SI 211: Numerical Analysis Homework 1

Due on Step 30, 2020, 23:59

1. Floating point numbers. What is the bit representations of the floating point number 5.25 using the IEEE standard for double precision numbers? Solve this problem with pen and paper first. Use a computer program to verify your result.

   solution:
   $5.25=(101.01)_2=(1.0101)_2*10^2$
   $2+1023=(10000000001)_2$ and 5.25>0 so the first position will be 0
   and we need to add 0 until 64 digital
   Finally,5.25=0100000000010101000000000000000000000000000000000000000000000000(forty eight 0)
   use bitstring(5.25) will get the same answer

2. Numerical evaluation error. Evaluate the function

$$f(x) = \frac{1-\cos(x)}{x^2}$$

   with a compute program of your choice using the standard IEEE double precision floating point format. Plot the numerical result on a logarithmic scale for $x \in \left[10^{-15}, 10^{-1}\right]$.

   solution:
   l'm sorry that i can't insert the graph here,so i put all ther graph at the end of the document please see graph 2 at the end

3. Taylor expansion. Consider again the function

$$f(x) = \frac{1-\cos(x)}{x^2}$$

   from the first homework problem. Can you approximate f by using a Taylor approximation? Does this help you to evaluate f with higher accuracy?

   solution
   f(x)=f(1^{-100})+f'(1^{-100})(x-1^{-100})+....+$\frac{f^{n+1}(eps)}{(n+1)!}x^{n+1}$,eps bewteen 1^{-100} and x
   And yes we can evaluate f with highter accuracy when x in [10^{-15},10^{-1}] as the error is proportional to $\frac{f^{n+1}(eps)}{(n+1)!}(x-1^{-100})^{n+1}$ which is c$(x-1^{-100})^n$,we can get highter accuracy when we choose n>=10 or even more

4. Numerical differentiation. Consider the five-point differentiation formula

$$f'(x) \approx \frac{1}{12h}\left[-25f\left(x_0\right)+48f\left(x_0+h\right)-36f\left(x_0+2h\right)+16f\left(x_0+3h\right)-3f\left(x_0+4h\right)\right]$$

(a) What is the mathematical approximation error of this formula?

solution:
please see graph 4.a at the end

(b) For which values of h would you expect that this formula leads to a minimum approximation error taking both the mathematical as well as the numerical approximation error into account?

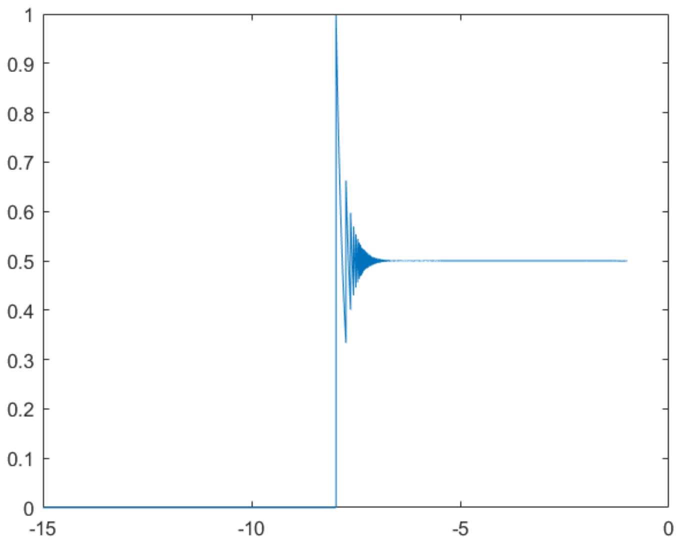
solution:
when h=$10^{-3.2}$ for h^4=$\frac{eps}{h}$,eps=10^{-16}

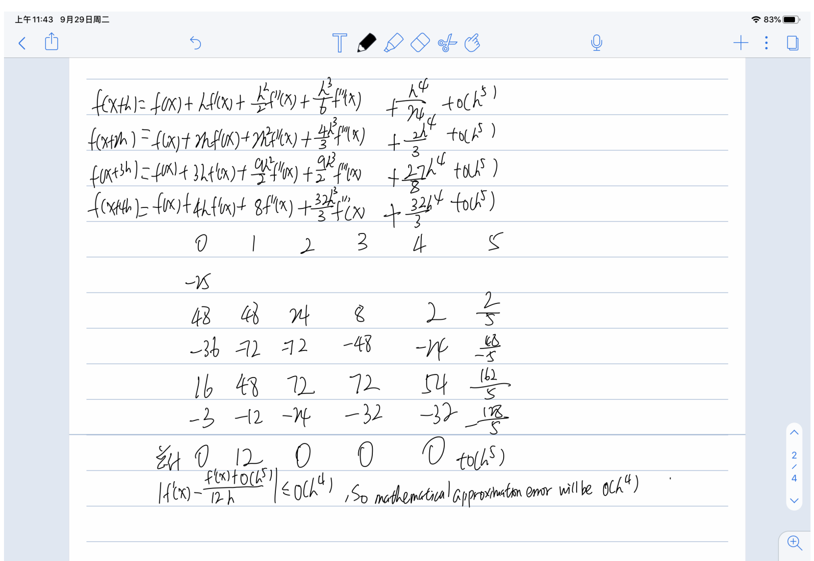
(c) Implement the above differentiation formula in a compute program of your choice and use it to find the derivative of the test function $f(x) = e^x$ at $x = 1$. Plot the total derivative evaluation error as a function of h and intepret your results.

solution:
please see graph 4.c at the end

f'(x)=[0.000728373716271324, -0.000219016598075328, -5.85377734729775e-6, 1.25165001074734e-6, -1.69435460772860e-7, -3.64215546611035e-9, -3.64215591019956e-9, 1.47406087336321e-10, 5.29709609509155e-12, -1.80833126250945e-12, -5.52793633090687e-9, -6.43277833574096
5,]
log(h)=[-12,-11,-10,-9,-8,-7,-6,-5,-4,-3,-2,-1]
We can see that when log(h) in [-11,-2],formular is almost the same as f'(x)=e when x=1 and the nearest value is at about h=3 based on (b) where eps=3.2 has the miniest error

graph2,graph4.a,graph4.c:

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$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \frac{h^4}{24} + o(h^5)$

$f(x+2h) = f(x) + 2hf'(x) + 2h^2f''(x) + \frac{4h^3}{3}f'''(x) + \frac{2h^4}{3} + o(h^5)$

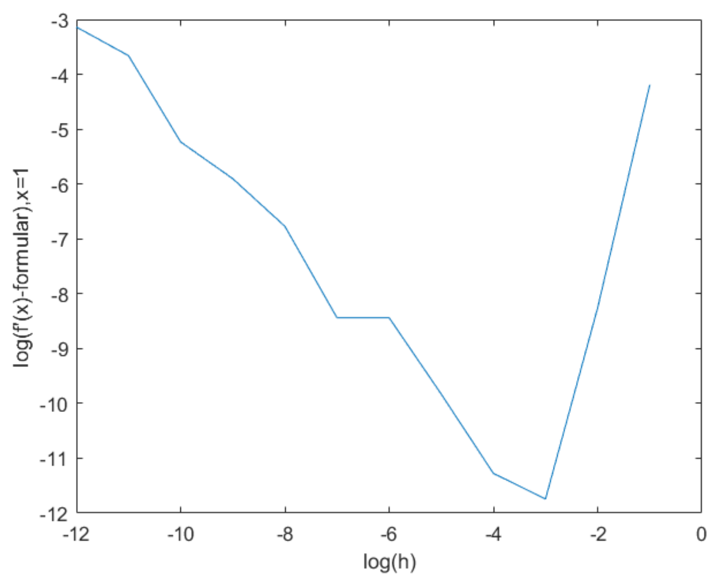
$f(x+3h) = f(x) + 3hf'(x) + \frac{9h^2}{2}f''(x) + \frac{9h^3}{2}f'''(x) + \frac{27}{8}h^4 + o(h^5)$

$f(x+4h) = f(x) + 4hf'(x) + 8f''(x) + \frac{32h^3}{3}f'''(x) + \frac{32h^4}{3} + o(h^5)$

0	1	2	3	4	5
-25					
48	48	24	8	2	$\frac{2}{5}$
-36	-72	-72	-48	-24	$-\frac{48}{5}$
16	48	72	72	54	$\frac{162}{5}$
-3	-12	-24	-32	-32	$-\frac{128}{5}$

总计 0 12 0 0 0 $+o(h^5)$

$\left|f'(x) - \frac{f'(x)+o(h^5)}{12h}\right| \le o(h^4)$, So mathematical approximation error will be $o(h^4)$



log(f'(x)-formular),x=1
-3
-4
-5
-6
-7
-8
-9
-10
-11
-12
-12
-10
-8
-6
-4
-2
0
log(h)