Digital Signal Processing Ⅰ

13rd Week EXPERIMENT

Report

(8th report of DSP1 course)

Subject	Digital Signal Processing Ⅰ
Professor	Joon-Hyuk Chang
Submission Date	May 31st, 2021
University	Hanyang University
School	College of Engineering
Department	Department of Computer Science & Engineering
Student ID	**Name**
2019009261	최가온(CHOI GA ON)

-Contents-

Ⅰ. Abstraction

1. z-Transform

1-1. Inversion of the z-Transform

Given the original X(z) with following form,

$$X(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a^N z^{-N}} \;,\quad R_{x-} < |z| < R_{x+}$$

express it as

$$X(z) = \frac{\widetilde{b_0} + \widetilde{b_1} z^{-1} + \cdots + \widetilde{b_{N-1}} z^{-(N-1)}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} + \sum_{k=0}^{M-N} C_k z^{-k}$$

The first term is the proper rational part, and the second term is the polynomial (finite-length) part. This can be obtained by performing polynomial division if $M \geq N$ using the deconv function.

Perform a partial fraction expansion on the proper rational part of $X(z)$. Then we can get X(z) of the following form,

$$X(z) = \sum_{k=1}^{N} \frac{R_k}{1 - p_k z^{-1}} + \sum_{k=0}^{M-N} C_k z^{-k}$$

where p_k is the k^{th} pole of $X(z)$ and R_k is the residue at p_k. It is assumed that the poles are distinct for which the residues are given by

$$R_k = \left. \frac{\widetilde{b_0} + \widetilde{b_1} z^{-1} + \cdots + \widetilde{b_{N-1}} z^{-(N-1)}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} (1 - p_k z^{-1}) \right|_{z=p_k}$$

This is a more general form, in which a pole p_k has multiplicity r, then its expansion is given by

$$\sum_{l=1}^{r} \frac{R_{k,l} z^{-(l-1)}}{(1 - p_k z^{-1})^l} = \frac{R_{k,1}}{1 - p_k z^{-1}} + \frac{R_{k,2} z^{-1}}{(1 - p_k z^{-1})^2} + \cdots + \frac{R_{k,r} z^{-(r-1)}}{(1 - p_k z^{-1})^r}$$

where the residues $R_{k,l}$ are computed using a more general formula.

$$\mathcal{Z}^{-1}\left[\frac{z}{z-p_k}\right] = \begin{cases} p_k^n u(n) & |z_k| \leq R_{x-} \\ -p_k^n u(-n-1) & |z_k| \geq R_{x+} \end{cases}$$

1-2. MATLAB Implementation

A MATLAB function residuez is available to compute the residue part and the direct (or polynomial) terms of a rational function in z^{-1}.

Let $X(z)$ be a rational function in which the numerator and the denominator polynomials are in ascending powers of z^{-1}.

$$X(z) = \frac{b_0 + b_1 z^{-1} + b^2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a^2 z^{-2} + \cdots + a_N z^{-N}} = \sum_{k=1}^{N} \frac{R_k}{1 - p_k z^{-1}} + \sum_{k=0}^{M-N} C_k z^{-k}$$

```
[R, p, C] = residuez(b, a)
```

argument: b, a　　　　　returned value: R, p, C

computes the residues, poles, and direct terms of $X(z)$ in which two polynomials $B(z)$ and $A(z)$ are given in two vectors b and a, respectively.

The returned column vector R contains the residues, column vector p contains the pole locations, and row vector C contains the direct terms.

```
[b, a] = residuez(R, p, C)
```

argument: R, p, C　　　　　returned value: b, a

converts the partial fraction expansion back to polynomials with coefficients in row vectors b and a.

Ⅱ. Exercises

In this part, there are four exercise questions. Each exercise consists of code and its result. All documents including Matlab code, result, and this report are uploaded in this website:

https://github.com/Gaon-Choi/ELE3076/tree/main/8_inverse_z-Transform

Example 4.7

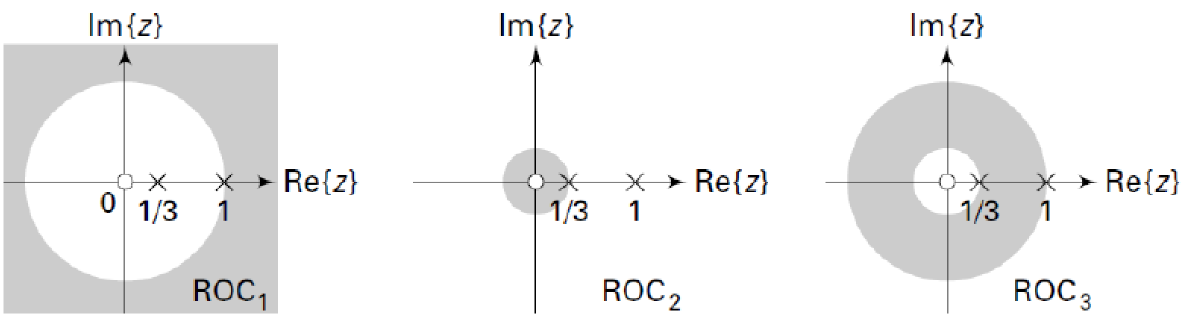
Find the inverse z-transform of $x(z) = \frac{z}{3z^2-4z+1}$.

$$X(z) = \frac{z}{3\left(z^2 - \frac{4}{3}z + \frac{1}{3}\right)} = \frac{\frac{1}{3}z^{-1}}{1 - \frac{4}{3}z^{-1} + \frac{1}{3}z^{-2}} = \frac{\frac{1}{3}z^{-1}}{(1-z^{-1})\left(1-\frac{1}{3}z^{-1}\right)}$$
$$= \frac{\frac{1}{2}}{1-z^{-1}} - \frac{\frac{1}{2}}{1-\frac{1}{3}z^{-1}}$$

or $X(z) = \frac{1}{2}\left(\frac{1}{1-z^{-1}}\right) - \frac{1}{2}\left(\frac{1}{1-\frac{1}{3}z^{-1}}\right)$

There are two poles; $z_1 = 1$ and $z_2 = \frac{1}{3}$

The ROC is not specified by one case. Thus, there are three possible cases of different ROCs as shown in the following figure.



i) ROC_1: $1 < |z| < \infty$

Then,

$$x_1(n) = \frac{1}{2}u(n) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n)$$

which is a right-sided sequence.

ii) ROC_2: $0 < |z| < \frac{1}{3}$

Then,

$$x_2(n) = \frac{1}{2}\{-u(-n-1\} - \frac{1}{2}\left\{-\left(\frac{1}{3}\right)^n u(-n-1)\right\} = \frac{1}{2}\left(\frac{1}{3}\right)^n u(-n-1) - \frac{1}{2}u(-n-1)$$

which is a left-sided sequence.

iii) ROC_3: $\frac{1}{3} < |z| < 1$

Then,

$$x_3(n) = -\frac{1}{2}u(-n-1) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n)$$

which is a two-sided sequence.

Example 4.8

To check our residue calculations, let us consider the rational function

$$X(z) = \frac{z}{3z^2 - 4z + 1}$$

given in Example 4.7.

First rearrange $X(z)$ so that it is a function in ascending powers of z^{-1}.

$$X(z) = \frac{z^{-1}}{3 - 4z^{-1} + z^{-2}} = \frac{0 + z^{-1}}{3 - 4z^{-1} + z^{-2}}$$

(MATLAB Code)

```
b = [0, 1];
a = [3, -4, 1];

[R, p, C] = residuez(b, a)
```

(Result)

According to the result of MATLAB code, we can obtain $X(z)$ as follows.

$$X(z) = \frac{\frac{1}{2}}{1 - z^{-1}} - \frac{\frac{1}{2}}{1 - \frac{1}{3}z^{-1}}$$

Reversely, to convert back to the rational function form,

(MATLAB Code)

```
b = [0, 1];
a = [3, -4, 1];
[R, p, C] = residuez(b, a);

[b, a] = residuez(R, p, C)
```

(Result)

```
명령 창
>> example4_8_2

b =

   -0.0000    0.3333


a =

    1.0000   -1.3333    0.3333

fx >>
```

so that

$$X(z) = \frac{0 + \frac{1}{3}z^{-1}}{1 - \frac{4}{3}z^{-1} + \frac{1}{3}z^{-2}} = \frac{z^{-1}}{3 - 4z^{-1} + z^{-2}} = \frac{z}{3z^2 - 4z + 1}$$

It is similar as before.

Example 4.9

Compute the inverse z-Transform of

$$X(z) = \frac{1}{(1-0.9z^{-1})^2(1+0.9z^{-1})} \quad , \qquad |z| > 0.9$$

We need to evaluate the denominator polynomial as well as the residues using the MATLAB script.

The denominator polynomial can be computed using the MATLAB's polynomial function poly, which computes the polynomial coefficients, given its roots.

(MATLAB Code)

```
b = 1; a = poly([0.9, 0.9, -0.9])
[R, p, C] = residuez(b, a)
```

(Result)

a =

1.0000 -0.9000 -0.8100 0.7290

R =

0.2500 + 0.0000i

0.2500 + 0.0000i

0.5000 - 0.0000i

p =

-0.9000 + 0.0000i

0.9000 + 0.0000i

0.9000 - 0.0000i

C =

    [ ]

From the result of the residue calculations and using the order of residues,

$$X(z) = \frac{0.25}{1-0.9z^{-1}} + \frac{0.5}{(1-0.9z^{-1})^2} + \frac{0.25}{1+0.9z^{-1}}, \quad |z| > 0.9$$

$$= \frac{0.25}{1-0.9z^{-1}} + \frac{0.5}{0.9}z\frac{0.9z^{-1}}{(1-0.9z^{-1})^2} + \frac{0.25}{1+0.9z^{-1}}, \quad |z| > 0.9$$

Using the inverse z-Transform,

$$x(n) = 0.25(0.9)^n u(n) + \frac{5}{9}(n+1)(0.9)^{n+1}u(n+1) + 0.25(-0.9)^n u(n)$$
$$= 0.75(0.9)^n u(n) + 0.5n(0.9)^n u(n) + 0.25(-0.9)^n u(n)$$

(MATLAB Code)

```
% check sequence
[delta, n] = impseq(0, 0, 7); x = filter(b, a, delta)
% answer sequence
x = (0.75) * (0.9).^n + (0.5) * n.*(0.9).^n + (0.25)*(-
0.9).^n
```

(Result)

x =

1.0000	0.9000	1.6200	1.4580	1.9683	1.7715	2.1258	1.9132

x =

1.0000	0.9000	1.6200	1.4580	1.9683	1.7715	2.1258	1.9132

It means the calculation is correct.

Example 4.10

Determine the inverse z-transform of

$$X(z) = \frac{1 + 0.4\sqrt{2}z^{-1}}{1 - 0.8\sqrt{2}z^{-1} + 0.64z^{-2}}$$

so that the resulting sequence is causal and contains no complex numbers.

We need to find the poles of $X(z)$ in the polar form to determine the ROC of the causal sequence.

(MATLAB Code)

```
b = [1, 0.4 * sqrt(2)];
a = [1, -0.8 * sqrt(2), 0.64];
[R, p, C] = residuez(b, a)

Mp = (abs(p))'   % pole magnitudes
Ap = (angle(p))'/pi   % pole angles in pi units
```

(Result)

R =

0.5000 - 1.0000i

0.5000 + 1.0000i

p =

0.5657 + 0.5657i

0.5657 - 0.5657i

C =

[ ]

Mp =

0.8000 0.8000

Ap =

0.2500 -0.2500

From these calculations,

$$X(z) = \frac{0.5 - j}{1 - 0.8e^{+j\frac{\pi}{4}}z^{-1}} + \frac{0.5 + j}{1 - 0.8e^{-j\frac{\pi}{4}}z^{-1}}, \quad |z| > 0.8$$

$$
\begin{aligned}
x(n) &= (0.5 - j)\, 0.8^n e^{+j\frac{\pi}{4}n} u(n) + (0.5 + j) 0.8^n e^{-j\frac{\pi}{4}n} u(n) \\
&= 0.8^n \left[0.5 \left\{ e^{+j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n} \right\} - j \left\{ e^{+j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n} \right\} \right] u(n) \\
&= 0.8^n \left[\cos\left(\frac{\pi n}{4}\right) + 2\sin\left(\frac{\pi n}{4}\right) \right] u(n)
\end{aligned}
$$

(MATLAB verification)

```
[delta, n] = impseq(0, 0, 6);
x = filter(b, a, delta) % check sequence

x = ((0.8).^n).*(cos(pi*n/4)+2*sin(pi*n/4))
```

x =

1.0000 1.6971 1.2800 0.3620 -0.4096 -0.6951 -0.5243

x =

1.0000 1.6971 1.2800 0.3620 -0.4096 -0.6951 -0.5243

It means that the calculation is correct.