e) = 0$ because this is the deviatoric part of the strain and we have

$$\boxed{\frac{\partial\varepsilon_s^e}{\partial\boldsymbol{\epsilon}} = \sqrt{\tfrac{2}{3}}\,\frac{\boldsymbol{e}^e}{\|\boldsymbol{e}^e\|} = \sqrt{\tfrac{2}{3}}\,\boldsymbol{n}} \quad \text{and} \quad \boxed{\frac{\partial\varepsilon_v^e}{\partial\boldsymbol{\epsilon}} = \boldsymbol{1}\,.}$$

Using these, we get

$$\frac{\partial p}{\partial\boldsymbol{\epsilon}} = \frac{p_0}{\widetilde{\kappa}}\,\exp\left[-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right]\left(\sqrt{6}\,\alpha\,\varepsilon_s^e\,\boldsymbol{n} - \beta\,\boldsymbol{1}\right). \tag{4}$$

The derivative of q with respect to $\boldsymbol{\epsilon}$ can be calculated in a similar way, i.e.,

$$\frac{\partial q}{\partial \boldsymbol{\epsilon}} = 3\mu \frac{\partial \varepsilon_s^e}{\partial \boldsymbol{\epsilon}} + 3\varepsilon_s^e \frac{\partial \mu}{\partial \boldsymbol{\epsilon}} = 3\mu \frac{\partial \varepsilon_s^e}{\partial \boldsymbol{\epsilon}} - 3\frac{p_0}{\widetilde{\kappa}}\,\alpha\,\varepsilon_s^e \exp\left[-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right]\frac{\partial \varepsilon_v^e}{\partial \boldsymbol{\epsilon}}\,.$$

Using the expressions in the boxes above,

$$\frac{\partial q}{\partial \boldsymbol{\epsilon}} = \sqrt{6}\,\mu\,\boldsymbol{n} - 3\frac{p_0}{\widetilde{\kappa}}\,\exp\left[-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right]\alpha\,\varepsilon_s^e\,\boldsymbol{1}\,. \tag{5}$$

Also,

$$\frac{\partial \boldsymbol{n}}{\partial \boldsymbol{\epsilon}} = \sqrt{\tfrac{2}{3}}\left[\frac{1}{\varepsilon_s^e}\frac{\partial \boldsymbol{e}^e}{\partial \boldsymbol{\epsilon}} - \frac{1}{(\varepsilon_s^e)^2}\,\boldsymbol{e}^e \otimes \frac{\partial \varepsilon_s^e}{\partial \boldsymbol{\epsilon}}\right].$$

Using the previously derived expression, we have

$$\frac{\partial \boldsymbol{n}}{\partial \boldsymbol{\epsilon}} = \sqrt{\tfrac{2}{3}}\,\frac{1}{\varepsilon_s^e}\left[\mathbf{I}^{(s)} - \tfrac{1}{3}\,\boldsymbol{1}\otimes\boldsymbol{1} - \sqrt{\tfrac{2}{3}}\,\frac{1}{\varepsilon_s^e}\,\frac{\boldsymbol{e}^e\otimes\boldsymbol{e}^e}{\|\boldsymbol{e}^e\|}\right]$$

or

$$\frac{\partial \boldsymbol{n}}{\partial \boldsymbol{\epsilon}} = \sqrt{\tfrac{2}{3}}\,\frac{1}{\varepsilon_s^e}\left[\mathbf{I}^{(s)} - \tfrac{1}{3}\,\boldsymbol{1}\otimes\boldsymbol{1} - \boldsymbol{n}\otimes\boldsymbol{n}\right]. \tag{6}$$

Plugging the expressions for these derivatives in the original equation, we get

$$\begin{aligned}\frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\epsilon}} = \frac{p_0}{\widetilde{\kappa}}\,\exp\left[-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right]\left(\sqrt{6}\,\alpha\,\varepsilon_s^e\,\boldsymbol{1}\otimes\boldsymbol{n} - \beta\,\boldsymbol{1}\otimes\boldsymbol{1}\right) + 2\mu\,\boldsymbol{n}\otimes\boldsymbol{n} - \sqrt{6}\frac{p_0}{\widetilde{\kappa}}\,\exp\left[-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right]\alpha\,\varepsilon_s^e\,\boldsymbol{n}\otimes\boldsymbol{1} + \\ \tfrac{2}{3}\,\frac{q}{\varepsilon_s^e}\left[\mathbf{I}^{(s)} - \tfrac{1}{3}\,\boldsymbol{1}\otimes\boldsymbol{1} - \boldsymbol{n}\otimes\boldsymbol{n}\right].\end{aligned}$$

Reorganizing,

$$\boxed{\begin{aligned}\frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\epsilon}} = \frac{\sqrt{6}\,p_0\,\alpha\,\varepsilon_s^e}{\widetilde{\kappa}}\,\exp\left[-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right](\boldsymbol{1}\otimes\mathbf{n} + \mathbf{n}\otimes\boldsymbol{1}) - \left(\frac{p_0\beta}{\widetilde{\kappa}}\,\exp\left[-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right] + \tfrac{2}{9}\,\frac{q}{\varepsilon_s^e}\right)\boldsymbol{1}\otimes\boldsymbol{1} + \\ 2\left(\mu - \tfrac{1}{3}\,\frac{q}{\varepsilon_s^e}\right)\boldsymbol{n}\otimes\boldsymbol{n} + \tfrac{2}{3}\,\frac{q}{\varepsilon_s^e}\,\mathbf{I}^{(s)}\,.\end{aligned}} \tag{7}$$

3.4 Tangent calculation: elastic-plastic

From the previous section recall that

$$\frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\epsilon}} = \boldsymbol{1}\otimes\frac{\partial p}{\partial \boldsymbol{\epsilon}} + \sqrt{\tfrac{2}{3}}\,\boldsymbol{n}\otimes\frac{\partial q}{\partial \boldsymbol{\epsilon}} + \sqrt{\tfrac{2}{3}}\,q\,\frac{\partial \boldsymbol{n}}{\partial \boldsymbol{\epsilon}}$$

where

$$\frac{\partial p}{\partial \boldsymbol{\epsilon}} = \frac{p_0}{\widetilde{\kappa}}\,\exp\left[-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right]\left(3\alpha\,\varepsilon_s^e\frac{\partial \varepsilon_s^e}{\partial \boldsymbol{\epsilon}} - \beta\,\frac{\partial \varepsilon_v^e}{\partial \boldsymbol{\epsilon}}\right), \qquad \frac{\partial q}{\partial \boldsymbol{\epsilon}} = 3\mu\,\frac{\partial \varepsilon_s^e}{\partial \boldsymbol{\epsilon}} - 3\frac{p_0}{\widetilde{\kappa}}\,\alpha\,\varepsilon_s^e\,\exp\left[-\frac{\varepsilon_v^e - \varepsilon_{v0}^e}{\widetilde{\kappa}}\right]\frac{\partial \varepsilon_v^e}{\partial \boldsymbol{\epsilon}} \quad \text{and}$$

$$\frac{\partial \boldsymbol{n}}{\partial \boldsymbol{\epsilon}} = \sqrt{\tfrac{2}{3}}\left[\frac{1}{\varepsilon_s^e}\,\frac{\partial \boldsymbol{e}^e}{\partial \boldsymbol{\epsilon}} - \frac{1}{(\varepsilon_s^e)^2}\,\boldsymbol{e}^e\otimes\frac{\partial \varepsilon_s^e}{\partial \boldsymbol{\epsilon}}\right].$$

The total strain is equal to the elastic strain for the purely elastic case and the tangent is relatively straightforward to calculate. For the elastic-plastic case we have

$$\boldsymbol{\epsilon}^e_{n+1} = \boldsymbol{\epsilon}^e_{\text{trial}} - \Delta\gamma\left[\frac{\partial f}{\partial \boldsymbol{\sigma}}\right]_{n+1}.$$

Dropping the subscript $n+1$ for convenience, we have

$$\frac{\partial \boldsymbol{\epsilon}^e}{\partial \boldsymbol{\epsilon}} = \frac{\partial \boldsymbol{\epsilon}^e_{\text{trial}}}{\partial \boldsymbol{\epsilon}} - \frac{\partial f}{\partial \boldsymbol{\sigma}}\otimes\frac{\partial \Delta\gamma}{\partial \boldsymbol{\epsilon}} - \Delta\gamma\,\frac{\partial}{\partial \boldsymbol{\epsilon}}\left[\frac{\partial f}{\partial \boldsymbol{\sigma}}\right] = \mathbf{I}^{(s)} - \left[\frac{2p - p_c}{3}\,\boldsymbol{1} + \sqrt{\tfrac{3}{2}}\,\frac{2q}{M^2}\,\boldsymbol{n}\right]\otimes\frac{\partial \Delta\gamma}{\partial \boldsymbol{\epsilon}} - \Delta\gamma\,\frac{\partial}{\partial \boldsymbol{\epsilon}}\left[\frac{2p - p_c}{3}\,\boldsymbol{1} + \sqrt{\tfrac{3}{2}}\,\frac{2q}{M^2}\,\boldsymbol{n}\right].$$