SI 211: Numerical Analysis Homework 2

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1. Jacobi matrix. Let us consider the multivariate function Compute the Jacobi matrix of f at x = $(1, 2)^T$

   solution:
   g(x)=
   $$\begin{pmatrix} 2x_1x_2 + x_2^2 & 2x_1x_2 + x_1^2 \\ 2x_1 + x_2 & x_1 \end{pmatrix} \tag{1}$$
   g((1,2))=
   $$\begin{pmatrix} 8 & 6 \\ 4 & 1 \end{pmatrix} \tag{2}$$

2. Polynomial interpolation. Assume that a function f : R → R satisfies f ((1) = 6, f (2) = 12 and f (4) = 66. Construct a polynomial function of the form p (x) = a0 + a1x + a2x3 such that p interpolates f at x=−1, 2, 4. Find the a0, a1 and a2

   solution:
   $$\begin{cases} f(-1) = p(-1) = a_0 - a_1 - a_2 = 6 \\ f(2) = p(2) = a_0 + 2a_1 + 8a_2 = 12 \\ f(4) = p(4) = a_0 + 4a_1 + 64a_2 = 66 \end{cases} \tag{3}$$
   $$\begin{cases} a_0 = 6 \\ a_1 = -1 \\ a_2 = 1 \end{cases} \tag{4}$$

3. Interpolation with rational functions. Let us assume that we have given points (x0, y0) = ((2, ;6) (x1, y1) = ((1, ;3) (x2, y2) = (1, 5) (x3, y3) = (2, 10) Construct a function q : R → R of the form q (x) = a_-1/x + a0 + a1x + a2x^2 such that q (xi) = yi for all i =0, 1, 2, 3. Find the scalar coefficients a_-1, a0, a1, a2.

   solution:
   $$\begin{cases} q(-2) = -0.5a_{-1} + a_0 - 2a_1 + 4a_2 = -6 \\ q(-1) = -a_{-1} + a_0 - a_1 + a_2 = -3 \\ q(1) = a_{-1} + a_0 + a_1 + a_2 = 5 \\ q(2) = 0.5a_{-1} + a_0 + 2a_1 + 4a_2 = 10 \end{cases} \tag{5}$$

$$\begin{cases} a_{-1} = 0 \\ a_0 = \frac{2}{3} \\ a_1 = 4 \\ a_2 = \frac{1}{3} \end{cases} \tag{6}$$

4. solution:
   question4 and question5 are photos which are at the end of the article

5. solution: the orange line is the origin function,the blue one is the polynomial
   as sin(x) has been quasinucleated well,so it is difficult to see the difference

4. Prove the formula of two-point cubic Hermite interpolation

$H_3(x_k)=y_k$, $H_3(x_{k+1})=y_{k+1}$

$H_3'(x_k)=m_k$, $H_3'(x_{k+1})=m_{k+1}$

Let $H_3(x)=\alpha_k(x)y_k+\alpha_{k+1}(x)y_{k+1}+\beta_k(x)m_k+\beta_{k+1}(x)m_{k+1}$ satisfies

$\alpha_k(x_k)=1$, $\alpha_k(x_{k+1})=0$, $\alpha_k'(x_k)=\alpha_k'(x_{k+1})=0$

$\alpha_{k+1}(x_k)=0$, $\alpha_{k+1}(x_{k+1})=0$, $\alpha_{k+1}'(x_k)=\alpha_{k+1}'(x_{k+1})=0$

$\beta_k(x_k)=\beta_k(x_{k+1})=0$, $\beta_k'(x_k)=1$, $\beta_k'(x_{k+1})=0$

$\beta_{k+1}(x_k)=\beta_{k+1}(x_{k+1})=0$, $\beta_{k+1}'(x_k)=0$, $\beta_{k+1}'(x_{k+1})=1$

$\therefore$ let $\alpha_k(x)=(ax+b)\left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2$

$\alpha(x_{k+1})=\alpha'(x_{k+1})=0$

$\alpha_k(x_k)=ax_k+b=1$

$\alpha_k'(x_k)=2\frac{ax_k+b}{x_k-x_{k+1}}+a=0$

$\Rightarrow \begin{cases} a=-\frac{2}{x_k-x_{k+1}} \\ b=1+\frac{2x_k}{x_k-x_{k+1}} \end{cases}$ $\Rightarrow \alpha_k(x)=\left(1+2\frac{x-x_k}{x_{k+1}-x_k}\right)\left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2$

$\alpha_{k+1}(x)=\left(1+2\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)\left(\frac{x-x_k}{x_{k+1}-x_k}\right)^2$

Let $\beta_k(x)=a(x-x_k)\left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2$

$\beta_k(x_k)=\beta_k(x_{k+1})=0$

$\beta_k'(x_k)=a=1 \Rightarrow \beta_k(x)=(x-x_k)\left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2$

$\beta_{k+1}(x)=(x-x_{k+1})\left(\frac{x-x_k}{x_{k+1}-x_k}\right)^2$

$\therefore H_3(x)=\left(1+2\frac{x-x_k}{x_{k+1}-x_k}\right)\left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2y_k+\left(1+2\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)\left(\frac{x-x_k}{x_{k+1}-x_k}\right)^2y_{k+1}+(x-x_k)\left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2m_k+(x-x_{k+1})\left(\frac{x-x_k}{x_{k+1}-x_k}\right)^2m_{k+1}$

$x_1=1, x_2=2, H_3(x_1)=y_1=1, H_3(x_2)=y_2=2, H_3'(x_1)=m_1=2, H_3'(x_2)=m_2=4, k=1$

$\therefore H_3(x)=\left(1+2\frac{x-1}{2-1}\right)\left(\frac{x-2}{1-2}\right)^2\cdot 1+\left(1+2\frac{x-2}{1-2}\right)\left(\frac{x-1}{2-1}\right)^2\cdot 2+(x-1)\left(\frac{x-2}{1-2}\right)^2\cdot 2+(x-2)\left(\frac{x-1}{2-1}\right)^2\cdot 4$

$=(2x-1)(x-2)^2-2(2x-5)(x-1)^2+2(x-1)(x-2)^2+4(x-2)(x-1)^2$

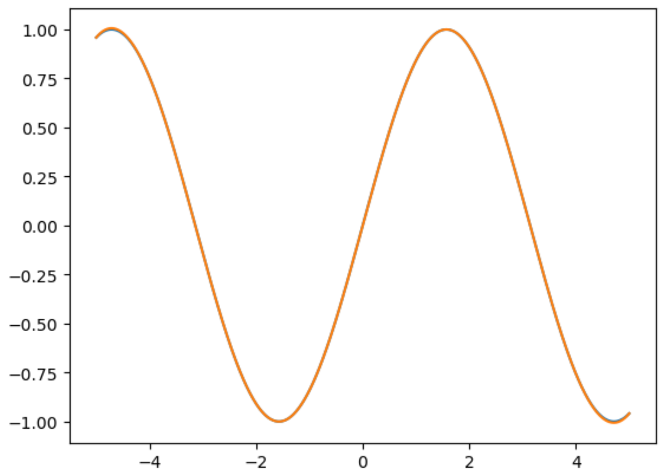
$=(x-2)^2(2x-1+2x-2)+(x-1)^2(4x-8-4x+10)$

$=(4x-3)(x-2)^2+2(x-1)^2$

$=(4x-3)(x^2-4x+4)+2x^2-4x+2$

$=4x^3-16x^2+16x-3x^2+12x-12+2x^2-4x+2$

$=4x^3-17x^2+24x-10$


1.00
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0.00
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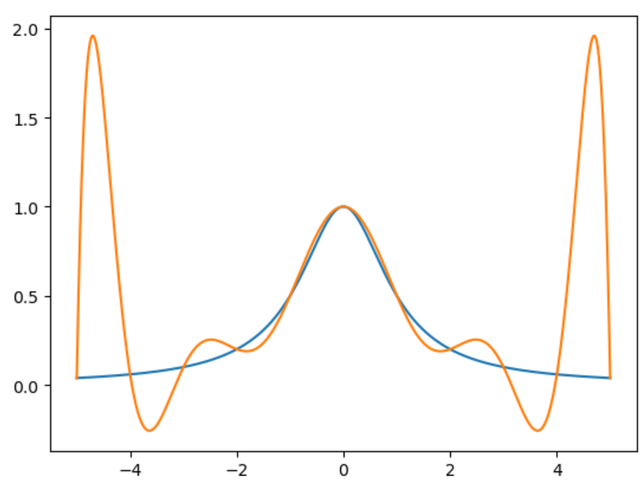
0
2
4


2.0
1.5
1.0
0.5
0.0
−4
−2
0
2
4

5. $|f(x)-p(x)| = R(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{j=0}^{n}(x-x_j) \quad \xi\in(a,b)$

① $f(x) = \sin(x)$ and give $x_0=-5, \cdots, x_{10}=5$ $\quad$ max for $(x+5)\cdots(x-5)$ $\quad$ or it can be chose as $\bar{x}-x=10$

$R(x) = \frac{\sin^{(11)}(\xi)}{11!}\prod_{j=0}^{10}(x-x_j) \le \frac{1}{11!}\times\prod_{j=0}^{\wedge}(x-x_j) \le \frac{416614.45}{11!} = 0.01044$ $\qquad$ where $R(x) = \frac{10^{10}}{11!}$

② $f(x) = \frac{1}{1+x^2}$

$R(x) = \frac{\left(\frac{1}{1+x^2}\right)^{(11)}(\xi)}{11!}\prod_{j=0}^{10}(x-x_j)$

As $|f^{n}(x)| \approx 2^n n!\, O(|x|^{-2-n})$

$R(x) \approx \frac{\left(\frac{1}{1+x^2}\right)^{(11)}(\xi)}{11!}\prod_{j=0}^{10}(x-x_j)$ $\qquad$ $\left(\frac{1}{1+x^2}\right)^{(n+1)}$ isn't uniformly converges, so the error's upper bound

can not be ensured without derivation