A Concise Representation Shunn's Two-Dimensional Variable-Density MMS Solutions

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For all the solutions presented in this document, we use a simple mixing model for two streams given by

$$\frac{1}{\rho} = \frac{1-f}{\rho_0} + \frac{f}{\rho_1} \tag{1}$$

where the mixture fraction f is transported via

$$\frac{\partial \rho f}{\partial t} + \nabla \cdot \mathbf{u} \rho f = \nabla \rho \Gamma \cdot \nabla f + R_f \tag{2}$$

This document simplifies the presentation and implementation of two important variable density analytical solutions using the MMS approach. Both can be found in Shunn et al. (2012).

1 Corrugated-Front MMS

For this problem, we set

$$\hat{x} = u_f t - x + a \cos[k(v_f t - y)]; \tag{3}$$

$$f = \frac{1 + \tanh(b\hat{x}e^{-wt})}{1 + \frac{\rho 0}{\rho 1} + (1 + \frac{\rho 0}{\rho 1}) \tanh(b\hat{x}e^{-wt})} \tag{4}$$

$$u = \frac{\rho_1 - \rho_0}{\rho} \left\{ -\omega\hat{x} + \frac{\omega\hat{x} - u_f}{\exp\left[2b\hat{x}\exp(-\omega t)\right] + 1} + \frac{\omega \ln\left[\exp(2b\hat{x}\exp(-\omega t) + 1)\right]}{2b\exp(-\omega t)} \right\} \tag{5}$$

$$v = v_f \tag{6}$$

The continuity equation does not require a source term. The mixture fraction equation requires the addition of a source term. Let

$$\begin{aligned} s_0 &= e^{tw}; \\ s_1 &= u_f t - x + a\cos[k(v_f t - y)]; \\ s_2 &= e^{2bs_1/s_0} \end{aligned}$$

$$
\begin{aligned}
R_f =& \frac{r_0 r_1 s_2}{s_0^2 (r_0 + r_1 s_2)^3} \times \\
& \Big\{-4Ab^2 r_0 - 2a^2 Ab^2 k^2 r_0 + 4Ab^2 r_1 s_2 + 2a^2 Ab^2 k^2 r_1 s_2 + 2br_0 r_1 s_0 u_f \\
& + 2br_1^2 s_0 s_2 u_f - 2br_0^2 s_0 t u_f w - 2br_0 r_1 s_0 s_2 t u_f w + 2br_0^2 s_0 w x \\
& + 2br_0 r_1 s_0 s_2 w x + 2abs_0 (r_0 + r_1 s_2)(Ak^2 - r_0 w) \cos[k(tv_f - y)] \\
& + 2a^2 Ab^2 k^2 (r_0 - r_1 s_2) \cos[2k(tv_f - y)] + r_0^2 s_0^2 w \ln[1 + s_2] \\
& - r_0 r_1 s_0^2 w \ln(1 + s_2) + r_0 r_1 s_0^2 s_2 w \ln(1 + s_2) \\
& - r_1^2 s_0^2 s_2 w \ln(1 + s_2)\Big\}
\end{aligned}
$$

2 Oscillating MMS

For this periodic problem, we set

$$
\begin{aligned}
\hat{x} =& x - u_f t \\
\hat{y} =& y - v_f t \\
\bar{x} =& \pi k \hat{x} \\
\bar{y} =& \pi k \hat{y} \\
\bar{t} =& \pi \omega t
\end{aligned}
$$

$$
f = \frac{\sin(\bar{x}) \sin(\bar{y}) \cos(\bar{t}) + 1}{\left(1 - \frac{\rho_0}{\rho_1}\right) \sin(\bar{x}) \sin(\bar{y}) \cos(\bar{t}) + \left(1 + \frac{\rho_0}{\rho_1}\right)}; \tag{7}
$$

$$
\rho u = \left(\frac{-\omega}{4k}\right) (\rho_1 - \rho_0) \cos(\bar{x}) \sin(\bar{y}) \sin(\bar{t}) \tag{8}
$$

$$
\rho v = \left(\frac{-\omega}{4k}\right) (\rho_1 - \rho_0) \sin(\bar{x}) \cos(\bar{y}) \sin(\bar{t}) \tag{9}
$$

The source term for the continuity equation is

$$
R_\rho = \frac{1}{2} \pi k (\rho_0 - \rho_1) \cos(\bar{t}) \left[u_f \cos(\bar{x}) \sin(\bar{y}) + v_f \sin(\bar{x}) \cos(\bar{y})\right] \tag{10}
$$

so that

$$
\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = R_\rho \tag{11}
$$

The source term for the mixture fraction is given by R_f such that

$$
\begin{aligned}
s_0 &= \cos(\pi\omega t),\\
s_1 &= \sin(\pi\omega t),\\
s_2 &= \cos(k\pi x),\\
s_3 &= \sin(k\pi x),\\
s_4 &= \cos(k\pi y),\\
s_5 &= \sin(k\pi y),\\
s_6 &= \sin[k\pi(x - tu_f)],\\
s_7 &= \sin[k\pi(y - tv_f)],\\
s_8 &= \cos[k\pi(x - tu_f)],\\
s_9 &= \cos[k\pi(y - tv_f)],\\
s_{10} &= \rho_0 + \rho_1,\\
s_{11} &= \rho_0 - \rho_1\\
A &\equiv \rho\Gamma = \text{constant}
\end{aligned}
$$

$$
\begin{aligned}
R_f = &-\frac{\pi\rho_1}{4(s_{10} - s_0 s_6 s_7 s_{11})^3}\times\\
&\Big\{16Ak^2\pi\rho_0 s_0^2 s_9^2 s_{11} s_6^2\\
&- 16Ak^2\pi\rho_0 s_0 s_{10} s_6 s_7 + 16Ak^2\pi\rho_0 s_0^2 s_8^2 s_{11} s_7^2\\
&+ 16Ak^2\pi\rho_0 s_0^2 s_{11} s_6^2 s_7^2 + 2ks_0 s_8 s_{10}^3 s_7 u_f\\
&- 6k\rho_0^2 s_0^2 s_8 s_{11} s_6 s_7^2 u_f - 12k\rho_0\rho_1 s_0^2 s_8 s_{11} s_6 s_7^2 u_f\\
&- 6k\rho_1^2 s_0^2 s_8 s_{11} s_6 s_7^2 u_f + 6ks_0^3 s_8 s_{10} s_{11}^2 s_6^2 s_7^3 u_f\\
&- 2ks_0^4 s_8 s_{11}^3 s_6^3 s_7^4 u_f + 2ks_0 s_9 s_{10}^3 s_6 v_f\\
&- 6k\rho_0^2 s_0^2 s_9 s_{11} s_6^2 s_7 v_f - 12k\rho_0\rho_1 s_0^2 s_9 s_{11} s_6^2 s_7 v_f\\
&- 6k\rho_1^2 s_0^2 s_9 s_{11} s_6^2 s_7 v_f + 6ks_0^3 s_9 s_{10} s_{11}^2 s_6^3 s_7^2 v_f\\
&- 2ks_0^4 s_9 s_{11}^3 s_6^4 s_7^3 v_f - 2\rho_0 s_0 s_1 s_9^2 s_{10} s_{11} s_6^2 w\\
&+ 4\rho_0 s_1 s_{10}^2 s_6 s_7 w + 2\rho_0 s_0^2 s_1 s_9^2 s_{11}^2 s_6^3 s_7 w\\
&- 5\rho_0 s_0 s_1 s_{10} s_{11} s_7^2 w + 3\rho_0 s_0 s_1 s_8^2 s_{10} s_{11} s_7^2 w\\
&- 3\rho_0 s_0 s_1 s_{10} s_{11} s_6^2 s_7^2 w + 2\rho_0 s_0^2 s_1 s_8^2 s_{11}^2 s_6 s_7^3 w\\
&+ 4\rho_0^2 s_0^2 s_1 s_{11} s_6^3 s_7^3 w - 4\rho_0\rho_1 s_0^2 s_1 s_{11} s_6^3 s_7^3 w\Big\}
\end{aligned}
$$

References

Shunn, L., Ham, F., and Moin, P. (2012). Verification of variable-density flow solvers using manufactured solutions. *Journal of Computational Physics*, 231:3801–3827.