SI231 - Matrix Computations, 2021 Fall

Solution of Homework Set #3

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Acknowledgements:

1) Deadline: **2021-11-16 23:59:59**
2) **Late Policy details** can be found on piazza.
3) Submit your homework in **Homework 3** on **Gradscope**. Entry Code: **2RY68R**. Make sure that you have correctly select pages for each problem. If not, you probably will get 0 point.
4) No handwritten homework is accepted. You need to write LaTeX. (If you have difficulty in using LaTeX, you are allowed to use **MS Word or Pages** for the first and the second homework to accommodate yourself.)
5) Use the given template and give your solution in English. Solution in Chinese is not allowed.
6) Your homework should be uploaded in the PDF format, and the naming format of the file is not specified.
7) For the calculation problems, you are highly required to write down your solution procedures in detail. And **all values must be represented by integers, fractions or square root**, floating points are not accepted.

I. QR decomposition via Gram-Schmidt Orthogonality

Problem 1. (15 points + 5 points)

Given a matrix $\mathbf{A} = \begin{bmatrix} 1 & 3 & 7 \\ 1 & 3 & 1 \\ 1 & -1 & 3 \\ -1 & 1 & 3 \end{bmatrix}$

1) Give the QR decomposition via Gram-Schmidt Orthogonality. You should write the derivation of finding the orthogonal matrix $\mathbf{Q}$ and upper triangular matrix $\mathbf{R}$.
2) Solve least squares problems $\min \|\mathbf{Ax} - \mathbf{b}\|_2$ via QR decomposition where $\mathbf{b} = \begin{bmatrix} 6 & 6 & 8 & 8 \end{bmatrix}^T$.

Solution:

Problem 2. (16 points + 4 points)

Consider the subspace $\mathcal{S}$ spanned by $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$,

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ \epsilon \\ \epsilon \\ \epsilon \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 0 \\ \epsilon \\ \epsilon \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 1 \\ \epsilon \\ 0 \\ \epsilon \end{bmatrix}, \quad \mathbf{a}_4 = \begin{bmatrix} 1 \\ \epsilon \\ \epsilon \\ 0 \end{bmatrix},$$

where ϵ is a small real number such that $1 + k\epsilon^2 \approx 1$ $(k \in \mathbb{N}^+)$. Use the **classical** Gram-Schmidt algorithm and the **modified** Gram-Schmidt algorithm respectively, find two sets of basis for $\mathcal{S}$ by hand (derivation is expected). Are the two sets of basis the same? If not, which one is the desired orthogonal basis? Report what you have found.

Solution:

II. QR DECOMPOSITION VIA HOUSEHOLDER REFLECTION

Problem 1. (15 points + 5 points)

Consider a matrix $\mathbf{A} \in \mathbb{R}^{4\times 3}$. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & -4 \\ 2 & -4 & -4 \\ 2 & -4 & -6 \\ 0 & 1 & 1 \end{bmatrix}$

1) Use Householder reflection to give the full QR decomposition of matrix $\mathbf{A}$, i.e. $\mathbf{A} = \mathbf{QR}$ while $\mathbf{QQ^T} = I$.
2) For $\mathbf{b} = \begin{bmatrix} 9 & 14 & -15 \end{bmatrix}^T \in \mathbb{R}^3$, solve the underdetermined system $\mathbf{A^T x} = \mathbf{b}$ via QR decomposition of $\mathbf{A}$.

Solution:

III. QR DECOMPOSITION VIA GIVENS ROTATION

Problem 1. (9 points + 9 points + 2 points)

Given a dense matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix} \tag{1}$$

and a sparse matrix

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 3 & 0 & 4 & 0 \\ 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \end{bmatrix} \tag{2}$$

1) Give the QR decomposition of $\mathbf{A}$ with $\mathbf{Q}$ being square.
2) Give the QR decomposition of $\mathbf{B}$ with $\mathbf{Q}$ being square.
3) Discuss when Givens rotation is better than Householder reflection and when Householder reflection is better than Givens rotation.

Solution:

IV. Projection

Problem 1. (3 points + 7 points + 5 points + 5 points)

Given matrix $\mathbf{A}$ as an $n \times n$ projector.

1) Prove that $\mathcal{R}(\mathbf{A}) \oplus \mathcal{N}(\mathbf{A}) = \mathbf{R}^n$.
2) Prove the matrix $\mathbf{A}^T$ is also a projector. If $\mathbf{A}$ is a orthogonal projector, prove that $\mathbf{A}^T$=$\mathbf{A}$.
3) Is the product of a series of projectors still a projector? For *Yes*, please give the proof; For *No*, please give an example.
4) If $\mathbf{A}$ is the orthogonal projector onto $\mathcal{N}(\mathbf{B})$ ($\mathbf{B}$ is an $m \times n$ matrix may not be full rank), please determine $\mathbf{A}$ using $\mathbf{B}$ and give your reason.

   Hint: $\mathbf{B}^\dagger$ is the pseudo inverse of $\mathbf{B}$ satisfies the following properties:

   1) $\mathbf{B}\mathbf{B}^\dagger\mathbf{B} = \mathbf{B}$
   2) $\mathbf{B}^\dagger\mathbf{B}\mathbf{B}^\dagger = \mathbf{B}^\dagger$
   3) $(\mathbf{B}\mathbf{B}^\dagger)^T = (\mathbf{B}\mathbf{B}^\dagger)$
   4) $(\mathbf{B}^\dagger\mathbf{B})^T = (\mathbf{B}^\dagger\mathbf{B})$

Solution: