Digital Signal Processing Ⅰ

14st Week EXPERIMENT

Report

(9th report of DSP1 course)

Subject	Digital Signal Processing Ⅰ
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Ⅰ. Abstraction

1. System representation in the z-domain

1-1. System representation in the z-domain

Similar to the frequency response function $H(e^{jw})$, we can define the z-domain function, $H(z)$, called the system function. However, unlike $H(e^{jw})$, $H(z)$ exists for systems that may not be BIBO stable.

The system function $H(z)$ is defined by:

$$H(z) = \mathcal{Z}[h(n)] = \sum_{-\infty}^{\infty} h(n)z^{-n}; \quad R_{h-} < |z| < R_{h+}$$

Using the convolution property of the z-transform, the output transform $Y(z)$ is given by:

$$Y(z) = H(z)X(z) \;\; : \;\; ROC_y = ROC_h \cap ROC_x$$

provided ROC_x overlaps with ROC_h.

1-2. System function from the DE representation

When LTI systems are described by a difference equation

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{l=0}^{M} b_l x(n-l)$$

the system function $H(z)$ can easily be computed.[1] Taking the z-transform of both sides, and using properties of the z-transform,

$$Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) = \sum_{l=0}^{M} b_z z^{-l} X(z)$$

---

[1] z_l: system zeros $\quad p_k: system\ poles$

or,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{l=0}^{M} b_l z^{-l}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)} = \frac{b_0 z^{-M}\left(z^M + \cdots + \frac{b_M}{b_0}\right)}{z^{-N}(z^N + \cdots + a_N)}$$

After factorization, we obtain

$$H(z) = b_0 z^{N-M} \frac{\prod_{k=1}^{N}(z - z_l)}{\prod_{k=1}^{N}(z - p_k)}$$

where z_ls are the system zeros and p_ks are the system poles. Thus $H(z)$ can also be represented in the z-domain using a pole-zero plot. This fact is useful in designing simple filters by proper placement of poles and zeros.

To determine zeros and poles of a rational $H(z)$, we can use the MATLAB function roots on both the numerator and the denominator polynomials. (Its inverse function poly determines polynomials coefficients from its roots, as discussed in the previous section.) It is also possible to use MATLAB to plot these roots for a visual display of a pole-zero plot. The function zplane(b, a) plots poles and zeros, given the numerator row vector b and the denominator row vector a. As before, the symbol o represents a zero and the symbol x represents a pole. The plot includes the unit circle for reference. Similarly, zplane(z, p) plots the zeros in column vector z and the poles in column vector p.

1-3. Transfer function representation

In the ROC of $H(z)$ includes a unit circle ($z = e^{jw}$), then we can evaluate $H(z)$ on the unit circle, resulting in a frequency response function or transfer function $H(e^{jw})$.

$$H\left(e^{jw}\right) = b_0 e^{j(N-M)w} \frac{\prod_{l=1}^{M}(e^{jw} - z_l)}{\prod_{k=1}^{N}(e^{jw} - p_k)}$$

The factor $(e^{jw} - z_l)$ can be interpreted as a vector in the complex z-plane from a zero z_l to the unit circle at $z = e^{jw}$, while the factor $(e^{jw} - p_k)$ can be interpreted as a vector from a pole p_k to the unit circle at $z = e^{jw}$.

Hence the magnitude response function

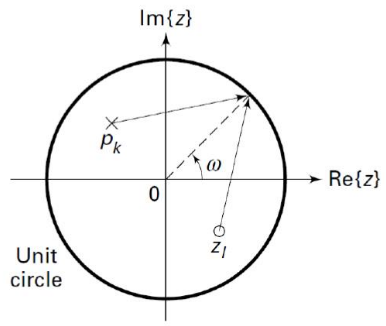
$$|H(e^{jw})| = |b_0| \frac{|e^{jw} - z_1| \ldots |e^{jw} - z_M|}{|e^{jw} - p_1| \ldots |e^{jw} - p_N|}$$

can be interpreted as a product of the lengths of vectors from zeros to the unit circle divided by the lengths of vectors from poles to the unit circle and scaled by $|b_0|$. Similarly, the phase response function

$$\angle H(e^{jw}) = [0\ or\ \pi] + [(N - M)w] + \sum_{1}^{M} \angle(e^{jw} - z_k) - \sum_{1}^{N} \angle(e^{jw} - p_k)$$

can be interpreted as a sum of a constant factor, a linear-phase factor, and a non-linear phase factor (angles from the "zero vectors" minus the sum of angles from the "pole vectors").

1-4. MATLAB Implementation



A function called freqz for this computation, which uses the preceding interpretation. In its simplest form, this function is invoked by

$$[H, w] = freqz(b, a, N)$$

which returns the N-point frequency vector w and the N-point complex frequency response vector H of the system, given its numerator and denominator coefficients in vectors b and a. The frequency response is evaluated at N points equally spaced around the upper half of the unit circle. Note that the b and a vectors are the same vectors we use in the $filter$ function or derived from the difference equation representation.

The second form

$$[H, w] = freqz(b, a, N, \text{"whole"})$$

uses N points around the whole unit circle for computation.

In yet another form

$$H = freq(b, a, w)$$

(Uses example)

$$H(z) = \frac{0.05634(1+z^{-1})(1-1.0166z^{-1}+z^{-2})}{(1-0.683z^{-1})(1-1.4461z^{-1}+0.7957z^{-2})}.$$

Express the numerator and denominator as polynomial convolutions. Find the frequency response at 2001 points spanning the complete unit circle.

```
b0 = 0.05634;
b1 = [1  1];
b2 = [1 -1.0166 1];
a1 = [1 -0.683];
a2 = [1 -1.4461 0.7957];

b = b0*conv(b1,b2);
a = conv(a1,a2);

[h,w] = freqz(b,a,'whole',2001);
```

Ⅱ. Exercises

In this part, there are four exercise questions. Each exercise consists of code and its result. All documents including Matlab code, result, and this report are uploaded in this website:

https://github.com/Gaon-Choi/ELE3076/tree/main/9_z-Transform2

Example 4.11

Given a causal system

$$y(n) = 0.9y(n-1) + x(n)$$

a. Determine $H(z)$ and sketch its pole-zero plot.

We need to convert the given formula to z-transform form.

$$Y(z) = 0.9z^{-1}Y(z) + X(z)$$

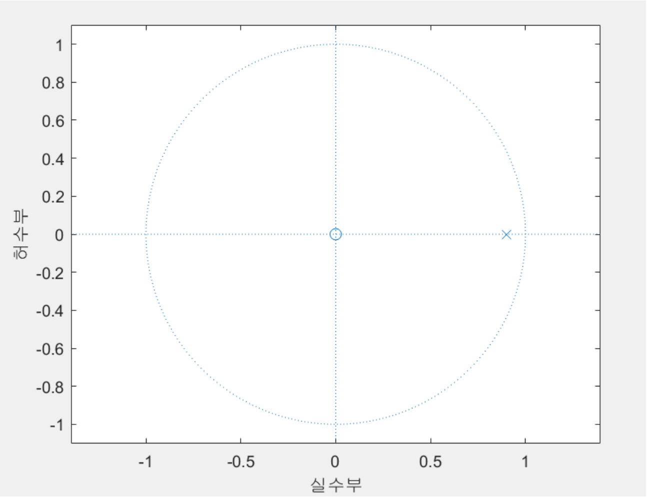
Since the system is causal, the system function goes outward.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 0.9z^{-1}}, \qquad |z| > 0.9$$

(Source code)

```
b = [1, 0]; a = [1, 0.9]; zplane(b, a)
```

(Result)

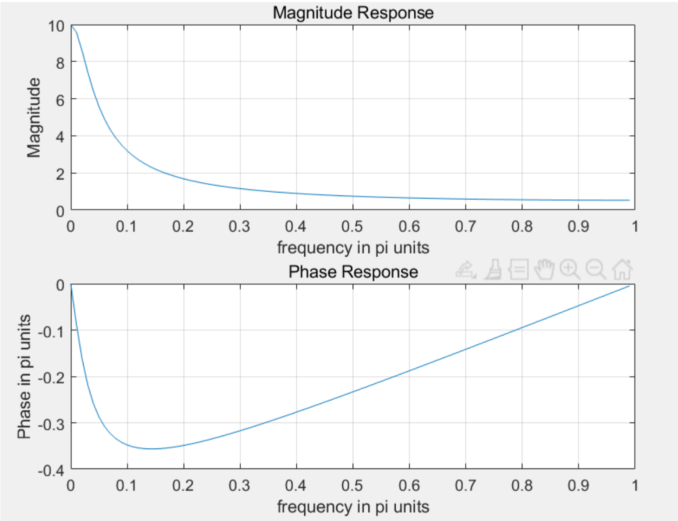


b. Plot $|H(e^{jw})|$ and $\angle H(e^{jw})$.

(Source code)

```
[H, w] = freqz(b, a, 100); magH = abs(H); phaH = angle(H);
subplot(2, 1, 1); plot(w/pi, magH); grid
xlabel("frequency in pi units"); ylabel("Magnitude");
title("Magnitude Response")
subplot(2, 1, 2); plot(w/pi, phaH/pi); grid
xlabel("frequency in pi units"); ylabel("Phase in pi units");
title("Phase Response")
```

(Result)



c. Determine the impulse response $h(n)$.

From the result, we got $H(z)$.

$$H(z) = \frac{1}{1 - 0.9z^{-1}}$$

To get the impulse response $h(n)$, we need to use inverse z-transform.

$$h(n) = (0.9)^n u(n)$$

Example 4.13

A causal LTI system is described by the following difference equation:

$$y(n) = 0.81y(n-2) + x(n) - x(n-2)$$

Determine

a. the system function $H(z)$.

From the given formula, we need to use z-transform.

$$Y(z) = 0.81z^{-2}Y(z) + X(z) - z^{-2}X(z)$$

and we can get system function $H(z)$ using the definition.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-z^{-2}}{1-0.81z^{-2}} = \frac{1-z^{-2}}{(1+0.9z^{-1})(1-0.9z^{-1})}, \quad |z| > 0.9$$

b. the unit impulse response $h(n)$.

(Source code)

```
b = [1, 0, 1]; a = [1, 0, 0.81]; [R, p, C] = residuez(b, a)
```

(Result)

R =

-0.1173

-0.1173

p =

-0.9000

0.9000

C =

1.2346

$$H(z) = 1.2346 - 0.1173\frac{1}{1+0.9z^{-1}} - 0.1173\frac{1}{1-0.9z^{-1}}, \quad |z| > 0.9$$

From $H(z)$, we need to convert it to $h(n)$ by inverse z-transform.

Then, we can get

$$h(n) = 1.2346\delta(n) - 0.1173\{1+(-1)^n\}(0.9)^n u(n)$$

c. the unit step response $v(n)$, that is, the response to the unit step u(n), and

$$\mathcal{Z}[u(n)] = U(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1$$

$$V(z) = H(z)U(z) = \left[\frac{(1+z^{-1})(1-z^{-1})}{(1+0.9z^{-1})(1-0.9z^{-1})}\right]\left[\frac{1}{1-z^{-1}}\right], \quad \{|z| > 0.9\} \cap \{|z| > 1\}$$

$$= \frac{1+z^{-1}}{(1+0.9z^{-1})(1-0.9z^{-1})}, \quad |z| > 0.9$$

Using factorization technique, we can represent $V(z)$,

$$V(z) = 1.0556\frac{1}{1-0.9z^{-1}} - 0.0556\frac{1}{1+0.9z^{-1}}, \quad |z| > 0.9$$

Now, we can get $v(n)$ by the inverse z-transform,

$$v(n) = 1.0556(0.9)^n u(n) - 0.0556(-0.9)^n u(n)$$

d. the frequency response function $H(e^{jw})$, and plot its magnitude and phase over $0 \le \omega \le \pi$.

Substituting $z = e^{jw}$ in $H(z)$,

$$H(e^{jw}) = \frac{1-e^{-j2w}}{1-0.81e^{-j2w}}$$

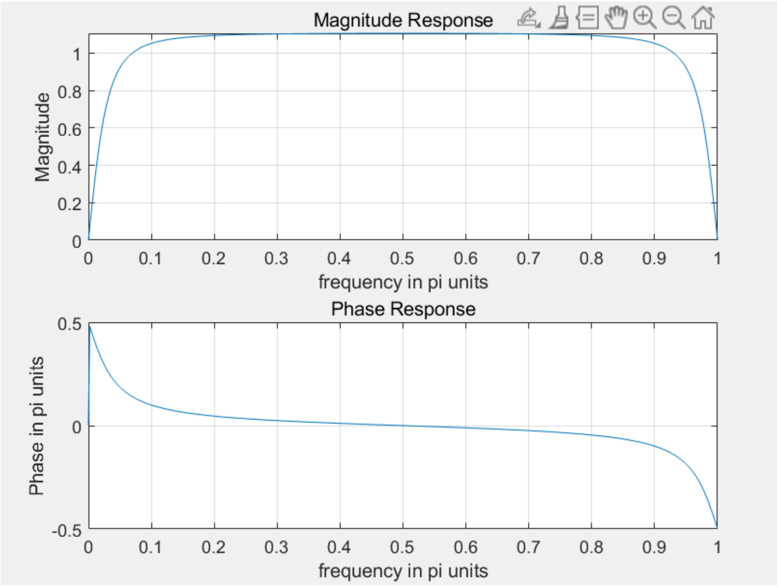
We will use the MATLAB script to compute and plot responses.

(Source code)

```
% from example 4_13_b
b = [1, 0, -1]; a = [1, 0, -0.81]; [R, p, C] = residuez(b, a)

w = [0:1:500] * pi / 500; H = freqz(b, a, w);
magH = abs(H); phaH = angle(H);
subplot(2, 1, 1); plot(w/pi, magH); grid
xlabel("frequency in pi units"); ylabel("Magnitude")
title("Magnitude Response")
subplot(2, 1, 2); plot(w/pi, phaH / pi); grid
xlabel("frequency in pi units"); ylabel("Phase in pi units")
title("Phase Response")
```

(Result)



Magnitude Response
Magnitude
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
frequency in pi units
Phase Response
Phase in pi units
0.5
0
-0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
frequency in pi units

Example 4.14

Solve

$$y(n) - \frac{3}{2}y(n-1) + \frac{1}{2}y(n-2) = x(n), \quad n \geq 0$$

where

$$x(n) = \left(\frac{1}{4}\right)^n u(n)$$

subject to $y(-1) = 4$ and $y(-2) = 10$.

(Source code)

```
n = [0:7]; x = (1/4).^n; xic = [1, -2];
b = 1; a = [1, -1.5, 0.5];
format long; y1 = filter(b, a, x, xic)
y2 = (1/3)*(1/4).^n+(1/2).^n+(2/3)*ones(1, 8) % MATLAB Check
```

(Result)

y1 =

1 ~ 6 번 열

2.000000000000000 1.250000000000000 0.937500000000000
0.796875000000000 0.730468750000000 0.698242187500000

7 ~ 8 번 열

0.682373046875000 0.674499511718750

y2 =

1 ~ 6 번 열

2.000000000000000 1.250000000000000 0.937500000000000
0.796875000000000 0.730468750000000 0.698242187500000

7 ~ 8 번 열

0.682373046875000 0.674499511718750

It means that the calculation is correct.

References

[1] Matlab Official Reference

https://kr.mathworks.com/help/signal/ref/freqz.html?lang=en