Digital Signal Processing Ⅰ

10th Week EXPERIMENT

Report

(7th report of DSP1 course)

Subject	Digital Signal Processing Ⅰ
Professor	Joon-Hyuk Chang
Submission Date	May 12th, 2021
University	Hanyang University
School	College of Engineering
Department	Department of Computer Science & Engineering
Student ID	**Name**
2019009261	최가온(CHOI GA ON)

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Ⅰ. Abstraction

1. z-Transform

1-1. Shortcomings of Fourier transform approach

The discrete-time Fourier transform approach can be used for representing discrete signals using complex exponential sequences. This representation clearly has advantages for LTI systems because it describes systems in the frequency domain using the frequency response function $H(e^{jw})$.

The computation of the sinusoidal steady-state response is greatly facilitated by the use of $H(e^{jw})$. Furthermore, response to any arbitrary absolutely summable sequence x(n) can easily be computed in the frequency domain by multiplying the transform $X(e^{jw})$ and the frequency response $H(e^{jw})$.

However, there two shortcomings to the Fourier transform approach. First, there are many useful signals in practice such as $u(n)$ and $nu(n)$ for which the discrete-time Fourier transform does not exist. Second, the transient response of a system due to initial conditions or due to changing inputs cannot be computed using the discrete-time Fourier transform approach.

1-2. Bilateral z-Transform

The z-transform of a sequence x(n) is given by

$$X(z) = \mathcal{Z}[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

where z is a complex variable. The set of z values for which $X(z)$ exists is called the region of convergence (ROC) and is given by

$$R_{x-} < |z| < R_{x+}$$

for some non-negative numbers $R_{x\_}$ and R_{x+}.

The inverse z-transform of a complex function $X(z)$ is given by

$$x(n) = \mathcal{Z}^{-1}[X(z)] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

where C is a counterclockwise contour encircling the origin and lying in the ROC.

1-3. Properties of the ROC

1. The ROC is always bounded by a circle since the convergence condition is on the magnitude $|z|$.

2. The sequence $x_1(n) = a^n u(n)$ is a special case of a right-sided sequence, defined as a sequence $x(n)$ that is zero for some $n < n_0$. The ROC for right-sided sequences is always outside of a circle of radius R_{x-}. If $n_0 \geq 0$, then the right-sided sequence is also called a causal sequence.

3. The sequence $x_2(n) = -b^n u(-n-1)$ is a special case of a left-sided sequence, defined as a sequence $x(n)$ that is zero for some $n > n_0$. If $n_0 \leq 0$, the resulting sequence is called an anti-causal sequence. The ROC for left-sided sequences is always inside of a circle of radius R_{x+}.

2. Important properties of the z-transform

2-1. Linearity

$$\mathcal{Z}[a_1x_1(n) + a_2x_2(n)] = a_1X_1(z) + a_2X_2(n); \quad ROC: ROC_{x1} \cap ROC_{x2}$$

2-2. Sample shifting

$$\mathcal{Z}[x(n-n_0)] = z^{-n_0}X(z); \qquad ROC: ROC_x$$

2-3. Frequency shifting

$$\mathcal{Z}[a^n x(n)] = X\left(\frac{z}{a}\right); \qquad ROC: ROC_x \; scaled \; by \; |a|$$

2-4. Folding

$$\mathcal{Z}[x(-n)] = X\left(\frac{1}{z}\right); \qquad ROC: Inverted \; ROC_x$$

2-5. Complex conjugation

$$\mathcal{Z}[x^*(n)] = X^*(z^*); \qquad ROC: ROC_x$$

2-6. Differentiation in the z-domain

$$\mathcal{Z}[nx(n)] = -z\frac{dX(z)}{dz}; \qquad ROC: ROC_x$$

2-7. Multiplication

$$\mathcal{Z}[x_1(n)x_2(n)] = \frac{1}{2\pi j}\oint_C \; X_1(v)X_2(z/v)\, v^{-1}dv;$$

$$ROC : ROC_{x1} \;\; \cap \;\; Inverted \; ROC_{x2}$$

where C is a closed contour that encloses the origin and lies in the common ROC.

2-8. Convolution

$$\mathcal{Z}[x_1(n) * x_2(n)] = X_1(z)X_2(z); \qquad ROC \; : \; ROC_{x1} \;\; \cap \;\; ROC_{x2}$$

Ⅱ. Exercises

In this part, there are four exercise questions. Each exercise consists of code and its result. All documents including Matlab code, result, and this report are uploaded in this website :

https://github.com/Gaon-Choi/ELE3076/tree/main/7_z-transform

Example 4.1

Let $x_1(n) = a^n u(n),\ 0 < |a| < \infty$. (This sequence is called a positive-time sequence.)

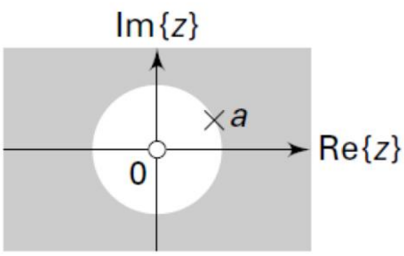
Then,

$$X_1(z) = \sum_{0}^{\infty} a^n z^{-n} = \sum_{0}^{\infty} \left(\frac{a}{z}\right)^n = \frac{1}{1 - az^{-1}};\ if\ \left|\frac{a}{z}\right| < 1$$

$$= \frac{z}{z-a},\quad ROC: |z| > |a|$$

i) zero (numerator) : $z = 0$

ii) pole (denominator) : $z = a$



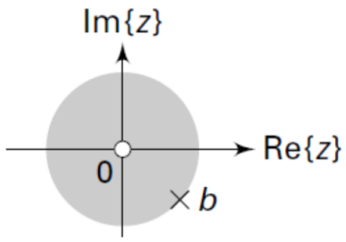
Example 4.2

Let $x_2(n) = -b^n u(-n-1),\ 0 < |b| < \infty$. (This sequence is called a negative-time sequence.)

$$X_2(z) = -\sum_{-\infty}^{-1} b^n z^{-n} = -\sum_{-\infty}^{-1} \left(\frac{b}{z}\right)^n = 1 - \sum_{0}^{\infty} \left(\frac{z}{b}\right)^n$$

$$= 1 - \frac{1}{1 - \frac{z}{b}} = \frac{z}{z-b},\quad ROC : 0 < |z| < |b|$$

The ROC is as follows.



If b = a, then $X_2(z) = X_1(z)$ except for their respective ROCs; that is, $ROC_1 \neq ROC_2$. This implies that the ROC is a distinguishing feature that guarantees the uniqueness of the z-transform.

Hence it plays a very important role in system analysis.

Example 4.3

Let $x_3(n) = x_1(n) + x_2(n) = a^n u(n) - b^n u(-n-1)$.

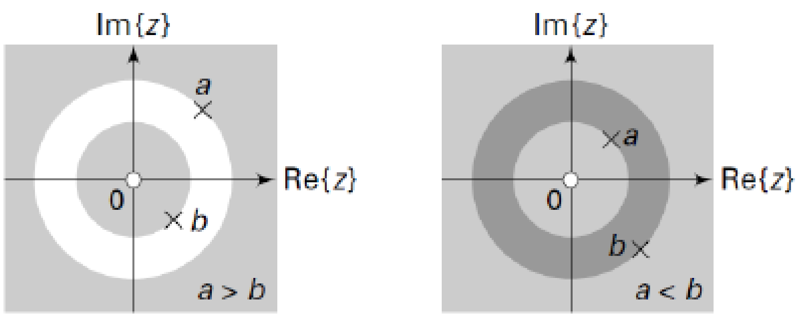
(This sequence is called a two-sided sequence.)

Then using the preceding two examples,

$$X_3(z) = \sum_{n=0}^{\infty} a^n z^{-n} - \sum_{-\infty}^{-1} b^n z^{-n}$$
$$= \left\{\frac{z}{z-a}, ROC_1: |z| > |a|\right\} + \left\{\frac{z}{z-b}, ROC_2: |z| < |b|\right\}$$
$$= \frac{z}{z-a} + \frac{z}{z-b}; \quad ROC_3 = ROC_1 \cap ROC_2$$

If |b| < |a|, then ROC_3 is a null space, and $X_3(z)$ does not exist.

If |a| < |b|, then the ROC_3 is |a| < |z| < |b|, and $X_3(z)$ exists in this region as shown in the next picture.



Example 4.5

Let $X_1(z) = z + 2 + 3z^{-1}$ and $X_2(z) = 2z^2 + 4z + 3 + 5z^{-1}$.

Determine $X_3(z) = X_1(z)X_2(z)$.

Note that

$$x_1(n) = \{\,1, 2, 3\,\}$$

$$x_2(n) = \{2, 4, 3, 5\,\}$$

(Matlab code)

```
x1 = [1, 2, 3]; n1 = [-1:1]; x2 = [2, 4, 3, 5]; n2 = [-2:1];
[x3, n3] = conv_m(x1, n1, x2, n2)
```

(Result)

Example 4.6

Using z-transform properties and the z-transform table, determine the z-transform of

$$x(n) = (n-2)(0.5)^{n-2}\cos\left[\frac{\pi}{3}(n-2)\right]u(n-2)$$

First, we need to calculate the z-transform of the x(n).

For simplicity, the sample-shift property is needed to compute the z-transform.

$$X(z) = \mathcal{Z}[x(n)] = z^{-2}\mathcal{Z}\left[n(0.5)^n\cos\left(\frac{\pi n}{3}\right)u(n)\right]$$

Applying the multiplication by a ramp property,

$$X(z) = z^{-2}\left\{-z\frac{d\mathcal{Z}\left[(0.5)^n\cos\left(\frac{\pi}{3}n\right)u(n)\right]}{dz}\right\}$$

with no change in the ROC.

From the z-transform table,

$$[a^n cosw_0 n]u(n) \leftrightarrow 1-\frac{(acosw_0)z^{-1}}{1-(2acosw_0)z^{-1}+a^2z^{-2}} \quad , \quad |z|>|a|$$

$$\mathcal{Z}\left[(0.5)^n\cos\left(\frac{\pi n}{3}\right)u(n)\right] = \frac{1-\left(0.5\cos\frac{\pi}{3}\right)z^{-1}}{1-2\left(0.5\cos\frac{\pi}{3}\right)z^{-1}+0.25z^{-2}} \quad , \quad |z|>0.5$$

$$= \frac{1-0.25z^{-1}}{1-0.5z^{-1}+0.25z^{-2}} \quad , \quad |z|>0.5$$

Hence,

$$X(z) = -z^{-1}\frac{d}{dz}\left\{1 - \frac{0.25z^{-1}}{1 - 0.5z^{-1} + 0.25z^{-2}}\right\}$$

$$= -z^{-1}\left\{\frac{-0.25z^{-2} + 0.5z^{-3} - 0.0625z^{-4}}{1 - z^{-1} + 0.75z^{-2} - 0.25z^{-3} + 0.0625z^{-4}}\right\}$$

$$= \frac{0.25z^{-3} - 0.5z^{-4} + 0.0625z^{-5}}{1 - z^{-1} + 0.75z^{-2} - 0.25z^{-3} + 0.0625z^{-4}}$$

(Matlab code)

```
b = [0, 0, 0, 0.25, -0.5, 0.0625]; a = [1, -1, 0.75, -0.25, 0.0625];
[delta, n] = impseq(0, 0, 7)

x = filter(b, a, delta) % check sequence

x = [(n-2).*(1/2).^(n-2).*cos(pi*(n-2)/3)].*stepseq(2, 0, 7) % original
sequence
```

(Result)

```
편집기 - C:₩Users₩USER₩Documents₩MATLAB₩example4_6.m
명령 창
>> example4_6

delta =

  1×8 logical 배열

   1   0   0   0   0   0   0   0


n =

     0     1     2     3     4     5     6     7


x =

         0         0         0    0.2500   -0.2500   -0.3750   -0.1250    0.0781


x =

         0         0         0    0.2500   -0.2500   -0.3750   -0.1250    0.0781
fx
```