1. $\forall x \in \mathbb{R}^n$, $\|x\|_\infty = \max\limits_{1\le i\le n}|x_i| = \sqrt{\max\limits_{1\le i\le n}|x_i|^2} = \sqrt{\max\limits_{1\le i\le n} x_i^2} \le \sqrt{x_1^2+\cdots+x_n^2} = \|x\|_2$ ①

$\|x\|_2 = \sqrt{x_1^2+\cdots+x_n^2} = \sqrt{|x_1|^2+\cdots+|x_n|^2} \le \sqrt{(|x_1|+\cdots+|x_n|)^2} = \sum\limits_{i=1}^{n}|x_i| = \|x\|_1$ ②

$\|x\|_1 = \sum\limits_{i=1}^{n}|x_i| \le n\cdot\max\limits_{1\le i\le n}|x_i| = n\|x\|_\infty$ ③

$\therefore \quad \|x\|_\infty \le \|x\|_2 \le \|x\|_1 \le n\|x\|_\infty$

2. "$\Rightarrow$"

$\forall \alpha \in \mathbb{R}$, $\|x+\alpha y\| = \sqrt{\langle x+\alpha y, x+\alpha y\rangle}$

$= \sqrt{\langle x,x\rangle + \alpha^2\langle y,y\rangle + 2\alpha\langle x,y\rangle}$ As $\langle x,y\rangle = 0$

$= \sqrt{\langle x,x\rangle + \alpha^2\langle y,y\rangle - 2\alpha\langle x,y\rangle}$

$= \sqrt{\langle x-\alpha y, x-\alpha y\rangle}$

$= \|x-\alpha y\|$

"$\Leftarrow$" $\forall \alpha \in \mathbb{R}$, $\|x+\alpha y\| = \sqrt{\langle x,x\rangle + \alpha^2\langle y,y\rangle + 2\alpha\langle x,y\rangle} = \sqrt{\langle x,x\rangle + \alpha^2\langle y,y\rangle - 2\alpha\langle x,y\rangle} = \|x-\alpha y\|$

$\langle x,x\rangle + \alpha^2\langle y,y\rangle + 2\alpha\langle x,y\rangle = \langle x,x\rangle + \alpha^2\langle y,y\rangle - 2\alpha\langle x,y\rangle$

$4\alpha\langle x,y\rangle = 0$

As $\forall \alpha \in \mathbb{R}$ is true, let $\alpha = 1$

then $4\langle x,y\rangle = 0$

$\langle x,y\rangle = 0$

Proved

3. Let $\varphi_n(x) = (x^2-1)^n = (x+1)^n(x-1)^n$, then $\varphi^{(k)}(\pm 1) = 0 \quad \forall k = 0,\cdots,n-1$ $\qquad \langle f,g\rangle = \int_{-1}^{1} f(x)g(x)dx$

$\therefore \forall m,n \in \mathbb{N}, m<n$, $\langle P_n, P_m\rangle = \langle P_m, P_n\rangle = \int_{-1}^{1} \frac{1}{2^n n!}[(x^2-1)^n]^{(n)} \cdot \frac{1}{2^m m!}[(x^2-1)^m]^{(m)} dx$ $\qquad \int_a^b f(x)dx = xf(x) - \int_a^b x\,d f(x)$

$= \frac{1}{2^n n!}\cdot\frac{1}{2^m m!}\int_{-1}^{1} \varphi_n^{(n)}(x)\varphi_m^{(m)}(x)dx$ $\qquad \int_{-1}^{1}\varphi_n^{(1)}(x)\varphi_m^{(1)}(x)dx$

$= \frac{1}{2^n n!}\frac{1}{2^m m!}\int_{-1}^{1}\varphi_m^{(m)}(x)d\varphi_n^{(n-1)}(x)$

$= \frac{1}{2^n n!}\frac{1}{2^m m!}\left[\underbrace{\varphi_m^{(m)}(x)\varphi_n^{(n-1)}(x)\Big|_{-1}^{1}}_{=0} - \int_{-1}^{1}\varphi_n^{(n-1)}(x)\varphi_m^{(m+1)}(x)dx\right]$

$= \frac{(-1)^n}{2^n n!}\frac{1}{2^m m!}\int_{-1}^{1}\varphi_n(x)\varphi_m^{(m+n)}(x)dx$

As $\varphi_n(x)$ is a polynomial of degree n

$\varphi_m(x)$ is a polynomial of degree m

$\therefore 1^\circ\ m \neq n$, as $m \le n \Rightarrow m < n$, then $\varphi_m^{(m+n)}(x) = 0$

$\therefore \langle P_n, P_m \rangle = 0$ when $n \neq m$

$2^\circ\ m = n\ \ \varphi_n^{(m)}(x) = \left[\frac{2n!}{n!} x^n + \cdots\right]^{(n)}$

$= \frac{1}{2^n n!} \cdot \frac{2n!}{n!}$

$\therefore \langle P_n, P_m \rangle = \frac{(-1)^n}{2^n n!} \frac{1}{2^n n!} \int_{-1}^{1} (x^2-1)^n \cdot 2n!\, dx$ let $x = \sin t$, $t \in [0, \frac{\pi}{2}]$

$= \frac{2n!}{2^m (n!)^2} \int_{-1}^{1} (1-x^2)^n dx$

$= \frac{2n!}{2^m (n!)^2} \int_0^{\frac{\pi}{2}} \cos^{2n} t\, d\sin t \cdot 2$

$= \frac{2n!}{2^m (n!)^2} \int_0^{\frac{\pi}{2}} \cos^{2n+1} t\, dt \cdot 2$

$= \frac{2n!}{2^m (n!)^2} \frac{(2n)!!}{(2n+1)!!} \cdot 2$

$= \frac{2n!}{(2n-1)!! 2^n} \cdot \frac{2n!!}{2^n} \cdot \frac{2}{2n+1} \cdot \frac{1}{(n!)^2}$

$= (n!)^2 \cdot \frac{2}{2n+1} \cdot \frac{1}{(n!)^2}$

$= \frac{2}{2n+1}$

$\therefore \forall n, m \in N,\ \langle P_n, P_m \rangle = \begin{cases} 0 & n \neq m \\ \frac{2}{2n+1} & n = m \end{cases}$

proved

4. $P_0 = \frac{1}{2^0 0!} (x^2-1)^0$

$= 1$

$P_1 = \frac{1}{2^1 1!} (x^2-1)'$

$= \frac{1}{2} \cdot 2x$

$= x$

$P_2 = \frac{1}{2^2 2!} [(x^2-1)^2]''$

$= \frac{1}{8} (x^4 - 2x^2 + 1)''$

$= \frac{1}{8} (4x^3 - 4x)'$

$= \frac{1}{8} (12x^2 - 4)$

$= \frac{3}{2} x^2 - \frac{1}{2}$

$\|P_0\|^2 = \int_{-1}^{1} 1\, dx = 2$ $\|P_2\|^2 = \int_{-1}^{1} P_2^2 dx = \frac{2}{5}$

$\|P_1\|^2 = \int_{-1}^{1} x^2 dx = \frac{2}{3}$

$f(x)=e^{x}\quad x\in[1,2]$

let $x=\frac{1}{2}t+\frac{3}{2},\ t\in[-1,1]$, $g(t)=e^{\frac{1}{2}t+\frac{3}{2}}\quad t\in[-1,1]$

$$p(t)=\frac{\langle g(t),P_0\rangle P_0}{\|P_0\|^2}+\frac{\langle g(t),P_1\rangle P_1}{\|P_1\|^2}+\frac{\langle g(t),P_2\rangle P_2}{\|P_2\|^2}$$

$$=e^{\frac{3}{2}}\left[\frac{1}{2}\int_{-1}^{1}e^{\frac{1}{2}t}dt+\frac{3t}{2}\int_{-1}^{1}te^{\frac{1}{2}t}dt+\frac{5}{2}\left(\frac{3}{2}t^2-\frac{1}{2}\right)\int_{-1}^{1}\left(\frac{3}{2}t^2-\frac{1}{2}\right)e^{\frac{1}{2}t}dt\right]$$

$$=e^{\frac{3}{2}}\left[1.04219+0.5126t+0.1272t^2-0.04242\right]$$

$$=0.5703t^2+2.2973t+4.4807$$

$x=2t-3$

$$p(x)=0.5703(2x-3)^2+2.2973(2x-3)+4.4807$$

$\therefore\ p(x)=2.2812x^2-2.2490x+2.7215\qquad \min\limits_{p\in P_2}\int_1^2|f(x)-p(x)|^2dx\approx 3.3866$

5\. $P_0=1$

let $P_1=1-x$

$$\langle P_m,P_n\rangle=\int_0^{+\infty}P_mP_ne^{-x}dx$$

$$\langle P_0,P_1\rangle=\int_0^{+\infty}(1-x)e^{-x}dx$$

$$=\int_0^{+\infty}e^{-x}dx-\int_0^{+\infty}xe^{-x}\qquad x\sim Expo(\lambda)\ \lambda=1,E(x)=1$$

$$=-e^{-x}\Big|_0^{+\infty}-1$$

$$=0$$

$$\|P_0\|^2=\int_0^{+\infty}e^{-x}dx$$

$$=1$$

$$\|P_1\|^2=\int_0^{+\infty}(1-x)^2e^{-x}dx$$

$$=\int_0^{+\infty}x^2e^{-x}-2xe^{-x}+e^{-x}dx$$

$$=\int_0^{+\infty}x^2e^{-x}dx+\int_0^{+\infty}2(1-x)e^{-x}dx-\int_0^{+\infty}e^{-x}dx$$

$$=-\int_0^{+\infty}x^2de^{-x}-1$$

$$=-x^2e^{-x}\Big|_0^{+\infty}+2\int_0^{+\infty}xe^{-x}dx-1$$

$$=1$$

$$p(x)=\frac{\langle f(x),P_0\rangle P_0}{\|P_0\|^2}+\frac{\langle f(x),P_1\rangle P_1}{\|P_1\|^2}$$

$$=\int_0^{+\infty}x^2e^{-x}dx+\int_0^{+\infty}x^2(1-x)e^{-x}dx\times(1-x)$$

$$=2+\left(2-\int_0^{+\infty}x^3e^{-x}dx\right)(1-x)$$

$$=2+\left(2+\int_0^{+\infty}x^3de^{-x}\right)(1-x)$$

$$=2+\left(2+x^3e^{-x}\Big|_0^{+\infty}-\int_0^{+\infty}e^{-x}\cdot 3x^2dx\right)(1-x)$$

$$=2+(2-2\times3)(1-x)$$

$$=4x-2$$

$\therefore\ p(x)=4x-2\qquad \min\limits_{p\in P_1}\int_0^{\infty}|f(x)-p(x)|^2e^{-x}dx=4$