COMP261 Lecture 21

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Data Compression 3 (or Using Predictions 1) :

Arithmetic Coding

Victoria
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o te Ūpoko o te Ika a Māui
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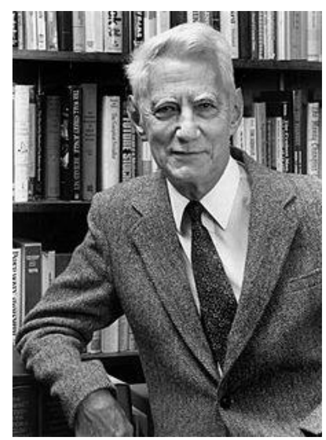
the problem: encoding data succinctly

- Opportunity #1: some symbols are used more

- Claude Shannon proved (1940's) there's a way to transmit symbol strings from alphabet X with an average of $H(X)$ bits/symbol, called the *entropy*:

$$H(X) = \sum_i P_i \log_2 \frac{1}{P_i}$$

- He showed it was possible, but not how to do it!
- Huffman Coding gets quite close

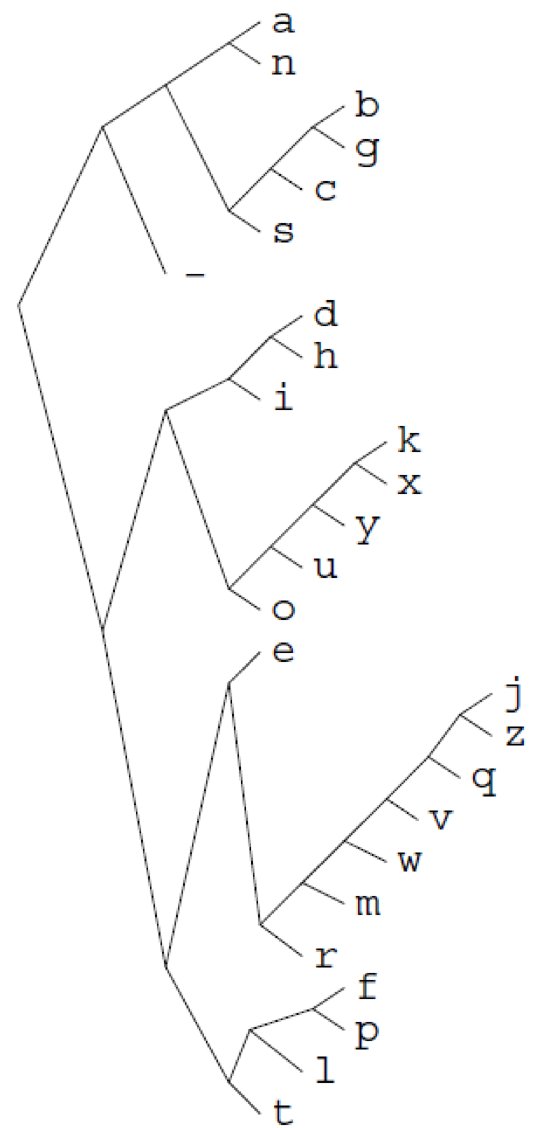


x	$P(x)$
a	0.0575
b	0.0128
c	0.0263
d	0.0285
e	0.0913
f	0.0173
g	0.0133
h	0.0313
i	0.0599
j	0.0006
k	0.0084
l	0.0335
m	0.0235
n	0.0596
o	0.0689
p	0.0192
q	0.0008
r	0.0508
s	0.0567
t	0.0706
u	0.0334
v	0.0069
w	0.0119
x	0.0073
y	0.0164
z	0.0007
–	0.1928

Huffman recap

- send each symbol as soon as it occurs (*symbol* code)
- optimal, given this restriction
- but wastes bits
- drop the restriction? (→ *stream* codes)

a_i	p_i	$\log_2 \frac{1}{p_i}$	l_i	$c(a_i)$
a	0.0575	4.1	4	0000
b	0.0128	6.3	6	001000
c	0.0263	5.2	5	00101
d	0.0285	5.1	5	10000
e	0.0913	3.5	4	1100
f	0.0173	5.9	6	111000
g	0.0133	6.2	6	001001
h	0.0313	5.0	5	10001
i	0.0599	4.1	4	1001
j	0.0006	10.7	10	1101000000
k	0.0084	6.9	7	1010000
l	0.0335	4.9	5	11101
m	0.0235	5.4	6	110101
n	0.0596	4.1	4	0001
o	0.0689	3.9	4	1011
p	0.0192	5.7	6	111001
q	0.0008	10.3	9	110100001
r	0.0508	4.3	5	11011
s	0.0567	4.1	4	0011
t	0.0706	3.8	4	1111
u	0.0334	4.9	5	10101
v	0.0069	7.2	8	11010001
w	0.0119	6.4	7	1101001
x	0.0073	7.1	7	1010001
y	0.0164	5.9	6	101001
z	0.0007	10.4	10	1101000001
–	0.1928	2.4	2	01



the problem: encoding data succinctly

- Opportunity #1: some symbols are used more
- Opportunity #2: the sequence isn't random
  - → Lempel-Ziv
  - → Arithmetic Coding, based on rather different ideas

- *reaches* the Shannon limit, for random ordered symbols, and
- *in conjunction with a predictive language model,* it does better still

x	$P(x)$
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z	0.0007
—	0.1928

think of bit strings as intervals

0.00

0.25

0.50

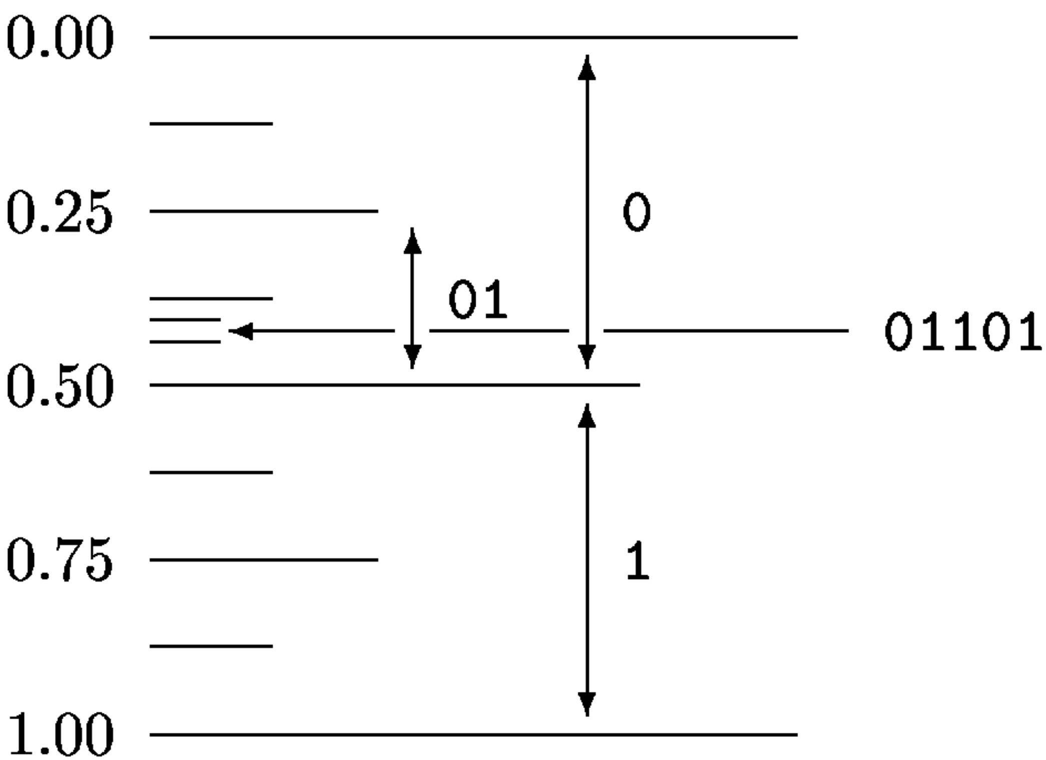
0.75

1.00

<table>
<tr><td rowspan="8">0</td><td rowspan="4">00</td><td rowspan="2">000</td><td>0000</td><td rowspan="16">The total symbol code budget</td></tr>
<tr><td>0001</td></tr>
<tr><td rowspan="2">001</td><td>0010</td></tr>
<tr><td>0011</td></tr>
<tr><td rowspan="4">01</td><td rowspan="2">010</td><td>0100</td></tr>
<tr><td>0101</td></tr>
<tr><td rowspan="2">011</td><td>0110</td></tr>
<tr><td>0111</td></tr>
<tr><td rowspan="8">1</td><td rowspan="4">10</td><td rowspan="2">100</td><td>1000</td></tr>
<tr><td>1001</td></tr>
<tr><td rowspan="2">101</td><td>1010</td></tr>
<tr><td>1011</td></tr>
<tr><td rowspan="4">11</td><td rowspan="2">110</td><td>1100</td></tr>
<tr><td>1101</td></tr>
<tr><td rowspan="2">111</td><td>1110</td></tr>
<tr><td>1111</td></tr>
</table>

...and ↔ think of intervals as bit-strings

- the interval corresponding to n-bits has width $1/2^n$
- to specify interval of size α, we will need about $\log_2 1/\alpha$ bits



eg: if α=**1/8**
we need
$\log_2 1/\alpha$ = **3 bits**

next slide considers sending symbols in a simple alphabet of just {a,b, □}

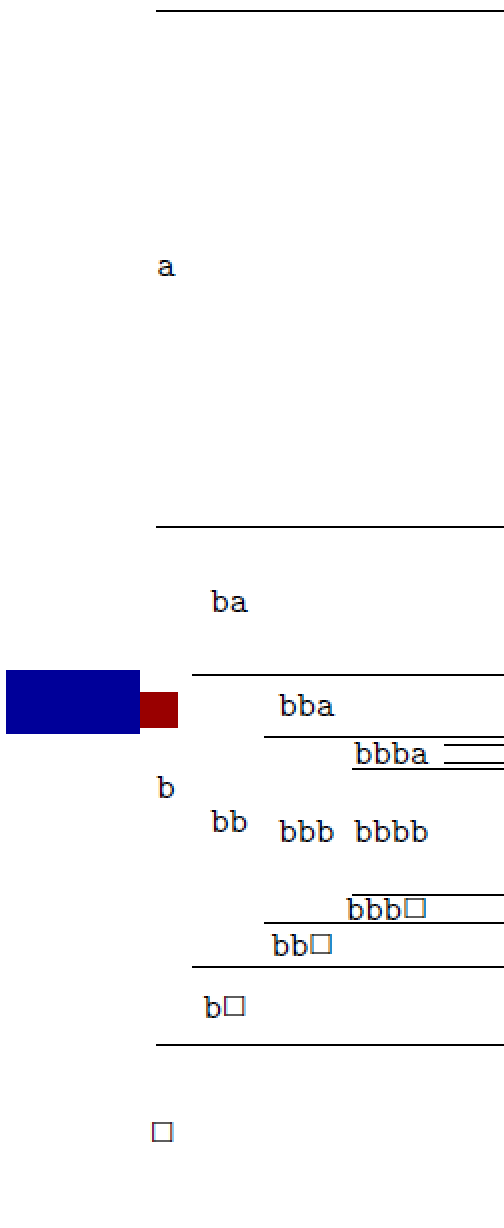
to send symbol string, send *interval* (as bit-string

- To send a string, I recursively partition up the interval [0,1] into segments...

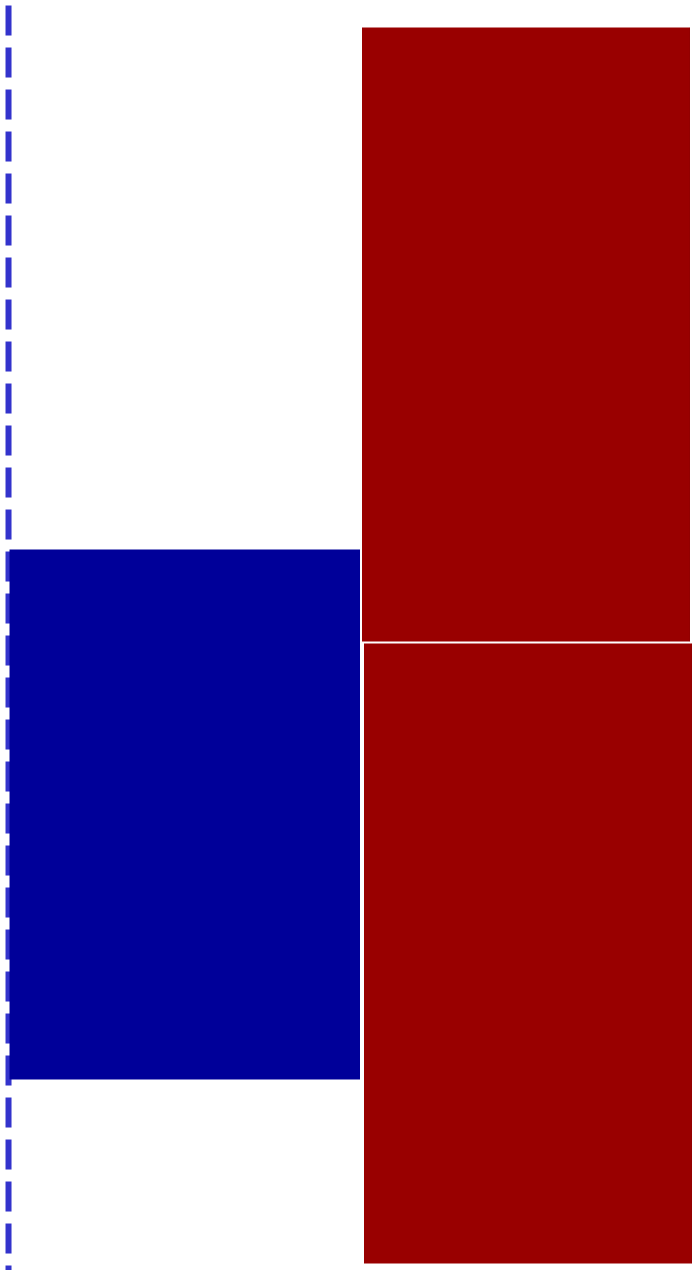
  (but <u>don't worry</u> about the partitioning scheme just yet!)

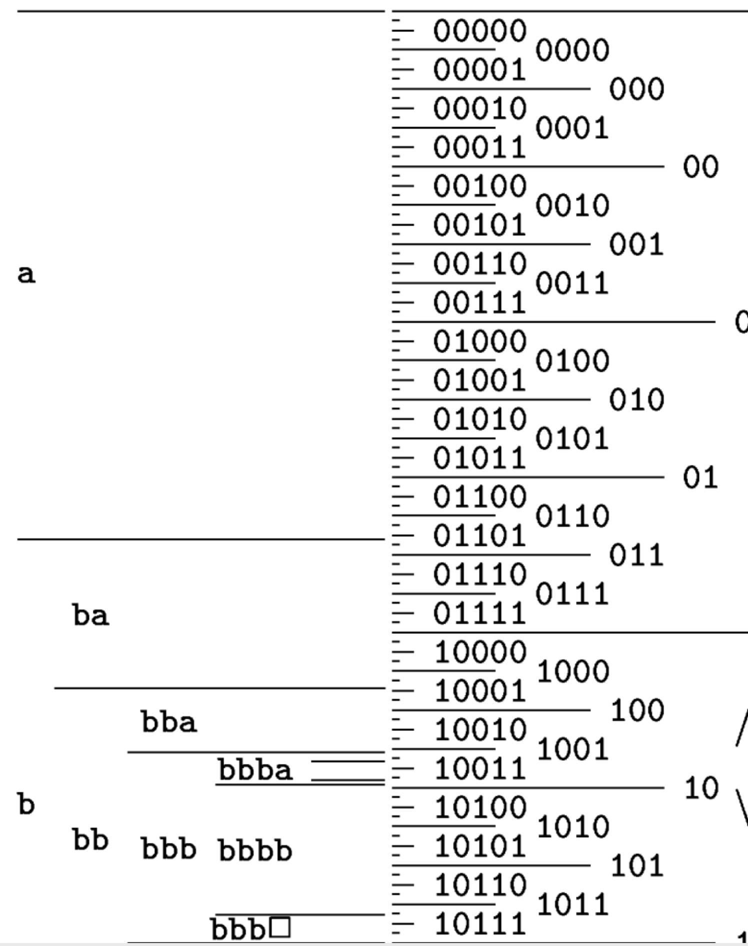
- I send you the binary string that corresponds to the **largest interval *enclosed by the string I want to send***.

- You should be able to *decode* this, provided you use <u>the same scheme</u> for partitioning as I did!



on-the-fly encoding: transmitting bbba□





b: not wholly enclosed by 0 or 1

(i.e. could be 01, 10, or 11)

→Don't transmit anything yet

on-the-fly encoding

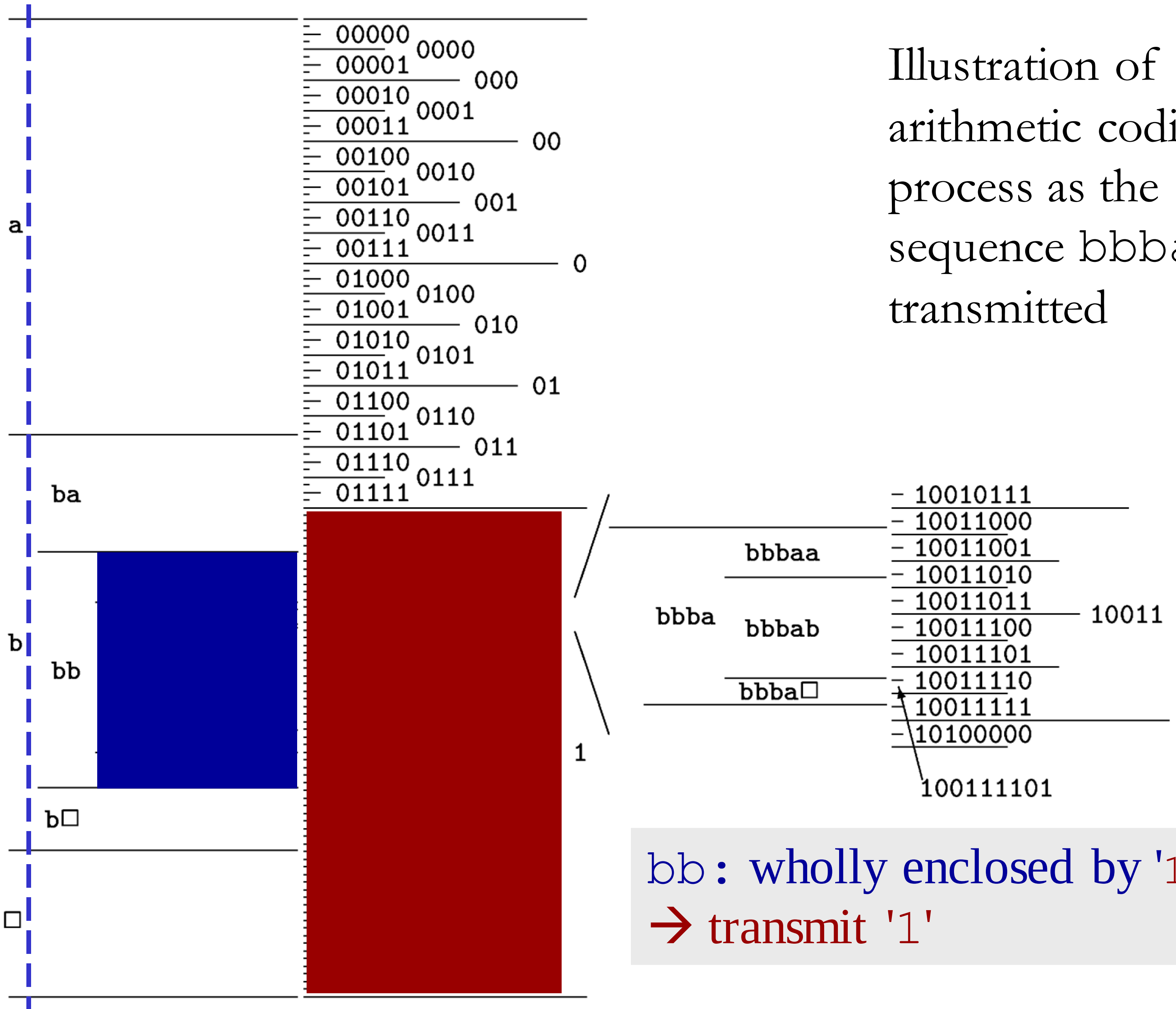


Illustration of the arithmetic coding process as the sequence bbba□ is transmitted

bb: wholly enclosed by '1' range,
→ transmit '1'

on-the-fly encoding

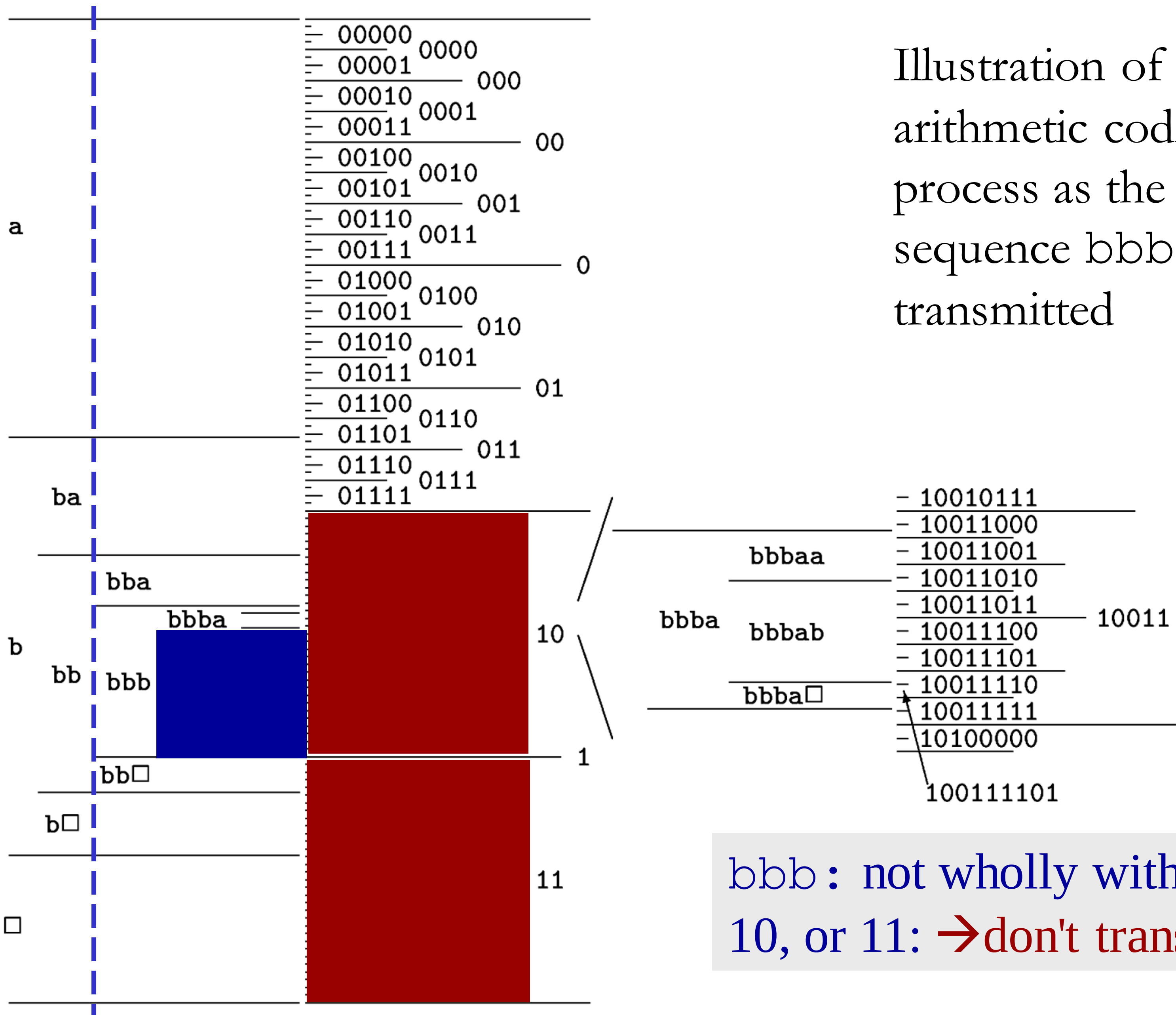


Illustration of the arithmetic coding process as the sequence bbba□ is transmitted

bbb: not wholly within either 10, or 11: →don't transmit yet

on-the-fly encoding

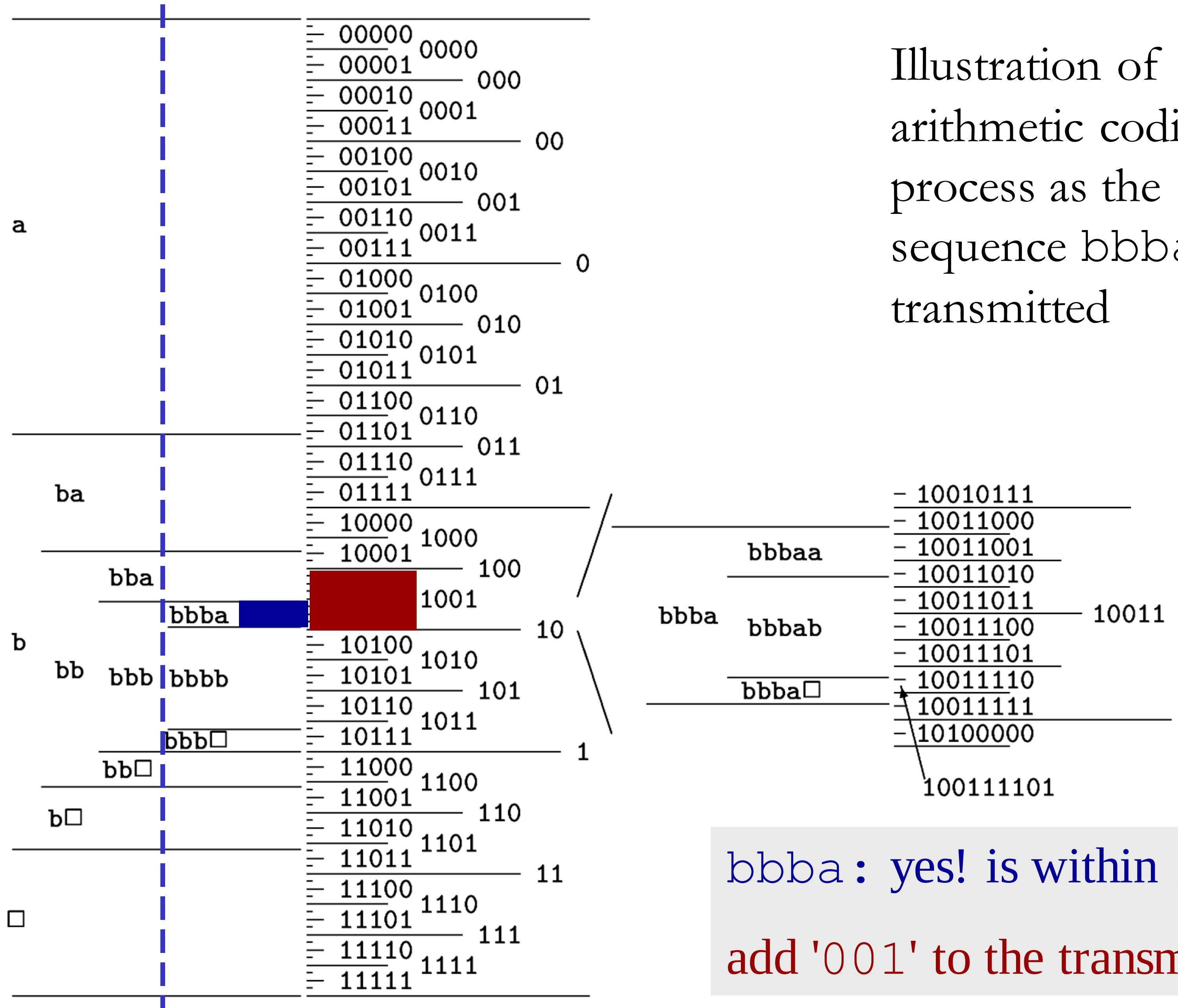


Illustration of the arithmetic coding process as the sequence bbba□ is transmitted

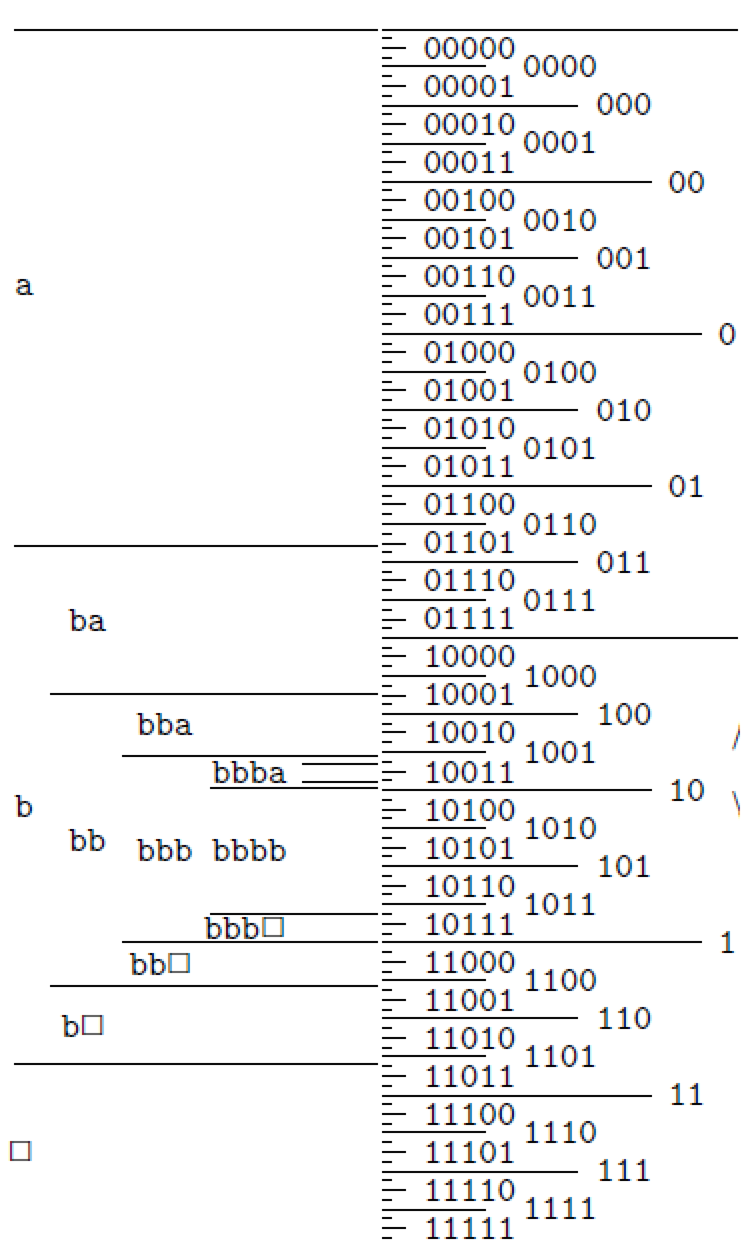
bbba: yes! is within 1001, so add '001' to the transmission

on-the-fly decoding

The first '1' arrives.

Could be b, or □.

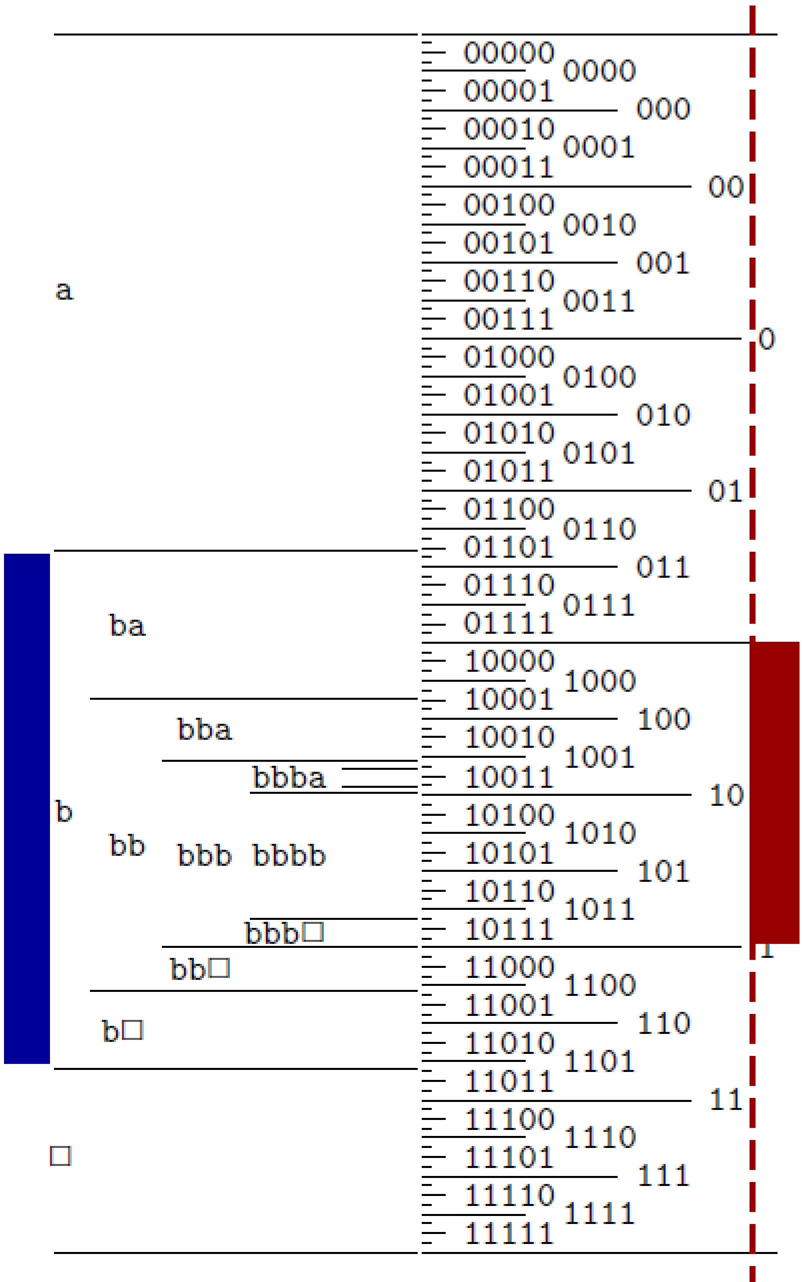
Don't emit anything yet



on-the-fly decoding

'10' has arrived

this is wholly enclosed by the 'b' interval, so now we can safely emit 'b'



3. what's the *best* partitioning scheme?

- suppose our scheme gives string **S** an interval of size α_s
- this is going to require $\log_2 1/\alpha_s$ bits
- expected message length will be $\sum_s P_s \log_2 \frac{1}{\alpha_s}$
- If we set $\alpha_s = P_s$ this matches the Shannon limit! (and any other scheme is worse)

So *this is the code that Shannon knew must exist*!

4. best partitioning for an entire string?

- thought: is there a recursive way to do the partitioning, which gives the right "real estate" to a whole **string**, not just individual symbols?
- remarkably, yes!
- based on the recursive "chain rule" of probabilities...

$$P(s_1, s_2) = P(s_1)P(s_2 \mid s_1)$$

$$P(s_1, s_2, s_3) = P(s_1)P(s_2 \mid s_1)P(s_3 \mid s_1, s_2)$$

$$\vdots$$

← details ***not*** examinable

- to do it, we need to build a predictive model of the language - Machine Learning, 400 level.

dasher

- 'dasher' started out life as a demonstration program to illustrate the process of arithmetic coding...

- a brand new way of writing:
  1. scratching squiggly shapes
  2. punching keys
  3. dasher

- http://en.wikipedia.org/wiki/Dasher - David MacKay
- For more on Arithmetic Coding see chapter 6 of David's (free) book

summary on Arithmetic Coding

- key insight is to make a *stream code*
- with a fixed partitioning, based on fixed symbol probabilities from a look-up table, we get to the Shannon limit for "random looking" text
- with partitioning based on dynamic symbol probabilities (via a learned *predictive model*) we get close to the entropy of the *strings in the language*, ie. the theoretical limit ☺