Calibration of the Heston model using Maximum Likelihood Estimation

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February 5, 2017

This document briefly outlines the Heston model calibration approach implemented in the R package

`https://github.com/mfrdixon/MLEMVD`.

1 Heston Model

Under the pricing measure Q, the Heston model describes the evolution of the log of stock price $s_t = ln\ S_t$ whose variance Y_t is given by a mean reverting square root process:

$$\begin{aligned} ds_t &= (a + bY_t)dt + \sqrt{Y_t}dW_1Q(t), & (1)\\ dY_t &= \kappa'(\theta' - Y_t)dt + \sigma\sqrt{Y_t}dW_2^Q(t), & (2)\end{aligned}$$

where

$$a = r - d, \qquad b = -\frac{1}{2}, \tag{3}$$

A key characteristic of the model is that the Wiener processes are correlated $dW_1^Q \cdot dW_2^Q = \rho dt$. This feature enables the model to exhibit the 'leverage effect'.

To find the model parameters from option prices first requires adjustment of the model to account for market risk, so that under the objective pricing measure P

$$\begin{aligned} ds_t &= (a + bY_t)dt + \sqrt{Y_t}dW_1Q(t), & (4)\\ dY_t &= \kappa(\theta - Y_t)dt + \sigma\sqrt{Y_t}dW_2^Q(t), & (5)\end{aligned}$$

where

$$a = r - d, \qquad b = \lambda_1(1-\rho^2) + \lambda_2\rho - \frac{1}{2}, \qquad \kappa = \kappa' - \lambda_2\sigma, \qquad \theta = \left(\frac{\kappa + \lambda_2\sigma}{\kappa}\right)\theta'. \tag{6}$$

The parameter set $\mathbf{p} := [\kappa, \theta, \sigma, \rho, \lambda_1, \lambda_2]$ and the additional non-linear constraint (the Feller condition) $2\kappa\theta - \sigma^2 > 0$ is imposed during the calibration to ensure that Y_t is positive.

2 Likelihood function estimation

Given a set of observed underlying and ATM constant maturity option prices $g_t := [S_t; C_t]$ sampled at dates $t_0, t_1, \ldots, t_n$, the likelihood function takes the form:

$$l_n(\mathbf{p}) := \frac{1}{n}\sum_{i=1}^{n} l_G(\Delta t_i, g(t_i)|g(t_{i-1})); \mathbf{p}) \tag{7}$$

where

$$l_G(\Delta, g|g_0; \mathbf{p}) := ln f_G(\Delta, g|g_0; \mathbf{p}) = -ln J_t(\Delta, g|g_0; \mathbf{p}) + l_X(\Delta, f^{-1}(g; \mathbf{p})|f^{-1}(g_0; \mathbf{p}); \mathbf{p}) \tag{8}$$

and l_X denotes the likelihood function of the partially observed state vector $x_t := [ln\ S_t, Y_t]$ evaluated at each date $t_0, t_1, \ldots, t_n$. Here $\Delta t_i := t_i - t_{i-1}$ denotes the time step between observations. J_t denotes the Jacobian of the option price with respect to Y_t, which is equivalently to vega.

3 Calibration

Given a sequence of observed underlying and corresponding near expiry, constant maturity, ATM options, we follow the these steps

- Initialize the unknown parameter vector to the model
- For each new parameter set $\mathbf{p}$ generated by the numerical optimization routine, compute the value of Y_t which satisfies the option price. Note this requires solving a nested one dimensional convex optimization, with linear bound constraints, so that $C_t \to \hat{Y}_t$.
- With $\hat{Y}_t$ and $\mathbf{p}$ compute ν of the option and thus the Jacobian term in Equation 8.
- Using $\hat{x}_t = [ln\ S_t, \hat{Y}_t]$ and $\mathbf{p}$ compute $l_X(\Delta, f^{-1}(g; \mathbf{p}) | f^{-1}(g_0; \mathbf{p}); \mathbf{p})$
- Minimize the log likelihood l_G over the parameters subject to the Feller condition.