SI231 - Matrix Computations, 2021 Fall

Solution of Homework Set #2

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Acknowledgements:

1) Deadline: **2021-11-01 23:59:59**
2) **Late Policy details** can be found on piazza.
3) Submit your homework in **Homework 2** on **Gradscope**. Entry Code: **2RY68R**. Make sure that you have **correctly select pages for each problem**. If not, you will **get 0 point**.
4) **No handwritten homework is accepted**, otherwise you will **get 0 point**. You need to write LaTeX. (If you have difficulty in using LaTeX, you are allowed to use **MS Word or Pages** for the first and the second homework to accommodate yourself.)
5) Use the given template and give your solution in English. Solution in Chinese is not allowed.
6) Your homework should be uploaded in the PDF format, and the naming format of the file is not specified.

I. Solve Linear Equations

Problem 1. (12 points + 9 points)

Given matrix $\mathbf{A} = \begin{bmatrix} 2 & 5 & -8 & 0 & 17 \\ 1 & 3 & -5 & 1 & 5 \\ -3 & -11 & 19 & -7 & -1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \in \mathbb{R}^{4\times 5}$ and matrix $\mathbf{B} = \begin{bmatrix} 2 & 1 & -5 & 1 \\ 1 & -3 & 0 & -6 \\ 0 & 2 & -1 & 2 \\ 1 & 4 & -7 & 6 \end{bmatrix} \in \mathbb{R}^{4\times 4}$.

1) Find out the column space, row space and null space of the matrix $\mathbf{A}$. (The answer is not unique, but the answer you get requires a detailed solving process, otherwise you will get zero points.)
2) For $\mathbf{b} = \begin{bmatrix} 9 & -1 & 4 & 15 \end{bmatrix}^T \in \mathbb{R}^4$ Solve the linear equation system $\mathbf{Bx} = \mathbf{b}$ with Gaussian Elimination, LU decomposition, and LU decomposition with partial pivoting, respectively.

(You are highly required to write down your solution procedures in detail. And all values must be represented by integers or fractions, floating point numbers are not accepted.)

Solution: 1.1) $\mathbf{A} = \begin{bmatrix} 2 & 5 & -8 & 0 & 17 \\ 1 & 3 & -5 & 1 & 5 \\ -3 & -11 & 19 & -7 & -1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -7 \\ -3 & -7 & 0 & 0 & 49 \\ 1 & 9 & 0 & -4 & -23 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 7 \\ -3 & -7 & 0 & 0 & -49 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2 & -7 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ so $\mathcal{R}(\mathbf{A})$ = span$\{\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -7 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\}$

1.2) $\mathbf{A} = \begin{bmatrix} 2 & 5 & -8 & 0 & 17 \\ 1 & 3 & -5 & 1 & 5 \\ -3 & -11 & 19 & -7 & -1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \Rightarrow \mathbf{A} = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & -1 & 2 & -2 & 7 \\ 0 & -2 & 4 & -4 & -14 \\ 0 & 4 & -8 & 4 & -8 \end{bmatrix} \Rightarrow \mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & -5 & 26 \\ 0 & -1 & 2 & -2 & 7 \\ 0 & 1 & -2 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & -5 & 26 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

so $\mathcal{R}(\mathbf{A^T})$ = span{ $\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & -2 & 0 & 3 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 & 1 & -5 \end{pmatrix}$}

1.3) Since $\mathcal{R}(\mathbf{A^T}) \oplus \mathcal{N}(\mathbf{A}) = R^5$ and $dim\mathcal{R}(\mathbf{A^T}) = 3$

$\therefore dim\mathcal{N}(\mathbf{A}) = 2$

let v1=$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \end{pmatrix}$, v2=$\begin{pmatrix} 0 & 1 & -2 & 0 & 3 \end{pmatrix}$ v3=$\begin{pmatrix} 0 & 0 & 0 & 1 & -5 \end{pmatrix}$,{v4,v5} be a basis of $\mathcal{N}(\mathbf{A})$

{v4,v5} $\perp$ {v1,v2,v3}

so first we construct v4=$\begin{pmatrix} -3 & 1 & 2 & 5 & 1 \end{pmatrix}$,v4 $\perp$ {v1,v2,v3}

then we construct v5 s.t. v5 $\perp$ {v1,v2,v3,v4},so {v4,v5} is a basis of $\mathcal{N}(\mathbf{A})$

v5=(a1,a2,a3,a4,a5) satisfies $\begin{cases} a1 + a3 + a5 = 0 \\ a2 - 2a3 + 3a5 = 0 \\ a4 - 5a5 = 0 \\ -3a1 + a2 + 2a3 + 5a4 + a5 = 0 \end{cases}$ it can be found that v5=$\begin{pmatrix} -19 & 73 & 26 & -35 & -7 \end{pmatrix}$ satisfies the equations.

$\therefore \mathcal{N}(\mathbf{A})$ =span{ $\begin{pmatrix} -3 & 1 & 2 & 5 & 1 \end{pmatrix}$, $\begin{pmatrix} -19 & 73 & 26 & -35 & -7 \end{pmatrix}$}

2.1) $[\mathbf{B}\ b] = \begin{bmatrix} 2 & 1 & -5 & 1 & 9 \\ 1 & -3 & 0 & -6 & -1 \\ 0 & 2 & -1 & 2 & 4 \\ 1 & 4 & -7 & 6 & 15 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 0 & -6 & -1 \\ 0 & 7 & -5 & 13 & 11 \\ 0 & 2 & -1 & 2 & 4 \\ 0 & 7 & -7 & 12 & 16 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 0 & -6 & -1 \\ 0 & 7 & -5 & 13 & 11 \\ 0 & 0 & 1 & -4 & 2 \\ 0 & 0 & -2 & -1 & 5 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & -3 & 0 & -6 & -1 \\ 0 & 7 & -5 & 13 & 11 \\ 0 & 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 0 & 0 & -7 \\ 0 & 7 & -5 & 0 & 24 \\ 0 & 0 & 1 & -0 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -0 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \therefore \text{x=} \begin{pmatrix} -1 \\ 2 \\ -2 \\ -1 \end{pmatrix}$

2.2) $\mathbf{B} = \begin{bmatrix} 2 & 1 & -5 & 1 \\ 1 & -3 & 0 & -6 \\ 0 & 2 & -1 & 2 \\ 1 & 4 & -7 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -5 & 1 \\ 0 & -\frac{7}{2} & \frac{5}{2} & -\frac{13}{2} \\ 0 & 2 & -1 & 2 \\ 0 & \frac{7}{2} & -\frac{9}{2} & \frac{11}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{4}{7} & 1 & 0 \\ \frac{1}{2} & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -5 & 1 \\ 0 & -\frac{7}{2} & \frac{5}{2} & -\frac{13}{2} \\ 0 & 0 & \frac{3}{7} & -\frac{12}{7} \\ 0 & 0 & -2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{4}{7} & 1 & 0 \\ \frac{1}{2} & -1 & -\frac{14}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -5 & 1 \\ 0 & -\frac{7}{2} & \frac{5}{2} & -\frac{13}{2} \\ 0 & 0 & \frac{3}{7} & -\frac{12}{7} \\ 0 & 0 & 0 & -9 \end{bmatrix} \therefore \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{4}{7} & 1 & 0 \\ \frac{1}{2} & -1 & -\frac{14}{3} & 1 \end{bmatrix} \quad \mathbf{U} = \begin{bmatrix} 2 & 1 & -5 & 1 \\ 0 & -\frac{7}{2} & \frac{5}{2} & -\frac{13}{2} \\ 0 & 0 & \frac{3}{7} & -\frac{12}{7} \\ 0 & 0 & 0 & -9 \end{bmatrix}$$

let Lz=b,we can find that $z = \begin{pmatrix} 9 \\ -\frac{11}{2} \\ \frac{6}{7} \\ 9 \end{pmatrix}$ then we solve the equations: Ux=z,then we can get $x = \begin{pmatrix} -1 \\ 2 \\ -2 \\ -1 \end{pmatrix}$

2.3) $\mathbf{B} = \begin{bmatrix} 2 & 1 & -5 & 1 \\ 1 & -3 & 0 & -6 \\ 0 & 2 & -1 & 2 \\ 1 & 4 & -7 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -5 & 1 \\ 0 & -\frac{7}{2} & \frac{5}{2} & -\frac{13}{2} \\ 0 & 2 & -1 & 2 \\ 0 & \frac{7}{2} & -\frac{9}{2} & \frac{11}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{4}{7} & 1 & 0 \\ \frac{1}{2} & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -5 & 1 \\ 0 & -\frac{7}{2} & \frac{5}{2} & -\frac{13}{2} \\ 0 & 0 & \frac{3}{7} & -\frac{12}{7} \\ 0 & 0 & -2 & -1 \end{bmatrix}$

$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ PB= $\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{2} & -1 & 1 & 0 \\ 0 & -\frac{4}{7} & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -5 & 1 \\ 0 & -\frac{7}{2} & \frac{5}{2} & -\frac{13}{2} \\ 0 & 0 & -2 & -1 \\ 0 & 0 & \frac{3}{7} & -\frac{12}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{2} & -1 & 1 & 0 \\ 0 & -\frac{4}{7} & -\frac{3}{14} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -5 & 1 \\ 0 & -\frac{7}{2} & \frac{5}{2} & -\frac{13}{2} \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & -\frac{27}{14} \end{bmatrix}$

$\therefore$ we have PB=LU,where $\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{2} & -1 & 1 & 0 \\ 0 & -\frac{4}{7} & -\frac{3}{14} & 1 \end{bmatrix}$, $\mathbf{U} = \begin{bmatrix} 2 & 1 & -5 & 1 \\ 0 & -\frac{7}{2} & \frac{5}{2} & -\frac{13}{2} \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & -\frac{27}{14} \end{bmatrix}$

let Lz=Pb,we can find that $z = \begin{pmatrix} 9 \\ -\frac{11}{2} \\ 5 \\ -\frac{27}{14} \end{pmatrix}$ then we solve the equations: Ux=z,then we can get $x = \begin{pmatrix} -1 \\ 2 \\ -2 \\ -1 \end{pmatrix}$

II. More about LU decomposition

Problem 1. (5 points × 3)

$\mathbf{A} \in \mathbb{R}^{n\times n}$ is a nonsingular matrix and we have $\mathbf{A} = \mathbf{LU}$, while $\mathbf{U} \in \mathbb{R}^{n\times n}$ is a upper triangular matrix and $\mathbf{L} \in \mathbb{R}^{n\times n}$ is a lower triangular matrix with $\mathbf{L}_{i,i} = 1$.

1) Prove every leading principal submatrix $\mathbf{A}_{\{1,\ldots,k\}}$ of $\mathbf{A}$ satisfies:
   $\det(\mathbf{A}_{\{1,\ldots,k\}}) \neq 0$
2) Prove the uniqueness of matrix $\mathbf{L}$ and $\mathbf{U}$
3) If $\mathbf{A}$ is symmetric, please illustrate the relationship between the matrix $\mathbf{L}$ and the matrix $\mathbf{U}$.

Solution: 1) $\because$ A is nonsingular

$\therefore$ det(A)=det(LU)=det(L)det(U) $\neq 0$

$\therefore$ det(L) $\neq 0$, det(U) $\neq 0$

Suppose $\exists$ k $\in [n-1]$: det($\mathbf{A_k}$)=0

$$\mathbf{A} = \begin{bmatrix} \mathbf{A_k} & a_1 \\ a_2 & a_3 \end{bmatrix} = \mathbf{LU} = \begin{bmatrix} \mathbf{L_k} & 0 \\ l_1 & l_2 \end{bmatrix} \begin{bmatrix} \mathbf{U_k} & u_1 \\ 0 & u_2 \end{bmatrix} \Rightarrow \mathbf{A_k} = \mathbf{L_k U_k}$$

$\because$ 0=det($\mathbf{A_k}$)=det($\mathbf{L_k}$)det($\mathbf{U_k}$), det(L_k)=1

$\therefore$ det(U_k)=0

let u_{ij} be the element at i row and j column of $\mathbf{U_k}$, i,j $\in$[k]

$\because det(\mathbf{U_k}) = \prod_{i=1}^{k} u_{ii}$

$\therefore \exists$ i $\in$[k]:u_{ii}=0 $\because det(\mathbf{U}) = \prod_{i=1}^{n} u_{ii}$

$\therefore det(\mathbf{U}) = 0$ Contradiction

$\therefore$every leading principal submatrix $\mathbf{A}_{\{1,\ldots,k\}}$ of $\mathbf{A}$ satisfies: $\det(\mathbf{A}_{\{1,\ldots,k\}}) \neq 0$ Q.E.D

2) $\because$ A is nonsingular

$\therefore$ det(A)=det(LU)=det(L)det(U) $\neq 0$

$\therefore$ det(L) $\neq 0$, det(U) $\neq 0$

$\therefore$ L and U are invertible let A=$\mathbf{LU}$=$\mathbf{L_1U_1}$ where $\mathbf{L_1}$,$\mathbf{U_1}$ satisfies the condition of the problem.

$\mathbf{UU_1^{-1}}$=$\mathbf{L^{-1}L_1}$ Based on the theorems mentioned in the class that the product of two upper(low) triangular matrices is an upper(low) triangular matrix; the inverse of the upper(low) triangular matrices is an upper(low) triangular matrix [Since these theorems have been proved in the class and they are also easy to prove,so I don't prove them here again]

$\because \mathbf{UU_1^{-1}}$ is an upper triangular matrix

$\therefore \mathbf{L^{-1}L_1}$ is both low and upper triangular matrix and the diagonal of both $\mathbf{L^{-1}}$ and $\mathbf{L_1}$ are all 1's,which means $\mathbf{L^{-1}L_1}$=$\mathbf{I}$

$\therefore \mathbf{UU_1^{-1}}$=$\mathbf{L^{-1}L_1}$=$\mathbf{I}$

$\therefore \mathbf{L_1}$=$\mathbf{L}$, $\mathbf{U_1}$=$\mathbf{U}$

$\therefore$ matrix L and U is unique Q.E.D

3) A=$\mathbf{LU}$=$\mathbf{LD\widetilde{U}}$ where $\mathbf{U} = \mathbf{D\widetilde{U}}$, D=$\begin{bmatrix} \mathbf{u_{11}} & 0 & \cdots & 0 & 0 \\ 0 & \mathbf{u_{22}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \mathbf{u_{nn}} \end{bmatrix}$

$\mathbf{A}$=$\mathbf{A^T}$=$\mathbf{\widetilde{U}^T D L^T}$

$\because$ A=$\mathbf{LU}$ is unique

$\therefore$ A=$\mathbf{LD\widetilde{U}}$ is unique

$\because$ $\mathbf{A}$=$\mathbf{A^T}$=$\mathbf{\widetilde{U}^T D L^T}$ where $\mathbf{\widetilde{U}^T}$ is a low triangle matrix $\mathbf{L^T}$ is an upper triangle matrix

$\therefore$ $\mathbf{\widetilde{U}}$= $\mathbf{L^T}$

$\therefore$ $\mathbf{U}$=$\mathbf{DL^T}$ where D=$\begin{bmatrix} \mathbf{u_{11}} & 0 & \cdots & 0 & 0 \\ 0 & \mathbf{u_{22}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \mathbf{u_{nn}} \end{bmatrix}$ Q.E.D

III. CHOLESKY DECOMPOSITION

Problem 1. (7 points $\times$ 2)

The Cholesky decomposition is a decomposition of a symmetric positive-definite matrix into the product of a lower triangular matrix and its transpose. The symmetric positive-definite matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ can be factored as

$$\mathbf{A} = \mathbf{L}\mathbf{L}^T,$$

where $\mathbf{L}$ is lower triangular matrix with positive diagonal elements.

1) Consider the matrix $\mathbf{A} = \begin{bmatrix} 4 & 4 & -8 & 4 \\ 4 & 5 & -6 & 6 \\ -8 & -6 & 24 & 4 \\ 4 & 6 & 4 & 25 \end{bmatrix}$, and give the Cholesky decomposition $\mathbf{A} = \mathbf{L}\mathbf{L}^T$.
2) Consider a symmetric positive-definite matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, and its Cholesky decomposition $\mathbf{A} = \mathbf{L}\mathbf{L}^\mathbf{T}$. Prove that $\kappa_F(\mathbf{L}) \le n\sqrt{\kappa_2(\mathbf{A})}$.[1]

   Hint: If matrix $\mathbf{A}$ is SPD, all its eigenvalues are positive, $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n > 0$. You can use the result that $\|\mathbf{A}\|_2 = \lambda_1$ and $\|\mathbf{A}^{-1}\|_2 = \frac{1}{\lambda_n}$

Solution: 1) Since we can decompose $\mathbf{A} = \mathbf{L}\mathbf{L}^\mathbf{T}$ we can get $\mathbf{L}$ step by step using symmetry. $\mathbf{A} = \begin{bmatrix} 4 & 4 & -8 & 4 \\ 4 & 5 & -6 & 6 \\ -8 & -6 & 24 & 4 \\ 4 & 6 & 4 & 25 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix} \begin{bmatrix} 2 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & * & 0 & 0 \\ -4 & * & * & 0 \\ 2 & * & * & * \end{bmatrix} \begin{bmatrix} 2 & 2 & -4 & 2 \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -4 & * & * & 0 \\ 2 & * & * & * \end{bmatrix} \begin{bmatrix} 2 & 2 & -4 & 2 \\ 0 & 1 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} =$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -4 & 2 & * & 0 \\ 2 & 2 & * & * \end{bmatrix} \begin{bmatrix} 2 & 2 & -4 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -4 & 2 & 2 & 0 \\ 2 & 2 & * & * \end{bmatrix} \begin{bmatrix} 2 & 2 & -4 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & * \\ 0 & 0 & 0 & * \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -4 & 2 & 2 & 0 \\ 2 & 2 & 4 & * \end{bmatrix} \begin{bmatrix} 2 & 2 & -4 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & * \end{bmatrix} =$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -4 & 2 & 2 & 0 \\ 2 & 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & -4 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \therefore \mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -4 & 2 & 2 & 0 \\ 2 & 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & -4 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2) First we proved as the same in HW1 that $\|\mathbf{A}\|_F \le \sqrt{n}\|\mathbf{A}\|_2$:

[1] $\kappa(\mathbf{A})$ associated with the linear equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ gives a bound on how inaccurate the solution $\mathbf{x}$ will be after approximation. $\kappa_F(\mathbf{A}) = \|\mathbf{A}\|_F\|\mathbf{A}^{-1}\|_F$ and $\kappa_2(\mathbf{A}) = \|\mathbf{A}\|_2\|\mathbf{A}^{-1}\|_2$

$\forall$ x $\in \mathbf{R^n}$: $\|\mathbf{x}\|_2 = 1, x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$\|\mathbf{A}\|_F{}^2 = \sum_{i=1}^n \|\mathbf{Ae_i}\|_2{}^2, \mathbf{e_i} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$

$\leq \sum_{i=1}^n \|\mathbf{A}\|_2{}^2 = \mathbf{n}\|\mathbf{A}\|_2{}^2$

So we have $\kappa_F(\mathbf{L}) = \|\mathbf{L}\|_F\|\mathbf{L^{-1}}\|_F \leq \mathbf{n}\|\mathbf{L}\|_2\|\mathbf{L^{-1}}\|_2 = \mathbf{n}\sqrt{\lambda_{max}(\mathbf{LL^T})}\sqrt{\lambda_{max}((\mathbf{L^{-1}})^\mathbf{T}\mathbf{L^{-1}})} = \mathbf{n}\sqrt{\lambda_{max}(\mathbf{A})}\sqrt{\lambda_{max}(\mathbf{A^{-1}})} = n\sqrt{\|\mathbf{A}\|_2}\sqrt{\|\mathbf{A^{-1}}\|_2)} = n\sqrt{\kappa_2(\mathbf{A})}$ Q.E.D

For short supplement, now prove that if matrix $\mathbf{A}$ is SPD, all its eigenvalues are positive, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ then we have $\mathbf{A^{-1}}$ is SPD,all its eigenvalues are positive, $0 < \frac{1}{\lambda_1} \leq \frac{1}{\lambda_2} \leq \cdots \leq \frac{1}{\lambda_n}$.

$\because$ $\mathbf{A}$ is SPD,then we have $\mathbf{A} = \mathbf{PDP^T}$, where P is orthonormal matrix,D is diagonal matrix with the diagonal from λ_1 to λ_n,$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$,then we have matrix B=$\mathbf{PD^{-1}P^T}$,the diagnoal of $\mathbf{D^{-1}}$ is from $\frac{1}{\lambda_1}$ to $\frac{1}{\lambda_n}$,$0 < \frac{1}{\lambda_1} \leq \frac{1}{\lambda_2} \leq \cdots \leq \frac{1}{\lambda_n}$.

$\mathbf{AB} = \mathbf{PDP^TPD^{-1}P^T} = \mathbf{PDD^{-1}P^T} = \mathbf{I}$

$\therefore \mathbf{B} = \mathbf{A^{-1}}$ and the eigenvalues are positive,$0 < \frac{1}{\lambda_1} \leq \frac{1}{\lambda_2} \leq \cdots \leq \frac{1}{\lambda_n}$.

IV. BANDED MATRIX

Problem 1. (5 points + 10 points)

$\mathbf{A} \in \mathbb{R}^{n\times n}$ is called a banded matrix if $a_{ij} = 0$ whenever $|i-j| > m$ for some positive integer m(called the bandwidth). If $\mathbf{A}$ is a nonsingular matrix with bandwidth m, and has LU decomposition $\mathbf{A} = \mathbf{LU}$,then $\mathbf{L}$ inherits the lower band structure of $\mathbf{A}$ with "lower bandwidth " m and $\mathbf{U}$ inherits the upper band structure of $\mathbf{A}$ with "upper bandwidth " m.

1) LU decomposition is particularly efficient in the case of banded matrices, consider a banded matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ with bandwidth m $(m \ll n)$. The solution of the set of equation $\mathbf{Ax} = \mathbf{b}$ can be determined in the steps

$$\mathbf{Ly} = \mathbf{b}, \qquad \mathbf{Ux} = \mathbf{y},$$

How many flops does this linear system require? (just calculate the forward and backwards substitution flops but need detail derivation)

2) Consider a symmetric positive-definite banded matrix

$$\mathbf{A} = \begin{bmatrix} a & b & 0 & \cdots & 0 \\ b & a & b & \cdots & 0 \\ 0 & b & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & a & b \\ 0 & 0 & \cdots & b & a \end{bmatrix}$$

where $a > 0$ and $a > 2|b|$.

Find the efficient Cholesky decomposition of the banded matrix $\mathbf{A}$, derive the complexity of your efficient Cholesky decomposition algorithm and try to complete the Algorithm 1.(You should write the derivation of finding the Cholesky decomposition of the matrix $\mathbf{A}$)

Solution: 1) First we write the common process of LU forward and backwards substitution:

$\mathbf{Ly} = \mathbf{b}, \mathbf{Ux} = \mathbf{y}$

$$\begin{cases} y_1 = b_1 \\ y_i = b_i - \displaystyle\sum_{k=1}^{i-1} l_{ik} y_k \quad i = 2, 3..., n \end{cases}, \quad \begin{cases} x_n = \dfrac{y_n}{u_{nn}} \\ x_i = \dfrac{x_i - \sum_{k=i+1}^{n} u_{ik} x_k}{u_{ii}} \quad i = n-1, n-2, ..., 1 \end{cases}$$

the forward process need 1+$\sum_{i=2}^{n}((i-1)+(i-1))$flops,the first 1 is calculating y_1

the first i-1 is the number of multiplication,the second i-1 is the number of addition.

So there are $n^2 - n + 1$ flops

the backward process need 2+$\sum_{i=1}^{n-1}((n-i)+(n-i)+1)$flops,the first 1 is calculating x_n

the first n-i is the number of multiplication,the second n-i is the number of addition,the last 1 is division.

So there are n^2 flops

Now consider the situation of band structure,if there is one 0,then there will be 2 less flops(one is multiplication,one is addition).So if the matrix A has bandwidth m,then there will be $(n-m-1)^2$ 0,so there will be

Algorithm 1 Cholesky decomposition for matrix $\mathbf{A}$

Input: The matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$

Output: Cholesky decomposition of $\mathbf{A}$;

1: $l_{11} = \sqrt{a_{11}}$
2: for i=2 **to** n
3: $l_{i1} = \frac{a_{i1}}{l_{11}}$
4: for j=2 **to** n-1
5: $l_{jj} = \sqrt{a_{jj} - \sum_{k=1}^{j-1} l_{jk}^2}$
6: for i=j+1 **to** n
7: $l_{ij} = \frac{a_{ij}-\sum_{k=1}^{j-1} l_{ik}l_{jk}}{l_{jj}}$
8: ENDFOR
9: ENDFOR
10: $l_{nn} = \sqrt{a_{nn} - \sum_{k=1}^{n-1} l_{nk}^2}$

$2(n-m-1)^2$ **flops less**.

In conclusion,the flops of matrix A with bandwidth m is $n^2-n+1+n^2-2(n-m-1)^2 = (3+4m)n - 2m^2-4m-1$**= O(mn)**

2) The algorithm is in the problem,below is the derivation of finding the algorithm:

Since A = LL$^{\mathbf{T}}$ has a good character—symmetry,so first look $l_{11} = \sqrt{a_{11}}, l_{i1} = \frac{a_{i1}}{l_{11}}$**,we get the first column of L**

then $l_{22} = \sqrt{a_{22}-l_{21}^2}$**,**$\mathbf{l}_{i2} = \frac{a_{i2}-l_{i1}l_{21}}{l_{22}}$**,we get the second column of L**
then $l_{33} = \sqrt{a_{33}-l_{31}^2-l_{32}^2}$**,**$\mathbf{l}_{i3} = \frac{a_{i3}-l_{i1}l_{31}-l_{i2}l_{32}}{l_{33}}$

So we can find the algorithm by induction:

$l_{11} = \sqrt{a_{11}}$

$l_{i1} = \frac{a_{i1}}{l_{11}}, i = 2, ..., n$

$for j = 2, ..., n-1$

$l_{jj} = \sqrt{a_{jj} - \sum_{k=1}^{j-1} l_{jk}^2}$
$l_{ij} = \frac{a_{ij}-\sum_{k=1}^{j-1} l_{ik}l_{jk}}{l_{jj}}, i = j+1, ..., n$
$l_{nn} = \sqrt{a_{nn} - \sum_{k=1}^{n-1} l_{nk}^2}$

Now we start to analyse the complexity of this method:

since we consider the complexity,so it is not important for 1 or 2 flops,the complexicity is about$\sum_{j=2}^{n-2}[2(j-1)+1+\sum_{i=j+1}^{n}(2(j-1)+1)]$**,which is O**$(\frac{n^3}{3})$**,this is the complexity of Cholesky decomposition**

When it used on A,we need less flops since A has a special structure.The same as problem1,one 0 means less 2 flops in circulation.The complexity is about$\sum_{j=2}^{n-2}[2m+\sum_{i=j+1}^{n}(2m+1)]$**,which is O**$(mn^2)$

V. Programming

Problem 1(5 points + 10 points)

In this problem, we explore the efficiency of the LU method together with the classical linear system solvers we have learnt in linear algebra.

1) Derive the complexity of the LU decomposition. Particularly, how many flops does the LU decomposition require? The corresponding pseudo code (in Matlab) is provided as follows[2]:

Algorithm 2 Pseudo-code of LU decomposition

```
function NAIVE_LU(A)
    n = size(A, 1)
    L = eye(n)
    U = A
    for k = 1 → n − 1 do
        for j = k + 1 → n do
            L(j, k) = U(j, k)/U(k, k)
            U(j, k : n) = U(j, k : n) − L(j, k) ∗ U(k, k : n)
        end for
    end for
    U = triu(U)
end function
```

2) Randomly generate a non-singular matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ and a vector $\mathbf{b} \in \mathbb{R}^{n \times 1}$, then program the following methods to solve $\mathbf{A}\mathbf{x} = \mathbf{b}$:
   - **LU decomposition.** We first find the LU decomposition of $\mathbf{A}$, then we solve $\mathbf{L}\mathbf{y} = \mathbf{b}$ and $\mathbf{U}\mathbf{x} = \mathbf{y}$.
   - **The inverse method:** Use the inverse of $\mathbf{A}$ to solve the problem, which can be written as,

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}\,.$$

   In your homework, you are required to submit the time-consuming plot (**one figure**) of given methods against the size of matrix $\mathbf{A}$ (i.e., n), where $n = 100, 200, \ldots, 1000$ (you can try larger n and see what will happen, be careful with the memory use of your PC!).

   Remarks:
   - You can use any language you like to program, but do not use built-in functions which are highly optimized to compute the LU decomposition or the matrix inverse (for example, Matlab function lu() and inv()). Otherwise, your results will contradict the complexity analysis, and your score will be discounted. You can implement the simplest version of these methods by yourself.

[2]$triu(\mathbf{U})$ is the Upper triangular part of the matrix $\mathbf{U}$

- In Matlab, to randomly generate a matrix or a vector, you can use randn function to generate normally distributed random numbers.

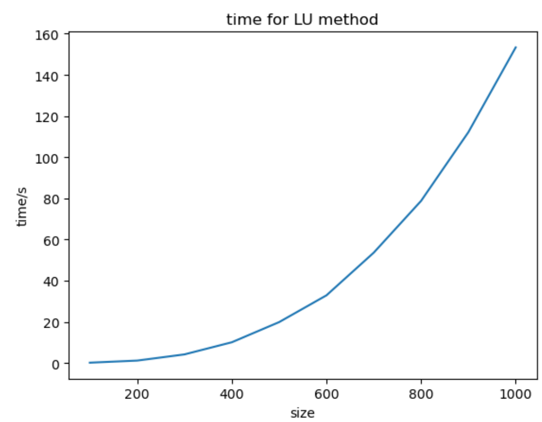
Remarks:

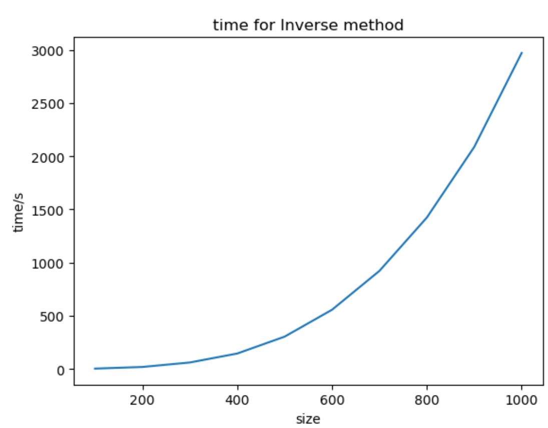
1) The definition of *flop* is: **The float operations of float numbers**. So the division($/$), multiplication($\times$), addition($+$) and subtraction($-$) should be taken into consideration. However, the assignment ($=$) is not an operation on float numbers by convention.
2) When handing in your homework in gradescope, package all your codes into **your_student_id+hw2_code.zip** and upload. In the package, you also need to include a file named README.txt/md to clearly identify the function of each file. Make sure that your codes can run and are consistent with your solutions

Solution: 1) The number of flops is $\sum_{k=1}^{n-1}\sum_{j=k+1}^{n}(1+(n-k+1)*2) = \sum_{k=1}^{n-1}(2n^2-4nk+2k^2+3n-3k) = (n-1)2n^2 - 4n\frac{(1+n-1)(n-1)}{2} + \frac{(n-1)n(2n-1)}{3} + 3n(n-1) - 3\frac{(1+n-1)(n-1)}{2} = \frac{2}{3}n^3 + \frac{1}{2}n^2 - n$

The complexity is $\mathbf{O}(\frac{2}{3}n^3)$

2)





VI. Roundoff Error

Problem 1 (10 points + 4 points + 6 points)

Given a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, consider the roundoff error in the process of solving $\mathbf{Ax} = \mathbf{b}$ by Gaussian elimination in three stages:

1. Decompose $\mathbf{A}$ into $\mathbf{LU}$, with roundoff error $\mathbf{E}$, $\bar{\mathbf{L}}$ and $\bar{\mathbf{U}}$ are computed instead, i.e.,
$$\mathbf{A} + \mathbf{E} = \bar{\mathbf{L}}\bar{\mathbf{U}}\,.$$
2. Solving $\mathbf{Ly} = \mathbf{b}$, numerically with roundoff error $\delta\bar{\mathbf{L}}$, $\hat{\mathbf{y}} = \mathbf{y} + \delta\mathbf{y}$ are computed instead.
$$(\bar{\mathbf{L}} + \delta\bar{\mathbf{L}})(\mathbf{y} + \delta\mathbf{y}) = \mathbf{b}\,.$$
3. Solving $\mathbf{Ux} = \mathbf{y}$, numerically with roundoff error $\delta\bar{\mathbf{U}}$, $\hat{\mathbf{x}} = \mathbf{x} + \delta\mathbf{x}$ are computed instead.
$$(\bar{\mathbf{U}} + \delta\bar{\mathbf{U}})(\mathbf{x} + \delta\mathbf{x}) = \hat{\mathbf{y}}\,.$$

Finally, we can get the computed solution $\hat{\mathbf{x}}$ and

$$\begin{aligned}\mathbf{b} =& (\bar{\mathbf{L}} + \delta\bar{\mathbf{L}})(\bar{\mathbf{U}} + \delta\bar{\mathbf{U}})(\mathbf{x} + \delta\mathbf{x})\\ =& (\mathbf{A} + \delta\mathbf{A})(\mathbf{x} + \delta\mathbf{x})\,.\end{aligned}$$

1) Prove the relative error of $\mathbf{x}$ has an upper bound as follows
$$\frac{\|\hat{\mathbf{x}} - \mathbf{x}\|}{\|\mathbf{x}\|} = \frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \le \frac{1}{1 - \kappa(\mathbf{A})\frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|}}\kappa(\mathbf{A})\frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|},$$
where $\kappa(\mathbf{A}) = \|\mathbf{A}\|\|\mathbf{A}^{-1}\|$ denotes the condition number of the matrix $\mathbf{A}$ (Suppose $\mathbf{A}$ and $\mathbf{A} + \delta\mathbf{A}$ are nonsingular and $\|\mathbf{A}^{-1}\|\|\delta\mathbf{A}\| < 1$).

   Hint: The following equation might be useful,
$$\|(\mathbf{I} - \mathbf{B})^{-1}\| = \|\sum_{k=0}^{\infty}\mathbf{B}^k\| \le \sum_{k=0}^{\infty}\|\mathbf{B}\|^k \le \frac{1}{1 - \|\mathbf{B}\|}\,.$$
where $\mathbf{I} - \mathbf{B}$ is nonsingular and $\|\mathbf{B}\| < 1$.

2) Given the system $\mathbf{Ax} = \mathbf{b}$
$$\begin{bmatrix}0.986 & 0.579\\ 0.409 & 0.237\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0.235\\ 0.107\end{bmatrix}.$$
Compute the condition number $\kappa_\infty(\mathbf{A})$[3] and the solution $\mathbf{x}$.

3) Continue with the same system in 2). Suppose the roundoff error
$$\delta\mathbf{A} = u\begin{bmatrix}2 & 6\\ 4 & 8\end{bmatrix},$$
where u is a small scalar. Fill the table below.

[3] $\kappa_\infty(\mathbf{A}) = \|\mathbf{A}\|_\infty\|\mathbf{A}^{-1}\|_\infty$

Table I: Compute the quantities

u	$\|\delta x_1\|$	$\|\delta x_2\|$	$\frac{1}{1-\kappa(\mathbf{A})\frac{\|\delta \mathbf{A}\|}{\|\mathbf{A}\|}}\kappa(\mathbf{A})\frac{\|\delta \mathbf{A}\|}{\|\mathbf{A}\|}$	$\frac{\|\delta \mathbf{x}\|_\infty}{\|\mathbf{x}\|_\infty}$
10^{-1}	1.5742327154971152	2.7710262939606376	-1.0018726759875538	0.9236754313202045
10^{-2}	1.8086336782220152	3.066487449595225	-1.0190477929762403	1.0221624831983993
10^{-4}	0.21460857510069253	0.3627449564601495	1.1505154639175457	0.12091498548671543
10^{-6}	0.0019024664891009735	0.003215572486617635	0.005378727998303526	0.0010718574955392022
10^{-8}	1.900309396507538e-05	3.211925579016395e-05	5.350238296545534e-05	1.0706418596721222e-05
10^{-10}	1.9002879136920114e-07	3.2118892523413933e-07	5.349954923561736e-07	1.0706297507804551e-07

Solution: 1) $\because (\mathbf{A}+\delta\mathbf{A})(\mathbf{x}+\delta\mathbf{x}) = \mathbf{b}, \mathbf{Ab}=\mathbf{b}$

$\therefore (\mathbf{A}+\delta\mathbf{A})\delta\mathbf{x} = -\delta\mathbf{A}\mathbf{x}$

$\because$ $\mathbf{A}$ and $\mathbf{A}+\delta\mathbf{A}$ are nonsingular

$\therefore \mathbf{A}+\delta\mathbf{A} = \mathbf{A}(\mathbf{I}+\mathbf{A^{-1}}\delta\mathbf{A})$

$\therefore \delta\mathbf{x} = -(\mathbf{I}+\mathbf{A^{-1}}\delta\mathbf{A})^{-1}\mathbf{A^{-1}}\delta\mathbf{A}\mathbf{x}$

$\because \|\mathbf{A}^{-1}\delta\mathbf{A}\| \leq \|\mathbf{A^{-1}}\|\|\delta\mathbf{A}\| < 1$

$\therefore \|\delta\mathbf{x}\| \leq \|(\mathbf{I}+\mathbf{A^{-1}}\delta\mathbf{A})^{-1}\|\|\mathbf{A^{-1}}\|\|\delta\mathbf{A}\|\|\mathbf{x}\|$

$\leq \frac{\|\mathbf{A^{-1}}\|\|\delta\mathbf{A}\|\|\mathbf{x}\|}{1-\|\mathbf{A^{-1}}\delta\mathbf{A}\|}$

$\leq \frac{\|\mathbf{A^{-1}}\|\|\delta\mathbf{A}\|\|\mathbf{x}\|}{1-\|\mathbf{A^{-1}}\|\|\delta\mathbf{A}\|}$

$\therefore \frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \frac{\|\mathbf{A}\|\|\mathbf{A^{-1}}\|\frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|}}{1-\|\mathbf{A}\|\|\mathbf{A^{-1}}\|\frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|}}$

$= \frac{1}{1-\kappa(\mathbf{A})\frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|}}\kappa(\mathbf{A})\frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|}$ Q.E.D

2) $\mathbf{A} = \begin{bmatrix} 0.986 & 0.579 \\ 0.409 & 0.237 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0.235 \\ 0.107 \end{bmatrix}$,Using the method $(\mathbf{A}|\mathbf{I})$,we can find $\mathbf{A^{-1}} = \begin{bmatrix} -75.7430 & 185.0431 \\ 130.7127 & -315.1167 \end{bmatrix}$

$\kappa_\infty(\mathbf{A}) = \|\mathbf{A}\|_\infty\|\mathbf{A}^{-1}\|_\infty = (0.986+0.579)(130.7127+315.116) = 697.723, \mathbf{x} = \mathbf{A^{-1}b} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

3) The answer has been written in the table,and below is the program:

$importnumpyasnp$

$A = np.array([[0.986, 0.579], [0.409, 0.237]])$

$b = np.array([[0.235], [0.107]])$

$Ainv = np.linalg.inv(A)$

$A_norm = np.linalg.norm(A, ord = np.inf)$

$Ainv_norm = np.linalg.norm(Ainv, ord = np.inf)$

$(x1, x2) = np.linalg.solve(A, b)$

$x = np.array([[x1[0]], [x2[0]]])$

$x_norm = np.linalg.norm(x, ord = np.inf)$

$deltaA = np.array([[2, 6], [4, 8]])$

$u = [1e-1, 1e-2, 1e-4, 1e-6, 1e-8, 1e-10]$

$deltax1 = []$

$deltax2 = []$

$upbound = []$

$lowbound = []$

$longterm = []$

$print(''A:'', A, ''Anorm:'', A_{n}orm, ''Ainv:'', Ainv, ''Ainvnorm:'', Ainv_{n}orm, ''x:'', (x1, x2))$

$foriinrange(len(u)):$

$tempu = u[i]$

$tempdeltaA = tempu * deltaA$

$tempA = A + tempdeltaA$

$(tempx1, tempx2) = np.linalg.solve(tempA, b)$

$tempx = np.array([[tempx1[0]], [tempx2[0]]])$

$deltax = x - tempx$

$deltax_{n}orm = np.linalg.norm(deltax, ord = np.inf)$

$lowbound.append(deltax_{n}orm/x_{n}orm)$

$deltax1.append(abs(x1[0] - tempx1[0]))$

$deltax2.append(abs(x2[0] - tempx2[0]))$

$tempdeltaA_{n}orm = np.linalg.norm(tempdeltaA, ord = np.inf)$

$templongterm = (Ainv_{n}orm * tempdeltaA_{n}orm)/(1 - Ainv_{n}orm * tempdeltaA_{n}orm)$

$longterm.append(templongterm)$

$print(''/////////////'')$

$print(deltax1)$

$print(''/////////////'')$

$print(deltax2)$

$print(''/////////////'')$

$print(longterm)$

$print(''/////////////'')$

$print(lowbound)$