

Chapter 1: Vector Spaces

Linear Algebra Done Right (4th Edition), by Sheldon Axler

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1A: \mathbb{R}^n and \mathbb{C}^n

Problem 1

Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbb{C}$.

Proof. Let $\alpha = a + bi, \beta = c + di$. Then we have that

$$\begin{aligned}\alpha + \beta &= (a + bi) + (c + di) \\ &= (c + di) + (a + bi) \\ &= \beta + \alpha\end{aligned}$$

□

Problem 3

Show that $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ for all $\alpha, \beta, \gamma \in \mathbb{C}$.

Proof. Choose arbitrary $\alpha, \beta, \gamma \in \mathbb{C}$. Denote $\alpha = a + bi, \beta = c + di, \gamma = e + fi$. Then we have that

$$\begin{aligned}(\alpha\beta)\gamma &= ((ac - bd) + (ad + bc)i)(e + fi) \\ &= (ace - bde - adf - bcf) + (ade + bce + acf - bdf)i\end{aligned}$$

At the same time, we have

$$\begin{aligned}\alpha(\beta\gamma) &= (a + bi)((ce - df) + (cf + de)i) \\ &= (ace - adf - bcf - bde) + (ade + acf + bce - bdf)i\end{aligned}$$

Hence, $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

□

Problem 5

Show that for any $\alpha \in \mathbb{C}$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$.

Proof. Denote $\alpha = a + bi$. By the property of a field, we know there exists unique $c = -a$ and $d = -b$ such that $\beta = c + di$ and $\alpha + \beta = 0$. Suppose for the sake of contradiction that β is not unique, then there exists $c' + d'i$ such that $(a + c') + (b + d')i = 0$ while $c' \neq a$ or $d' \neq d$, contradicting the uniqueness of additive inverse property. □

Problem 8

Find two distinct squared roots of i .

Proof. Suppose $\alpha = a + bi$'s square equals one.

$$(a + bi)^2 = a^2 - b^2 + 2abi = 1$$

Then $|a| = |b|$, $2ab = 1$. So we get the solution to be

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$$

□

Problem 10

Show that $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z}) \forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{F}^n$.

Proof.

$$\begin{aligned} (\mathbf{x} + \mathbf{y}) + \mathbf{z} &= (x_1 + y_1, \dots, x_n + y_n) + (z_1, \dots, z_n) \\ &= (x_1, \dots, x_n) + (y_1 + z_1, \dots, y_n + z_n) \\ &= \mathbf{x} + (\mathbf{y} + \mathbf{z}) \end{aligned}$$

□

Problem 14

Show that $\gamma(\mathbf{x} + \mathbf{y}) = \gamma\mathbf{x} + \gamma\mathbf{y} \forall \gamma \in \mathbb{F}, \mathbf{x}, \mathbf{y} \in \mathbb{F}^n$.

Proof.

$$\begin{aligned} \gamma(\mathbf{x} + \mathbf{y}) &= \gamma(x_1 + y_1, \dots, x_n + y_n) \\ &= (\gamma(x_1 + y_1), \dots, \gamma(x_n + y_n)) \\ &= (\gamma x_1 + \gamma y_1, \dots, \gamma x_n + \gamma y_n) \\ &= \gamma(x_1, \dots, x_n) + \gamma(y_1, \dots, y_n) \\ &= \gamma\mathbf{x} + \gamma\mathbf{y} \end{aligned}$$

□

1B: Definition of Vector Space

Theorem 1. *A vector space is a set that is closed under vector addition and scalar multiplication. It also has the following properties:*

- commutativity
- associativity
- additive identity
- additive inverse
- multiplicative identity
- (scalar) distributive

Notation: \mathbb{F}^S .

Explanation: If S is a set, then \mathbb{F}^S denotes the set of functions from S to \mathbb{F} (scalar function). e.g. $f \in \mathbb{F}^S$.

Comment: \mathbb{F}^S is a vector space; One can think of $f \in \mathbb{F}^n$ as $f: \{1, 2, \dots, n\} \rightarrow \mathbb{F}$.

Problem 1

Prove that $-(-\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$.

Proof. We know $-(-\mathbf{v})$ is the unique additive inverse of $-\mathbf{v}$. At same time by definition, $\mathbf{v} + (-\mathbf{v}) = 0$ and thus by commutativity $(-\mathbf{v}) + \mathbf{v} = 0$. This shows that \mathbf{v} is the unique additive inverse of $(-\mathbf{v})$, and such that $-(-\mathbf{v}) = \mathbf{v}$. \square

Problem 2

Suppose $a \in \mathbb{F}$, $\mathbf{v} \in V$, and $a\mathbf{v} = 0$. Prove that $a = 0$ or $\mathbf{v} = 0$.

Proof. Suppose for the sake of contradiction that $a \neq 0$ and $\mathbf{v} \neq 0$ but $a\mathbf{v} = 0$.

$$\begin{aligned}\mathbf{v} &= 1\mathbf{v} \\ \mathbf{v} &= \frac{1}{a} \cdot a\mathbf{v} \\ \mathbf{v} &= \frac{1}{a}0\end{aligned}$$

This forms a contradiction. \square

Problem 3

Suppose $\mathbf{v}, \mathbf{w} \in V$. Explain why there exists a unique $\mathbf{x} \in V$ such that $\mathbf{v} + 3\mathbf{x} = \mathbf{w}$.

Proof. Suppose there exists \mathbf{x}' which also satisfies the condition. Then we have

$$\mathbf{v} + 3\mathbf{x} = \mathbf{w} \quad \mathbf{v} + 3\mathbf{x}' = \mathbf{w} \quad (1)$$

This gives that $\mathbf{x} = (\mathbf{w} - \mathbf{v})/3 = \mathbf{x}'$ which shows \mathbf{x} is unique. \square

Problem 4

The empty set is not a vector space, why?

Proof. There is no additive identity in the empty set. \square

Problem 7

Suppose S is a nonempty set. Let V^S denotes the set of functions from S to V . Define a natural addition and scalar multiplication on V^S , and show that V^S is a vector space with these definitions.

Proof. Let $f, g \in V^S: S \rightarrow V$. Define the addition and multiplication to be that

$$f + g(x) = f(x) + g(x)$$

We have that

- commutativity: $f + g(x) = f(x) + g(x) = g(x) + f(x) = g + f(x)$
- associativity: $(f + g) + h(x) = f(x) + g(x) + h(x) = f + (g + h)(x)$
- additive identity: Define $0: S \rightarrow 0 \in V$, then $f + 0(x) = 0 + f(x) = f(x)$
- additive inverse: for every $f \in V^S$, define $g(x) = -f(x)$ which exists by the property of vector space and thus we have that $g + f = 0$ for every x and thus that it exists.
- multiplicative identity: same as above
- (scalar) distributive: $a(f + g)(x) = a(f(x) + g(x)) = af(x) + ag(x)$

\square

1C: Subspaces

Definition 2 (subspace). A subset \mathcal{U} of V is called subspace of V if \mathcal{U} is also a vector space with the same additive identity, addition, and scalar multiplication as on V .

Remark 3. The set $\{0\}$ is the smallest subspace of V , and V itself is the largest subspace of V .

Remark 4. The subspace of \mathbb{R}^2 are precisely $\{0\}$, all lines in \mathbb{R}^2 containing the origin, \mathbb{R}^2 .

Definition 5 (Sum of subspace). Suppose V_1, \dots, V_m are subspaces. The sum of them, denoted by $V_1 + \dots + V_m$, is the set of all possible sums of element of V_1, \dots, V_m . Specifically,

$$V_1 + \dots + V_m = \{v_1 + \dots + v_m : v_1 \in V_1, \dots, v_m \in V_m\}$$

Lemma 6. Suppose V_1, \dots, V_m are subspaces of V . Then $V_1 + \dots + V_m$ is the smallest subspace of V containing V_1, \dots, V_m .

Definition 7 (Direct Sum). Suppose V_1, \dots, V_m are subspaces of V .

- The sum $V_1 + \dots + V_m$ is called a direct sum if each element of $V_1 + \dots + V_m$ can be written only as a sum of v_1, \dots, v_m , where each $v_k \in V_k$.
- If $V_1 + \dots + V_m$ is a direct sum, then $V_1 \oplus \dots \oplus V_m$ denotes $V_1 + \dots + V_m$, with \oplus serving as the indication that this is a direct sum.

Example. Suppose V_k is a subspace of \mathbb{F}^n of those vectors whose coordinates are all zero but k -th coordinate. Then we have

$$\mathbb{F}^n = V_1 \oplus \dots \oplus V_m$$

Example (Sum that is not a direct sum). Suppose

$$V_1 = \{(x, y, 0) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$$

$$V_2 = \{(0, 0, z) \in \mathbb{F}^3 : z \in \mathbb{F}\}$$

$$V_3 = \{(0, y, y) \in \mathbb{F}^3 : y \in \mathbb{F}\}$$

Then $F^3 = V_1 + V_2 + V_3$ because for every $(x, y, z) \in \mathbb{F}^3$,

$$(x, y, z) = (x, y, 0) + (0, 0, z) + (0, 0, 0)$$

However, $F^3 \neq V_1 \oplus V_2 \oplus V_3$ since

$$\begin{aligned} (0, 0, 0) &= (0, -1, 0) + (0, 0, -1) + (0, 1, 1) \\ &= (0, 0, 0) + (0, 0, 0) + (0, 0, 0) \end{aligned}$$

Theorem 8. Suppose V_1, \dots, V_m are subspaces of V . Then $V_1 + \dots + V_m$ is a direct sum if and only if the only way to write 0 as a sum of $v_1 + \dots + v_m$, where $v_k \in V_k$, is by taking each v_k to equal 0 .

Theorem 9. Suppose that \mathcal{U} and \mathcal{W} are subspaces of V . Then

$$\mathcal{U} + \mathcal{W} \text{ is a direct sum} \iff \mathcal{U} \cap \mathcal{W} = \{0\}$$

Problem 1

Verify the following examples to be valid subspaces:

1. If $b \in \mathbb{F}$, then

$$\{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4 + b\}$$

is a subspace of \mathbb{F}^4 if and only if $b = 0$.

2. The set of continuous real-valued functions on the interval $[0, 1]$ is a subspace of $\mathbb{R}^{[0,1]}$.
3. The set of differentiable real-valued functions on \mathbb{R} is a subspace of $\mathbb{R}^{\mathbb{R}}$.
4. The set of differentiable real-valued functions f on the interval $(0, 3)$ such that $f'(2) = b$ is a subspace of $\mathbb{R}^{(0,3)}$ if and only if $b = 0$.
5. The set of all sequences of complex numbers with limit 0 is a subspace of \mathbb{C}^∞ .

Proof. 1. $\Rightarrow (0, 0, 0, 0)$ is an element and thus $0 = 0 + b$, $b = 0 \Leftarrow$ Easy to verify.

2. 0 function is cts; cts functions are closed under addition and scalar multiplication.
3. 0 function is differentiable; differentiable functions are closed under addition and scalar multiplication.
4. For this to be closed under addition, one needs to restrict that $f'(2) + g'(2) = b + b = b$ and thus $b = 0$.
5. $\lim_{n \rightarrow \infty} a(S_1 + S_2) = \lim_{n \rightarrow \infty} aS_1 + \lim_{n \rightarrow \infty} aS_2 = 0 + 0 = 0$. At the same time, the 0 sequence has limit 0 .

□

Problem 4

Suppose $b \in \mathbb{R}$. Show that the set of continuous real-valued functions f on the interval $[0, 1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbb{R}^{[0,1]}$ if and only if $b = 0$.

Proof. $\Rightarrow \int_0^1 f + g = \int_0^1 f + \int_0^1 g = 2b = b$ so $b = 0$.
 $\Leftarrow 0$ is in the set; closed under addition/multiplication. \square

Problem 5

Prove that \mathbb{R}^2 is not a subspace of \mathbb{C}^2 over the field \mathbb{C} .

Proof. This does not hold for scalar multiplication since we've defined scalar to be complex numbers. To see this, Let $a = (x + yi) \in \mathbb{C}$ and $\mathbf{z} = (z_1, z_2) \in \mathbb{R}^2$. We have that $a\mathbf{z} = (z_1(x + yi), z_2(x + yi)) \notin \mathbb{R}^2$. \square

Problem 7

Prove or disprove: If \mathcal{U} is a nonempty subset of \mathbb{R}^2 such that \mathcal{U} is closed under addition and under taking additive inverse ($-u \in \mathcal{U}$), then \mathcal{U} is a subspace in \mathbb{R}^2 .

Proof. No. $\mathcal{U} = \{(x_1, x_2) : x_1, x_2 \in \mathbb{Z}\}$. Then $\frac{1}{2}(1, 1) \notin \mathcal{U}$. \square

Problem 9

A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is called *periodic* if there exists a positive number p s.t. $f(x) = f(x + p)$ for all $x \in \mathbb{R}$. Is the set of periodic functions from \mathbb{R} to \mathbb{R} a subspace of $\mathbb{R}^{\mathbb{R}}$?

Proof. No, problem occurs with the addition. Suppose we have $f(x) = f(x + p)$ and $g(x) = g(x + q)$. Then $(f + g)(x) = f(x) + g(x) = f(x + p) + g(x + q) \neq (f + g)(x + l)$ for some fixed l for all p, q . \square

Problem 11

Prove that the intersection of every collection of subspaces of V is a subspace of V .

Proof. Let $\bigcap_i V_i$ denote the collection of subspaces of V . Then we know $0 \in \bigcap_i V_i$. Let $a \in \mathbb{F}, \mathbf{x}, \mathbf{y} \in \bigcap_i V_i$. We have that $a(\mathbf{x} + \mathbf{y}) \in \bigcap_i V_i$ and thus finish the proof. \square

Problem 12

Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Proof. Let V_1, V_2 be two subspaces of V .

\Rightarrow Suppose for the sake of contradiction that there exists $v_1 \in V_1$ s.t. $v_1 \notin V_2$ and $v_2 \in V_2$ s.t. $v_2 \notin V_1$. Then by assumption we have that $v_1 + v_2 \in V_1 \cup V_2$. Here we can also show that $v_1 + v_2 \notin V_1$ because if it does, $v_1 + v_2 + (-v_1) = v_2 \in V_1$. Similarly, $v_1 + v_2 \notin V_2$. Thus we've reached a contradiction.

\Leftarrow This direction is trivial. \square

Problem 14

Suppose

$$\mathcal{U} = \{(x, -x, 2x) \in \mathbb{F}^3 : x \in \mathbb{F}\} \text{ and } \mathcal{W} = \{(x, x, 2x) \in \mathbb{F}^3 : x \in \mathbb{F}\}$$

Describe $\mathcal{U} + \mathcal{W}$.

Proof.

$$(x, -x, 2x) + (y, y, 2y) = (x + y, -x + y, 2(x + y))$$

One can think of this as $\mathcal{U} + \mathcal{W} = \{(a, b, 2a) : a, b \in \mathbb{F}\}$. \square

Problem 15

Suppose \mathcal{U} is a subspace of V , what is $\mathcal{U} + \mathcal{U}$?

Proof. $\mathcal{U} + \mathcal{U} = \mathcal{U}$.

Take $\mathbf{u}_1 + \mathbf{u}_2 \in \mathcal{U} + \mathcal{U}$, then $\mathbf{u}_1 + \mathbf{u}_2 \in \mathcal{U}$. Conversely, take $\mathbf{u} \in \mathcal{U}$, then $\mathbf{u} = \mathbf{u} + 0 \in \mathcal{U} + \mathcal{U}$. \square

Problem 16

Is the operation of addition on the subspaces of V commutative ($\mathcal{U} + \mathcal{W} = \mathcal{W} + \mathcal{U}$)?

Proof. Take $u \in \mathcal{U}, w \in \mathcal{W}$, then $u + w = w + u$, implying the conclusion. \square

Problem 18

Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses?

Proof. Yes, the zero subspace *i.e.* $\{0\}$ is the additive identity. The subspace that have additive inverses is only $\{0\}$. \square

Problem 19

Prove or disprove: If V_1, V_2, \mathcal{U} are subspaces of V such that

$$V_1 + \mathcal{U} = V_2 + \mathcal{U}$$

then $V_1 = V_2$.

Proof. Counterexample: Consider when $\mathcal{U} = V_1 \sqcup V_2$, then the relation holds while $V_1 \neq V_2$. \square

Problem 20

Suppose

$$\mathcal{U} = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$$

Find a subspace $\mathcal{W} \in \mathbb{F}^4$ s.t. $\mathbb{F}^4 = \mathcal{U} \oplus \mathcal{W}$.

Proof. Define $\mathcal{W} = \{(0, a, b, 0) \in \mathbb{F}^4 : a, b \in \mathbb{F}\}$ to be a subspace of \mathbb{F}^4 .

Then first $\mathcal{W} + \mathcal{U} \subseteq \mathbb{F}^4$. Take $(q, w, e, r) \in \mathbb{F}^4$, we can have $(q, w, e, r) = (q, q, r, r) + (0, w - q, e - r, r) \in \mathcal{U} + \mathcal{W}$. We have $\mathbb{F}^4 = \mathcal{U} + \mathcal{W}$. Furthermore, take $(x, x, y, y) \in \mathcal{U}, (0, a, b, 0) \in \mathcal{W}$. For the element to be in the intersection, we need to have $(x, x, y, y) = (0, a, b, 0)$ which implies that $x = y = a = b = 0$ and thus $\mathcal{W} \cap \mathcal{U} = \{0\}$. \square

Problem 21

Suppose

$$\mathcal{U} = \{x, y, x + y, x - y, 2x\} \in \mathbb{F}^5 : x, y \in \mathbb{F}$$

Find a subspace $\mathcal{W} \in \mathbb{F}^5$ s.t. $\mathbb{F}^5 = \mathcal{U} \oplus \mathcal{W}$.

Proof. Define

$$\mathcal{W} = \{(0, 0, m, n, z) : m, n, z \in \mathbb{F}\}$$

Then $(a, b, c, d, e) = (a, b, a + b, a - b, 2a) + (0, 0, c - (a + b), d - (a - b), e - 2a)$. The rest follows exactly as in P20. \square

Problem 23

Prove or disprove: If V_1, V_2, \mathcal{U} are subspaces of V s.t.

$$V = V_1 \oplus \mathcal{U} \text{ and } V = V_2 \oplus \mathcal{U},$$

then $V_1 = V_2$.

Proof. Counterexample: Let $\mathcal{U} = \{(x, x) : x \in \mathbb{F}\}, V_1 = \{(x, 0) : x \in \mathbb{F}\}, V_2 = \{(0, x) : x \in \mathbb{F}\}$. \square

Problem 24

A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is called *even* if $f(-x) = f(x)$ and *odd* if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Let V_e denote the set of real-valued even functions on \mathbb{R} and let V_o denote the set of real-valued odd functions on \mathbb{R} . Show that $\mathbb{R}^{\mathbb{R}} = V_e \oplus V_o$.

Proof. \Leftarrow Trivial direction.

$$\Rightarrow f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

\square