SI231 - Matrix Computations, 2021 Fall

Homework Set #1

Prof. Yue Qiu

Acknowledgements:

1) Deadline: **2021-10-12 23:59:59**
2) **Late Policy details** can be found on piazza.
3) Submit your homework in **Homework 1** on **Gradscope**. Entry Code: **2RY68R**. Make sure that you have correctly select pages for each problem. If not, you probably will get 0 point.
4) No handwritten homework is accepted. You need to write LaTeX. (If you have difficulties in using LaTeX, you are allowed to use **MS Word or Pages** for the first and the second homework to accommodate yourself.)
5) Use the given template and give your solution in English. Solution in Chinese is not allowed.
6) Your homework should be uploaded in the PDF format, and the naming format of the file is not specified.

I. Vector space and subspace

Problem 1. (6 points $\times$ 3)

1) Let $\mathcal{X}$ and $\mathcal{Y}$ be two subspaces of a vector space $\mathcal{V}$:
   a) Prove that the intersection $\mathcal{X} \cap \mathcal{Y}$ is also a subspace of $\mathcal{V}$.
   b) Show that the union of $\mathcal{X} \cup \mathcal{Y}$ need not to be a subspace of $\mathcal{V}$.
2) Prove or give a counterexample:
   a) If $\mathcal{U}_1$, $\mathcal{U}_2$, and $\mathcal{W}$ are subspaces of $\mathcal{V}$ such that $\mathcal{U}_1 + \mathcal{W} = \mathcal{U}_2 + \mathcal{W}$, then $\mathcal{U}_1 = \mathcal{U}_2$.
   b) If $\mathcal{U}_1$, $\mathcal{U}_2$, and $\mathcal{W}$ are subspaces of $\mathcal{V}$ such that $\mathcal{V} = \mathcal{U}_1 \oplus \mathcal{W}$ and $\mathcal{V} = \mathcal{U}_2 \oplus \mathcal{W}$, then $\mathcal{U}_1 = \mathcal{U}_2$.[1].
3) Let $\mathbf{U} = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_r\}$ and $\mathbf{V} = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_r, \mathbf{v}\}$ be two sets of vectors from the same vector space, prove that $span(\mathbf{U}) = span(\mathbf{V})$ if and only if $\mathbf{v} \in span(\mathbf{U})$.

Solution:

[1] Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two subspaces of $\mathbb{R}^n$, if $\mathcal{S}_1 \cap \mathcal{S}_2 = \{\mathbf{0}\}$ and $\mathcal{S}_1 + \mathcal{S}_2 = \mathbb{R}^n$, we define the **direct sum** $\mathbb{R}^n = \mathcal{S}_1 \oplus \mathcal{S}_2$.

I

1.(1)(a) ① $\forall u,v \in \mathcal{X} \cap \mathcal{Y}$, $u \in \mathcal{X}, v \in \mathcal{X}$ $\therefore u+v \in \mathcal{X}$

$u \in \mathcal{Y}, v \in \mathcal{Y}$ $\therefore u+v \in \mathcal{Y}$

$\therefore u+v \in \mathcal{X} \cap \mathcal{Y}$

② $\forall a \in F, u \in \mathcal{X} \cap \mathcal{Y}$, $v \in \mathcal{X}$ $\therefore av \in \mathcal{X}$

$v \in \mathcal{Y}$ $\therefore av \in \mathcal{Y}$

$\therefore av \in \mathcal{X} \cap \mathcal{Y}$ Q.E.D.

(b). Counter example: $V = \mathbb{R}^2$, $\mathcal{X} = \{(a,0) \mid a \in \mathbb{R}\}$, $\mathcal{Y} = \{(0,b) \mid b \in \mathbb{R}\}$

$\mathcal{X}, \mathcal{Y}$ are subspace of V

but let $u = (1,0) \in \mathcal{X}$, $v = (0,1) \in \mathcal{Y}$. $u,v \in \mathcal{X} \cup \mathcal{Y}$, $u+v = (1,1) \notin \mathcal{X} \cup \mathcal{Y}$ Q.E.D.

(2)(a) Counter example: $V = W = \mathbb{R}^2$, $U_1 = \mathbb{R}^2$, $U_2 = \{(a,0) \mid a \in \mathbb{R}\}$

$U_1 + W = U_2 + W = \mathbb{R}^2$, but $U_1 \neq U_2$

(b) Counter example: $V = \mathbb{R}^2$, $W = \{(a,0) \mid a \in \mathbb{R}\}$, $U_1 = \{(0,b) \mid b \in \mathbb{R}\}$, $U_2 = \{(c,2c) \mid c \in \mathbb{R}\}$

$W + U_1 = V$, $W \cap U_1 = \{0\}$ $\Rightarrow V = U_1 \oplus W$

$W + U_2 = V$, $W \cap U_2 = \{0\}$ $\Rightarrow V = U_2 \oplus W$

but $U_1 \neq U_2$ eg. $(0,1) \in U_1$, $(0,1) \notin U_2$ Q.E.D.

(3) "$\Rightarrow$" $\because \operatorname{span}\{U\} = \operatorname{span}\{V\}$

$\therefore v \in \operatorname{span}\{V\} = \operatorname{span}\{U\}$

"$\Leftarrow$" $\because v \in \operatorname{span}\{U\}$, let $v = \sum_{i=1}^{r} \alpha_i u_i$, $\alpha_i \in F, i \in [r]$

since $U \subseteq V$

$\therefore \operatorname{span}\{U\} \subseteq \operatorname{span}\{V\}$ ①

$\forall w \in \operatorname{span}\{V\}$, $w = \sum_{i=1}^{r} \beta_i u_i + \beta_{r+1} v_i$, $\beta_i \in F, i \in [r+1]$

$= \sum_{i=1}^{r} \beta_i u_i + \sum_{i=1}^{r} \beta_{r+1} \alpha_i u_i$

$= \sum_{i=1}^{r} (\beta_i + \beta_{r+1} \alpha_i) u_i \in \operatorname{span}\{U\}$

$\therefore \operatorname{span}\{V\} \subseteq \operatorname{span}\{U\}$ ②

①② $\Rightarrow \operatorname{span}\{U\} = \operatorname{span}\{V\}$ Q.E.D.

II. BASIS, DIMENSION AND RANK

Problem 1. (5 points $\times$ 2) For matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, we have $\mathcal{V} = \{\mathbf{X} \in \mathbb{R}^{n\times n} | \mathbf{AX} = \mathbf{XA}\}$,

1) Prove that $\mathcal{V}$ is a linear subspace of the linear space $\mathbb{R}^{n\times n}$;
2) If $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$, please give a basis and the dimension of $\mathcal{V}$.

Solution:

II

1\. (1) 1° $\forall X, Y \in V = \{X \in \mathbb{R}^{n\times n} | AX = XA\}$

$AX = XA$, $AY = YA$

$A(X+Y) = (X+Y)A$

$\therefore X+Y \in V$

2° $\forall X \in V, c \in F$, $A(cX) = cAX = (cX)A$

$\therefore cX \in V$

$1^\circ 2^\circ \Rightarrow V$ is a linear subspace of V

(2). $V = \{X \in \mathbb{R}^{2\times 2} | \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} X = X \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\}$

let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$

$$\begin{cases} a+c = a+2b \Rightarrow c = 2b \\ b+d = a-b \\ 2a-c = c+2d \Rightarrow a = c+d \\ 2b-d = c-d \Rightarrow 2b = d \end{cases} \Rightarrow a = 2c = 2d = 4b \quad \therefore X = \begin{pmatrix} 4b & b \\ 2b & 2b \end{pmatrix}$$

$\therefore B = \{\begin{pmatrix} 4 & 1 \\ 2 & 2 \end{pmatrix}\}$ is a basis of V, $\dim V = 1$

Problem 2. (5 points) The linear space $\mathcal{S}$ contains the following polynomials: $f_1(t) = 1 + 4t - 2t^2 + t^3$, $f_2(t) = -1 + 9t - 3t^2 + 2t^3$, $f_3(t) = -5 + 6t + t^3$, $f_4(t) = 5 + 7t - 5t^2 + 2t^3$. Please give the rank of the quadruple $(f_1(t), f_2(t), f_3(t), f_4(t))$ and its maximal linearly independent set.

Solution:

2. $f_1(t) = 1+4t-2t^2+t^3 \longrightarrow (1,4,-2,1)_B$ Basis $B = \{1, t, t^2, t^3\}$

$f_2(t) = -1+9t-3t^2+2t^3 \longrightarrow (-1,9,-3,2)_B$

$f_3(t) = -5+6t+t^3 \longrightarrow (-5,6,0,1)_B$

$f_4(t) = 5+7t-5t^2+2t^3 \longrightarrow (5,7,-5,2)_B$

$$\begin{pmatrix} 1 & -1 & -5 & 5 \\ 4 & 9 & 6 & 7 \\ -2 & -3 & 0 & -5 \\ 1 & 2 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & -5 & 5 \\ 0 & 13 & 26 & -13 \\ 0 & -5 & -10 & 5 \\ 0 & 3 & 6 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & -5 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 2 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & -5 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ $\mathrm{Rank}\big((f_1(t), f_2(t), f_3(t), f_4(t)\big) = 2$

Maximal linearly independent set $= \{1+4-2t^2+t^3, -1+9t-3t^2+2t^3\}$

Problem 3. (5 points $\times$ 2) For any matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathcal{S}_1 = \{\mathbf{A} \in \mathbb{R}^{n\times n} | \mathbf{A}^T = \mathbf{A}\}$ and $\mathcal{S}_2 = \{\mathbf{A} \in \mathbb{R}^{n\times n} | \mathbf{A}^T = -\mathbf{A}\}$ are two subspaces of $\mathbb{R}^{n\times n}$,

1) Prove that $\mathbb{R}^{n\times n} = \mathcal{S}_1 \oplus \mathcal{S}_2$.
2) If n = 3, please give a basis of $\mathcal{S}_1$ and the dimension of $\mathcal{S}_2$.

Solution:

3.(1) 1° $\forall A \in \mathbb{R}^{n\times n}$, $A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$

let $U = \frac{1}{2}(A+A^T)$, $V = \frac{1}{2}(A-A^T)$, $A = U+V$

$U^T = \frac{1}{2}(A+A^T)^T = \frac{1}{2}(A^T+A) = U$ $\therefore U \in S_1$

$V^T = \frac{1}{2}(A-A^T)^T = \frac{1}{2}(A^T-A) = -V$ $\therefore V \in S_2$

$\therefore A = U+V \in S_1+S_2$

$\therefore \mathbb{R}^{n\times n} \subseteq S_1+S_2$

$\because S_1+S_2 \subseteq \mathbb{R}^{n\times n}$

$\therefore \mathbb{R}^{n\times n} = S_1+S_2$

2° $\forall A \in S_1 \cap S_2$: $A^T = A = -A$

$\therefore A = 0$, $S_1 \cap S_2 = \{0\}$

$\therefore \mathbb{R}^{n\times n} = S_1 \oplus S_2$

(2) $\because A^T = A$

$\therefore$ Basis of S_1: $B = \left\{ \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&1&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0\\1&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&0&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&1\\0&1&0 \end{pmatrix} \right\}$

$\because 9 = \dim(\mathbb{R}^{3\times 3}) = \dim(S_1 \oplus S_2) = \dim S_1 + \dim S_2 = 6 + \dim S_2$

$\therefore \dim S_2 = 3$

III. Four fundamental subspaces

Problem 1. (2 points + 5 points) For an $n \times m$ real matrix $\mathbf{A}$.

1) Determine the relationship of $dim(\mathcal{R}(\mathbf{A}))$, $dim(\mathcal{N}(\mathbf{A}))$, and $rank(\mathbf{A})$.
2) Prove that $\mathcal{N}(\mathbf{A}) \oplus \mathcal{R}(\mathbf{A}^T) = \mathbb{R}^m$.

Solution:

III

1.(1) $dim(R(A))+dim(N(A))=m$

$dim(R(A))=rank(A)$

(2) $\forall u \in N(A) \cap R(A^T)$

$\exists v \in \mathbb{R}^n: u=A^T v$

$u^T=v^T A$

$u^T u=v^T A u=0$

$\therefore u=0$

$\therefore N(A)+R(A^T)=N(A)\oplus R(A^T)$

$\because N(A)\subseteq \mathbb{R}^m, R(A^T)\subseteq \mathbb{R}^m$

$\therefore N(A)\oplus R(A^T)\subseteq \mathbb{R}^m$

$dim(N(A)\oplus R(A^T))=dim(N(A))+dim(R(A^T))$

$=dim(N(A))+dim(R(A))$

$=m$

$\therefore N(A)\oplus R(A^T)=\mathbb{R}^m$

Problem 2. (3 points + 5 points $\times$ 3) Given matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times m}$ and $rank([\mathbf{A}, \mathbf{B}]) = n$.

1) Determine the relationship between $dim(\mathcal{N}(\mathbf{A}^T))$ and $dim(\mathcal{R}(\mathbf{B}))$.
2) If $\mathbf{A}^T\mathbf{B} = 0$, determine the relationship between $\mathcal{N}(\mathbf{A}^T)$ and $\mathcal{R}(\mathbf{B})$.
3) Please determine the rank of $\begin{pmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{B} \end{pmatrix}$.
4) Please determine the **Supremacy** and **Infimum** of the rank of $\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}$ using m or n.

   (All Matrix $\mathbf{A}$'s, and $\mathbf{B}$'s that satisfy the mentioned condition)

Solution:

2. (1) $dim(N(A^T)) \leq dim(R(B))$

Proof: $\because R([A\ B]) = R(A) + R(B)$

$\therefore rank[A, B] = n \Leftrightarrow dim(R(A) + R(B)) = n$

$n = dim(N(A^T)) + dim(R(A^T)) = dim(N(A^T)) + dim(R(A)) = dim(R(A) + R(B)) \leq dim(R(A)) + dim(R(B))$

$\therefore dim(N(A^T)) \leq dim(R(B))$

(2) $N(A^T) = R(B)$

Proof: $\forall y \in R(B), \exists x \in \mathbb{R}^m: y = Bx$

$A^T y = A^T B x = 0$

$\therefore y \in N(A^T)$

$\therefore R(B) \subseteq N(A^T)$

$\because dim(N(A^T)) \leq dim(R(B))$

$\therefore N(A^T) = R(B)$

(3) $rank(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}) = n$

$rank(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}) = rank(A) + rank(B)$

$= dim(R(A)) + dim(R(B))$

$= dim(R(A^T)) + dim(N(A^T))$

$= n$

(4) $\inf rank\begin{pmatrix} A \\ B \end{pmatrix} = \min\{m, n\}$, $\sup rank\begin{pmatrix} A \\ B \end{pmatrix} = n$

$rank\begin{pmatrix} A \\ B \end{pmatrix} = dim(R(A^T) + R(B^T))$

$= dim(R(A^T)) + dim(R(B^T)) - dim(R(A^T) \wedge R(B^T))$

$= dim(R(A^T)) + dim(N(A^T)) - dim(R(A^T) \wedge R(B^T))$

$= n - dim(R(A^T) \wedge R(B^T))$

when $m \geq n$, let $A = (I\ 0)$, $B = 0$, then $rank([A B]) = n$ & $A^T B = 0$

when $m < n$, let $A = (e_1 \cdots e_m)$, $B = (e_{m+1}, \cdots e_n, 0)$, then $rank([A B]) = n$ & $A^T B = 0$

$\therefore \min dim(R(A^T) \wedge R(B^T)) = 0 \Rightarrow \sup rank\begin{pmatrix} A \\ B \end{pmatrix} = n$

when $m \geq n$, $\because A^T B = 0 \Rightarrow \forall i, j = 1, \cdots m: a_i^T b_j = 0 \Rightarrow rank\begin{pmatrix} A \\ B \end{pmatrix} = n$

when $m < n$, let $rank(R(A)) = m_1$, $rank(R(B)) = n - m_1$,

$\max dim(R(A^T) \wedge R(B^T)) = n - m_1 - m + m_1 = n - m \Rightarrow \inf rank\begin{pmatrix} A \\ B \end{pmatrix} = m$

IV. Vector norm and matrix norm

Problem 1. (5 points $\times$ 3) The Frobenius norm of a $\mathbb{R}^{n\times m}$ matrix $\mathbf{A}$ defined as the square root of the sum of the absolute squares of its elements,

$$\|\mathbf{A}\|_F \equiv \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2},$$

it also equal to the square root of the matrix trace of $\mathbf{A}^T\mathbf{A}$, where $\mathbf{A}^T$ is the transpose of $\mathbf{A}$,

$$\|\mathbf{A}\|_F = \sqrt{\mathbf{Tr}(\mathbf{A}^T\mathbf{A})}.$$

1) Show that Frobenius norm is a matrix norm.
   Hint: You may use the Cauchy-Schwarz inequality
   $$\|\mathbf{AB}\|_F \leq \|\mathbf{A}\|_F\|\mathbf{B}\|_F$$
2) The spectral norm of a matrix $\mathbf{A}$ is the largest singular value of $\mathbf{A}$ (the square root of the largest eigenvalue of the matrix $\mathbf{AA}^T$,
   $$\|\mathbf{A}\|_2 = \sqrt{\lambda_{max}(\mathbf{AA}^T)}.$$
   Show that $\|\mathbf{A}\|_2 \leq \|\mathbf{A}\|_F \leq \sqrt{n}\|\mathbf{A}\|_2$
3) Suppose $\mathbf{A} = \mathbf{xy}^T$, where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, show that
   $$\|\mathbf{A}\|_F^2 = \|\mathbf{x}\|_2^2\|\mathbf{y}\|_2^2$$

Solution:

IV
1. (1) 1° $\|A\|_F \geq 0$, $\|A\|_F = 0 \Leftrightarrow A = 0$ $\quad A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}_{n\times m}$

Proof: $\|A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}^2} \geq 0$

$\|A\|_F = 0 \Leftrightarrow \forall i,j: a_{ij} = 0 \Leftrightarrow A = 0$

2° $\forall c \in \mathbb{R}$: $\|cA\|_F = |c|\|A\|_F$

Proof: $\|cA\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}(ca_{ij})^2}$

$= |c|\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}^2}$

$= |c|\|A\|_F$

3° $\forall A, B \in \mathbb{R}^{n\times m}$: $\|A+B\|_F \leq \|A\|_F + \|B\|_F$

Proof: $\|A+B\|_F^2 = Tr((A+B)^T(A+B))$

$= Tr(A^TA) + Tr(B^TB) + 2Tr(A^TB)$

$= \|A\|_F^2 + \|B\|_F^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}b_{ij}$ $\quad$ let $a = (a_{11}, \cdots, a_{1m}, a_{21}, \cdots a_{2m}, \cdots, a_{nm})$, $b = (b_{11}, \cdots b_{1m}, b_{21}, \cdots b_{2m}, \cdots, b_{nm})$

$= \|A\|_F^2 + \|B\|_F^2 + 2ab^T$

$\leq \|A\|_F^2 + \|B\|_F^2 + 2\|a\|_2\|b\|_2$ $\quad$ it's clear that $\|A\|_F = \|a\|_2$, $\|B\|_F = \|b\|_2$

$= \|A\|_F^2 + \|B\|_F^2 + 2\|A\|_F\|B\|_F$

$= (\|A\|_F + \|B\|_F)^2$

$\therefore \|A+B\|_F \leq \|A\|_F + \|B\|_F$

(2). $\|A\|_2 = \sup_{\|x\|_2=1} \|Ax\|_2$

1° $\forall x \in \mathbb{R}^m$, $\|x\|_2=1$: $\|Ax\|_2^2 = \sum_{i=1}^{n} (a_i^T x)^2 \quad a_i = \begin{pmatrix} a_{i1} \\ \vdots \\ a_{im} \end{pmatrix}$

$$\le \sum_{i=1}^{n} (\|a_i\|_2 \|x\|_2)^2$$

$$= \sum_{i=1}^{n} \|a_i\|_2^2$$

$$= \|A\|_F^2$$

$\therefore \|A\|_2 = \sup_{\|x\|_2=1} \|Ax\|_2 \le \|A\|_F$

2° $\forall x \in \mathbb{R}^m$: $\|x\|_2 = 1$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$, $|x_j| \le 1$, $j = 1, \cdots, m$

$\|A\|_F^2 = \sum_{j=1}^{m} \|Ae_j\|_2^2 \quad e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} i$

$$\le \sum_{i=1}^{m} \|A\|_2^2$$

$$= m\|A\|_2^2$$

$\frac{1}{\sqrt{m}} \|A\|_F \le \|A\|_2$

1° 2° $\Rightarrow$ $\|A\|_2 \le \|A\|_F \le \frac{1}{\sqrt{m}} \|A\|_2$ Q.E.D

(3) $A = xy^T$, $x, y \in \mathbb{R}^n$

$\|A\|_F^2 = \|xy\|_F^2$

$$= \left\| \begin{pmatrix} x_1y_1 & \cdots & x_1y_n \\ \vdots & & \vdots \\ x_ny_1 & \cdots & x_ny_n \end{pmatrix} \right\|_F^2$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i y_j)^2$$

$$= \sum_{i=1}^{n} x_i^2 \sum_{j=1}^{n} y_j^2$$

$= \|x\|_2^2 \|y\|_2^2$ Q.E.D.

V. Projector and projection

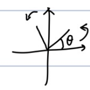
Problem 1. (2 points+5 points $\times$ 3) A rotation matrix $\mathbf{R} \in \mathbb{R}^{n\times n}$ is an orthogonal matrix ($\mathbf{R}\mathbf{R}^T = \mathbf{I_n}$).[2]

1) According to the above definition, find all rotation matrices in $\mathbb{R}^{2\times 2}$.
2) Let $\mathbf{R}_1$ and $\mathbf{R}_2$ be the rotation matrices in $\mathbb{R}^{2\times 2}$, if $\mathbf{R}_1$ is rotation through α_1 and $\mathbf{R}_2$ is rotation through α_2. Consider: is $\mathbf{R}_1\mathbf{R}_2$ the rotation matrix. If the answer is "yes", what is the angle of rotation, or else explain why the answer is "no".
3) For arbitrarily rotation matrix $\mathbf{R} \in \mathbb{R}^{n\times n}$, if $\mathbf{S} = (\mathbf{R} - \mathbf{I_n})(\mathbf{R} + \mathbf{I_n})^{-1}$, show that $\mathbf{S}$ is a skew symmetric matrix ($\mathbf{S}^T = -\mathbf{S}$)
4) If $\mathbf{S} \in \mathbb{R}^{n\times n}$ is a skew symmetric matrix, show that $\mathbf{R} = (\mathbf{I_n} - \mathbf{S})^{-1}(\mathbf{I_n} + \mathbf{S})$ is a rotation matrix.

Solution:

V

1.(1) $R = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \theta \in \mathbb{R}$



(2). Yes and $\alpha = \alpha_1 + \alpha_2$

(3). $S = (R - I_n)(R + I_n)^{-1}$, $RR^T = I_n \Rightarrow R^{-1} = R^T$

$(R + I_n)S = R - I_n$

$R(S - I_n) = -I_n(S + I_n)$

$S - I_n = -R^{-1}(S + I_n)$

$S^T - I_n = -(S + I_n)^T (R^{-1})^T$

$S^T = -(S + I_n)^T R + I_n$

$S^T = -S^T R - R + I_n$

$S^T(I_n + R) = -R + I_n$

$S^T = -(R - I_n)(I_n + R)^{-1} = -S$ Q.E.D.

(4) $R = (I_n - S)^{-1}(I_n + S)$

$R^T = [(I_n - S)^{-1}]^T (I_n + S)^T$

$R^T = (I_n - S)(I_n + S)^{-1}$

$RR^T = (I_n - S)^{-1}(I_n + S)(I_n - S)(I_n + S)^{-1}$

$= (I_n - S)^{-1}(I_n - S^2)(I_n + S)^{-1}$

$= (I_n - S)^{-1}(I_n - S)(I_n + S)(I_n + S)^{-1}$

$= I_n$

$\therefore$ R is a rotation Q.E.D.

[2] $\mathbf{I_n}$ is the identity matrix of size $n \times n$