Technical Note: Benchmark CVA Application

December 2, 2018

Abstract

This technical note describes a benchmark portfolio for CVA methodology.

1 Review of CVA estimation

The calculation of these credit charges requires the knowledge of the uncertain futures exposures to each counterparty and this is complicated, in large part, by their dependence on the distributions of future product values. We review CVA estimation partly following [Capriotti2017] but also using a statistical technique known as 'proxy pricing' to recycle the least square estimates in the CVA.

For a given portfolio of contracts with a counterparty, the expected loss associated with the counterparty defaulting is given by the unilateral CVA

$$CVA = -\mathbb{E}[\mathbf{1}_{\tau \leq T} \frac{L}{N(\tau)} V^{+}(\tau)] \tag{1.1}$$

$V(t)$ is the net present value of the portfolio, L is the loss given default of the counterparty, $N(t)$ is the chosen numeraire at time t which can be taken to be a discount function and τ is the default time between the current time t_0 and the longest maturity in the portfolio.

CVA requires the conditional future exposure V(t) on a set of n dates determined by a discretisation time grid $t_0, t_1, \ldots, T$. The process $Y(t) \in \mathbb{R}^d$ consists of d risk factors and its evolution is approximated by a Euler-Maruyama scheme over $N = T/h$ steps

$$X^{(i)}(n+1) = F_n(X^{(i)}(n), \theta) \tag{1.2}$$

and X_n is the Euler approximation $X_n \approx Y(hn), n > 0$ and $X_n = Y(0)$. Note that X_n is evaluated over each path, but we have chosen to drop the path index for ease of notation. Each counterparty's credit risk is modeled exogenously under a stochastic default intensity model, which when approximated with an Euler scheme gives

$$\lambda^{(i)}(n+1) = G_n(\lambda^{(i)}(n), \theta_\lambda) \tag{1.3}$$

and default probabilities in period $[nh, (n+1)h)$ for the counterparty and the difference of the exponential survival probabilities

$$P(nh \leq \tau < (n+1)h) = \exp\{-\int_{s=0}^{nh} \lambda(s)ds\} - \exp\{-\int_{s=0}^{(n+1)h} \lambda(s)ds\} \tag{1.4}$$

which is approximated by the discrete time evolution of $\lambda(s)$

$$\Delta p_n := p_n - p_{n+1} := \exp\{-h\sum_{j=0}^{n-1} \lambda_j\} - \exp\{-h\sum_{j=o}^{n} \lambda_j\}. \tag{1.5}$$

1.1 LSMC with Proxy Instrument Pricing

The key idea is to try to find a functional expression that can approximate the value of the instruments[1] in the portfolio conditional on the value of the state variables $X_n \in \mathbb{R}^d$.

Basic setup: We assume that we have generated a set of forward simulated discretized paths i on a strictly increasing time grid. As these paths contain the information of the value of the regression variables as well as the realized cash flows, $c_n^{(i)}$ from the derivative, we are able to determine the instrument value through backward induction.

Letting $H_n^{(i)}$ denote the holding value of the instrument at time point t_n along path i we have

$$H_n^{(i)} = \frac{N_n^{(i)}}{N_{n+1}^{(i)}} \left(\nu_{n+1}^{(i)} + c_{n+1}^{(i)}\right), \tag{1.6}$$

where $N_n^{(i)}$ is just the numeraire along path i at time t_n. The second assumption we make in the Least Squares Monte Carlo proxy algorithm is that the continuation value

$$C_n(x) = \mathbb{E}_n[H_n(X_{n+1}) \mid X_n = x] := N_n^{(i)}\mathbb{E}_n[\frac{\nu_{n+1}}{N_{n+1}} + \frac{c_{n+1}}{N_{n+1}} \mid X_n = x],\ n = N-1, \ldots, 1. \tag{1.7}$$

under the risk-neutral measure can be approximated by a parametric function:

$$H_n = f_{\theta_n}(X_n). \tag{1.8}$$

[1] Note that an alternative is to form a functional expression for the entire portfolio, but this has severe limitations in accuracy.

so that the conditional expectation is

$$\hat{C}_n = \sum_{d=1}^{D} \hat{\theta}_{d,n} \cdot \psi_d(X_n), \tag{1.9}$$

The advantage of this method lies in using the realized discounted cash flows to represent the derivative value, making the method suitable for most derivative products.

The $H_n(x)$ are regressed against all the simulated risk factors X_n using D basis functions $\{\psi_d(x)\}$ where the optimal regression parameters $\hat{\theta}_n$ are the result of the minimization

$$\hat{\theta}_n = \underset{\theta_1,\ldots,\theta_D}{\operatorname{argmin}} \frac{1}{M} ||C_n - \sum_{d=1}^{D} \theta_{d,n} \cdot \psi_d(X_t)||_2^2, \tag{1.10}$$

and where M denotes the number of in-the-money paths of the option. For each callable option, there is one set of regression coefficients $\hat{\theta}_n$ across all paths at time steps.

By backward induction, the value of the option at time n is given by comparing the estimated continuation value, $\hat{C}_n$ with the exercise value E_n (i.e. the intrinsic value of the derivative)

$$V_n^{(i)}(X_n^{(i)}) = \begin{cases} \mathrm{E}_n^{(i)}, \; E_n^{(i)} > \hat{C}_n \\ \mathrm{N}_n^{(i)} V_{n+1}^{(i)} / N_{n+1}^{(i)}, \text{ otherwise.} \end{cases} \tag{1.11}$$

Once the coefficients have been estimated, the approximation Eq.1.9 can be used to give the future exposures. This entails another forward simulation to avoid excessive out-of-sample bias. For $n = N, \ldots, 2$, we apply backward induction and the steps above to give the $V_2, \ldots, V_{n-1}$ by reusing the stored coefficients. This approach is well suited to CVA, which requires simulation of market portfolios in time, and hence the value of each derivative in the portfolio at future times. Note that the approach is not identical to [Longstaff2001], which does not recycle the coefficients, in a separate simulation step, in order to estimate the future exposure values. The details of the LSMC proxy algorithm are given in Table 1 below.

Finally, the pathwise CVA is given by

$$CVA^{(i)} = -\sum_{n=1}^{N} L_n^{(i)} \Delta p_{n-1}^{(i)} \frac{V_n^{(i)+}}{N_n^{(i)}} \tag{1.12}$$

which is averaged over all paths to obtain an estimate of the expected portfolio loss associated with a counterparty defaulting.

2 Example Application

The benchmark xIBOR portfolio holds a number of Bermudan swaptions on different interest rate swaps (IRS). Bermudan means that the derivative is exercisable (equivalently: callable) at

Pre-simulation:	Simulate M_1 sample paths of the state variables $X_n^{(i)}$ and generate the realized cash flows $c_n^{(i)}$ along each path i and each time point t_n.
Backward induction:	Compute $H_n^{(i)}$. Regress $H_n^{(i)}$ on $X_n^{(i)}$ to obtain $\hat{\theta}_n$.
Main simulation:	Simulate M_2 sample paths of the state variables $X_n^{(i)}$. Evaluate $\hat{C}_n^{(i)}$ along each path i and time point t_n to obtain the approximation of the exposure $\mathrm{V}_n^{(i)+} := \max(V_n^{(i)}(X_n^{(i)}), 0)$.
CVA:	Evaluate Eq. 1.12 to estimate the contribution of the instrument to the portfolio CVA.

Table 1: A Summary of the LSMC proxy method for CVA estimation. In general, each step should be repeated for each instrument in the portfolio.

multiple discrete time points, usually separated by, e.g., annual or semi-annual periods. Exercise of a Bermudan derivative is a trade-off between taking the option gains now or holding onto the option at possibly more favorable option gains later. Inherently, values of Bermudan interest rate derivatives therefore depend on multiple interest rates

The reference rates for the IRSs include US LIBOR, EIBOR, CADLIBOR and many others. These are swaps whose floating legs are based on LIBOR. A Gaussian short rate model is used to simulate each underlying short rate, the associated discount bond prices are found and hence the swap is valued. Each swaption is then priced under defaultable cash flows.

Although unrealistic, we ignore the correlation between the short rates. For further simplicity, we assume a Poisson default model with a fixed hazard rate. We begin by calculating the net present values of each Bermudan swaption. For convenience, this pricing model has a closed form solution which is compared with the LMSC for accuracy in the price estimate. To calculate the expected future exposures of the portfolios, the procedure is repeated for each future date, where we have stepped forward by one month. Note that a finer resolution can be easily used.

A Generalized Least Squares

Dropping the t subscript, the least squares problem can be written in the canonical form as

$$\min_{\alpha} |A \cdot \alpha - \mathbf{b}|^2 \tag{A.13}$$

where the design matrix $A \in \mathbf{R}^I$ and $A_{ij} := \psi_j(X_{,i})$ and $\mathbf{b} \in \mathbb{R}^I$ and $b_i := C_i$. Robust solution of the least-squares problem uses the singular value decomposition

$$\alpha^* = \sum_{d=1}^{D} \left(\frac{\mathbf{U}_{(i)} \cdot \mathbf{b}}{\sigma_i} \right) \mathbf{V}_{(i)}, \tag{A.14}$$

where $\mathbf{U}_{(i)}, \mathbf{V}_{(i)}$ are the column vectors of the orthogonal matrices $U \in \mathbb{R}^I$ and $V \in \mathbb{R}^D$ such that $A = U^T$.

The weights σ_i are the singular values of A given by the principal diagonal of the diagonal matrix $\Sigma \in \mathbb{R}^I$. For American option pricing, I is typically large and D is small resulting in a tall-skinny design matrix A. Singular value decomposition is the computational bottleneck at each time step of the simulation.