Chapter 2: Finite-Dimensional Vector Spaces

Linear Algebra Done Right (4th Edition), by Sheldon Axler

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2A: Span and Linear Independence

Definition 1 (Linear Combination)**.** *A* linear combination *of a list* $v_1, \dots, v_m$ *of vectors in* V *is a vector of the form*

$$a_1 v_1 + \cdots + a_m v_m$$

where $a_1, \dots, a_m \in \mathbb{F}$.

Definition 2 (Span)**.** *The set of all linear combinations of list of vectors* $v_1, \dots, v_m$ *in* V *is called the* span *of* $v_1, \dots, v_m$*, denoted by* $span(v_1, \dots, v_m)$. *In other words,*

$$span(v_1, \dots, v_m) = \{a_1 v_1 + \cdots a_m v_m \colon a_1, \dots, a_m \in \mathbb{F}\}$$

The span of the empty list $()$ *is defined to be* $\{0\}$.

Theorem 3. *The span of a list of vectors in* V *is the smallest subspace of* V *containing all vectors in the list.*

Definition 4 (Spans)**.** *If* $span(v_1, \dots, v_m)$ *equals* V*, we say the list* $v_1, \dots, v_m$ spans V.

Definition 5 (Finite-dimensional vector space)**.** *A vector space is called* finite-dimensional *if some list of vectors in it spans the space.*

Definition 6 (polynomial)**.** *A function* $p \colon \mathbb{F} \to \mathbb{F}$ *is called a* polynomial *with coefficients in* $\mathbb{F}$ *if there exist* $a_0, \dots, a_m \in \mathbb{F}$ *s.t.*

$$(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

for all $z \in \mathbb{F}$.

$\mathcal{P}(\mathbb{F})$ *is the set of all polynomials with coefficients in* $\mathbb{F}$.

Definition 7 (Linear independence)**.** *A list of vectors* $v_1, \dots, v_m$ *in* V *is called* linearly independent *if the only choice of* $a_1, \dots, a_m \in \mathbb{F}$ *that makes*

$$a_1 v_1 + \cdots + a_m v_m = 0$$

is $a_1 = \cdots = a_m = 0$.

The empty list $()$ *is also declared to be linearly independent.*

Lemma 8. *Suppose* $v_1, \dots v_m$ *is a linearly dependent list in* V. *Then there exists* $k \in \{1, 2, \dots, m\}$ *s.t.*

$$v_k \in span(v_1, \dots, v_{k-1})$$

Furthermore, if k *satisfies the condition above and the* k^{th} *term is removed from* $v_1, \dots v_m$*, then the span of the remaining list equals* $span(v_1, \dots, v_m)$.

Lemma 9 (length of linearly independent list $\leq$ length of spanning list)**.** *In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.*

Problem 1

Find a list of four distinct vectors in $\mathbb{F}^3$ whose span equals

$$\{(x, y, z) \in \mathbb{F}^3 : x + y + z = 0\}$$

Proof. Example: $(1, 0, -1), (0, -1, 1), (1, -1, 0), (1, -2, 1)$. $\square$

Problem 2

Prove or give a counterexample: If v_1, v_2, v_3, v_4 spans V, then the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also spans V.

Proof. Take any $v \in V$, then we have $v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + (a_1 + a_2 + a_3 + a_4)v_4$. Conversely, any linear combination of these vectors still belong to V. $\square$

Problem 3

Suppose $v_1, \ldots, v_m$ is a list of vectors in V. For $k \in \{1, \ldots, m\}$, let

$$w_k = v_1 + \cdots + v_k$$

Show that $\text{span}(v_1, \ldots, v_m) = \text{span}(w_1, \ldots, w_m)$

Proof. Take $v = \sum_{i=1}^m a_iv_i$ from l.h.s, then we can write

$$\begin{aligned}
v &= a_1v_1 + \cdots a_mv_m \\
&= (a_1 - a_m)v_1 + \cdots (a_{m-1} - a_m)v_{m-1} + a_mw_m \\
&= (a_1 - a_m - a_{m-1})v_1 + \cdots (a_{m-2} - a_{m-1} - a_m)v_{m-2} + (a_{m-1} - a_m)w_{m-1} + a_mw_m \\
&= \sum_{i=1}^m \left(a_i - \sum_{j=i+1}^m a_j \right) w_i \in \text{r.h.s}
\end{aligned}$$

Conversely, take $w = \sum_{i=1}^m b_iw_i = \sum_{i=1}^m (b_i \sum_{j=1}^i c_j)v_j \in$ l.h.s.. $\square$

Problem 4

(a) Show that a list of length one in a vector space is linearly independent if and only if the vector in the list is not 0.
(b) Show that a list of length two in a vector space is linearly independent if and only if neither of the two vectors in the list is a scalar multiple of the other.

Proof. (a) In order for the only way to write $av = 0$ is to ensure $v \neq 0$.

(b) $av_1 + bv_2 = 0$. $\Rightarrow$ the only solution is $a = b = 0$ so v_1 cannot be a multiple of v_2; vice versa. $\Leftarrow$ same reason. $\square$

Problem 8

Suppose v_1, v_2, v_3, v_4 is linearly independent in V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is also linearly independent.

Proof.

$$a_1(v_1-v_2)+a_2(v_2-v_3)+a_3(v_3-v_4)+a_4v_4 = a_1v_1+(-a_1+a_2)v_2+(-a_2+a_3)v_3+(-a_3+a_4)v_4$$

The only solution is $a_1 = 0, -a_1 + a_2 = 0, -a_2 + a_3 = 0, -a_3 + a_4 = 0$. $\square$

Problem 9

Prove or give a counter example: If $v_1, \dots, v_m$ is a linearly independent list of vectors in V, then

$$5v_1 - 4v_2, v_2, v_3, \dots, v_m$$

is also linearly independent.

Proof.

$$\begin{aligned} &a_1(5v_1 - 4v_2) + a_2v_2 + \dots a_mv_m \\ &= 5a_1v_1 + (-4a_1 + a_2)v_2 + \dots a_mv_m \end{aligned}$$

The only solution is $5a_1 = -4a_1 + a_2 = \dots = a_m = 0$. $\square$

Problem 10

Prove or give a counterexample: If $v_1, \dots, v_m$ is a linearly independent list of vectors in V and $\lambda \in \mathbb{F}$ with $\lambda \neq 0$, then $\lambda v_1, \dots, \lambda v_m$ is linearly independent.

Proof.

$$\sum_i^m \lambda a_i v_i = 0$$

The only solution is $a_i = 0$ for all i. $\square$

Problem 12

Suppose $v_1, \dots, v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \dots, v_m + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \dots, v_m)$.

Proof. We know the only solution to $\sum_i^m a_i v_i$ is all $a_i = 0$. Now we have that non-zero a_i for solving $\sum_i^m a_i v_i + \sum_i^m a_i w = 0$. Thus $w = \frac{\sum_i^m a_i v_i}{\sum_i^m a_i}$ which completes the proof. $\square$

Problem 13

Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Show that

$$v_1, \ldots, v_m, w \text{ is linearly independent } \iff w \notin \text{ span}(v_1, \ldots, v_m)$$

Proof. $\Rightarrow$ By contradiction, if w in the span then it can be written as linear combination for some nonzero a_i and thus they are linearly dependent.

$\Leftarrow$ Similarly. $\square$

Problem 17

Prove that V is infinite-dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $v_1, \ldots, v_m$ is linearly independent for every positive integer m.

Proof. $\Rightarrow$ There doesn't exist any finite list of vectors that span the space. For the sake of contradicting assumes for every sequence $v_1, \ldots$ of vectors in V $\exists$ m such that $v_1, \ldots, v_m$ is linearly dependent. Then this means we can construct the basis for the vector space as follows: select $v_i \in V$ s.t. v_i and $v_1, \ldots, v_{i-1}$ are linearly independent. This means that there exits m s.t. $v_1, \ldots, v_m$ that spans V, forming a contradiction.

$\Leftarrow$ Suppose for contradiction. Then there exists a span, contradicting the linear independence claim for every m. $\square$

Problem 18

Prove $\mathbb{F}^\infty$ is infinite-dimensional.

Proof. We apply Problem 17. Construct v_i to be the vector that has 1 on the i-th coordinate and 0 elsewhere. clearly $v_1, v_2, \ldots$ are s.t. $v_1, \ldots, v_m$ is linearly independent for every positive integer m. $\square$

2B: Bases

Definition 10 (basis)**.** *A* basis *of V is a list of vectors in V that is linearly independent and spans V.*

Theorem 11 (criterion for basis)**.** *A list $v_1, \ldots, v_m$ of vectors in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form*

$$v = \sum_{i}^{m} a_i v_i$$

where $a_i \in \mathbb{F}$.

Lemma 12 (every spanning list contains a basis)**.** *Every spanning list in a vector space can be reduced to a basis of the vector space.*

Lemma 13. *Every finite-dimensional vector space has a basis.*

Lemma 14. *Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.*

Lemma 15. *Suppose V is finite-dimensional and $\mathcal{U}$ is a subspace of V. Then there is a subspace W of V such that $V = \mathcal{U} \oplus W$.*

Problem 1

Find all vector spaces that have exactly one basis.

Proof. The only answer is $\{0\}$. Otherwise, for any basis v one can get av for $a \neq 0, a \neq 1$. $\square$

Problem 4

1. Let $\mathcal{U}$ be the subspace of $\mathbb{C}^5$ defined by $\mathcal{U} = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : 6z_1 = z_2, z_3 + 2z_4 + 3z_5 = 0\}$ Find a basis of $\mathcal{U}$.
2. Extend the basis to a basis in $\mathbb{C}^5$.
3. Find a subspace $\mathcal{W}$ of $\mathbb{C}^5$ s.t. $\mathbb{C}^5 = \mathcal{U} \oplus \mathcal{W}$.

Proof.

1. $(z_1, 6z_1, -2z_4 - 3z_5, z_4, z_5) \Rightarrow \{$(1,6,0,0,0), (0,0,-2,1,0), (0,0,-3,0,1)$\}$
2. $\{(1, 6, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, -3, 0, 1), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0)\}$
3. $\mathcal{W} = \mathrm{span}(\{(0, 1, 0, 0, 0), (0, 0, 1, 0, 0)\})$

$\square$

Problem 5

Suppose V is finite-dimensional and $\mathcal{U}, \mathcal{W}$ are subspaces of V such that $V = \mathcal{U} + \mathcal{W}$. Prove that there exists a basis of V consisting of vectors in $\mathcal{U} \cup \mathcal{W}$.

Proof. Let $\{v_i\}_{i=1}^m$ denote the basis for the vector space V. By definition we have $v_i = u_i + w_i$ for some u_i, w_i. Then we have the spanning set of the vector space V $\sum_i^m a_i(u_i + w_i)$, which can be reduced to a basis by the lemma. $\square$

Problem 7

Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of V.

Proof. We know v_1, v_2, v_3, v_4 is linearly independent and spans V.

$a_1(v_1+v_2)+a_2(v_2+v_3)+a_3(v_3+v_4)+a_4v_4 = a_1v_1+(a_1+a_2)v_2+(a_2+a_3)v_3+(a_3+a_4)v_4$

which shows the linear independence. For proving spanning, let $v \in V$ then

$$v = \sum_{i=1}^{4} a_i v_i = a_1(v_1 + v_2) + (a_2 - a_1)(v_2 + v_3) + (a_3 - a_2)(v_3 + v_4) + (a_4 - a_3)v_4$$

$\square$

Problem 8

Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and $\mathcal{U}$ is a subspace of V such that $v_1, v_2 \in \mathcal{U}$ and $v_3 \notin \mathcal{U}$ and $v_4 \notin \mathcal{U}$, then v_1, v_2 is a basis of $\mathcal{U}$.

Proof. Take $V = \mathbb{R}^4$ and the standard basis. Consider $\mathcal{U} = \{(x_1, x_2, x_3, kx_3)\}$, then we disprove the claim. $\square$

Problem 10

Suppose $\mathcal{U}$ and $\mathcal{W}$ are subspaces of V s.t. $V = \mathcal{U} \oplus \mathcal{W}$. Suppose also that $u_1, \ldots u_m$ is a basis of $\mathcal{U}$ and $w_1, \ldots, w_n$ is a basis of $\mathcal{W}$. Prove that

$$u_1, \ldots, u_m, w_1, \ldots, w_n$$

is a basis of V.

Proof. We know that this set is linearly independent (otherwise violating the direct sum assumption) so it sufficies to prove the spanning. Let $v \in V$, then $v = u + w = \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j w_j$. $\square$

2C: Dimension

Lemma 16 (basis length does not depend on basis)**.** *Any two bases of a finite-dimensional vector space have the same length.*

Definition 17 (dimension)**.**

- *The* dimension *of a finite-dimensional vector space is the length of any basis of the vector space.*
- *The dimension of a finite-dimensional vector space V is denoted by dim V.*

Corollary 18. *If V is finite-dimensional and $\mathcal{U}$ is a subspace of V, then* $\dim \mathcal{U} \leq \dim V$.

Corollary 19. *Suppose V is finite-dimensional. Then every linearly independent list of vectors in V of length* $\dim V$ *is a basis of V.*

Corollary 20. *Suppose that V is finite-dimensional and $\mathcal{U}$ is a subspace of V such that* $\dim \mathcal{U} = \dim V$. *Then $\mathcal{U} = V$.*

Corollary 21. *Suppose that V is finite-dimensional. Then every spanning list of vectors in V of length dim V is a basis of V.*

Theorem 22 (dimension of a sum)**.** *If V_1 and V_2 are subspaces of a finite-dimensional vector space, then*

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$$

Problem 1

Show that the subspaces of $\mathbb{R}^2$ are precisely $\{0\}$, all lines in $\mathbb{R}^2$ containing the origin and $\mathbb{R}^2$.

Proof. We know $\dim(\mathbb{R}^2) = 2$ so the subspace dimension is either 0 ($\{0\}$) or 1 (then this means it has to be lines crossing the origin). $\square$

Problem 5

(a) Let $\mathcal{U} = \{p \in \mathcal{P}_4(\mathbb{F})\colon p(2) = p(5)\}$. Find a basis of $\mathcal{U}$.
(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbb{F})$.
(c) Find a subspace $\mathcal{W}$ of $\mathcal{P}_4(\mathbb{F})$ s.t. $\mathcal{P}_4(\mathbb{F}) = \mathcal{U} \oplus \mathcal{W}$.

Proof. (a) This means that

$$a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 = a_0 + 5a_1 + 25a_2 + 125a_3 + 625a_4$$

Solving this gives that $a_1 = -7a_2 - 39a_3 - 203a_4$. So we can write that

$$\begin{aligned} p(x) &= a_0 + (-7a_2 - 39a_3 - 203a_4)x + a_2x^2 + a_3x^3 + a_4x^4 \\ &= a_0 + a_2(x^2 - 7x) + a_3(x^3 - 39x) + a_4(x^4 - 203x) \end{aligned}$$

The basis now becomes $\{1, x^2 - 7x, x^3 - 39x, x^4 - 203x\}$
(b) $\{1, x, x^2 - 7x, x^3 - 39x, x^4 - 203x\}$
(c) $\mathcal{W} = \{x\}$ $\square$

Problem 8

Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that

$$\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m - 1$$

Proof. We claim that

$$v_{i+1} - v_i \in \operatorname{span}(v_1 + w, \ldots, v_m + w) \text{ for all } m \geq i \geq 2$$

as $v_{i+1} - v_i = (v_{i+1} + w) - (v_i + w)$. Thus $v_2 - v_1, \ldots, v_m - v_{m-1}$ is in the $\operatorname{span}(v_1 + w, \ldots, v_m + w)$. We've proved that $v_2 - v_1, \ldots, v_m - v_{m-1}$ is linearly independent and this list has $m - 1$ vectors and thus we've proved the claim. $\square$

Problem 9

Suppose m is a positive integer and $p_0, p_1, \ldots, p_m \in \mathcal{P}(\mathbb{F})$ are such that each p_k has degree k. Prove that $p_0, p_1, \ldots, p_m$ is a basis of $\mathcal{P}_m(\mathbb{F})$.

Proof. It's easy to see that this list is linearly independent. Take any element in $\mathcal{P}_m(\mathbb{F})$, we can decompose that by degrees and get each component to be some multiple of $p_i's$. $\square$

Problem 10

Suppose m is a positive integer. For $0 \leq k \leq m$, let

$$p_k(x) = x^k(1-x)^{m-k}$$

Show that $p_0, \ldots, p_m$ is a basis of $\mathcal{P}_m(\mathbb{F})$.

Proof. It suffices to prove that $p_k(x)$ are linearly independent. We know from binomial theorem that

$$(x+y)^m = \sum_{i=0}^{m} \binom{m}{i} x^i y^{m-i}$$

Applying this identity here we get that

$$p_k(x) = x^k \sum_{i=0}^{m-k} \binom{m-k}{i} 1^{m-i}(-1)^i x^i = \sum_{i=0}^{m-k} a_i x^{k+i}$$

where $a_i = \binom{m-k}{i}(-1)^i$. For the polynomial to be identically 0 ($\sum_{k=0}^{m} c_k p_k(x) = 0$), for each power x^i we need the coefficient to be 0:

$$\sum_{k=0}^{\min(i,m)} c_k \binom{m-k}{i-k} (-1)^{i-k} = 0$$

We can prove the only solution for this is all $c_k = 0$ by induction on m.

The base case is trivial. Assume the statement holds for $k = m-1$, then we try to prove for $p_m(x)$, where we have $P(x) - c_m p_m(x) = \sum_{k=0}^{m-1} c_k p_k(x) = 0$. This means that $c_m = 0$ and we've proved the linear independence. $\square$

Problem 11

Suppose $\mathcal{U}$ and $\mathcal{W}$ are both four-dimensional subspaces of $\mathbb{C}^6$. Prove that there exist two vectors in $\mathcal{U} \cap \mathcal{W}$ such that neither of these vectors is a scalar multiple of the other.

Proof. $\dim(\mathcal{U} \cap \mathcal{W}) = \dim(\mathcal{U}) + \dim(\mathcal{W}) - \dim(\mathcal{U} + \mathcal{W}) \geq 8 - 6 = 2$

So there must exist two linearly independent vectors in the intersection. $\square$

Problem 12

Suppose that $\mathcal{U}$ and $\mathcal{W}$ are subspaces of $\mathbb{R}^8$ such that $\dim \mathcal{U} = 3, \dim \mathcal{W} = 5$ and $\mathcal{U} + \mathcal{W} = \mathbb{R}^8$. Prove that $\mathbb{R}^8 = \mathcal{U} \oplus \mathcal{W}$.

Proof. Similar to problem 11, we can get that $\dim(\mathcal{U} \cap \mathcal{W}) = 0$. □

Problem 14

Suppose V is a ten-dimensional vector space and V_1, V_2, V_3 are subspaces of V with $\dim V_1 = \dim V_2 = \dim V_3 = 7$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

Proof.

$$\dim((V_1 \cap V_2) + V_3) = \dim(V_1 \cap V_2) + \dim V_3 - \dim(V_1 \cap V_2 \cap V_3)$$

and also that

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$$

This gives that

$$\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_2) - \dim((V_1 \cap V_2) + V_3)$$

Note that from question we know $\dim V_1 + \dim V_2 + \dim V_3 = 21 > 20 = 2 \dim V$

Hence we know

$$\dim(V_1 \cap V_2 \cap V_3) > (\dim(V) - \dim(V_1 + V_2)) + (\dim(V) - \dim((V_1 \cap V_2) + V_3)) > 0$$

We've thus proved the claim. □

Problem 16

Suppose V is finite-dimensional and $\mathcal{U}$ is a subspace of V with $\mathcal{U} \neq V$. Let $n = \dim V$ and $m = \dim \mathcal{U}$. Prove that there exist $n - m$ subspaces of V, each of dimension $n - 1$, whose intersection equals $\mathcal{U}$.

Proof. To show existence, we can start with the basis for $\mathcal{U} : u_1, \ldots, u_m$. We extend the basis to $V : \{u_1, \ldots, u_m, v_1, \ldots, v_{n-m}\} \coloneqq K$. Construct the subspace $V_i = \operatorname{span}\{K \setminus \{v_i\}\}$. Then we know that $\bigcap_i V_i = \operatorname{span}\{\{u_1, \ldots, u_m\}\}$. □

Problem 17

Suppose that $V_1, \ldots, V_m$ are finite-dimensional subspaces of V. Prove that $V_1 + \cdots + V_m$ is finite-dimensional and

$$\dim(V_1 + \cdots + V_m) \leq \dim V_1 + \cdots + \dim V_m$$

Proof. We prove by induction on m. The base case is trivial. Assume the statement hold for k, then for $k+1$, we have that (Denote $V_1 + \ldots + V_k = M_k$)

$$\dim(M_k + V_{k+1}) \leq \dim(V_1) + \cdots + \dim(V_{k+1})$$

which is finite. □

Problem 18

Suppose V is finite-dimensional with $\dim V = n \geq 1$. Prove that there exist one-dimensional subspaces $V_1, \ldots, V_n$ of V such that

$$V = V_1 \oplus \cdots V_n$$

Proof. We know there are basis $\{v_1, \ldots, v_n\}$ for V. Hence we can construct

$$V_i = \operatorname{span}\{v_i\}$$

□

Problem 19

Prove or give a counter example:

$$\begin{aligned}\dim(V_1 + V_2 + V_3) = &\dim V_1 + \dim V_2 + \dim V_3 \\ &- \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) \\ &+ \dim(V_1 \cap V_2 \cap V_3)\end{aligned}$$

Proof. We know that

$$\begin{aligned}&\dim((V_1 + V_2) + V_3) = \dim(V_1 + V_2) + \dim(V_3) - \dim((V_1 + V_2) \cap V_3) \\ &= \dim(V_1) + \dim(V_2) + \dim(V_3) - \dim(V_1 \cap V_2) - \dim((V_1 + V_2) \cap V_3)\end{aligned}$$

Here we can get that

$$\begin{aligned}\dim((V_1 + V_2) \cap V_3) &= \dim((V_1 \cap V_3) + (V_2 \cap V_3)) \\ &= \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3) - \dim(V_1 \cap V_2 \cap V_3)\end{aligned}$$

By substituting this back, we can get the desired solution. □