SI231 - Matrix Computations, 2021 Fall

Solution of Homework Set #3

Prof. Yue Qiu

Acknowledgements:

1) Deadline: **2021-11-16 23:59:59**
2) **Late Policy details** can be found on piazza.
3) Submit your homework in **Homework 3** on **Gradscope**. Entry Code: **2RY68R**. Make sure that you have correctly select pages for each problem. If not, you probably will get 0 point.
4) No handwritten homework is accepted. You need to write LaTeX. (If you have difficulty in using LaTeX, you are allowed to use **MS Word or Pages** for the first and the second homework to accommodate yourself.)
5) Use the given template and give your solution in English. Solution in Chinese is not allowed.
6) Your homework should be uploaded in the PDF format, and the naming format of the file is not specified.
7) For the calculation problems, you are highly required to write down your solution procedures in detail. And **all values must be represented by integers, fractions or square root**, floating points are not accepted.

I. QR DECOMPOSITION VIA GRAM-SCHMIDT ORTHOGONALITY

Problem 1. (15 points + 5 points)

Given a matrix $\mathbf{A} = \begin{bmatrix} 1 & 3 & 7 \\ 1 & 3 & 1 \\ 1 & -1 & 3 \\ -1 & 1 & 3 \end{bmatrix}$

1) Give the QR decomposition via Gram-Schmidt Orthogonality. You should write the derivation of finding the orthogonal matrix $\mathbf{Q}$ and upper triangular matrix $\mathbf{R}$.
2) Solve least squares problems $\min \|\mathbf{Ax} - \mathbf{b}\|_2$ via QR decomposition where $\mathbf{b} = \begin{bmatrix} 6 & 6 & 8 & 8 \end{bmatrix}^T$.

Solution: (1) $\tilde{q_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$, $q_1 = \frac{\tilde{q_1}}{||\tilde{q_1}||_2} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$

$$\tilde{q_2} = a_2 - P_{\text{span}\{q_1\}}(a_2) = a_2 - \langle q_1, a_2 \rangle q_1 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ 1 \end{pmatrix} - [\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ -1 \\ 1 \end{pmatrix}] \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -1 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} =$$

$$\begin{pmatrix} 2 \\ 2 \\ -2 \\ 2 \end{pmatrix}, \; q_2 = \frac{\tilde{q_2}}{||\tilde{q_2}||_2} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$\tilde{q_3} = a_3 - P_{\text{span}\{q_1,q_2\}}(a_3) = a_3 - \langle q_1, a_3\rangle q_1 - \langle q_1 2, a_3\rangle q_2 = \begin{pmatrix} 7 \\ 1 \\ 3 \\ 3 \end{pmatrix} - -[\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 7 \\ 1 \\ 3 \\ 3 \end{pmatrix}] \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} -$$

$$[\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 7 \\ 1 \\ 3 \\ 3 \end{pmatrix}] \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 3 \\ 3 \end{pmatrix} - 4 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} - 4 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 3 \\ 3 \end{pmatrix}, \; q_3 = \frac{\tilde{q_3}}{||\tilde{q_3}||_2} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$\therefore Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \; R = \begin{pmatrix} 2 & 2 & 4 \\ 0 & 4 & 4 \\ 0 & 0 & 6 \end{pmatrix}$$

(2) $x_{\text{ls}} = R^{-1}Q^T b = \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{4}{3} \end{pmatrix}$

Problem 2. (16 points + 4 points)

Consider the subspace $\mathcal{S}$ spanned by $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$,

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ \epsilon \\ \epsilon \\ \epsilon \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 0 \\ \epsilon \\ \epsilon \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 1 \\ \epsilon \\ 0 \\ \epsilon \end{bmatrix}, \quad \mathbf{a}_4 = \begin{bmatrix} 1 \\ \epsilon \\ \epsilon \\ 0 \end{bmatrix},$$

where ϵ is a small real number such that $1 + k\epsilon^2 \approx 1$ ($k \in \mathbb{N}^+$). Use the **classical** Gram-Schmidt algorithm and the **modified** Gram-Schmidt algorithm respectively, find two sets of basis for $\mathcal{S}$ by hand (derivation is expected). Are the two sets of basis the same? If not, which one is the desired orthogonal basis? Report what you have found.

Solution: Classical Gram-Schmidt:

$$q_1 = \frac{a_1}{||a_1||_2} = \begin{pmatrix} 1 \\ \epsilon \\ \epsilon \\ \epsilon \end{pmatrix}$$

$$\tilde{q_2} = a_2 - \langle q_1, a_2 \rangle q_1 = \begin{pmatrix} 1 \\ 0 \\ \epsilon \\ \epsilon \end{pmatrix} - (1+2\epsilon^2) \begin{pmatrix} 1 \\ \epsilon \\ \epsilon \\ \epsilon \end{pmatrix} = \begin{pmatrix} 0 \\ -\epsilon \\ 0 \\ 0 \end{pmatrix}, \; q_2 = \frac{\tilde{q_2}}{||\tilde{q_2}||_2} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\tilde{q_3} = a_3 - \langle a_3, q_1 \rangle q_1 - \langle a_3, q_2 \rangle q_2 = \begin{pmatrix} 1 \\ \epsilon \\ 0 \\ \epsilon \end{pmatrix} - (1+2\epsilon^2) \begin{pmatrix} 1 \\ \epsilon \\ \epsilon \\ \epsilon \end{pmatrix} - (-\epsilon) \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\epsilon \\ -\epsilon \\ 0 \end{pmatrix}, \; q_3 = \frac{\tilde{q_3}}{||\tilde{q_3}||_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix}$$

$$\tilde{q_4} = a_4 - \langle a_4, q_1 \rangle q_1 - \langle a_4, q_2 \rangle q_2 - \langle a_4, q_3 \rangle q_3 = \begin{pmatrix} 1 \\ \epsilon \\ \epsilon \\ 0 \end{pmatrix} - (1+2\epsilon^2) \begin{pmatrix} 1 \\ \epsilon \\ \epsilon \\ \epsilon \end{pmatrix} - (-\epsilon) \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} - (-\sqrt{2}\epsilon) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2\epsilon \\ -\epsilon \\ -\epsilon \end{pmatrix},$$

$$q_4 = \frac{\tilde{q_4}}{||\tilde{q_4}||_2} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 \\ -2 \\ -1 \\ -1 \end{pmatrix}$$

$\because \langle q_2, q_3 \rangle = \frac{\sqrt{2}}{2} \neq 0$

$\therefore \{q_1, q_2, q_3, q_4\}$ is not an orthonormal basis of S

Modified Gram-Schmidt:

$$q_1 = \frac{a_1}{||a_1||_2} = \begin{pmatrix} 1 \\ \epsilon \\ \epsilon \\ \epsilon \end{pmatrix}$$

$$\tilde{q_2} = a_2 - \langle q_1, a_2 \rangle q_1 = \begin{pmatrix} 1 \\ 0 \\ \epsilon \\ \epsilon \end{pmatrix} - (1+2\epsilon^2) \begin{pmatrix} 1 \\ \epsilon \\ \epsilon \\ \epsilon \end{pmatrix} = \begin{pmatrix} 0 \\ -\epsilon \\ 0 \\ 0 \end{pmatrix}, \ q_2 = \frac{\tilde{q_2}}{||\tilde{q_2}||_2} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\tilde{q_3} = a_3 - \langle a_3, q_1 \rangle q_1 = \begin{pmatrix} 1 \\ \epsilon \\ 0 \\ \epsilon \end{pmatrix} - (1+2\epsilon^2) \begin{pmatrix} 1 \\ \epsilon \\ \epsilon \\ \epsilon \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\epsilon \\ 0 \end{pmatrix}$$

$$\tilde{q_3} = \tilde{q_3} - \langle \tilde{q_3}, q_2 \rangle q_2 = \begin{pmatrix} 0 \\ 0 \\ -\epsilon \\ 0 \end{pmatrix} - 0 = \begin{pmatrix} 0 \\ 0 \\ -\epsilon \\ 0 \end{pmatrix}, \ q_3 = \frac{\tilde{q_3}}{||\tilde{q_3}||_2} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

$$\tilde{q_4} = a_4 - \langle a_4, q_1 \rangle q_1 = \begin{pmatrix} 1 \\ \epsilon \\ \epsilon \\ 0 \end{pmatrix} - (1+2\epsilon^2) \begin{pmatrix} 1 \\ \epsilon \\ \epsilon \\ \epsilon \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\epsilon \end{pmatrix}$$

$$\tilde{q_4} = \tilde{q_4} - \langle \tilde{q_4}, q_2 \rangle q_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\epsilon \end{pmatrix} - 0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\epsilon \end{pmatrix}$$

$$\tilde{q_4} = \tilde{q_4} - \langle \tilde{q_4}, q_3 \rangle q_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\epsilon \end{pmatrix} - 0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\epsilon \end{pmatrix}, \ q_4 = \frac{\tilde{q_4}}{||\tilde{q_4}||_2} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$\{q_1, q_2, q_3, q_4\}$ is an orthonormal basis of S

So Classical Gram-Schmidt and Modified Gram-Schmidt don't get the same basis.

II. QR DECOMPOSITION VIA HOUSEHOLDER REFLECTION

Problem 1. (15 points + 5 points)

Consider a matrix $\mathbf{A} \in \mathbb{R}^{4\times 3}$. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & -4 \\ 2 & -4 & -4 \\ 2 & -4 & -6 \\ 0 & 1 & 1 \end{bmatrix}$

1) Use Householder reflection to give the full QR decomposition of matrix $\mathbf{A}$, i.e. $\mathbf{A} = \mathbf{QR}$ while $\mathbf{QQ^T} = I$.
2) For $\mathbf{b} = \begin{bmatrix} 9 & 14 & -15 \end{bmatrix}^T \in \mathbb{R}^3$, solve the underdetermined system $\mathbf{A^T x} = \mathbf{b}$ via QR decomposition of $\mathbf{A}$.

Solution: (1) $a_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \end{pmatrix}, e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$v_1 = a_1 + ||a_1||_2 e_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 2 \\ 0 \end{pmatrix}$$

$$H_1 = I_{4,4} - \frac{2v_1 v_1^T}{||v_1||_2^2} = I - \frac{1}{12}\begin{pmatrix} 16 & 8 & 8 & 0 \\ 8 & 4 & 4 & 0 \\ 8 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & 0 \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A^{(1)} = H_1 A = \begin{pmatrix} -3 & 5 & 8 \\ 0 & -2 & 2 \\ 0 & -2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$a_2 = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}, e_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$v_2 = a_2 + ||a_2||_2 e_2 = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$$

$$\tilde{H}_2 = I_{3,3} - \frac{2v_2 v_2^T}{||v_2||_2^2} = I - \frac{1}{15}\begin{pmatrix} 25 & 10 & -5 \\ 10 & 4 & -2 \\ -5 & -2 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{11}{15} & \frac{2}{15} \\ \frac{1}{3} & \frac{2}{15} & \frac{14}{15} \end{pmatrix}$$

$$H_2 = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{H}_2 \end{pmatrix},\ A^{(2)} = H_2 A^{(1)} = \begin{pmatrix} -3 & 5 & 8 \\ 0 & 3 & -1 \\ 0 & 0 & -\frac{6}{5} \\ 0 & 0 & \frac{8}{5} \end{pmatrix}$$

$$a_3 = \begin{pmatrix} -\frac{6}{5} \\ \frac{8}{5} \end{pmatrix}, e_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$v_3 = a_3 - ||a_3||_2 e_3 = \begin{pmatrix} -\frac{6}{5} \\ \frac{8}{5} \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{16}{5} \\ \frac{8}{5} \end{pmatrix}$$

$$\tilde{H}_3 = I_{2,2} - \frac{2v_3v_3^T}{||v_3||_2^2} = I - \frac{5}{32} \begin{pmatrix} \frac{256}{25} & -\frac{128}{25} \\ -\frac{128}{25} & \frac{64}{25} \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

$$H_3 = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{H}_3 \end{pmatrix},\ A^{(3)} = H_3 A^{(2)} = \begin{pmatrix} -3 & 5 & 8 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\therefore Q = (H_3H_2H_1)^T = \begin{pmatrix} -\frac{1}{3} & \frac{8}{9} & -\frac{2}{9} & -\frac{2}{9} \\ -\frac{2}{3} & -\frac{2}{9} & \frac{5}{9} & -\frac{4}{9} \\ -\frac{2}{3} & -\frac{2}{9} & -\frac{4}{9} & \frac{5}{9} \\ 0 & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix},\ R = A^{(3)} = \begin{pmatrix} -3 & 5 & 8 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

(2) $x_{ls} = (AA^T)^{-1}Ab = (QRR^TQ^T)^{-1}QRb = \begin{pmatrix} \frac{203}{27} \\ \frac{136}{27} \\ -\frac{116}{27} \\ \frac{85}{9} \end{pmatrix}$

and we verify that $A^T x_{ls} = \begin{pmatrix} 9 \\ 14 \\ -15 \end{pmatrix} = b$

$\therefore x_{ls}$ is the exact solution of $A^T x = b$

III. QR DECOMPOSITION VIA GIVENS ROTATION

Problem 1. (9 points + 9 points + 2 points)

Given a dense matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix} \tag{1}$$

and a sparse matrix

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 3 & 0 & 4 & 0 \\ 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \end{bmatrix} \tag{2}$$

1) Give the QR decomposition of $\mathbf{A}$ with $\mathbf{Q}$ being square.
2) Give the QR decomposition of $\mathbf{B}$ with $\mathbf{Q}$ being square.
3) Discuss when Givens rotation is better than Householder reflection and when Householder reflection is better than Givens rotation.

Solution: (1)$J_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $A^{(1)} = J_1 A = \begin{pmatrix} \sqrt{2} & -\frac{\sqrt{2}}{2} & -\frac{3\sqrt{2}}{2} \\ 0 & \frac{5\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 1 & -2 & 3 \end{pmatrix}$

$J_2 = \begin{pmatrix} \frac{\sqrt{6}}{3} & 0 & \frac{\sqrt{3}}{3} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{6}}{3} \end{pmatrix}$, $A^{(2)} = J_2 A^{(1)} = \begin{pmatrix} \sqrt{3} & -\sqrt{3} & 0 \\ 0 & \frac{5\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{6}}{2} & \frac{3\sqrt{6}}{2} \end{pmatrix}$

$J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{5\sqrt{7}}{14} & -\frac{\sqrt{21}}{14} \\ 0 & \frac{\sqrt{21}}{14} & \frac{5\sqrt{7}}{14} \end{pmatrix}$, $A^{(3)} = J_2 A^{(2)} = \begin{pmatrix} -\sqrt{3} & \sqrt{3} & 0 \\ 0 & \sqrt{14} & -\frac{\sqrt{14}}{7} \\ 0 & -0 & \frac{-3\sqrt{14}+15\sqrt{42}}{28} \end{pmatrix}$

$\therefore A^{(3)} = J_3 J_2 J_1 A$

$\therefore R = A^{(3)} = \begin{pmatrix} -\sqrt{3} & \sqrt{3} & 0 \\ 0 & \sqrt{14} & -\frac{\sqrt{14}}{7} \\ 0 & -0 & \frac{-3\sqrt{14}+15\sqrt{42}}{28} \end{pmatrix}$, $Q = (J_3 J_2 J_1)^T = \begin{pmatrix} \frac{\sqrt{12}}{6} & \frac{3\sqrt{14}}{14} & -\frac{\sqrt{42}}{42} \\ -\frac{\sqrt{12}}{6} & \frac{\sqrt{14}}{7} & \frac{2\sqrt{42}}{21} \\ \frac{\sqrt{12}}{6} & -\frac{\sqrt{14}}{14} & \frac{5\sqrt{42}}{42} \end{pmatrix}$

(2) $J_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$, $B^{(1)} = J_1 B = \begin{pmatrix} \sqrt{2} & 0 & -\frac{\sqrt{2}}{2} & 0 & 2\sqrt{2} \\ 0 & 3 & 0 & 4 & 0 \\ 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \end{pmatrix}$

$$J_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}, B^{(2)} = J_2 B^{(1)} = \begin{pmatrix} \sqrt{2} & 0 & -\frac{\sqrt{2}}{2} & 0 & 2\sqrt{2} \\ 0 & 3\sqrt{2} & 0 & 2\sqrt{2} & \sqrt{2} \\ 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2\sqrt{2} & \sqrt{2} \end{pmatrix}$$

$$J_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, B^{(3)} = J_3 B^{(2)} = \begin{pmatrix} \sqrt{2} & 0 & -\frac{\sqrt{2}}{2} & 0 & 2\sqrt{2} \\ 0 & 3\sqrt{2} & 0 & 2\sqrt{2} & \sqrt{2} \\ 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & -2\sqrt{2} & \sqrt{2} \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\therefore B^3 = J_3 J_2 J_1 B$$

$$\therefore R = B^3 = \begin{pmatrix} \sqrt{2} & 0 & -\frac{\sqrt{2}}{2} & 0 & 2\sqrt{2} \\ 0 & 3\sqrt{2} & 0 & 2\sqrt{2} & \sqrt{2} \\ 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & -2\sqrt{2} & \sqrt{2} \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, Q = (J_3 J_2 J_1)^T = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \end{pmatrix}$$

(3) When the matrix is dense, Householder reflection is better.When the matrix is sparse, Givens rotation is better.The reason is that Givens rotation always need more steps but less calculation in each step.So Householder reflection is better when the matrix is dense and when the matrix is sparse, the steps of Givens rotation will almost the same or just a little more than House reflection. So in this situation, Givens rotation will be better.

IV. Projection

Problem 1. (3 points + 7 points + 5 points + 5 points)

Given matrix $\mathbf{A}$ as an $n \times n$ projector.

1) Prove that $\mathcal{R}(\mathbf{A}) \oplus \mathcal{N}(\mathbf{A}) = \mathbf{R}^n$.
2) Prove the matrix $\mathbf{A}^T$ is also a projector. If $\mathbf{A}$ is a orthogonal projector, prove that $\mathbf{A}^T$=$\mathbf{A}$.
3) Is the product of a series of projectors still a projector? For *Yes*, please give the proof; For *No*, please give an example.
4) If $\mathbf{A}$ is the orthogonal projector onto $\mathcal{N}(\mathbf{B})$ ($\mathbf{B}$ is an $m \times n$ matrix may not be full rank), please determine $\mathbf{A}$ using $\mathbf{B}$ and give your reason.

   Hint: $\mathbf{B}^\dagger$ is the pseudo inverse of $\mathbf{B}$ satisfies the following properties:

   1) $\mathbf{B}\mathbf{B}^\dagger\mathbf{B} = \mathbf{B}$
   2) $\mathbf{B}^\dagger\mathbf{B}\mathbf{B}^\dagger = \mathbf{B}^\dagger$
   3) $(\mathbf{B}\mathbf{B}^\dagger)^T = (\mathbf{B}\mathbf{B}^\dagger)$
   4) $(\mathbf{B}^\dagger\mathbf{B})^T = (\mathbf{B}^\dagger\mathbf{B})$

Solution: (1) $1^{'} \forall x \in R(A) \cap N(A) : \exists\ y \in \mathcal{R}^n : x = Ay$

$\therefore 0 = Ax = A^2 y = Ay = x$

$\therefore R(A) + N(A) = R(A) \oplus N(A)$

$2^{'} R(A) \oplus N(A) \subseteq \mathcal{R}^n$ is obvious

$3^{'} \forall x \in \mathcal{R}^n : x = Ax + (x - Ax)$

$Ax \in R(A)$, $x - Ax \in N(A)$ since $A(x - Ax) = Ax - A^2 x = Ax - Ax = 0$

$\therefore x \in R(A) \oplus N(A)$

$\therefore \mathcal{R}^n \subseteq R(A) \oplus N(A)$

$1^{'}2^{'}3^{'} \Rightarrow R(A) \oplus N(A) = \mathcal{R}^n$

(2)$1^{'} (A^T)^2 = A^T A^T = (A^2)^T = A^T$

$\therefore A^T$ is also a projector

$2^{'} \forall x \in \mathcal{R}^n : (Ax)^T (x - Ax) = 0$

$x^T A^T (x - Ax) = 0$

$x^T (A^T - A^T A) x = 0$

$\therefore A^T = A^T A$

also since $x^T (A^T - A^T A) x = 0$

$\therefore x^T (A^T - A^T A)^T (x^T)^T = 0$

$\therefore x^T (A - A^T A) x = 0$

$\therefore A = A^T A = A^T$ Q.E.D

(3) No,it may not be a projector.

Counter-example: let $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, P_1^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = P_1$

$P_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, P_2^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = P_2$, so P_1 and P_2 are both projectors

$P_2 P_1 v = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

$(P_2 P_1)^2 v = P_2 P_1 (P_2 P_1 v) \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} \neq P_2 P_1 v$

$\therefore P_2 P_1$ is not a projector

[By the way, sometimes it can still be a projector, for example:

$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, P_1^2 = P_1,\ P_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, P_2^2 = P_2$

$(P_2 P_1)^2 = P_2 P_1 = 0$]

(4) let P be an orthogonal projector onto $R(B^T)$, by solving least square problems we will have $BB^T x = Bb$

Since B may not be full-rank, so we have $x = (B^T)^\dagger b, P = B^T (B^T)^\dagger$

$\because P$ is an orthogonal projector onto $R(B^T)$

$\therefore R(P) = R(B^T)$

$\because R(P) \oplus N(P) = R(B^T) \oplus N(B) = \mathcal{R}^n, R(P) = R(B^T)$

$\therefore R(I - P) = N(P) = N(B)$

$\because (I - P)^2 = I - P, (I - P)^T = I - P^T = I - P$

$\therefore I - P$ is an orthogonal projector onto $R(I - P) = N(B)$, by uniqueness of orthogonal projector, we have

$A = I - P = I - B^T (B^T)^\dagger$