Numerical Analysis, 2020 Fall

Homework 6

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Problem1

1. Suppose $(I-A)$ is not invertible

$\therefore \exists x_0 \neq 0, (I-A)x = 0$

$\therefore \exists x_0 \neq 0, Ax_0 = x_0 \Rightarrow \frac{\|Ax_0\|}{\|x_0\|} = 1$

$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} \geq \frac{\|Ax_0\|}{\|x_0\|} = 1$ contridiction

$\therefore I-A$ is invertible

$(I-A)(I-A)^{-1} = I$

$(I-A)^{-1} = I + A(I-A)^{-1}$

$\therefore \|(I-A)^{-1}\| \leq \|I\| + \|A(I-A)^{-1}\| \leq \|I\| + \|A\|\|(I-A)^{-1}\| = 1 + \|A\|\|(I-A)^{-1}\|$

$\therefore \|(I-A)^{-1}\| \leq \frac{1}{1-\|A\|}$

Problem2

2. $Ax=b$ $\quad A=\begin{pmatrix}2.0002 & 1.9998\\ 1.9998 & 2.0002\end{pmatrix},\ b=\begin{pmatrix}4\\4\end{pmatrix}$

$x=(1,1)$ $\quad A^{-1}=\begin{pmatrix}1250.125 & -1249.875\\ -1249.875 & 1250.125\end{pmatrix}$

$A(x+\delta x)=b+\delta b$ $\quad \delta b=(2\times10^{-4},-2\times10^{-4})$

$y=x+\delta x=(1.5,0.5)$, $\delta x=(0.5,-0.5)$

$\mathrm{Cond}_\infty(A)=\|A^{-1}\|_\infty\|A\|_\infty$ $\qquad \frac{\|\delta x\|_\infty}{\|x\|_\infty}=\frac{0.5}{1}=\frac{1}{2}$

$=4\times2500$ $\qquad \frac{\|\delta b\|}{\|b\|_\infty}=0.5\times10^{-4}$

$=10^4$

As $\mathrm{Cond}_\infty(A)$ is very large, we can imply that the system of equations is ill-condition

which means little change of b (δb) will cause big change of x (δx) [$\delta b=(2\times10^{-4},-2\times10^{-4})$, $\delta x=(\frac{1}{2},-\frac{1}{2})$]

And the anwer also verify the theorm:

if A is non-singular, $Ax=b\neq0$, $A(x+\delta x)=b+\delta b$, then

$$\left.\begin{array}{l} A(x+\delta x)=b+\delta b \Rightarrow \delta x=A^{-1}\delta b \Rightarrow \|\delta x\|=\|A^{-1}\delta b\|\le\|A^{-1}\|\|\delta b\| \\ Ax=b\Rightarrow \|b\|\le\|A\|\|x\| \end{array}\right\} \Rightarrow \frac{\|\delta x\|}{\|x\|}\le\|A^{-1}\|\|A\|\frac{\|\delta b\|}{\|b\|}$$

in this question $\underbrace{\frac{\|\delta x\|_\infty}{\|x\|_\infty}}_{=0.5}=\underbrace{\mathrm{Cond}_\infty(A)}_{=10^4}\times\underbrace{\frac{\|\delta b\|_\infty}{\|b\|_\infty}}_{=0.5\times10^{-4}}$

Problem3

the program has been implemented in LR-decomposition.py

3. $A=LU$

$Ax=b\Rightarrow LUx=b$ $\qquad L=\begin{pmatrix}1 & & & 0\\ L_{21} & 1 & & \\ \vdots & \vdots & \ddots & \\ L_{n1} & L_{n2} & \cdots & 1\end{pmatrix}$ $\qquad U=\begin{pmatrix}U_{11} & \cdots & U_{1n}\\ & \ddots & \vdots\\ 0 & & U_{nn}\end{pmatrix}$

$\begin{cases}Ly=b\\ Ux=y\end{cases}$

$\because$ ① $U_{1i}=a_{1i}$ $i=1,\cdots,n$, $L_{i1}=\frac{a_{i1}}{U_{11}}$, $i=2,\cdots,n$

② $U_{ri}=a_{ri}-\sum_{k=1}^{r-1}L_{rk}U_{ki}$ $\quad i=r,\cdots,n$ $\quad r=2,\cdots,n$

$L_{ir}=\frac{(a_{ir}-\sum_{k=1}^{r-1}L_{ik}U_{kr})}{U_{rr}}$ $\quad i=r+1,\cdots,n$, 且 $r\neq n$

③ $\begin{cases}y_1=b_1\\ y_i=b_i-\sum_{k=1}^{i-1}L_{ik}y_k & i=2,\cdots,n\end{cases}$

$\begin{cases}x_n=\frac{y_n}{U_{nn}}\\ x_i=\frac{(y_i-\sum_{k=i+1}^{n}U_{ik}x_k)}{U_{ii}} & ,i=n-1,n-2,\cdots,1\end{cases}$

$\begin{pmatrix}1 & -2 & 1\\ 2 & 1 & -3\\ 4 & -7 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\5\\-1\end{pmatrix}\Rightarrow y=\begin{pmatrix}0\\5\\-2\end{pmatrix}\Rightarrow x=\begin{pmatrix}3\\2\\1\end{pmatrix}$

Problem4

(4) $A_{m\times m}$ is a symmetry positive band-structured matrix with bandwidth $2n+1$

$$A=\begin{pmatrix} a_{11} & \cdots & a_{n1} & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ a_{n1} & & & & & a_{nm} \\ 0 & \ddots & & & & \vdots \\ \vdots & & & & & \\ 0 & \cdots & 0 & a_{mn} & \cdots & a_{mm} \end{pmatrix} \quad \begin{array}{ll} a_{ij}=a_{ji} & i=1,\cdots m \\ a_{ij}=0 & \text{for } j<i-n+1 \text{ or } j>i+n-1 \end{array}$$

$$A=LDL^T=\begin{pmatrix} 1 & & & 0 \\ l_{21} & 1 & & \\ l_{31} & & \ddots & \\ \vdots & & & \\ l_{m1} & \cdots & \cdots & 1 \end{pmatrix}\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}\begin{pmatrix} 1 & & 0 \\ l_{21} & \ddots & \\ \vdots & & \ddots \\ l_{m1} & \cdots & 1 \end{pmatrix}^T$$

$\because$ A is symmetry positive

$\therefore$ All the leading principle minor D_i of $A>0$, $D_i=\begin{vmatrix} a_{11} & \cdots & a_{1i} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{ii} \end{vmatrix}$, $i=1,\cdots m$

$\therefore$ $d_1=D_1>0$

$d_i=\frac{D_i}{D_{i-1}}>0 \quad i=2,\cdots m$

$$A=\begin{pmatrix} d_1 & - & - & - \\ l_{21}d_1 & l_{21}^2 d_1+d_2 & & \\ l_{31}d_2 & l_{31}l_{21}d_1+l_{32}d_2 & \ddots & \\ \vdots & & & \ddots \end{pmatrix}$$

$\because$ A's bandwidth $=2n+1$

$\therefore$ $\forall\ j\geq i-n+1$ or $j\leq i+n-1$, $a_{ij}\neq 0$

$\therefore$ $l_{21}d_1\neq 0$, $\quad l_{21}\neq 0$

$l_{31}d_2\neq 0 \Rightarrow l_{31}\neq 0$

$\vdots \qquad \vdots$

$l_{n1}d_1\neq 0 \qquad l_{n1}\neq 0$

$l_{(n+1)1}d_1=0 \qquad l_{n+1}=0$

$\vdots \qquad \vdots$

$l_{m1}d_1=0 \qquad l_m=0$

$\because$ L is also band-structured matrix

$\therefore$ the bandwidth of L is $n+1$