Numerical Optimization, 2020 Fall

Homework 3

Due on 14:59 OCT 10, 2020

请尽量使用提供的 tex 模板, 单纯形法的表格可手绘拍照加入文档.

1 单纯形法

以下均考虑非退化线性规划问题即可。

(i) 考虑一线性规划问题的规范型如下:

以(p,q)元转轴后，新规范形的系数

Pivot, 意味着以此消元

$$\begin{array}{llllll}
1\cdot x_1 & & & +y_{1q}x_q+\ldots+y_{1n}x_n & = \bar{b}_1 \\
 & 1\cdot x_p & & +\boxed{y_{pq}}x_q+\ldots+y_{pn}x_n & = \bar{b}_p \\
 & & \ddots & \vdots & \vdots \\
 & & 1\cdot x_m & +y_{mq}x_q+\ldots+y_{mn}x_n & = \bar{b}_m
\end{array}$$

Pivot后, 意味此列出基

记进基变量的下标为 q, 转轴前分量 j 对应的 reduced cost 为 r_j, 转轴后对应的 reduced cost 为 r'_j。试证明 reduced cost 的更新公式为 $r'_j = r_j - \frac{y_{pj}}{y_{pq}} r_q$(参考 Lecture 3 中 17 页)。 [20pts]

解:

(1) 为方便证明，将B(1),…B(m)记为1,…,m

因为 $p\in\{B(1)\ldots B(m)\}$, $q\in n/\{B(1),\cdots B(m)\}$, p,q的任意性,

所以将p看作B(p), q看作q且q∉1,…m是不失一般性的

$r_j = c_j - z_j$

$z_j = c_1y_{1j}+\cdots+c_py_{pj}+c_{p+1}y_{p+1j}+\cdots+c_my_{mj} = (c_1,\cdots,c_p,\cdots,c_m)\begin{pmatrix} y_{1j} \\ \vdots \\ y_{pj} \\ \vdots \\ y_{mj}\end{pmatrix}$

原先是 $x_p = \cdots - y_{pj}x_j$, $c_px_p = \cdots - c_py_{pj}x_j$

现在是 $x_q = \cdots - y'_{pj}x_j$, $c_qx_q = \cdots - c_qy'_{pj}x_j$

$r'_j = c_j - z'_j$

$z'_j = c_1y'_{1j}+\cdots+c_{p-1}y'_{p-1j}+c_qy'_{pj}+c_{p+1}y'_{p+1j}+\cdots+c_my'_{mj} = (c_1,\cdots,c_q,\cdots,c_m)\begin{pmatrix} y'_{1j} \\ \vdots \\ y'_{pj} \\ \vdots \\ y'_{mj}\end{pmatrix}$

特别地 $r_q = c_q - z_q$

$z_q = (c_1,\cdots,c_p,\cdots,c_m)\begin{pmatrix} y_{1q} \\ \vdots \\ y_{mq}\end{pmatrix}$

$r'_j = c_j - z'_j$

$= c_j - (c_1,\cdots,c_q,\cdots,c_m)\begin{pmatrix} y'_{1j} \\ \vdots \\ y'_{pj} \\ \vdots \\ y'_{mj}\end{pmatrix} = c_j - (c_1,\cdots,0,\cdots,c_m)\begin{pmatrix} y'_{1j} \\ \vdots \\ 0 \\ \vdots \\ y'_{mj}\end{pmatrix} - c_qy'_{pj}$

$r_j - \frac{y_{pj}}{y_{pq}}r_q = c_j - z_j - \frac{y_{pj}}{y_{pq}}(c_q - z_q)$

$= c_j - \frac{y_{pj}}{y_{pq}}c_q - (c_1,\cdots,c_p,\cdots,c_m)\begin{pmatrix} y_{1j}-\frac{y_{pj}}{y_{pq}}y_{1q} \\ \vdots \\ y_{pj}-\frac{y_{pj}}{y_{pq}}y_{pq}=0 \\ \vdots \\ y_{mj}-\frac{y_{pj}}{y_{pq}}y_{mq}\end{pmatrix} = c_j - (c_1,\cdots,0,\cdots,c_m)\begin{pmatrix} y'_{1j} \\ \vdots \\ 0 \\ \vdots \\ y'_{mj}\end{pmatrix} - c_qy'_{pj}$

$\therefore r'_j = r_j - \frac{y_{pj}}{y_{pq}}r_q$

(ii) 单纯形表中右下角的-f 对应当前基本可行解的目标函数值的相反数。试证明, 经过一次转轴后更新的-f 对应更新后基本可行解对应的目标函数值的相反数 (参考 Lecture 3 中 20 页)。 [20pts]

Simplex Method in Tableau Format

单纯形表(tableau)：**BFS**对应规范形的表格＋

既约费用系数和**BFS**目标值的相反数

	x_1	$\cdots$	x_p	$\cdots$	x_m	x_{m+1}	x_{m+2}	$\cdots$	x_q	$\cdots$	x_n	$B^{-1}b$
	1	$\cdots$	0	$\cdots$	0	$y_{1,m+1}$	$y_{1,m+2}$	$\cdots$	y_{1q}	$\cdots$	y_{1n}	$\bar{b}_1$
		$\ddots$				$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
	0	$\cdots$	1	$\cdots$	0	$y_{p,m+1}$	$y_{p,m+2}$	$\cdots$	y_{pq}	$\cdots$	y_{pn}	$\bar{b}_p$
				$\ddots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
	0	$\cdots$	0	$\cdots$	1	$y_{m,m+1}$	$y_{m,m+2}$	$\cdots$	y_{mq}	$\cdots$	y_{mn}	$\bar{b}_m$
r^T	0	$\cdots$	0	$\cdots$	0	r_{m+1}	r_{m+2}	$\cdots$	r_q	$\cdots$	r_n	$\boxed{-f}$

单纯形表可以提供计算需要的所有信息！ why?

解：

(2)

	$x_1 \cdots$	$x_p \cdots$	x_m	$x_{m+1} \cdots$	$x_q \cdots$	x_n	$B^{-1}b$
初	$1 \cdots$	$0 \cdots$	0	$y_{1m+1} \cdots$	$y_{1q} \cdots$	y_{1n}	$\bar{b}_1$
				$y_{pm+1} \cdots$	$y_{pq} \cdots$	y_{pn}	
表							
	$0 \cdots$	$0 \cdots$	1	$y_{mm+1} \cdots$		y_{mn}	$\bar{b}_m$
c^T	$c_1 \cdots$	$c_p \cdots$	c_m	$c_{m+1} \cdots$	$c_q \cdots$	c_n	0

1° 证明由初始BFS构成的单纯形表的f为对应初始BFS的目标函数值的相反数

仍以 $x_1, \cdots x_m$ 作为 $x_{B(1)} \cdots x_{B(m)}$，依规定不失任意性

目标：将 $c_1, \cdots c_m$ 消为0

因此 $f = 0 - c_1\bar{b}_1 - c_2\bar{b}_2 - \cdots - c_m\bar{b}_m$ $\qquad f_0 = c_1x_1 + \cdots + c_mx_m + \cdots + c_nx_n$

$= -(c_1, \cdots c_m)\begin{pmatrix} \bar{b}_1 \\ \vdots \\ \bar{b}_m \end{pmatrix}$ $\qquad = c_1\bar{b}_1 + \cdots c_m\bar{b}_m$

$f = -f_0$ 得证

	$x_1 \cdots$	$x_p \cdots$	x_m	$x_{m+1} \cdots$	$x_q \cdots$	x_n	$B^{-1}b$
	$1 \cdots$	$0 \cdots$	0	$y_{1m+1} \cdots$	$y_{1q} \cdots$	y_{1n}	$\bar{b}_1$
				$y_{pm+1} \cdots$	$y_{pq} \cdots$	y_{pn}	
	$0 \cdots$	$0 \cdots$	1	$y_{mm+1} \cdots$		y_{mn}	$\bar{b}_m$
r^T	$0 \cdots$	$0 \cdots$	0	$r_{m+1} \cdots$	$r_q \cdots$	r_n	$(-f_0)/f$

2° 证明经过一次转轴更新的-f对应更新后基本可行解对应目标函数的相反数

设 x_q 进基，x_p 出基

$f' = f - r_q \cdot \frac{\bar{b}_p}{y_{pq}}$ (为了使 $r_q=0$，所以最后一行减去 $\frac{r_q}{y_{pq}}$ 倍P行) $f_0' = f_0 + (c_q - z_q)\frac{\bar{b}_p}{y_{pq}}$ ($f_0 = c_1\bar{b}_1 + \cdots + c_m\bar{b}_m - \sum_{j=m+1}^{n}(c_j - z_j)x_j$

$= f_0 + r_q\frac{\bar{b}_p}{y_{pq}}$ $\qquad \because$ q进基，P出基，所以 x_q 由0变为了 $\frac{\bar{b}_p}{y_{pq}}$

$\frac{\bar{b}_p}{y_{pq}}$ 即是使得 $x_1, \cdots x_m$ 中一值至0的最小值，

$f' = f - r_q \cdot \frac{\bar{b}_p}{y_{pq}} = -(f_0 + r_q\frac{\bar{b}_p}{y_{pq}}) = -f_0'$，$f' = -f_0'$ $\qquad$ 即 $\min\left\{\frac{\bar{b}_i}{y_{iq}} \middle| y_{iq} > 0, i = 1, \cdots m\right\}$，此时

得证 $\qquad$ 对应 $x_p = \bar{b}_p - \frac{\bar{b}_p}{y_{pq}}y_{pq} = 0$)

2 修正单纯形法

2.1 证明题

试证明 Lecture 4 中 20 页 λ 的更新公式为: $\hat{\lambda}^T = \lambda^T + \frac{r_q}{y_{pq}}\boldsymbol{u}_p$。 [20pts]

解:

2.(1) 仍以 $x_1, \cdots x_m$ 作为 $x_{B(1)} \cdots x_{B(m)}$, 依假定不失任意性

$$\lambda^T + \frac{r_q}{y_{pq}}u_p = (c_1, \cdots, c_p, \cdots, c_m)\begin{pmatrix} u_1 \\ \vdots \\ u_p \\ \vdots \\ u_m \end{pmatrix} + \frac{c_q - z_q}{y_{pq}}u_p$$

$$= c_1u_1 + \cdots + c_{p-1}u_{p-1} + c_{p+1}u_{p+1} + \cdots + c_mu_m + \left[\cancel{x_p} + \frac{c_q}{y_{pq}} - \frac{y_{1q}c_1 + y_{p-1q}c_q + \cancel{y_{pq}c_q} + y_{p+1q}c_q + \cdots + y_mc_m}{y_{pq}}\right]u_p$$

$$= c_1u_1 + \cdots + c_{p-1}u_{p-1} + c_{p+1}u_{p+1} + \cdots + c_mu_m + \left[\frac{c_q}{y_{pq}} - c_1\frac{y_{1q}}{y_{pq}} - \cdots - c_{p-1}\frac{y_{p-1q}}{y_{pq}} - c_{p+1}\frac{y_{p+1q}}{y_{pq}} - \cdots - \frac{y_mc_m}{y_{pq}}\right]u_p \quad ①$$

$$\hat{\lambda}^T = \hat{c}_B^T\hat{B}^{-1}$$

$$= \hat{c}_B^T E_{pq} B^{-1}$$

$$= (c_1, \cdots, c_{p-1}, c_q, c_{p+1}, \cdots, c_m)(e_1, \cdots, e_{p-1}, v, e_{p+1}, \cdots, e_m)\begin{pmatrix} u_1 \\ \vdots \\ u_p \\ \vdots \\ u_m \end{pmatrix}$$

$$= (c_1, \cdots, c_q, \cdots, c_m)\begin{pmatrix} u_1 - \frac{y_{1q}}{y_{pq}}u_p \\ \vdots \\ \frac{1}{y_{pq}}u_p \\ \vdots \\ u_m - \frac{y_{mq}}{y_{pq}}u_p \end{pmatrix} \quad \left(v_i = \begin{cases} -\frac{y_{iq}}{y_{pq}} & i \neq p \\ \frac{1}{y_{pq}} & i = p \end{cases}\right)$$

$$= c_1u_1 + \cdots + c_{p-1}u_{p-1} + c_{p+1}u_{p+1} + \cdots + c_mu_m + \frac{c_q}{y_{pq}}u_p - c_1\frac{y_{1q}}{y_{pq}}u_p - \cdots - c_{p-1}\frac{y_{p-1q}}{y_{pq}}u_p - c_{p+1}\frac{y_{p+1q}}{y_{pq}}u_p - \cdots - c_m\frac{y_{mq}}{y_{pq}}u_p$$

$$= c_1u_1 + \cdots + c_{p-1}u_{p-1} + c_{p+1}u_{p+1} + \cdots + c_mu_m + \left[\frac{c_q}{y_{pq}} - c_1\frac{y_{1q}}{y_{pq}} - \cdots - c_{p-1}\frac{y_{p-1q}}{y_{pq}} - c_{p+1}\frac{y_{p+1q}}{y_{pq}} - \cdots - c_m\frac{y_{mq}}{y_{pq}}\right]u_p \quad ②$$

① = ②

$\therefore \hat{\lambda}^T = \lambda^T + \frac{r_q}{y_{pq}}u_p$ 得证

2.2 计算题

试用两阶段法求解如下线性规划问题 (详见 Lecture 4 第 13 页), 给出各个步骤的单纯形表。 [40pts]

$$\begin{aligned}
\textbf{minimize} \quad & x_1 - x_2 \\
\textbf{subject to} \quad & -x_1 + 2x_2 + x_3 = 2 \\
& -4x_1 + 4x_2 - x_3 = 4 \\
& -5x_1 + 6x_2 = 6 \\
& x_1 - x_3 = 0 \\
& x_1,\ x_2,\ x_3 \geq 0
\end{aligned}$$

解:

辅助问题: $\min \sum_{i=4}^{7} x_i$

s.t. $-x_1+2x_2+x_3+x_4=2$

$-4x_1+4x_2-x_3+x_5=4$

$-5x_1+6x_2+x_6=6$ 初始BFS: $(0,0,0,2,4,6,0)$, 以 (x_4,x_5,x_6,x_7) 作为一组基

$x_1-x_3+x_7=0$

$x_1,x_2,\cdots,x_7 \geq 0$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
	-1	2	1	1	0	0	0	2
	-4	4	-1	0	1	0	0	4
	-5	6	0	0	0	1	0	6
	1	0	-1	0	0	0	1	0
C^T	0	0	0	1	1	1	1	0

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
	$-\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	1
	-2	0	-3	-2	1	0	0	0
	-2	0	-3	-3	0	1	0	0
	1	0	-1	0	0	0	1	0
R^T	3	0	7	6	0	0	0	0

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
	0	1	$\frac{5}{4}$	1	$-\frac{1}{4}$	0	0	1
	1	0	$\frac{3}{2}$	1	$-\frac{1}{2}$	0	0	0
多余	0	0	0	-1	-1	1	0	0
	0	0	$-\frac{5}{2}$	-1	$\frac{1}{2}$	0	1	0
R^T	0	0	$\frac{5}{2}$	3	$\frac{3}{2}$	0	0	0

去0行以及x_6

	x_1	x_2	x_3	x_4	x_5	x_7	b
	0	1	$\frac{5}{4}$	1	$-\frac{1}{4}$	0	1
	1	0	$\frac{3}{2}$	1	$-\frac{1}{2}$	0	0
	0	0	$-\frac{5}{2}$	-1	$\frac{1}{2}$	1	0
R^T	0	0	$\frac{5}{2}$	3	$\frac{3}{2}$	0	0

	x_1	x_2	x_3	x_4	x_5	x_7	b
	0	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1
	1	0	0	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{3}{5}$	0
	0	0	1	$\frac{2}{5}$	$-\frac{1}{5}$	$-\frac{2}{5}$	0
R^T	0	0	0	3	2	1	0

$(x_1,x_2,x_3,x_4,x_5,x_7)=(0,1,0,0,0,0)$

BFS: $(0,1,0)$

	x_1	x_2	x_3	b
	0	1	0	1
	1	0	0	0
	0	0	1	0
c^T	1	-1	0	0

$\min \ x_1 - x_2$

s.t. $-x_1+2x_2+x_3=2$

$-4x_1+4x_2-x_3=4$

$-5x_1+6x_2=6$ (冗余去掉)

$x_1-x_3=0$

$x_1,x_2,x_3\geq 0$

	x_1	x_2	x_3	b
	0	1	0	1
	1	0	0	0
	0	0	1	0
R^T	0	0	0	1

∴ 最优解: (0,1,0) 目标函数: -1